Octahedron: a holomorphic symbolic representation of the complex sphere

Petr Kůrka*

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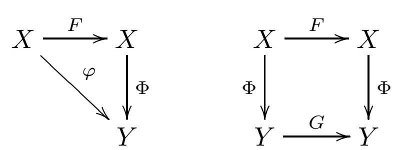
1 Introduction

Codings of real numbers are based on the fact that each compact metric space is a factor of the Cantor space of symbolic sequences. The binary coding of the unit interval $\mathbb{I} = [0, 1]$ is a factor map $\Phi_2 : \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{I}$ defined by $\Phi_2(u) = \sum_{n=0}^{\infty} u_n 2^{-n-1}$, where $\mathbb{Z}_2 = \{0, 1\}$ is the binary alphabet. This can be regarded as a dynamical system $F : \mathbb{Z}_2^* \times \mathbb{I} \to \mathbb{I}$, or a continuous action of the monoid $\mathbb{Z}_2^*$ on $\mathbb{I}$. The action is generated by maps $F_0, F_1 : \mathbb{I} \to \mathbb{I}$ given by $F_i(x) = (x+i)/2$, and $F_u = F_{u_0} \circ \cdots \circ F_{u_{k-1}}$ for any finite word $u \in \mathbb{Z}_2^k$. For each infinite word $u \in \mathbb{Z}_2^{\mathbb{N}}$, $\Phi_2(u)$ is the unique number contained in all $F_{u_{[0,k)}}(\mathbb{I})$. This can be generalized to any contractive A^*-action (see Edgar [2], or Barnsley [1]). If all $(F_a)_{a \in A}$ are contractions on a compact metric space X, then there exists a unique attractor $Y \subseteq X$ with $Y = \bigcup_{a \in A} F_a(Y)$, and a factor map $\Phi : A^{\mathbb{N}} \to Y$.

Binary coding is not very convenient, since continuous maps on $\mathbb{I}$ cannot be lifted to continuous maps on the symbolic space. For $u = 0^\infty$ and $v = 01^\infty$, the first digit of $\Phi_2(u) + \Phi_2(v) = \frac{1}{2}$ cannot be determined from any finite prefices of u and v. To be able to perform continuous operations in the symbolic space, we need a factor map with the extension property.

Theorem 1 (Weihrauch [6], Kůrka [3]) *For a Cantor space X and a compact metric space Y there exists a factor map $\Phi : X \to Y$ with the extension property. This means that for any continuous map $\varphi : X \to Y$ there exists a continuous map $F : X \to X$ such that $\Phi \circ F = \varphi$.*



Then, any continuous map $G : Y \to Y$ can be lifted to a continuous map $F : X \to X$ such that $\Phi \circ F = G \circ \Phi$. An example of a factor map with the extension property is $\Phi_3 : \mathbb{Z}_3^{\mathbb{N}} \to [0, 1]$ given by $\varphi(u) = \sum_{n=0}^{\infty} u_n 2^{-n-2}$, where $\mathbb{Z}_3 = \{0, 1, 2\}$. This is again the attractor of a $\mathbb{Z}_3^*$-action determined by maps $F_i(x) = (x + i)/4$. The reason why Φ_3 has the extension property is that the interiors of intervals $F_i(\mathbb{I}) = [\frac{i}{4}, \frac{i+2}{4}]$, cover the space $\mathbb{I}$.

The construction of a symbolic representation for the whole set $\mathbb{R}$ of real numbers poses another problem that $\mathbb{R}$ is not compact, so only a noncompact subset of the symbolic space could be used for the coding. A reasonable alternative is to construct a coding for a compactification of $\mathbb{R}$, e.g., for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Even more promising seems to be to look for codings of the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The complex sphere is the ultimate accomplished arithmetical structure, which is perfect in the sense that nothing could be added to it or taken away from it without destroying its beautiful geometric, algebraic, and analytical properties (see e.g., Penrose [5] or Kůrka [4]).

The only holomorphic (differentiable) self-maps of the complex sphere are the rational functions, i.e., quotients of polynomials. Among them, Möbius transformations are distinguished by being

*Center for Theoretical Study, Academy of Sciences and Charles University in Prague, Jilská 1, CZ-11000 Praha 1, Czechia & Centre de Physique Théorique, CNRS Luminy, Case 907, Marseille Cedex 9, France

conformal isomorphisms. In fact, the conformal geometry of the complex sphere is created by the group of Möbius transformations. Thus, when we try to construct a symbolic representation of the complex sphere, a dynamical system based on Möbius transformations is the most obvious choice. The aparent problem is that Möbius transformations are not contracting and that they are surjective, so the forward images of the space cannot converge to a point.

However, instead of convergence of sets, we can use convergence of measures, and inquire whether the images of the uniform measure converge to a point measure. This approach has an additional advantage, that finite numbers (i.e., finite words in the alphabet of digits) can be interpreted as imprecise numbers. The preciseness of a number increases with its length. As generators, we use local contractions, which are contractive only in neighbourhoods of their attractive fixed points. We consider holomorphic dynamical systems generated by local contractions to a finite set of regularly spaced points of the complex sphere. A natural choice for these points are vertices of a Platonic solid. Among the five Platonic solid, the regular octahedron is distinguished by the property that its six vertices represent important arithmetical constants: 0, 1, i, -1, $-i$ and ∞.

