COMPLETE CCC BOOLEAN ALGEBRAS, THE ORDER SEQUENTIAL TOPOLOGY, AND A PROBLEM OF VON NEUMANN

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Abstract. Let B be a complete *ccc* Boolean algebra and let τ_s be the topology on B induced by the algebraic convergence of sequences in B.

1. Either there exists a Maharam submeasure on B or every nonempty open set in (B, τ_s) is topologically dense.
2. It is consistent that every weakly distributive complete *ccc* Boolean algebra carries a strictly positive Maharam submeasure.
3. The topological space (B, τ_s) is sequentially compact if and only if the generic extension by B does not add independent reals.

We also give examples of *ccc* forcings adding a real but not independent reals.

1. Introduction. We investigate combinatorial properties of complete *ccc* Boolean algebras (for basic definitions and facts about Boolean algebras see [9]). The focus is on properties related to the existence of a Maharam submeasure and on forcing properties. In particular, we address the question of when the forcing adds independent reals. The work is continuation of [3] and [2] and is related to the problems of von Neumann and Maharam.

The problem of von Neumann from The Scottish Book ([14], Problem 163) asks whether every weakly distributive complete *ccc* Boolean algebra carries a countably additive measure.

Von Neumann's problem can be divided into two distinctly different questions. Weak distributivity is a consequence of a property possibly weaker than measurability, namely the existence of a continuous strictly positive submeasure (a Maharam submeasure); the Control Measure Problem of [10] asks whether every complete Boolean algebra that carries a continuous submeasure must also carry a measure.

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For the exact formulation of this question see [5]. It should be noted that the Control Measure Problem is equivalent to a Π^1_2 statement and is therefore absolute for inner models and generic extensions.

The second question is the following modified von Neumann problem: does every weakly distributive *ccc* complete Boolean algebra carry a strictly positive Maharam submeasure? This statement is not provable in ZFC, as the algebra associated with a Suslin tree is a counterexample; moreover, a forcing notion constructed by Jensen under $V = L$ in [7] yields a counterexample that is countably generated. In Section 3 we show that it is consistent that the modified von Neumann problem holds.

D. Maharam [10] characterized algebras that carry a continuous submeasure as those on which the sequential topology τ_s is metrizable. In [3] this is improved to the condition that B is *ccc* and (B, τ_s) is a Hausdorff space. In Section 2 we prove the following

Decomposition Theorem. *Let B be a complete ccc Boolean algebra. Then there are disjoint elements $d, m \in B$ such that $d \vee m = \mathbf{1}$ and*

(i) *In the space $(B \restriction d, \tau_s)$ the closure of every nonempty open set is the whole space.*
(ii) *The Boolean algebra $B \restriction m$ carries a strictly positive Maharam submeasure.*

A sequence $\langle a_n : n \in \omega\rangle$ in a topological space *converges* to a point a ($\lim_n a_n = a$) if for every open neighborhood U of a, all but finitely many of the a_n's belong to U. If the space is Hausdorff then every sequence converges to at most one point (the *limit* of a_n). A topological space X is *Fréchet* if for every set $A \subset X$, every point in the closure of A is the limit of some sequence in A. A space is *sequentially compact* if every sequence has a convergent subsequence.

Let B be a complete Boolean algebra. B is *weakly distributive* (more exactly (ω, ω)-weakly distributive) if for every sequence $\langle P_n : n \in \omega\rangle$ of countable maximal antichains, there exists a maximal antichain Q with the property that each $q \in Q$ meets only finitely many elements of each P_n. Equivalently, B is weakly distributive if and only if in any generic extension $V[G]$ by B, every function $f : \omega \to \omega$ is bounded by some $g : \omega \to \omega$ in the ground model V (i.e. $f(n) \le g(n)$ for all $n \in \omega$).

B adds an independent real if there exists some $X \subset \omega$ in $V[G]$, the generic extension by B, such that neither X nor its complement has an infinite subset Y such that $Y \in V$. Note that both Cohen and random forcing add independent reals; the latter is weakly distributive while the former is not. Neither Sacks forcing [13] (weakly distributive) nor Miller forcing [11] (not weakly distributive) add independent reals. In Section 5 we present *ccc* variants of these forcings.

A sequence $\langle a_n : n \in \omega\rangle$ in B *converges* (algebraically) to $\mathbf{0}$ if there exists a decreasing sequence $b_0 \ge b_1 \ge \dots \ge b_n \ge \dots$ with $\bigwedge_{n\in\omega} b_n = \mathbf{0}$ such that $a_n \le b_n$ for all $n \in \omega$. A sequence $\langle a_n : n \in \omega\rangle$ converges to $a \in B$ ($\lim a_n = a$) if the sequence $\langle a_n \vartriangle a : n \in \omega\rangle$ of symmetric differences converges to $\mathbf{0}$. The *order sequential topology* τ_s on B is defined as follows: the closure of a set $A \subset B$ is the smallest set $\overline{A} \supset A$ with the property that the limit of every convergent sequence in $\overline{A}$ is in $\overline{A}$. The space (B, τ_s) is T_1 and every topologically convergent sequence has a unique limit. Moreover, a sequence $\langle a_n : n \in \omega\rangle$ in B converges to a topologically if and only if every subsequence of $\langle a_n : n \in \omega\rangle$ has a subsequence that converges to a algebraically.

In [3] it is proved that the space (B, τ_s) is Fréchet if and only if B is weakly distributive and satisfies the $\mathfrak{b}$-chain condition, where $\mathfrak{b}$ is the *bounding number*: the least cardinality of a family $\mathcal{F}$ of functions from ω to ω such that $\mathcal{F}$ is unbounded; i.e. for every $g : \omega \to \omega$ there is some $f \in \mathcal{F}$ such that $g(n) \leq f(n)$ for infinitely many n's.

A *submeasure* on B is a nonnegative real valued function μ such that

(i) $\mu(\mathbf{0}) = 0$
(ii) $\mu(a) \leq \mu(b)$ whenever $a \leq b$
(iii) $\mu(a \vee b) \leq \mu(a) + \mu(b)$

A *strictly positive submeasure* has $\mu(a) = 0$ only if $a = \mathbf{0}$. A *Maharam submeasure* is continuous; i.e. $\lim \mu(a_n) = 0$ for every decreasing sequence $\langle a_n : n \in \omega \rangle$ such that $\bigwedge_n a_n = \mathbf{0}$.

