THE DISCRETE CHARM OF NON-STANDARDNESS

Kateřina Trlifajová

The absence of the theory of real numbers was one of the main barriers to a development of modern mathematics until the late nineteenth century. Although studies in analysis, differential geometry and algebra, all of which utilized real numbers, were quite advanced, mathematicians often operated on the basis of their intuitive understanding.

There were at least three main reasons for creating a rigorous foundation of the reals.

1. Arithmetization of real numbers means understanding the continuum. Reals represent in a way a bridge between discreteness and continuity, numbers and space, arithmetic and geometry. They express in a numerical way what points, lines and space are.

2. The lacking theory of reals would have to serve as an exact foundation of the calculus. For more than two hundred and fifty years this powerful instrument had been used. Many new branches stemmed from this source: a theory of differential equations, differential geometry, a calculus on variation, functions of complex variables and many others. Mathematicians did not have a correct mathematical basis for their investigation, they were partially guided by an intuitive and physical insight. Their insight was based on the original Leibniz and Newton calculus as a computation with ideal infinitely small entities, inifinitesimals. It was necessary to compute carefully, there was no exact theory of infinitesimals or exhaustive rules how to deal with them. Their investigation was thus accompanied by errors or confusion of the creative process.

3. The creation of non-Euclidean geometry caused that geometry lost its status of an absolute truth. It still seemed that mathematics based on the ordinary arithmetic must correspond to reality. For instance, Gauss distinguished arithmetics from geometry in that only the former was purely *a priori,* only laws of arithmetics were necessary and true. The foundation of the general number system would avoid any doubts about the truth of arithmetics.

The fulfilment of the third requirement appeared to be impossible in the twentieth century. It was Gődel who published a major and dismaying discovery in 1931. Gődel's famous Incompleteness Theorem demonstrated that in any system rich enough to contain the formal elementary arithmetic there were always theorems that could never be proven or disproven. Arithemtic acquired a similar position as geometry. It is relatively true but it cannot lay a claim to an absolute truth.

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After 1931, we can speak about the truth in mathematics in two ways: as the consistency of a given theory and as the correspondence of a theory with our insight of the reality. The former must naturaly hold for any mathematical theory, the later is considered to be less important. In this paper, we will investigate how later constructions of real numbers satisfy the first two requierements and how they correspond to our intuition of reality.

Bolzano's theory

The question of the structure of real numbers touched Bolzano in his paper in 1813. He tried to prove by purely analytical ways (that is arithmetic here) that if a function $f(x)$ is continuous in the interval $[a, b]$, $f(a) > 0$, $f(b) < 0$ then there is always a c such that $a < c < b$ and $f(c) = 0$. This assertion is not too surprising, mathematicians have used it quite often, but its reasoning was only geometrical. If a continuous line is positive on one side of the interval and negative on the other side then it must cross the x coordinate necessarily.

Bolzano formulated the so-called Bolzano-Cauchy condition for the convergence of the sequence $\{a_n\}$ as follows.

$$(\forall k > 0)(\exists n > 0)(\forall m > 0)(|a_n - a_{n+m}| < 1/k)$$

He could not prove it because there was no arithmetic construction of real numbers available. He did not have a name for the point - real number - to which the Bolzano-Cauchy sequence converges. Bolzano was aware of this gap in his proof. It was one of the reasons why he tried to build his own theory of real numbers around 1836. Although Bolzano's conception of infinity is unusual, this theory is correct, it can be interpreted both in a standard and in a non-standard way. However, his study remained only in manuscripts, as many of Bolzano's work was forbidden, and thus had no influence for more than a century.

Cantor's conception of completion

Cantor is very famous as the inventor of the set theory, a mathematical theory of actual infinity. The first step toward this theory was the foundation ot the theory of real numbers. His construction is well known, it is the so-called completion of rational numbers: he added to all Bolzano-Cauchy sequences (he called them "fundamental") their limits, in the sense that Cantor associated fundamental sequences with symbolic limits. If two fundamental sequences $\{a_n\}$, $\{b_n\}$ had the property that for any $\epsilon > 0$ there was an n such that for all $m > n$ one could say that $|a_n - a_m| < \epsilon$, Cantor defined that these sequences had the same limit. The limits represent real numbers. Their geometric interpretation is that of points in a line. If we denote rational numbers by Q, their completion by $\hat{Q}$ and real numbers by R we can symbolically describe the Cantor's construction:

$$Q \longrightarrow \hat{Q} \cong R$$

The investigation of the structure of sets of real numbers led Cantor to the discovery of ordinal and cardinal numbers, and to the creation of a set theory.