We first consider the case of extended real line $\overline{\mathbb{R}}$, and construct holomorphic dynamical systems consisting of local contractions to vertices of regular polygons. The coding should satisfy some constraints, otherwise the contraction to a point would be cancelled by a contraction to an opposite point. These constraints define subshifts on the set of vertices of the polygon. We consider two such subshifts, the walk subshift, and the slow walk subshifts, and show that a symbolic representation can be based on them. The most interesting case is the square, whose vertices are numbers 0, 1, ∞ and -1. This system can be extended to the complex sphere adding vertices in i and $-i$ to obtain the regular octahedron. The unique contraction quotient for which the system works is the golden mean number $q = (3-\sqrt{5})/2$.

2 Real Möbius transformations

Consider the stereographic projection P of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to the unit circle $\{(x,y) \in \mathbb{R}^2 :\ x^2+y^2=1\}$ given by

$$P(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right), \quad P^{-1}(x,y) = \frac{x}{1-y}.$$

Parametrize the unit circle by $(\cos(t-\frac{\pi}{2}), \sin(t-\frac{\pi}{2}))$, where $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, so $t, t' \in \mathbb{R}$ are indentified in $\mathbb{T}$ iff $(t-t')/2\pi \in \mathbb{Z}$. The lowest point $(0,-1)$ of the circle is parametrized by $t=0$ We have mutually inverse maps $\mathbf{x} : \mathbb{T} \to \overline{\mathbb{R}}$ and $\mathbf{t} : \overline{\mathbb{R}} \to \mathbb{T}$ given by

$$\mathbf{x}(t) = \frac{\cos(t-\frac{\pi}{2})}{1-\sin(t-\frac{\pi}{2})} = \tan\frac{t}{2}, \quad \mathbf{t}(x) = 2\arctan x + 2\pi k.$$

Consider orientation preserving Möbius transformations $m_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d}$ of $\overline{\mathbb{R}}$ with positive determinant $ad-bc>0$. On $\mathbb{T}$ we get corresponding **circle Möbius transformation** $M_{(a,b,c,d)} = \mathbf{t} \circ m_{(a,b,c,d)} \circ \mathbf{x}$ which can be regarded as increasing continuous functions $M : \mathbb{R} \to \mathbb{R}$ with $M(t+2\pi) = M(t)+2\pi$. We have

$$\begin{aligned} M_{(a,b,c,d)}(t) &= 2\arctan\frac{a\tan\frac{t}{2}+b}{c\tan\frac{t}{2}+d} + 2k\pi, \\ M'(a,b,c,d)(t) &= \frac{(ad-bc)(1+\mathbf{x}^2(t))}{(a\cdot\mathbf{x}(t)+b)^2+(c\cdot\mathbf{x}(t)+d)^2}. \end{aligned}$$

Denote by $C_q(t) = 2\arctan(q\tan\frac{t}{2})$ the **contraction to** 0 with quotient $q<1$. This is the Möbius transfromation which corresponds to the contraction $x \mapsto qx$ of the real line. We have $C'_q(t) = q(1+\mathbf{x}^2(t))/(1+q^2\mathbf{x}^2(t))$ and

$$|C'_q(t)| \le 1 \iff \mathbf{x}(t) \le 1/\sqrt{q} \iff |t| \le \pi - \alpha_q, \quad \text{where } \alpha_q = 2\arctan\sqrt{q} < \pi.$$

We denote by $U_0 = [\alpha_q - \pi, \pi - \alpha_q]$ the **contraction interval** of C_q and by $V_0 = [-\alpha_q, \alpha_q]$ the **expansion interval** of $C_q^{-1} = C_{\frac{1}{q}}$ respectively. Note that $V_0 \subset U_0$ and $C_q(U_0) = V_0$, so

$C_q(\pi - \alpha_q) = \alpha_q$. We consider also contractions $C_{q,\alpha}$ to points $\alpha \in \mathbb{T}$ given by

$$C_{q,\alpha}(t) = \alpha + C_q(t-\alpha) = 2\arctan \frac{(q+\tan^2\frac{\alpha}{2})\tan\frac{t}{2} + (1-q)\tan\frac{\alpha}{2}}{(1-q)\tan\frac{\alpha}{2}\tan\frac{t}{2} + (1+q\tan^2\frac{\alpha}{2})}.$$

Lemma 2 *There exists an increasing continuous function $\psi_q : [0, 2(\pi - \alpha_q)] \to [0, 2\alpha_q]$ such that $\psi_q(0) = 0$, $0 < \psi_q(t) < t$ for $t > 0$ and $|C_q(W)| < \psi_q(|W|)$ for each interval $W \subseteq U_0$. Moreover, there exists a constant $c > 0$ such that for any $W \subseteq V_0$ with $|W| \le c$ we have $|C_q^{-1}(W)| > \psi_q^{-1}(|W|)$.*

Here $|W|$ is the length of the interval W. The proof follows from the fact that $C_q'(t) < 1$ for each $t \in \mathrm{int}(U_0)$.