If B carries a strictly positive Maharam submeasure then B is *ccc* and weakly distributive, and if B is atomless then B adds an independent real.

By [10] and [3], the following properties are equivalent: B carries a strictly positive Maharam submeasure; (B, τ_s) is metrizable; (B, τ_s) is regular; B is *ccc* and (B, τ_s) is Hausdorff.

2. Decomposition theorem. We shall now use the results and techniques from [3] to prove the Decomposition theorem.

Theorem 2.1. *Let B be a complete ccc Boolean algebra. Then there are disjoint elements $d, m \in B$ such that $d \vee m = \mathbf{1}$ and*

(i) *In the space $(B \restriction d, \tau_s)$ the closure of every nonempty open set is the whole space.*
(ii) *The Boolean algebra $B \restriction m$ carries a strictly positive Maharam submeasure.*

The elements d and m are uniquely determined, and either can be $\mathbf{0}$. If $m \neq \mathbf{0}$ then B carries a nontrivial continuous submeasure while if $m = \mathbf{0}$ then every continuous real valued function on B is constant.

Proof. First we prove the theorem in the case when B is weakly distributive. Let B be a weakly distributive complete *ccc* Boolean algebra. By [3] the space (B, τ_s) is Fréchet. Let

$$\mathcal{N} = \{U : U \text{ is an open neighborhood of } \mathbf{0} \text{ and is downward closed}\}$$

where *downward closed* means that $a < b \in U$ implies $a \in U$. It is proved in ([3] Lemma 3.6) that if (B, τ_s) is Fréchet then $\mathcal{N}$ is a neighborhood base of $\mathbf{0}$, that $U \vee V = U \vartriangle V$ for $U, V \in \mathcal{N}$, and that the closure $cl(A)$ of each downward closed set A is $\bigcap\{A \vee U : U \in \mathcal{N}\}$ and is also downward closed.

Now let

$$D = \bigcap\{cl(U) : U \in \mathcal{N}\} \text{ and } d = \bigvee D.$$

D is both downward closed and topologically closed, and it follows from the remarks above that

$$D = \bigcap\{U \vee V : U, V \in \mathcal{N}\} = \bigcap\{U \vee U : U \in \mathcal{N}\}.$$

If $a \notin D$ then for some $U \in \mathcal{N}$, $a \notin U \vee U$ and hence U and $a \vartriangle U$ are disjoint; in other words, a is Hausdorff separated from $\mathbf{0}$. It follows that if we let $m = -d$, then the space $(B \restriction m, \tau_s)$ is a Hausdorff space. By [3], $B \restriction m$ carries a strictly positive Maharam submeasure. It remains to show that in $(B \restriction d, \tau_s)$, every nonempty open set is dense.

Lemma 2.2. *D is closed under $\vee$.*

Proof. Let $U \in \mathcal{N}$. By ([3] Lemma 4.11), there exists a $U_1 \in \mathcal{N}$ such that $U_1 \vee U_1 \vee U_1 \subset U \vee U$. Similarly, there exists a $U_2 \in \mathcal{N}$ such that $U_2 \vee U_2 \vee U_2 \subset U_1 \vee U_1$, and letting $V = U_1 \cap U_2$ we get $V \vee V \vee V \vee V \subset U \vee U$. Hence $D = \bigcap\{V \vee V \vee V \vee V : V \in \mathcal{N}\}$. It follows that $D \vee D = D$. □

Now let $\{a_n : n \in \omega\}$ be a maximal antichain in D. The sequence $\{\bigvee_{k=0}^{n} a_k : n \in \omega\}$ is in D and converges to d. Since D is closed, we have $d \in D$, and so $B \restriction d = D$.

For every $U \in \mathcal{N}$, $cl(U) \supset D$. Now let G be an arbitrary topologically open set in $B \restriction d$. There exist $a \in D$ and $U \in \mathcal{N}$ such that $G \supset (a \bigtriangleup U) \cap D$. Since $cl(a \bigtriangleup U) \supset a \bigtriangleup D = D$, we have $cl(G) \supset D$ and the theorem follows for the weakly distributive case.

In the general case, there exists an element $d_1 \in B$ such that $B \restriction -d_1$ is weakly distributive, and such that $B \restriction d_1$ is nowhere weakly distributive. There exists an infinite matrix $\{a_{kl}\}$ such that each row is a partition of d_1 and for every nonzero $x \le d_1$ there is some $k \in \omega$ such that $x \wedge a_{kl} \neq \mathbf{0}$ for infinitely many l.

Let d_2 and m, with $d_2 \vee m = -d_1$, be the decomposition of the weakly distributive algebra $B \restriction -d_1$, so that $B \restriction m$ carries a strictly positive Maharam submeasure and $(B \restriction d_2, \tau_s)$ has the property that every nonempty open set is dense in the space. Let $d = d_1 \vee d_2$, and let us prove that in $(B \restriction d, \tau_s)$, every nonempty open set is dense.

Let U be an open neighborhood of $\mathbf{0}$ in $B \restriction d$. The space $(B \restriction d_2, \tau_s)$ is a closed subspace of $(B \restriction d, \tau_s)$, and $V = U \cap B \restriction d_2$ is an open neighborhood of $\mathbf{0}$ in $B \restriction d_2$.

Let $c \le d$ be arbitrary; we shall prove that c is in the closure of U. Let $c_1 = c \wedge d_1$ and $c_2 = c \wedge d_2$. Since c_2 is in the closure of V and $B \restriction d_2$ is Fréchet, there exists a sequence $\langle z_n : n \in \omega\rangle$ in V that converges to c_2. We shall prove that $c_1 \vee z_n \in cl(U)$ for each $n \in \omega$, and then it follows that $c = \lim_n(c_1 \vee z_n)$ is in $cl(U)$.

Thus let $n \in \omega$ be fixed. For every k and every l let $y_{kl} = c_1 \wedge \bigvee_{i \ge l} a_{ki}$. Since the sequence $\langle y_{0l} : l \in \omega\rangle$ converges to $\mathbf{0}$, we have $\lim_l(y_{0l} \vee z_n) = z_n$, and since $z_n \in U$, there exists some l_0 such that $y_{0l_0} \vee z_n \in U$. Let $x_0 = y_{0l_0}$.