It was accepted as the right theory of mathematical infinity by the end of the nineteenth century.

Cantor's rejection of infinitesimals

A hope had risen for some mathematicians. Mathematical analysis could indeed be based on infinitesimals because the theory of infinite numbers intrinsically justifies infinitely small numbers as their inverse values.

But Cantor denied this idea very firmly. Ordinal or cardinal numbers cannot serve as inverse values of infinitesimals and Cantor knew it very well. They can be added and multiplied and it would be difficult to find consistent arithmetical laws for their inverse values. Infinitesimals contradict the Archimedean Axiom, this axiom follows from the completeness of real numbers. If they were admited as a new sort of numbers sandwiched between rational and irrational numbers they would only complicate on already enough complicated problem of the continuum hypothesis.

Cantor was sure that his characterization of the infinite, and of real numbers, was the only characterization possible. He compared the theory of infinitesimals to the same level as the attempts to square the circle, it means as impossible. His arguments were so persuasive that Bertrand Russell argued in his *Principles of Mathematics* that mathematicians, fully understanding the nature of real numbers, could safely conclude that the non-existence of infinitesimals was firmly established (Russell, 1903,335). He was wise enough to add, however, that if it were ever possible to speak of infinitesimal numbers, it would have to be in a radically new sense.

Foundations of the calculus

The classical model of the continuum - either by Cantor or by Dedekind is correct and consistent. It agrees with our first request on real numbers. Continuum is composed from infinitely many (symbolic) real numbers. Reals are arithmetic expressions of points of a line. They satisfy all we expect from real numbers: linearly ordering, density and completeness. Moreover they are Archimedean.

Calculus is based on so-called $\epsilon - \delta$ approach that is generally accepted until now. "For every $\epsilon > 0$, there is a $\delta > 0$ such that … " is a typical phrase by which definitions of limits, continuity, differentiation, convergence and divergence of infinite series and others begin. Surprisingly, these formulations remind of a potential infinity, they do not take the advantage of Cantor's infinite numbers representing the actual infinity.

But we can have objections against the transparency of this approach. The idea of infinitely small seems to appeal more naturally to our intuition. The use of infinitesimals was widespread during the rise of the differential and integral analysis nearly for three centuries. It is simpler and clearer to compute with infinitesimals than to describe the computation in terms of limits.

The question now is whether it is possible to find a theory of real numbers which is

1. a consistent model of the continuum,

2. a consistent base for the calculus based on infinitesimals and
3. appropriate to our intuition.

This is not only a question of truth, as a consistency, it is also a question of truth as a comprehensibility and an insight.

Non-standard constructions

Non-standard real numbers are extensions of standard reals that contain infinitesimals. Our examples are Robinson's Non-StandardAanalysis, Vopěnka's Alternative Set Theory and Nelson's Internal Set Theory. Their constructions are based on a similar principle.

Russell was right that if infinitesimals could be accepted it must be done in an entirely new sense not only as quantities sandwiched between standard reals. There is introduced a class of new entities, non-standard numbers, a *substratum,* we denote it by S. The structure of real numbers is defined on it.

Real numbers are not considered only to be symbols denoting limits or cuts - but they are bubbled into *monads.*[1] They keep all their characteristic properties: **linear ordering, density** $((\forall x, y \in R)(\exists z \in R)(x < z < y))$**, completeness** (all Bolzano-Cauchy sequences of real numbers have their limits in R) and the **Archimedean Axiom** $((\forall x, y \in R)(\exists n \in N)(\frac{1}{n}|x| < |y| < n|x|))$**.**

The *substratum* S is the class of non-standard numbers. The natural numbers N and the rational numbers Q, or their isomorphic representatives, are subsets of S.