3 Measures

Given a compact metric space X, we consider the space $\mathfrak{M}(X)$ of Borel probability measures with weak* topology, i.e., $\lim_{n\to\infty} \mu_n = \mu$ iff $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ for each continuous function f. A continuous function $F : X \to Y$ extends to a continuous map $F : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ by $(F\mu)(U) = \mu(F^{-1}(U))$. Denote by δ_x the Dirac **point measure** concentrated on x, i.e., $\delta(U) = 1$ iff $x \in U$. Measures which are absolutely continuous with respect to the Lebesgue measure have **densities**, which are nonnegative functions with unit integral. In particular the **uniform Lebesgue measure** κ on $\mathbb{T}$ has the constant probability density $h(t) = 1/2\pi$. The corresponding measure $\mathbf{x}\kappa$ on $\overline{\mathbb{R}}$ has the Cauchy density $h(\mathbf{t}(x)) \cdot \mathbf{t}'(x) = \frac{1}{\pi(1+x^2)}$. The probability density of $M_{(a,b,c,d)}\kappa$ is

$$\begin{aligned} h_{(a,b,c,d)}(t) &= h(M^{-1}_{(a,b,c,d)}(t)) \cdot (M^{-1}_{(a,b,c,d)})'(t) \\ &= \frac{1}{2\pi} \cdot \frac{(ad-bc)(1+\mathbf{x}^2(t))}{(d\cdot \mathbf{x}(t) - b)^2 + (c \cdot \mathbf{x}(t) - a)^2} \end{aligned}$$

A probability measure on the circle (uderstood now as a subset of the complex plane $\mathbb{C}$) can be characterized by its **mean** $\mathsf{E}(\mu) := \int_{\mathbb{T}} d\mu$ which is a complex number in the unit disk. For a measure with density h we get $\mathsf{E}(h) = \int_{-\pi}^{\pi} h(t)e^{it}\, dt$. In particular for the densities of Möbius transfromations, the real and imaginary parts are

$$\begin{aligned} \Re\mathsf{E}(h_{(a,b,c,d)}) &= \frac{ad-bc}{\pi}\int_{-\infty}^{+\infty} \frac{(1-x^2)\, dx}{[(dx-b)^2 + (cx-a)^2](1+x^2)} \\ &= \frac{(c^2+d^2-a^2-b^2)[(a-d)^2+(b+c)^2]}{(c^2+d^2-a^2-b^2)^2 + 4(ac+bd)^2} \end{aligned}$$

$$\begin{aligned} \Im\mathsf{E}(h_{(a,b,c,d)}) &= \frac{ad-bc}{\pi}\int_{-\infty}^{+\infty} \frac{2x\, dx}{[(dx-b)^2 + (cx-a)^2](1+x^2)} \\ &= \frac{2(ac+bd)[(a-d)^2+(b+c)^2]}{(c^2+d^2-a^2-b^2)^2 + 4(ac+bd)^2} \end{aligned}$$

The argument $\arg \mathsf{E}(h)$ characterizes the mean of the Möbius transformation on $\mathbb{T}$ and the absolute value $|\mathsf{E}(h)|$ characterizes the preciseness of the distribution. In particular for the uniform measure with density $h(t) = 1/2\pi$ we have $\mathsf{E}(h) = 0$. For the point measure δ_t we have $\mathsf{E}(\delta_t) = e^{it}$, so $|\mathsf{E}(\delta_t)| = 1$. In the sequel we use the following obvious lemma.

Lemma 3 *Let $(M_n : \mathbb{T} \to \mathbb{T})_{n\ge 0}$ be a sequence of circle Möbius transformations. Assume that there exists $t \in \mathbb{T}$ and $c > 0$ such that for each interval $I \ni t$ we have $\liminf_{n\to\infty}(M_n\kappa)(I) > c$. Then $\lim_{n\to\infty}(M_n\kappa)(I) = 1$ and $\lim_{n\to\infty} M_n\kappa = \delta_t$.*

4 Subshifts and their actions

For a finite alphabet A, denote by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A. The length of a word $u = u_0 \dots u_{n-1} \in A^n$ is denoted by $|u| := n$ and the word of zero length is λ. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$), if there exists k such that $v_{k+i} = u_i$ for $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. With the operation of concatenation and empty word λ, A^* is the free monoid over A.

We denote by $A^{\mathbb{N}}$ the Cantor space of infinite sequences of letters of A equipped with the metric $d(x, y) := 2^{-n}$, where $n = \min\{i \geq 0 : x_i \neq y_i\}$. The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{N}}$, which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. For a subshift Σ there exists a set $D \subseteq A^*$ of forbidden words such that $\Sigma = \mathcal{S}_D := \{x \in A^{\mathbb{N}} : \forall u \sqsubseteq x, u \notin D\}$. A subshift is of **finite type (SFT)**, if there exists a finite set $D \subset A^*$ such that $\Sigma = \mathcal{S}_D$. The **order** $\mathfrak{o}(\Sigma)$ is the length of the longest word of D. A subshift is uniquely determined by its **language** $\mathcal{L}(\Sigma) := \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$. We denote by $\mathcal{L}^n(\Sigma) := \mathcal{L}(\Sigma) \cap A^n$.