Next we consider the sequence $\langle y_{1l} \vee x_0 \vee z_n : l \in \omega\rangle$. This sequence converges to $x_0 \vee z_n \in U$ and so there exists some l_1 such that $x_1 \vee z_n \in U$ where $x_1 = y_{1l_1} \vee x_0$. We proceed by induction and obtain a sequence $\langle l_k : k \in \omega\rangle$ and an increasing sequence $\langle x_k : k \in \omega\rangle$ with $x_k \vee z_n \in U$ for each k. The sequence $\langle x_k : k \in \omega\rangle$ converges to c_1 because otherwise, if $b \neq \mathbf{0}$ is the complement of $\bigvee_k x_k$ in c, then $b \le \bigwedge_k \bigvee_{i<l_k} a_{ki}$ and so b meets only finitely many elements in each row of the matrix. Hence $c_1 \vee z_n = \lim_k(x_k \vee z_n) \in cl(U)$. □

Let us note that in the case $d = \mathbf{1}$ we have $U \vee U = B$ for every open neighborhood $U \in \mathcal{N}$.

3. Weak distributivity. If B is a complete *ccc* Boolean algebra then B is weakly distributive if and only if the space (B, τ_s) is Fréchet. In this Section we present yet another necessary and sufficient condition for weak distributivity.

Definition 3.1. [2] I_s is the collection of all sets $A \subset B$ such that A is either finite, or is the range of a sequence in B that converges to $\mathbf{0}$ (algebraically).

I_s is an ideal of sets, $I_s \subset \mathcal{P}(B)$. An ideal I of sets is a *P-ideal* if for any sequence $\langle A_n : n \in \omega\rangle$ of sets in I there exists a set $A \in I$ such that $A_n - A$ is finite for every $n \in \omega$. We shall prove the following equivalence which is implicit in [12]:

Theorem 3.2. (S. Quickert) *A complete ccc Boolean algebra B is weakly distributive if and only if the ideal I_s is a P-ideal.*

First we give a different description of I_s (this is the definition used by Quickert in [12]):

Lemma 3.3. *Let B be a complete ccc Boolean algebra. Then $A \in I_s$ if and only if there exists a maximal antichain W such that every $w \in W$ is incompatible with all but finitely many $a \in A$.*

Proof. First let $A \in I_s$, $A = \{a_n : n \in \omega\}$ where $\lim a_n = \mathbf{0}$. Let $b_n \geq a_n$ be such that $\langle b_n : n \in \omega\rangle$ is decreasing and $\bigwedge_n b_n = \mathbf{0}$. We may assume that $b_0 = \mathbf{1}$, and set $w_n = b_n - b_{n+1}$, for each $n \in \omega$. The set $W = \{w_n : n \in \omega\}$ is a maximal antichain and each w_n is incompatible with all a_k, $k \geq n+1$.

Conversely let A satisfy the condition of the lemma, with $W = \{w_n : n \in \omega\}$ an antichain that witnesses it. If A is infinite then it is necessarily countable, say $A = \{a_n : n \in \omega\}$. We claim that $\langle a_n : n \in \omega\rangle$ converges to $\mathbf{0}$. For each n, let $b_n = \bigvee_{k=n}^{\infty} w_k$; the sequence $\langle b_n : n \in \omega\rangle$ is decreasing and $\bigwedge_n b_n = \mathbf{0}$. From the condition on W it follows that there is an increasing sequence $k_0 < k_1 < \cdots < k_n \ldots$ such that for every $n \in \omega$, $b_n \geq a_k$ for all $k \geq k_n$. This implies that $\langle a_n : n \in \omega\rangle$ converges to $\mathbf{0}$. □

Proof of the theorem 3.2. First let us assume that B is weakly distributive, and let $\langle A_n : n \in \omega\rangle$ be a sequence of sets in I_s. For each $n \in \omega$, let W_n be a witness to $A_n \in I_s$. By weak distributivity there is a maximal antichain $W = \{w_n : n \in \omega\}$ such that for each n and each k, w_n meets only finitely many elements of W_k. It follows that for each n and each k, w_n meets only finitely many elements of A_k. Let

$$A = \{a : \exists n \text{ such that } \ a \in A_n \text{ and } a \text{ is incompatible with every } w_i,\ i \leq n\}.$$

For any given n, if $a \in A_n - A$ then a is compatible with some w_i, $i \leq n$; since for each i there are only finitely many such a, the set $A_n - A$ is finite. To complete the proof we show that W witnesses that $A \in I_s$: For each i, if w_i meets $a \in A$ then $a \notin A_n$ for all $n \geq i$ and hence $a \in A_k$ for some $k < i$. Therefore w_i meets only finitely many $a \in A$.

Conversely, assume that I_s is a P-ideal, and let $\langle W_n : n \in \omega\rangle$ be a sequence of maximal antichains. Since every maximal antichain is itself in I_s, there exists a set $A \in I_s$ such that $W_n - A$ is finite for every $n \in \omega$. Let W be a witness to $A \in I_s$. Each $w \in W$ meets only finitely many $a \in A$, and since $W_n - A$ is finite, w meets only finitely many elements of W_n, for each n. This proves that B is weakly distributive. □

The following principle was formulated by S. Todorčević:

P-Ideal Dichotomy (PID). *Let S be an infinite set. Then for every P-ideal $\mathcal{I} \subset [S]^{\leq\omega}$ either*

(i) $\exists Y \subset S$ *uncountable such that* $[Y]^{\leq\omega} \subset \mathcal{I}$, *or*
(ii) $\exists \{S_n : n \in \omega\}$ *such that* $\bigcup_n S_n = S$ *and* $\forall n \in \omega \quad \forall I \in \mathcal{I} \quad |S_n \cap I| < \omega$.

The principle PID follows from the Proper Forcing Axiom and is also consistent with GCH [15]. For related principles with many interesting applications, see [1].