We can define that a non-standard number $a \in S$ is
1. **infinitely small** (an infinitesimal) iff $(\forall n \in N)(|a| < \frac{1}{n})$,
2. **finite** iff $(\exists n \in N)(|a| < n)$,
3. **infinite** iff $(\forall n \in N)(|a| > n)$,
4. two non-standard numbers $a, b \in S$ are **infinitely close** , denoted by $a \doteq b$, iff their difference is infinitely small,
5. $Mon(a) \ = \ \{b \in S; b \doteq a\}$.

$Mon(a)$ is called the **monad** of a. It contains all non-standard numbers that are infinitely close to a. The monad of zero $Mon(0)$ contains all infinitely small numbers, we denote it also by S_i.

The relation $\doteq$ is an equivalence. The class of finite non-standard numbers is denoted by S_f. We factorize the class S_f by the equivalence $\doteq$. The classes of this factorization are monads that respresent real numbers.

$$S_f / \doteq \ \cong \ R$$

We can also describe this construction from an algebraic point of view. Non-standard numbers S form a commutative ring. Finite non-standard numbers S_f a subring in S. Infinitely small non-standard numbers S_i form a maximal ideal in S, the monad of zero.

[1] A. Robinson used this poetic name having borrowed it from Leibniz.

A well-known algebraic theorem says that a factorization of a commutative ring modulo its maximal ideal yields a field. Thus, factorizing S_f modulo S_i, we obtain a field, namely the field of real numbers R.

$$S_f/S_i \quad \cong \quad S_f/\doteq \quad \cong \quad R$$

If we define an ordering on $S_f/S_i = R$ we can prove all necessary properties of real numbers: linearity, density, completeness, Archimedean Axiom.

The following is the symbolic scheme of the construction.

$$Q \;\longrightarrow\; S \;\supseteq\; S_f \;\supseteq\; S_i \;\longrightarrow\; S_f/S_i \cong R$$

Comparing the standard and the non-standard construction above, we receive the following commutative diagram.

$$\begin{array}{ccc} Q & \longrightarrow & S \supseteq S_f \supseteq S_i \\ & \searrow \qquad \swarrow & \\ & \hat{Q} \cong R \cong S_f/S_i & \end{array}$$

This is a good philosophical model of a continuum. The basic example of a continuum is a line. Points of a line are expressed as monads. All points (monads) together fill a line. Any of its part is infinitely divisible, as the old Aristotle's characteristic of continuum requests. Concurrently in the accordance with old ideas of Democritus and Zeno, the line is assembled from infinitely many infinitely small points. These points are not only symbolic, they have their own "body" composed from non-standard entities.

The last but not least task remains: the construction of the universe of non-standard numbers S. There are several ways to acheive it. They differ in accents they put on: a mathematical construction, a philosophical reasoning, a simplicity.

Robinson's non-standard analysis

The first mathematical construction of a non-standard theory of real numbers was created in 1963 by Abraham Robinson. In fact, he did not give an arithmetization of a continuum, a construction of reals from rationals. But it is a construction of non-standard reals from standard reals. Robinson works in the Zermello and Fraenkel Set Theory with the Axiom of Choice (ZFC). He uses a free (non-trivial) ultrafilter U on natural numbers N, the existence of which is guarranted by the Axiom of Choice.

This is the Robinson's construction in brief. We proceed from real numbers R. R^N is the set of all the real sequences, or equivalently, the set of all functions from N to R. The equivalence relation $\sim$ on sequences a, b in R^N is introduced via the ultrafilter:

$$a \sim b \iff \{i \in N | f(i) = g(i)\} \in U$$

R^N modulo the equivalence relation $\sim$ yields the ultraproduct $R^N/U = R^*$, called the non-standard reals.

$$R^* = R^N/U$$

The *substratum* S is the class of all non-standard real numbers R^*. It is a commutative ring, even a field.

We define naturally an embedding of R into R^* that assigns every r from R the constant function r^*.[2]

Thanks to the propeties of an ultrafilter, we can extend the arithmetical operations $+$, $.$ and the ordering $<$ from R on R^* via the ultrafilter U. Both R and R^* are complete linear y ordered fields.