By a **dynamical system** we mean a Σ-action over a compact metric space X, i.e., a continuous map $F : \mathcal{L}(\Sigma) \times X \to X$ satisfying $F_\lambda = \mathrm{Id}_X$ and $F_{uv} = F_u \circ F_v$, (the discrete topology is assumed on $\mathcal{L}(\Sigma)$). The action is given by generators $(F_a)_{a\in A}$ and $F_u = F_{u_0} \circ \cdots \circ F_{u_{n-1}}$ for any $u \in \mathcal{L}(\Sigma)$. As alphabets we use groups $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ of integers modulo n with circle distance defined by $d(a, b) = \min\{|a - b|, n - |a - b|\}$.

Definition 4 *A* **holomorphic arithmetics** *with arity $n > 2$ and quotient $0 < q < 1$ is a Σ-action $F : \mathcal{L}(\Sigma) \times \mathbb{T} \to \mathbb{T}$, where $F_i = C_{q,\frac{2\pi i}{n}}$, and $\Sigma \subseteq \mathbb{Z}_n^{\mathbb{N}}$ is a subshift. Denote by*

$$U_i = [-\pi + \alpha_q + \tfrac{2\pi i}{n}, \pi - \alpha_q + \tfrac{2\pi i}{n}], \quad V_i = [-\alpha_q + \tfrac{2\pi i}{n}, \alpha_q + \tfrac{2\pi i}{n}]$$

the **contraction intervals** *of F_i and the* **expansion intervals** *of F_i^{-1} respectively. The* **cylinder** *of $u \in \mathbb{Z}_n^k$ is $V_u = F_u(U_{u_{k-1}})$.*

We have $F_i'(t) \leq 1$ iff $t \in U_i$ and $(F_i^{-1})'(t) \geq 1$ iff $t \in V_i$. Moreover, $F_i(U_i) = V_i$.

5 Walk subshifts

Definition 5 *For $n \geq 2$ the* **walk subshift** *$\mathcal{W}_n \subseteq \mathbb{Z}_n^{\mathbb{N}}$ is a SFT of order 2 with forbidden words $D = \{ij \in \mathbb{Z}_n^2 : d(i, j) > 1\}$.*

Thus only transitions $i \to (i-1)$, $i \to i$, and $i \to (i+1)$ to neghbouring letters are alowed in a walk subshift. For $n = 2, 3$, $\mathcal{W}_n$ is the full shift.

Proposition 6 *Let $n \geq 4$, and let $F : \mathbb{Z}_n^* \times \mathbb{T} \to \mathbb{T}$ be the n-ary arithmetics with quotient $q < 1$. Then the contraction and expansion conditions*

$$\begin{aligned} F_i(U_i) &\subseteq U_{i-1} \cap U_i \cap U_{i+1}, &(1)\\ F_i^{-1}(V_i) &\subseteq V_{i-1} \cup V_i \cup V_{i+1} &(2) \end{aligned}$$

are satisfied iff $q = \tan^2 \frac{\pi(n-2)}{4n}$, i.e., $\alpha_q = \pi(\frac{1}{2} - \frac{1}{n})$. In this case the conditions (1) and (2) hold with equality and $V_i \cap V_{i+1} \neq \emptyset$. If moreover $n \geq 5$, then $\mathrm{int}(V_i) \cap \mathrm{int}(V_{i+1}) \neq \emptyset$.

Proof: Since $F_i(U_i) = V_i$, the two conditions read

$$\begin{aligned} [-\alpha_q, \alpha_q] &\subseteq [-\pi + \alpha_q + \tfrac{2\pi}{n}, \pi - \alpha_q - \tfrac{2\pi}{n}],\\ [-\pi + \alpha_q, \pi - \alpha_q] &\subseteq [-\alpha_q - \tfrac{2\pi}{n}, \alpha_q + \tfrac{2\pi}{n}] \end{aligned}$$

This yields $\alpha_q \leq \pi(\frac{1}{2} - \frac{1}{n}) \leq \alpha_q$, so we obtain the equality and $q = \tan^2 \frac{\alpha_q}{2} = \tan^2 \frac{\pi(n-2)}{4n}$. The intervals V_i and V_{i+1} intersect iff $\alpha_q \geq \pi/n$ which gives $n \geq 4$. For $n \geq 5$, the inequality $\alpha_q > \pi/n$ is strict, so the interiors of V_i intersect as well.

Theorem 7 *Let $n \geq 4$ and let $F : \mathcal{L}(\mathcal{W}_n) \times \mathbb{T} \to \mathbb{T}$ be the holomorphic arithmetics with quotient $q = \tan^2\left(\frac{\pi}{4} - \frac{\pi}{2n}\right)$. There exists a factor map $\Phi_n : \mathcal{W}_n \to \mathbb{T}$ such that for each $u \in \mathcal{W}_n$,*

$$\lim_{n\to\infty} F_{u_{[0,n)}}\kappa = \delta_{\Phi_n(u)}, \quad \bigcap_{k\geq 0} V_{u_{[0,k)}} = \{\Phi_n(u)\}.$$

If $n \geq 5$, then Φ_n has the extension property.