Lemma 3.4. [12] *If B is ccc then there exists no uncountable $X \subset B$ such that $[X]^\omega \subset I_s$.*

Proof. Let $X \subset B^+$ be uncountable, where $B^+ = B - \{\mathbf{0}\}$. First we claim that there exists some $b \in B^+$ such that for every nonzero $a \leq b$ the set $X_a = \{x \in X : x \wedge a \neq \mathbf{0}\}$ is uncountable. To see this assume that for every $b \in B^+$ there is some nonzero $a \leq b$ such that the set X_a is at most countable; thus the set $D = \{a \in B^+ : X_a \text{ is at most countable}\}$ is dense in B. By *ccc* D has a countable subset $W \subset D$ such that $\bigvee W = \mathbf{1}$. Now $X = \bigcup_{a \in W} X_a$; a contradiction.

Let $b \in B^+$ be as in the claim. If $Y \subset X$ is countable than we can find a countable set $Z \subset (X - Y)$ such that $\bigvee Z \geq b$. This is because $\bigvee(X - Z) \geq b$ (by the claim), and so a countable $Z \subset (X - Y)$ with $\bigvee Z = \bigvee(X - Y)$ exists by *ccc*.

Now let $X_0 \in [X]^\omega$ be such that $\bigvee X_0 \geq b$ and by induction let $X_n \in [X - \bigcup_{i<n} X_i]^\omega$ such that $\bigvee X_n \geq b$. Clearly $\bigcup X_n \in [X]^\omega$; we claim that $\bigcup X_n \notin I_s$. Otherwise by lemma 3.3 there is an antichain W such that every $w \in W$ is incompatible with all but finitely many $x \in \bigcup X_n$. To obtain a contradiction it is enough to choose some $w \in W$ such that $w \wedge b \neq \mathbf{0}$. □

Corollary 3.5. *Let B be weakly distributive, ccc complete Boolean algebra. PID implies that every singleton is a G_δ set in (B, τ_s).*

Proof. It is enough to show that $\{\mathbf{0}\}$ is a G_δ set in (B, τ_s). Assuming PID for I_s, it follows that $B = \bigcup_{n=0}^\infty S_n$ with each S_n meeting only finitely many elements of each $A \in I_s$. It follows that $\mathbf{0}$ is not in the closure of $S_n - \{\mathbf{0}\}$ for any n. Let $U_n = B - \mathrm{cl}(S_n - \{\mathbf{0}\})$. Each U_n is an open neighborhood of $\mathbf{0}$ and $\bigcap_{n\in\omega} U_n = \{\mathbf{0}\}$, hence $\{\mathbf{0}\}$ is G_δ in (B, τ_s). □

Theorem 3.6. *Assuming PID, every weakly distributive ccc complete Boolean algebra carries a strictly positive Maharam submeasure.*

Proof. Let $m, d \in B$ be given by the decomposition theorem. If $m = \mathbf{1}$ the space (B, τ_s) is completely metrizable. Suppose now that $d > \mathbf{0}$. By the corollary 3.5 there is a family $\{U_n : n \in \omega\}$ of open neighborhoods of $\mathbf{0}$ such that $\bigcap_{n\in\omega} U_n = \{\mathbf{0}\}$. We may assume that $U_{n+1} \subset U_n$, and since B is weakly distributive, the space is Fréchet and we may assume that each U_n is downward closed. By the decomposition theorem, d is in the closure of every nonempty open set, and since the space is Fréchet, there exists for each n a sequence $\{a_k^n\}_k$ in U_n that converges to d. By weak distributivity there exists a function $k(n)$ such that the sequence $\{b_n\}_n = \{a_{k(n)}^n\}_n$ converges to d. Since $d > \mathbf{0}$ there exists a $c > \mathbf{0}$ such that $b_n \geq c$ for eventually all n, say all $n \geq n_0$. Since each U_n is downward closed and $b_n \in U_n$, it follows that $c \in U_n$ for all $n \geq n_0$, a contradiction. □

Let us say that B has the *G_δ-property* if $\{\mathbf{0}\}$ is a G_δ-set in (B, τ_s). In the proof of theorem 3.6 we applied PID by using the G_δ-property. Thus we proved the following equivalence in ZFC.

Theorem 3.7. *Let B be a complete Boolean algebra. Then B carries a strictly positive Maharam submeasure if and only if*

(i) *B is weakly distributive, and*
(ii) *B has the G_δ property.* □

(Note that the G_δ-property implies *ccc*.) The Cohen algebra (for adding any number of Cohen reals), which is not weakly distributive, also has the G_δ-property.

4. Independent reals. We shall now give a necessary and sufficient condition for a complete *ccc* Boolean algebra to add independent reals.

Theorem 4.1. *Let B be a complete ccc Boolean algebra. B does not add independent reals if and only if (B, τ_s) is sequentially compact.*

Proof. We identify infinite sequences $\langle a_n : n \in \omega \rangle$ in B with Boolean names for subsets of ω, namely $a_n = \| n \in \dot{A} \|$

First let $\langle a_n : n \in \omega \rangle$ be a name for an independent real (without loss of generality we assume that it is independent with Boolean value $\mathbf{1}$). We shall prove that $\langle a_n : n \in \omega \rangle$ has no topologically convergent subsequence. Toward a contradiction, assume that it does. Then it has an algebraically convergent subsequence and since an independent real intersected with an infinite ground model set is independent on that set, we may as well assume that $\langle a_n : n \in \omega \rangle$ itself is convergent. Let a be the limit of $\langle a_n : n \in \omega \rangle$ and let $b_n = a_n \triangle a$, for each $n \in \omega$.

The sequence $\langle b_n : n \in \omega \rangle$ converges to $\mathbf{0}$. We claim that $\langle b_n : n \in \omega \rangle$ is a name for a finite set. To see this, let $\langle \tilde{b}_n : n \in \omega \rangle$ be a decreasing sequence with $b_n \leq \tilde{b}_n$ and $\bigwedge_n \tilde{b}_n = \mathbf{0}$. If G is a generic filter on B, then only finitely many $\tilde{b}_n$'s can be in G, hence $\langle \tilde{b}_n : n \in \omega \rangle$ is a name for a finite set, and so is $\langle b_n : n \in \omega \rangle$.

Now the constant sequence $\langle a : n \in \omega \rangle$ is a name for either ω (with Boolean value a) or $\emptyset$ (with value $-a$), and since $a_n = a \triangle b_n$, the B-valued real $\langle a_n : n \in \omega \rangle$ is the symmetric difference of either ω or $\emptyset$ and a finite set. Hence the real $\langle a_n : n \in \omega \rangle$ is either finite or cofinite, and hence not independent.