The extension of any $A \subseteq R$ to $A^* \subseteq R^*$ is defined in the following way.

$$f \in A^* \iff \{n \in N | f(n) \in A\} \in U$$

As described above we define for $a \in R^*$ that
1. a is **infinitesimal** iff $(\forall r \in R)(|a| < r)$,[3]
2. a is **finite** iff $(\exists r \in R)(|a| < r)$,
3. a is **infinite** iff $\frac{1}{a}$ is infinitesimal,[4]
4.two elements $a, b \in R^*$ are **infinitely close**$(a \doteq b)$ iff their difference $|a - b|$ is infinitesimal.

(We use real numbers instead of natural numbers in these definitions, but the result would be the same.)

Let R_f denote the set of all finite elements of R^*. We can show that for every finite non-standard number a from R^* there is exactly one standard real number $r \in R$ such that the difference $a - r^*$ is infinitely small. We call this unique element of $r \in R$ the standard part of a.

The relation of $\doteq$ is an equivalence. By factorizing R_f modulo this equinalence or modulo R_i we obtain the structure of real numbers R.

$$R_f/R_i \cong R$$

The construction can be described symbolically by the following way.

$$N \subseteq R \longrightarrow R^N \longrightarrow R^N/U = R^* \supseteq R_f \supseteq R_i \longrightarrow R_f/R_i \cong R$$

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The main result about ultrafilter extensions the theorem of Los. Its immediate corollary is the general transfer principle. It says that the same properties of the first-order logic hold for elements of R and R^*.

Transfer Principle. Let $\phi(X_1, ..., X_m, x_1, ..., x_n)$ be a formula of the first-order logic. Then for any $A_1, ..., A_m \subseteq R$ and $r_1, ..., r_n \in R$, $\phi(A_1, ...A_m, r_1, ...r_n)$ is true in R iff $\phi(A_1^*, ..., A_m^*, r_1^*, ..., r_n^*)$ is true in R^*.

This principle enables us to define the same structures in R and in R^* and guarantees their correspondence. This is a very good foundation for the calculus. It is possible to define continuity of functions, limits and differentiation by a simple and natural way now.

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[2]That is, the function from N to R such that $r(n) = r$ for all $n \in N$.

[3]For instance, the function $f(n)_U = \frac{1}{n}$ defines a class of the ultraproduct that represents an infinitesimal, the function $f'(n)_U = \frac{1}{n^2}$ represents another infinitesimal.

[4]The functions $g(n)_U = n, g'(n)_U = n^2$ represent infinite numbers.

The non-standard constructions of reals from rationals

The Robinson's construction is not a construction of reals from rationals. Nevertheless, the same way can be used for this case. The role of the *substratum* S, plays the class of non-standard rational numbers now.

We proceed from the rational Q and the natural numbers N. Q^N is the set of all sequences of rational numbers. The ultraproduct Q^N/U represents the class of non-standard rational numbers Q^*

$$Q^N/U \;=\; Q^*$$

We can define the embedding of Q into Q^*, infinitesimals Q_i, infinite numbers, finite rational numbers Q_f, infinite closeness $\doteq$ by the same way with the help of an ultrafilter U. By the factorization of finite rational numbers modulo infinitesimals or equivalently by the equivalence of infinite closeness, we receive the structure of real numbers $Q_f/\doteq \;=\; Q_f/Q_i \;\cong\; R$. We can describe it symbolically as follows.

$$N \subseteq Q \longrightarrow Q^N \longrightarrow Q^N/U = Q^* \supseteq Q_f \longrightarrow Q_f/\doteq \;\cong\; R$$

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This approach can be compared to the standard construction of Cantor. He proceeds also from the sequences of rational numbers from Q^N. He deals only with the set of all Bozano-Cauchy sequences, it is a commutative ring, denoted by B. Two sequences have the same limit if their difference is a sequence converging to zero, the set of all these sequences is a maximal ideal, denoted by C. We can say that Cantor makes a factorization of B modulo C immediately. He excludes infinitely small entities at the beginning. He receives a commutative field B/C, i.e. real numbers R. Cantor did not need any ultrafilters, the Bozano-Cauchy condition sufficed:

$$N \subseteq Q \longrightarrow Q^N \supseteq B \supseteq C \longrightarrow B/C \cong R$$

Cantor's completion can be interpreted in terms of non-standardness.