Proof: We use the contraction and expansion conditions of Proposition 6. Assume $u \in \mathcal{W}_n$ and $k > 0$. Since $u_{[k-1,k-2]} \in \mathcal{L}(\mathcal{W}_n)$, we have $F_{u_{k-1}}(U_{u_{k-1}}) \subseteq U_{u_{k-2}}$. Applying $F_{u_{[0,k-2)}}$ we get $V_{u_{[0,k)}} \subseteq V_{u_{[0,k-1)}}$. Using the contraction function ψ of Lemma 2, we get

$$|V_{u_{[0,k)}}| \leq \psi(|F_{u_{[1,k)}}(U_{u_{k-1}})|) \leq \psi^2(|F_{u_{[2,k)}}(U_{u_{k-1}})|) \leq \cdots \leq \psi^k(|U_{u_{k-1}}|).$$

Since the only fixed point of ψ is 0, we get $\lim_{k\to\infty} |V_{u_{[0,k)}}| = 0$, so $\bigcap_k V_{u_{[0,k)}}$ is a singleton containg the unique element $\Phi_n(u)$. Clearly $\Phi_n : \mathcal{W}_n \to \mathbb{T}$ is continuous. Since

$$(F_{u_{[0,k)}}\kappa)(V_k) = \kappa((F_{u_{[0,k)}})^{-1}F_{u_{[0,k)}}(U_{u_0})) = \kappa(U_{u_0}) = 2(\pi - \alpha),$$

we have $\lim_{n\to\infty} F_{u_{[0,k)}}\kappa = \delta_{\Phi_n(u)}$ by Lemma 3.
To prove that Φ_n is surjective, take any $t_0 \in \mathbb{T}$. Choose u_0 such that $t_0 \in V_{u_0}$. Since $F^{-1}(V_{u_0}) = V_{u_0-1} \cup V_{u_0} \cup V_{u_0+1}$, there exists u_1 such that $u_{[0,1]} \in \mathcal{L}(\mathcal{W}_n)$ and $t_1 = F^{-1}_{u_0}(t_0) \in \mathrm{int}(V_{u_1})$, etc. We get a sequence $t_k := F^{-1}_{u_{[0,k)}}(t_0) \in V_{u_k}$, and $F_{u_{[0,k)}}(t_k) = t_0$. If $I \ni t_0$ is any interval containing t_0, then for all sufficiently large k we have $(F_{u_{[0,k-1)}}\kappa)(I) \geq 2\alpha_q$, so $\lim_{k\to\infty} F_{u_{[0,k-1)}}\kappa = \delta_{t_0}$. If $n \geq 5$, then $\{\mathrm{int}(V_i) :\ i \in A\}$ is an open cover of $\mathbb{T}$. Moreover, for any $u \in \mathcal{L}(\mathcal{W}_n)$, $\{\mathrm{int}(V_{ua}) :\ a \text{ such that } ua \in \mathcal{L}(\mathcal{W}_n)\}$ is an open cover of $\mathrm{int}(V_u)$. We show that $\Phi_n : \mathcal{W}_n \to \mathbb{T}$ has the extension property. Let $\varphi : \mathcal{W}_n \to \mathbb{T}$ be continuous and $u \in \mathcal{W}_n$. There exists k_0 and v_0 such that $\varphi([u_{[0,k_0)}]) \subseteq \mathrm{int}(V_{v_0})$. There exists k_1 and v_1 such that $v_{[0,1]} \in \mathcal{L}(\mathcal{W}_n)$ and $\varphi([u_{[0,k_1)}]) \subseteq \mathrm{int}(V_{v_{[0,1]}})$. Continuing in this manner, we construct $F(u) = v \in \mathcal{W}_n$ such that $\Phi_n F(u) = \varphi(u)$.

6 Slow walk subshifts

For $n = 4$ the neighbouring intervals V_i intersect only in their endpoints, so Φ_4 does not have the extension property. To get it, we define a smaller subshift in which we cannot go around the circle too fast. We keep the expansion condition while we relax the contraction condition.

Definition 8 *For $n \geq 4$ the* **slow walk subshift** *$\mathcal{S}_n \subseteq \mathbb{Z}_n^{\mathbb{N}}$ is a SFT of order 3 with forbidden words $D = \{ij \in \mathbb{Z}_n^2 :\ d(i,j) > 1\} \cup \{ijk \in \mathbb{Z}_n^3 :\ d(i,k) > 1\}$.*

Proposition 9 *Let $n \geq 4$, and let $F : \mathcal{L}(\mathcal{S}_i) \times \mathbb{T} \to \mathbb{T}$ be n-ary arithmetics with quotient $q < 1$. Then the conditions*

$$F_0^{-1}(V_0) \subseteq V_{-1} \cup V_0 \cup V_1 \tag{3}$$
$$F_0(V_0) \subseteq U_{-1} \cap U_1, \tag{4}$$
$$F_0^{-1}(V_0 \cap U_1) \subseteq U_0 \cap (V_0 \cup V_1), \tag{5}$$
$$F_0(U_0 \cap V_1) \subseteq U_1 \tag{6}$$

are satisfied iff $q = q_n = (c_n^2 - c_n\sqrt{c_n^2 + 4} + 2)/2$, where $c_n = \tan\frac{\pi}{n}$. In this case $|V_i \cap V_{i+1}| > 0$ and conditions (4) and (5) are satisfied with equality.