Conversely, let $\langle a_n : n \in \omega \rangle$ be a sequence that has no convergent subsequence. We shall produce a name for an independent real (or rather independent with nonzero Boolean value).

First we claim that $\langle a_n : n \in \omega \rangle$ has a subsequence $\langle c_n : n \in \omega \rangle$ with the property that $\limsup x_n = \limsup c_n$ for every subsequence $\langle x_n : n \in \omega \rangle$ of $\langle c_n : n \in \omega \rangle$. This is proved as follows: let $a_n^0 = a_n$. Suppose that $\langle a_n^0 : n \in \omega \rangle$ has a subsequence $\langle a_n^1 : n \in \omega \rangle$ with $\limsup a_n^0 > \limsup a_n^1$, which has a subsequence $\langle a_n^2 : n \in \omega \rangle$ with $\limsup a_n^1 > \limsup a_n^2$ and so on. At limit stages we produce a subsequence by diagonalization. Since B satisfies *ccc*, the process stops after countably many steps, and we obtain $\langle c_n : n \in \omega \rangle$ with the desired property.

We repeat this argument for $\liminf$, and so we may assume that $\liminf x_n = \liminf c_n$ for every subsequence $\langle x_n : n \in \omega \rangle$. Since $\langle c_n : n \in \omega \rangle$ is not convergent we have $\liminf c_n < \limsup c_n$. Let $c = \liminf c_n$ and let $b_n = c_n - c$, for each $n \in \omega$. We have $\liminf x_n = \mathbf{0}$ and $\limsup x_n = u > \mathbf{0}$ for every subsequence $\langle x_n : n \in \omega \rangle$ of $\langle b_n : n \in \omega \rangle$.

It follows that for every d with $\mathbf{0} < d \leq u$, $d \wedge b_n \neq \mathbf{0}$ and $d - b_n \neq \mathbf{0}$ for all but finitely many n: otherwise, there is a subsequence $\langle x_n : n \in \omega \rangle$ such that either $x_n \geq d$ for all n or $x_n \leq -d$ for all n, contradicting $\liminf x_n = \mathbf{0}$ and $\limsup x_n = u$. We claim that u forces that $\langle b_n : n \in \omega \rangle$ is an independent real. Let G be a generic on B with $u \in G$, and let $X = \{n : b_n \in G\}$. If X is not independent then there exists an infinite $A \subset \omega$ (in V) such that either $A \subset X$ or $A \cap X = \emptyset$. In the former case let $d = u \wedge \bigwedge_{n \in A} b_n$; in the latter, let $d = \bigwedge_{n \in A} u - b_n$. In either case, $d \in G$ and so $\mathbf{0} < d \leq u$ and either $d \leq b_n$ or $d \leq -b_n$ for infinitely many n, a contradiction. □

Remarks. (i) In [2] there is another, algebraic, characterization of Boolean algebras B that do not add independent reals: the existence of an *almost regular* embedding of the Cantor algebra (i.e. the countable atomless Boolean algebra) into B.

(ii) In sequential T_1 spaces, sequential compactness is equivalent to countable compactness (every countable open cover has a finite subcover). Therefore a complete *ccc* Boolean algebra B does not adds independent reals if and only if (B, τ_s) is countably compact.

(iii) If $B = \mathcal{P}(\omega)$ then (B, τ_s) is a compact space. We don't know whether there exists an atomless Boolean algebra B such that (B, τ_s) is compact. Note that B has to be *ccc*.

5. Examples. We present three examples of complete *ccc* Boolean algebras that do not add independent reals. All three examples are only consistent, not in ZFC.

5.1 The first example is a complete Boolean algebra $B = \mathcal{P}(\kappa)/I$ where I is certain σ-saturated ideal on κ. The example is due to Główczyński who showed in [6] that B is weakly distributive, countably generated and does not carry a strictly positive Maharam submeasure. We show that B does not add independent reals.

We use the known properties of the sequential topology on the power set algebra $\mathcal{P}(\kappa)$, cf. [3]:

(i) $(\mathcal{P}(\kappa), \tau_s)$ is Hausdorff
(ii) $(\mathcal{P}(\kappa), \tau_s)$ is regular if and only if $\kappa = \omega$
(iii) $(\mathcal{P}(\kappa), \tau_s)$ is Fréchet if and only if $\kappa < \mathfrak{b}$
(iv) $(\mathcal{P}(\kappa), \tau_s)$ is sequentially compact if and only if $\kappa < \mathfrak{s}$ ($\mathfrak{s}$ is the splitting number)

If I is σ-ideal on $\mathcal{P}(\kappa)$ then it is a closed subset in $(\mathcal{P}(\kappa), \tau_s)$ and the sequential topology on $B = \mathcal{P}(\kappa)/I$ is the quotient topology of τ_s. Hence if $(\mathcal{P}(\kappa), \tau_s)$ is Fréchet (or sequentially compact) then so is (B, τ_s).

Now let κ be a measurable cardinal and let $V[G]$ be a generic extension of V by *ccc* forcing in which Martin's Axiom holds and $2^{\aleph_0} > \kappa$. The measure on κ in V generates a nonprincipal σ-saturated σ-ideal I on κ in $V[G]$. Let $B = \mathcal{P}(\kappa)/I$. B is an atomless complete *ccc* Boolean algebra, and since $\mathfrak{b} = \mathfrak{s} = 2^{\aleph_0} > \kappa$ (from MA) it follows that (B, τ_s) is Fréchet and sequentially compact. Hence B is weakly distributive and does not add independent reals.

This example illustrates that in theorem 3.6 PID cannot be replaced by MA.

5.2. The second example is Jensen's forcing [7] that produces a minimal nonconstructible real. The corresponding Boolean algebra (constructed in L) is *ccc* and weakly distributive. A slight modification of Jensen's construction guarantees that the forcing does not add independent reals.

We proceed under the assumption of $V = L$, and assume that $\{S_\alpha : \alpha < \omega_1\}$ is a diamond sequence, namely such that for every $A \subset L_{\omega_1}$, the set $\{\alpha < \omega_1 : A \cap L_\alpha = S_\alpha\}$ is stationary.