Vopěnka's philosophical approach

A similar construction of real numbers appears in Vopěnka's Alternative Set Theory. Vopěnka does not work in ZFC or in any other modification of Cantorian Set Theory and he does not use ultrafilters.

His concept of infinity is based on the phenomenological notion of the horizon encompassed by so-called *semisets.*

He employs two types of natural numbers, the finite (standard) natural numbers N and all natural numbers N^* that involve also infinite numbers $N^* - N$. Both kinds of numbers, N and N^*, are models of the Peano arithmetic.

Finite numbers are "before the horizon", they are accessible in a way. The class of all finite numbers N is a typical example of a semiset. Infinite numbers are "beyond the horizon".

The finite rational numbers Q form the quotient field of N, the rational numbers Q^* form the quotient field of all natural numbers N^*. Because elements of $N^* - N$ are infinite, their inverse values in Q^* are infinitely small. While infinite numbers are "beyond the horizon of the distance" infinitely small numbers are "beyond the horizon of the depth". Two elements of Q^* are infinitely close iff their distance is infinitely small. The relation of infinitely closeness is an equivalence relation. Classes of this equivalence are semisets, they are called monads.

Again, we shall deal only with rational numbers that are smaller than any finite number $n \in N$, we denote them by Q_f. It is acommutative ring. Infinitely small rational numbers Q_i form a maximal ideal. By the factorization of Q_f modulo Q_i or equivalently by the equivalence of infinite closeness, we obtain the commutative field of real numbers. Their elements are represented by monads of infinitely close rational numbers.

$$N \subseteq N^* \longrightarrow Q^* \supseteq Q_f \supseteq Q_i \longrightarrow Q_f/Q_i = Q_f/\doteq \cong R$$

The important difference in comparing with the Robinson's construction is that the transfer principle does not hold for elements of Q^* and R. The Q^* is an elementary extension of Q but not of R. Not all functions that are defined in R can be directly defined in Q^*. Non-standard rational numbers are not complete. This is the reason why this model is not suitable for the foundation of the calculus.

Axiomatic non-standard theory

Properties of the ultraproducts are wonderful, but not so straightforward. It is nearly impossible to imagine free ultrafilters. If our point of view is a clearness and a good insight into the continuum we can use an axiomatic approach.

The example of a non-standard theory based axiomatically is provided by the Internal Set Theory (IST) described in 1977 by E. Nelson. It axiomatizes a basis of Abraham Robinson's non-standard analysis. It works in ZFC and constructs a theory extending ordinary mathematics that can serve as a good base for calculus.

IST adds only one predicate "standard" and three new axioms. The transfer principle, the idealization principle and the standardization principle. Using this new predicate, all the necessary notions are introduced. (Variables are defined on the set of real numbers R.) For instance a real number x is
infinitesimal iff for all standard real $y > 0$ there is $|x| < y$,
finite iff for some standard real number y there is $|x| < y$,
x is **infinitely close** to y iff $x - y$ is infinitesimal.

The axiomatic approach was also used by Vopěnka in his book *Calculus infinitesimalis.* He introduced four principles that are similar to Nelson's axioms. Then, he built a calculus based on infinitesimals.

Conclusion

There are many possibilities of introducing non-standard real numbers. The mathematical means applied for this purpose are quite different: ultrafilters, semisets, and fundamental sequences.

A rare consensus prevails among mathematicians concerning the shape of the continuum. Although the conception of infinity could differ, formal mathematical propeties of real numbers are the same: linearity, density, completeness, the Archimedean Axiom.

However a *substratum* on which real numbers are defined varies. In fact, Cantor' theory has no *substratum* and consequently no space for infinitesimals. Robinson's theory is a perfect base for non-standard analysis however it employs the non-intuitive notion of an ultrafilters to extend the standard continuum. Vopěnka's and other constructions based on non-standard rational numbers are well-founded philosophically, but they are not suitable for non-standard analysis. Axiomatic systems are the simplest, nevertheless they do not provide us with a constructions of the continuum, they formally describe its properties.

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