Proof: Condition (3) gives $\alpha \geq \frac{\pi}{2} - \frac{\pi}{n}$ analogously as in Proposition 6. It follows

$$\begin{aligned}
U_0 \cap V_1 &= [\alpha - \pi, \pi - \alpha] \cap [-\alpha + \tfrac{2\pi}{n}, \alpha + \tfrac{2\pi}{n}] = [-\alpha + \tfrac{2\pi}{n}, \pi - \alpha], \\
U_1 \cap V_0 &= [\alpha - \pi + \tfrac{2\pi}{n}, \pi - \alpha + \tfrac{2\pi}{n}] \cap [-\alpha, \alpha] = [\alpha - \pi + \tfrac{2\pi}{n}, \alpha], \\
U_0 \cap (V_0 \cup V_1) &= [-\alpha + \pi, \pi - \alpha] \cap [-\alpha, \alpha + \tfrac{2\pi}{n}] = [-\alpha, \pi - \alpha]
\end{aligned}$$

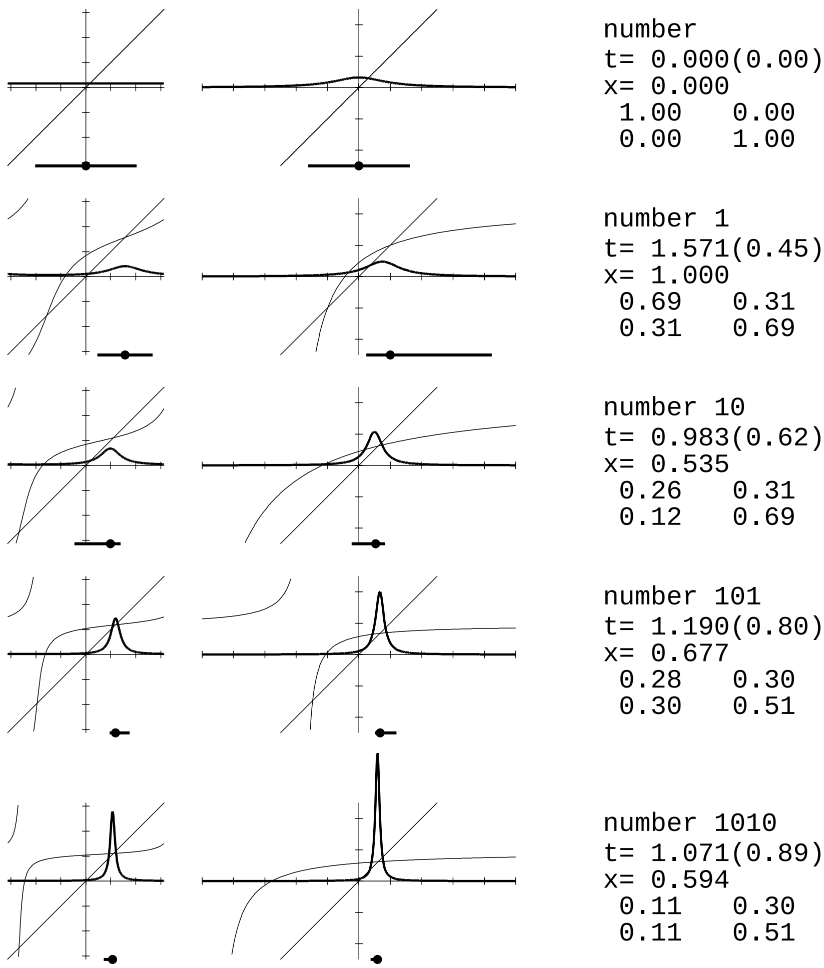


Figure 1: Some finite holomorphic numbers, their densities and cylinders

Condition (4) gives $[F_0(-\alpha), F_0(\alpha)] \subseteq [-\pi+\alpha+\frac{2\pi}{n}, \pi-\alpha-\frac{2\pi}{n}]$, so

$$F_0(\alpha) \leq \pi(1-\tfrac{2}{n}) - \alpha \implies q\tan\tfrac{\alpha}{2} \leq \tan(\tfrac{\pi}{2}-\tfrac{1}{n}-\tfrac{\alpha}{2}).$$

We have $\tan\frac{\alpha}{2} = \sqrt{q}$ and $\tan\frac{\pi}{n} = c_n$. Using formulas $\tan(\frac{\pi}{2}-x) = 1/\tan x$ and $\tan(x+y) = (\tan x + \tan y)/(1-\tan x\tan y)$, we get

$$q\sqrt{q} \leq \frac{(1/c_n)-\sqrt{q}}{1+\sqrt{q}/c_n} = \frac{1-\sqrt{q}c_n}{c_n+\sqrt{q}} \implies q^2+\sqrt{q}(q+1)c_n - 1 \leq 0.$$