Let $Seq = \{0,1\}^{<\omega}$ be the set of all finite $0-1$ sequences. We shall construct a forcing notion P consisting of perfect trees $T \subset Seq$; the ordering of P is by inclusion. P will be the union of a continuous ω_1-sequence of countable sets

$$P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \dots \quad \alpha < \omega_1$$

where every P_α is closed under taking restrictions $T \restriction s = \{t \in T : t \subset s \text{ or } t \subset s\}$ (where $s \in T$). Let $P_0 = \{T_0 \restriction s : s \in Seq\}$ where $T_0 = Seq$. At limit stages, $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$.

We now describe the construction of $P_{\alpha+1}$ from P_α. Let $\mathcal{X}_\alpha$ be the set of all X such that for some $\beta \le \alpha$, $X = S_\beta$ and is a predense set in P_β, along with all $Q_\beta = P_{\beta+1} - P_\beta$ for $\beta < \alpha$. The inductive condition is that each $X \in \mathcal{X}_\alpha$ is predense in P_α. (This inductive condition remains true at the limit stages.) Enumerate the countable set $\mathcal{X}_\alpha$ so that each X occurs infinitely often in the enumeration: $\mathcal{X}_\alpha = \{X_n^\alpha : n \in \omega\}$. For each $p \in P_\alpha$ we construct a perfect tree $T = T(\alpha, p) \subset p$ and then let $Q_\alpha = \{T(\alpha,p) \restriction s : p \in P_\alpha,\ s \in T(\alpha,p)\}$ and $P_{\alpha+1} = P_\alpha \cup Q_\alpha$. The tree T will be the fusion of a collection $\{p_\sigma : \sigma \in Seq\}$ where each p_σ is in P_α and:

(i) $p_\emptyset \subset p$,
(ii) $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ are both stronger then p_σ and have incompatible stems,
(iii) $T = \bigcap_{n\in\omega} \bigcup_{|\sigma|=n} p_\sigma$.

If it is not the case that S_α is a P_α-name for a subset of ω, we let $p_\emptyset = p$, and for each $\sigma \in Seq$, if $|\sigma| = n$, find $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ in P_α that satisfy (ii) such that

(iv) both $p_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1}$ are stronger than some $x \in X_n^\alpha$.

If $S_\alpha = \dot{A}$ is a P_α name for a subset of ω and some $q \subset p$ forces infinitely many n into $\dot{A}$, we let $p_\emptyset = q$ and again find p_σ, $\sigma \in Seq$, that satisfy (ii) and (iv).

Thus assume

$$\forall q \subset p\ \exists k\ \forall n \ge k\ \exists r \subset q\ r \Vdash n \notin \dot{A}$$

[We wish to point out that q and r ranges over P_α, and the forcing relation $\Vdash$ refers to the forcing P_α.]

At stage n, first find for each σ with $|\sigma| = n$ conditions $\bar{p}_{\sigma^\frown 0}$ and $\bar{p}_{\sigma^\frown 1}$ that satisfy (ii) and (iv), and then find some $a_n \in \omega$ sufficiently large so that $a_n > a_{n-1}$ and for each σ with $|\sigma| = n$ there exist conditions $p_{\sigma^\frown 0} \subset \bar{p}_{\sigma^\frown 0}$ and $p_{\sigma^\frown 1} \subset \bar{p}_{\sigma^\frown 1}$ that all force $a_n \notin \dot{A}$. Then let $Y_p^\alpha = \{a_n : n \in \omega\}$, let T be the fusion of $\{p_\sigma : \sigma \in seq\}$, and finally, $Q_\alpha = \{T(\alpha,p) \restriction s : p \in P_\alpha,\ s \in T(\alpha,p)\}$.

We claim that each $X \in \mathcal{X}_\alpha$ is predense in $P_{\alpha+1} = P_\alpha \cup S_\alpha$: Let $q \in P_{\alpha+1}$. As X is predense in P_α, we may assume that $q \in Q_\alpha$, $q = T(\alpha,p) \restriction s$.

Let $\{p_\sigma : \sigma \in Seq\}$ be the fusion collection for $T(\alpha,p)$, and let $n > |s|$ be such that $X = X_n^\alpha$. There exists some σ with $|\sigma| = n+1$ such that $t \supset s$ where t is the stem of p_σ, and by (iv), p_σ is stronger then some $x \in X$. Hence $q \restriction t < p_\sigma < x$, and so q and x are compatible in $P_{\alpha+1}$.

It follows that every $X \in \mathcal{X}_\alpha$ is predense in every P_β, $\beta \ge \alpha$, and therefore in P.

Now it follows that P satisfies *ccc*: Let X be a maximal antichain in P. For a closed unbounded set of α's, $X \cap \alpha$ is a maximal antichain in P_α. Therefore there exists an α such that $X \cap \alpha = S_\alpha$ is a maximal antichain in P_α and hence $X \cap \alpha \in \mathcal{X}_\alpha$. Thus $X \cap \alpha$ is predense, and hence a maximal antichain, in P, and so $X = X \cap \alpha$ and X is countable.

It is well known that forcing with perfect trees does not add unbounded reals and so P is weakly distributive. We shall now prove that P does not add independent reals. Let $\dot{A}$ be a name for a subset of ω, and let p be a condition. We prove that there exists a stronger condition q and some infinite set $Y \subset \omega$ such that either $q \Vdash Y \subset \dot{A}$ or $q \Vdash Y \cap \dot{A} = \emptyset$. Thus assume that there is no $q \subset p$ that forces infinitely many n into $\dot{A}$.

For each n, let X_n be a maximal antichain whose members all decide $n \in \dot{A}$, and let γ be large enough so that $X_n \subset P_\gamma$ for each n. Note that for every $\alpha \ge \gamma$

and every $q \in P_\alpha$, $q \Vdash n \in \dot{A}$ has the same meaning in P_γ as in P. So let $\alpha \geq \gamma$ be such that $\dot{A} = S_\alpha$. Let $\{p_\sigma : \sigma \in Seq\}$ be the fusion collection for $T(\alpha, p)$. As there is no $q \subset p$ in P_α that forces infinitely many n into $\dot{A}$, there exists an infinite set $Y = Y_p^\alpha = \{a_n : n \in \omega\}$ such that for every n and every σ with $|\sigma| = n+1$, $p_\sigma \Vdash a_n \notin \dot{A}$. It follows that the condition $T(\alpha, p)$ forces $Y \cap \dot{A} = \emptyset$. Hence P does not add independent reals.