Condition (5) gives

$$[\alpha-\pi+\tfrac{2\pi}{n}, \alpha] = U_1\cap V_0 \subseteq F(U_0\cap(V_10\cup V_1)) = [F_0(-\alpha), F_0(\pi-\alpha)]$$

We get $\alpha \leq F_0(\pi-\alpha)$ which is satisfied with equality and $\pi-\frac{2\pi}{n}-\alpha \leq F_0(\alpha)$, so

$$\frac{(1/c_n)-\sqrt{q}}{1+\sqrt{q}/c_n} = \frac{1-c_n\sqrt{q}}{c_n+\sqrt{q}} \leq q\sqrt{q} \implies q^2+\sqrt{q}(q+1)c_n-1 \geq 0$$

Together with condition (4) we get $q^2+\sqrt{q}(q+1)c_n-1=0$, and therefore $q+c_n\sqrt{q}-1=0$. This is a quadratic equation with solution $\sqrt{q} = (-c_n+\sqrt{c_n^2+4})/2$ and $q_n = (c_n^2 - c_n\sqrt{c_n^2+4}+2)/2$. Condition (6) gives

$$[F_0(-\alpha+\tfrac{2\pi}{n}), F_0(\pi-\alpha)] = F_0(U_0\cap V_1) \subseteq U_1 = [\alpha-\pi+\tfrac{2\pi}{n}, \pi-\alpha+\tfrac{2\pi}{n}]$$

Thus $F_0(\pi-\alpha) \le \pi - \alpha + \frac{2\pi}{n}$ which is satisfied, and $F_0(\alpha - \frac{2\pi}{n}) \le \pi(1-\frac{2\pi}{n}) - \alpha$. We get

$$q\frac{\sqrt{q}-c_n}{1+\sqrt{q}c_n} = q\tan(\tfrac{\alpha}{2}-\tfrac{\pi}{n}) \le \tan(\tfrac{\pi}{2}-\tfrac{\pi}{n}-\tfrac{\alpha}{2}) = \frac{1-\sqrt{q}c_n}{c_n+\sqrt{q}}$$

We get $q^2 \le 1$ which is satisfied. To show $|V_i \cap V_{i+1}| > 0$ we need $\alpha > \frac{\pi}{n}$, which gives

$$\tan\alpha = \frac{2\sqrt{q}}{1-q} > c_n = \frac{1-q}{\sqrt{q}}$$

or $c_n = \tan\frac{\pi}{n} < \sqrt{2}$, which is satisfied for $n \ge 4$.

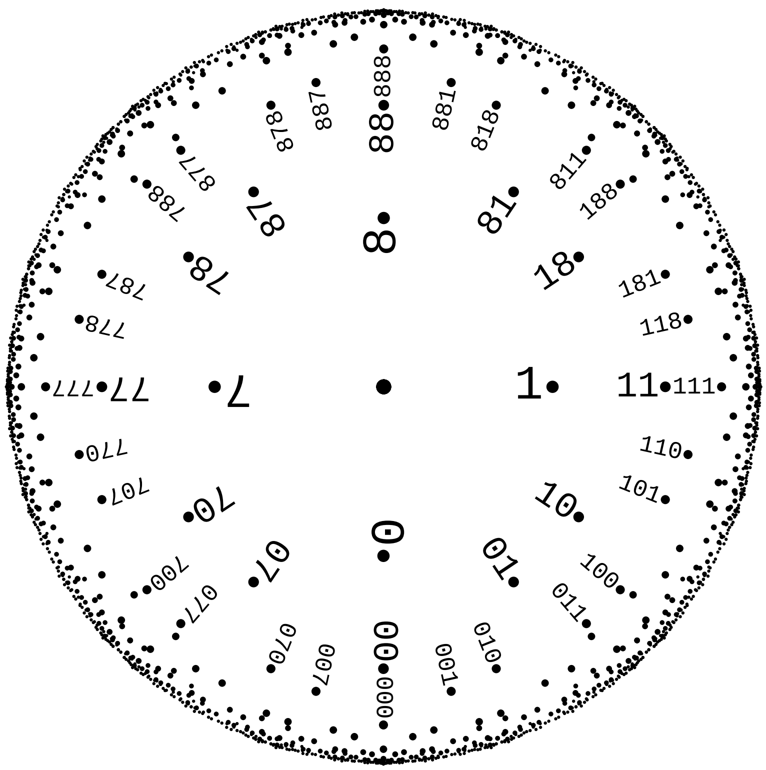


Figure 2: Means of finite holomorphic numbers

Theorem 10 *Let $n \ge 4$ and let $F : \mathcal{L}(\mathcal{S}_n) \times \mathbb{T} \to \mathbb{T}$ be the holomorphic arithmetics with quotient q_n from Proposition 9. There exists a factor map $\Psi_n : \mathcal{S}_n \to \overline{\mathbb{R}}$ with the extension property such that for each $u \in \mathcal{S}_n$,*

$$\lim_{n\to\infty} F_{u_{[0,n)}}\kappa = \delta_{\Psi_n(u)}, \quad \bigcap_{k>0} V_{u_{[0,k)}} = \{\Psi_n(u)\}.$$