5.3. The third example is a complete *ccc* Boolean algebra that is not weakly distributive and does not add independent reals. Again, we work under the assumption of $V = L$. While the previous example is a *ccc* version of Sacks forcing [13], this example is a *ccc* variant of Miller's forcing [11] with superperfect trees. We show that Miller's argument for the absence of independent reals can be used in the context of this Jensen - style construction.

Let $\{S_\alpha : \alpha \in \omega_1\}$ be a diamond sequence for L_{ω_1}, and let $Seq = \omega^{<\omega}$. Forcing conditions will be superperfect trees $T \subset Seq$ and P will be constructed via a continuous sequence

$$P_0 \subset P_1 \subset \cdots \subset P_\alpha \subset \ldots, \quad \alpha \in \omega_1$$

of countable sets closed under restrictions, with $P_0 = \{Seq \upharpoonright s : s \in Seq\}$ and $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ for limit α's.

At stage α of the construction, let $\mathcal{X}_\alpha$ be the set of all S_β, $\beta \leq \alpha$, that are predense in P_β and all $Q_\beta = P_{\beta+1} - P_\beta$, $\beta < \alpha$. By induction, each $X \in \mathcal{X}_\alpha$ is predense in P_α. Enumerate $\mathcal{X}_\alpha$ so that each X occurs infinitely often, $\mathcal{X}_\alpha = \{X_n^\alpha : n \in \omega\}$. For each $p \in P_\alpha$ we construct a superperfect tree $T(\alpha, p) \subset p$ and let $Q_\alpha = \{T(\alpha, p) \upharpoonright s : p \in P_\alpha,\ s \in T(\alpha, p)\}$, and $P_{\alpha+1} = P_\alpha \cup Q_\alpha$.

Assume that $S_\alpha = \dot{A}$ is a name for a subset of ω; along with $T(\alpha, p)$ we construct an infinite set $Y_p^\alpha = \{a_n : n \in \omega\}$ such that (under the right circumstances) the condition $T(\alpha, p)$ will force (in P) either $Y_p^\alpha \subset \dot{A}$ or $Y_p^\alpha \cap \dot{A} = \emptyset$. At the same time, $T(\alpha, p)$ will be compatible with every $X \in \mathcal{X}_\alpha$, to guarantee that X remains predense in $P_{\alpha+1}$. [If S_α is not a name for a subset of ω then we only handle second requirement at stage α.]

We recall that $s \in T$ is a *splitting node* if $s^\frown k \in T$ for infinitely many k, and that s is an n^{th} splitting node if moreover $|\{t : t \subset s \text{ is a splitting node}\}| = n$.

Step 1. Let U be a nonprincipal ultrafilter on ω. We construct a superperfect tree $T \subset p$, and for each splitting node $s \in T$ a set $A_s \subset \omega$, and for each successor $s^\frown k$ of s in T a condition $p_{s^\frown k} \in P_\alpha$ such that $s^\frown k \subset \text{stem}(p_{s^\frown k})$ and

(1) $T \upharpoonright (s^\frown k) \subset p_s$,
(2) if s is an n^{th} splitting node then $p_{s^\frown k} \subset x$ for some $x \in X_n^\alpha$,
(3) either $\forall s\ A_s \in U$ or $\forall s\ -A_s \in U$,
(4) $\forall m\ \exists k_0\ \forall k \geq k_0$ if $s^\frown k \in T$ then $p_{s^\frown k} \Vdash \dot{A} \cap m = A_s \cap m$.

To construct T, let $T_0 = p$ and let $s = \text{stem}(p)$ be the 1^{st} splitting node of T_0. For each successor $s^\frown k$ of s in T_0, find $p_{s^\frown k}$ with $s^\frown k \subset \text{stem}(p_{s^\frown k})$ such that (2) holds for X_0^α and such that $p_{s^\frown k}$ decides $\dot{A} \cap k$. Then thin out the successor $s^\frown k$ successively so that $0 \in \dot{A}$ is decided the same way by all, $1 \in \dot{A}$ is decided the same way by all starting with the second one, $2 \in \dot{A}$ by all starting with the third one etc. When finished, let $A_s = \{m :$ eventually all $p_{s^\frown k}$ force $m \in \dot{A}\}$, and let $T_1 = \bigcup\{p_{s^\frown k} : s^\frown k \text{ are the retained successors of } s\}$.

Next consider all 2^{nd} splitting nodes s of T_1 and repeat the construction of T_1 from T_0, using X_1^α. Repeating this ω times, we get trees T_n, $n \in \omega$, and let $\tilde{T} = \bigcap_{n=0}^\infty T_n$; $\tilde{T}$ is a superperfect tree. Let $\tilde{T} = C_1 \cup C_2$ where $C_1 = \{s : A_s \in U\}$ and $C_2 = \{s : -A_s \in U\}$. Either C_1 or C_2 contains a superperfect tree T. The tree T satisfies (1) - (4).

Step 2. Assume that $\forall s\ A_s \in U$ (the argument is similar in the opposite case). Let $T_0 = T$, $F_0 = \{s_0\}$ where $s_0 = \text{stem}(T_0)$ and let $a_0 \in A_s$. All but finitely many successors $s^\frown k$ in T_0 have the property that $p_{s^\frown k} \Vdash a_0 \in \dot{A}$, and so remove the finitely many successors, resulting in a tree $T_1 \subset T_0$ with stem s_0.

Let $F_1 \subset T_1$ be the $\{s_0, s_1\}$ where s_1 is the leftmost 2^{nd} splitting node of T_1, and let $a_1 > a_0$ be such that $a_1 \in A_{s_0} \cap A_{s_1}$. (The set is nonempty because in U.) Remove finitely many successors of s_1 so that for all the remaining ones, $p_{s_1^\frown k} \Vdash a_1 \in \dot{A}$. Also remove finitely many successors of s_0, with the same result. (There is no problem with the successors of s_0 below s_1, because every node above it is either below or above s_1.) The resulting T_2 is such that $F_1 \subset T_2 \subset T_1$.

Now let $F_2 \subset T_2$ be the finite set $F_2 \supset F_1$ that is obtained by adding to F_1 the leftmost 3^{rd} splitting node s_2 above s_1 and the second leftmost 2^{nd} splitting node s_3 (above s_0). Let $a_2 > a_1$ be such that $a_2 \in \bigcup_{s \in F} A_s$. Remove finitely many successors of s_3, finitely many successors of s_2, finitely many successors of s_1 and finitely many successors of s_0 (in that order) so that all the remaining $p_{s_i^\frown k}$ force $a_2 \in \dot{A}$. This produces T_3, and $F_2 \subset T_3 \subset T_2$. We continue in this fashion, and let $T(\alpha, p) = \bigcap_{n=0}^\infty T_n$, $Y_p^\alpha = \{a_n : n \in \omega\}$. This completes the construction of $P_{\alpha+1}$.

As in the second example, the forcing P satisfies *ccc* because every $X \in \mathcal{X}_\alpha$ is predense in every P_β, $\beta \geq \alpha$. The conditions $Seq \restriction s$ witness that the generic function $f : \omega \to \omega$ is unbounded and therefore the forcing P is not weakly distributive.

If $\dot{A}$ is a name for a subset of ω and $p \in P$, then for some sufficiently large α, $\dot{A} = S_\alpha$ and $q \Vdash n \in \dot{A}$ has the same meaning in P_α as in P. The construction of $T(\alpha, p)$ and Y_p^α yields that the condition $T(\alpha, p)$ either forces $Y_p^\alpha \subset \dot{A}$ or forces $Y_p^\alpha \cap \dot{A} = \emptyset$.

6. Additional comments (September 30, 2004).

After we posted our paper on the e-Print Archive [www.arxiv.org/abs/math.LO/0312473 (December 28, 2003)] we learned about several results related to our work, obtained independently by others.

In [4], I. Farah and J. Zapletal studied weakly distributive *ccc* forcings that are "suitably definable". Among others, they proved that every such Boolean algebra carries a strictly positive Maharam submeasure. Their result (obtained by methods different from ours) follows from our proof: the P-Ideal Dichotomy for "suitably definable" ideals is provable in ZFC (this has been confirmed by S. Todorcevic).

In December 2003, B. Velickovic presented two lectures at CRM in Barcelona [16] where he used methods similar to ours to show that under PID every weakly distributive *ccc* atomless Boolean algebra adds independent reals. This result also follows from our theorem, as every atomless algebra that carries a strictly positive Maharam submeasure adds independent reals. We stated this fact in the introduction without a proof and so we include the proof here for the sake of completeness. This was also proved in [16].

Lemma 6.1. *If B carries a strictly positive Maharam submeasure then B adds an independent real.*

Proof. Let μ be a strictly positive Maharam submeasure on B.

Case 1. The submeasure μ is uniformly exhaustive. By [8], B is a measure algebra which is known to add independent reals.

Case 2. There exists an $\varepsilon > 0$ and a sequence $\langle P_n : n \in \omega \rangle$ of finite antichains with $|P_n| \geq n$, and $\mu(a) \geq \varepsilon$ for each $a \in P_n$. We can find infinitely many functions f_k, $k \in \omega$, such that $f_k(n) \in P_n$ for every $n \in \omega$ and when $k \neq l$ then $f_k(n) \neq f_l(n)$ for eventually all n. Since B does not add independent reals, by theorem 4.1 τ_s is sequentially compact and so we can find convergent subsequences g_k of f_k, with $\mathrm{dom}(g_{k+1}) \subset \mathrm{dom}(g_k)$. If $a_k = \lim_n g_k(n)$ then the a_k's are mutually disjoint, and $\mu(a_k) \geq \varepsilon$ for each $k \in \omega$ (by continuity of μ). But $\lim_k \mu(a_k)$ should be zero, by continuity; a contradiction.

□

We would like to add the following consequence of the Decomposition theorem.

Lemma 6.2. *Let B be a complete ccc atomless Boolean algebra. If B has the G_δ property then B adds independent reals.*

Proof. Let m and d be as in theorem 2.1. If $m > \mathbf{0}$ then $B \restriction m$ adds independent reals by lemma 6.1. Let $d = \mathbf{1}$ and suppose that B does not add independent reals, i.e. (B, τ_s) is sequentially compact. We first claim that $\mathcal{N} = \{U : U$ is an open neighborhood of $\mathbf{0}$ and is downward closed$\}$ is a neighborhood base of $\mathbf{0}$: Let V be an open neighborhood of $\mathbf{0}$. The set $X_V = \{a \in B : \exists b \leq a \quad b \notin V\}$ is upward closed and contains $B - V$. We claim that X_V is topologically closed: Let $a = \lim_n a_n$, with $a_n \in X_V$. For each n let $b_n \in B - V$ be such that $b_n \leq a_n$. Let b be the limit of some convergent subsequence of $\langle b_n : n \in \omega \rangle$. We have $b \in B - V$ and $b \leq a$, hence $a \in X_V$.

Let $\{\mathbf{0}\} = \bigcap_{n\in\omega} V_n$, where $V_{n+1} \subset V_n$ and each V_n is a downward closed, open set. Clearly $-V_n = \{-v : v \in V_n\}$ is an open neighborhood of $\mathbf{1}$ and an upward closed set. By theorem 2.1(i) $V_n \cap -V_n$ is nonempty so let $a_n \in V_n \cap -V_n$. The sequence $\langle a_n : n \in \omega \rangle$ has no convergent subsequence: Let $I \in [\omega]^\omega$ then $\bigwedge_{i\in I} a_i \in V_n$ for any $n \in \omega$; hence $\bigwedge_{i\in I} a_i = \mathbf{0}$ and similarly $\bigvee_{i\in I} a_i = \mathbf{1}$. □

Corollary 6.3. *Suppose B is an infinite, ccc complete Boolean algebra and B has the G_δ property. Then B does not add independent reals if and only if B is isomorphic to $\mathcal{P}(\omega)$.* □

As a final comment we wish to mention that in June 2004, S. Todorcevic obtained the following remarkable improvement of our theorem 3.7. A Boolean algebra B satisfies the *σ-finite cc* if $B = \bigcup_{n\in\omega} A_n$ where for every n, every antichain in A_n is finite. (Note that every B with the G_δ-property has the σ-finite *cc*).

Theorem 6.4. (S. Todorcevic) *Let B be a complete Boolean algebra. Then B carries a strictly positive Maharam submeasure if and only if*

(i) *B is weakly distributive, and*
(ii) *B satisfies the σ-finite cc.*

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