Proof: By (4) we have $F_1(U_0 \cap V_1) \subseteq U_0 \cap V_1$ and by (6) we have $F_0(U_0 \cap V_1) \subseteq U_1 \cap V_0$. Moreover, both F_0 and F_1 are contracting on $U_0 \cap V_1$. If $u \in \mathbb{Z}_n^{\mathbb{N}}$, then $F_{u_{k-1}}(V_{u_{k-1}}) \subseteq U_{u_{k-2}} \cap V_{u_{k-1}}$, $F_{u_{[k-2,k-1]}}(V_{u_{k-1}}) \subseteq U_{u_{k-3}} \cap V_{u_{k-2}}$, so for any $0 < j < k$ we have $F_{[j,k)}(V_{u_{k-1}}) \subseteq V_{u_j} \cap U_{u_{j-1}}$. Since $|F_{u_{[0,k-1)}}(V_{u_{k-1}})| \le \psi^k(|V_{u_{k-1}}|)$, the intersection $\cap_n V_{u_{k-1}}$ contains a unique point $\Psi_n(u)$ and $\lim_{k\to\infty} F_{u_{[0,k)}}\kappa = \delta_{\Psi_n(u)}$. Conversely let $t_0 \in \mathbb{T}$ and let u_0 be such that $t_0 \in V_{u_0}$. By (3) there exists u_1 such that $u_{[0,1]} \in \mathcal{L}(\mathcal{S}_n)$ and $t_1 = F_{u_0}^{-1}(t_0) \in U_{u_0} \cap V_{u_1}$. By (5) there exists u_2 such that $u_{[0,2]} \in \mathcal{L}(\mathcal{S}_n)$ and $t_2 = F_{u_1}^{-1}(t_1) \in U_{u_1} \cap V_{u_2}$, etc., so $t_k = F_{u_{[0,k)}}^{-1}(t_0) \in U_{u_{k-1}} \cap V_{u_k}$. Since F_0^{-1} is expansive on $V_0 \cap U_1$, for any interval $I \ni t_0$ we get $|F_{u_{[0,k)}}^{-1}(I)| \ge 2\alpha$ for all sufficiently large k. It follows that $\lim_{n\to\infty} F_{u_{[0,k)}}\kappa = \delta_{t_0}$.

Figure 1 shows Möbius transformations of some finite holomorphic numbers. The graphs on the left display the circle Möbius transformations, their densities and cylinders (botom thick line). The graphs on the right display the same data in the extended real line. The mean of a number is given by its argument, followed by radius in parenthesis. Figure 2 displays some finite holomorphic numbers placed in the unit disk according to the mean of their densities. We can see that shorter numbers are less precise and situated more to the centre of the circle. Since $\mathbf{x}(0) = 0$, $\mathbf{x}(\pi/2) = 1$, $\mathbf{x}(\pi) = \infty$ and $\mathbf{x}(3\pi/2) = -1$, we use alphabet $\{0, 1, 8, 7\}$ insted of $\mathbb{Z}_4$. Here 8 stands for ∞ and 7 for -1.

7 The octahedron subshift

In the complex domain, the stereographic projection

$$P(x+iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right), \quad P^{-1}(x,y,z) = \frac{x+iy}{1-z}$$

maps the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to the unit sphere $\mathbb{S} = \{(x,y,z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 = 1\}$. In $\overline{\mathbb{C}}$ we have Möbius transformations $m_{(a,b,c,d)}(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers with $ad - cb \neq 0$. Corresponding sphere Möbius transformations are $M_{(a,b,c,d)} = P \circ m_{(a,b,c,d)} \circ P^{-1}$. We have again contractions $C_{q,\alpha}$ to points $\alpha \in \mathbb{S}$ and their contraction and expansion disks U_α and V_α. In Figure 3, the expansion disks V_i are displayed both in the extended complex plane (left) and in the sphere (right).

The 4-ary holomorphic arithmetics, whose vertices are numbers 0, 1, ∞ and -1 can be extended to the complex sphere $\overline{\mathbb{C}}$ by adding two more vertices i and $-i$. The resulting six digits are vertices of a regular octahedron. We use octahedron alphabet $\mathbb{O} = \{0, 1, \mathrm{I}, 7, \mathrm{J}, 8\}$ which corresponds to vertices $\{0, 1, i, -1, -i, \infty\}$. Denote by $E(\mathbb{O}) = \{\{0,1\}, \{0,\mathrm{I}\}, \{0,7\}, \{0,\mathrm{J}\}, \{1,\mathrm{I}\}, \ldots\}$ the set of twelve edges of the octahedron.

Definition 11 *The octahedron subhift* $\mathcal{O} \subset \mathbb{O}^{\mathbb{N}}$ *is a SFT of order* 3 *with forbidden words* $D = \{ijk \in \mathbb{O}^3 : \; \{i,j,k\} \notin \mathbb{O} \cup E(\mathbb{O})\}$.

Thus $u \in \mathbb{O}^{\mathbb{N}}$ belongs to $\mathcal{O}$ iff each subword of u of length 3 belongs to an edge of the octahedron Transitions like 018 or 01I are forbidden.

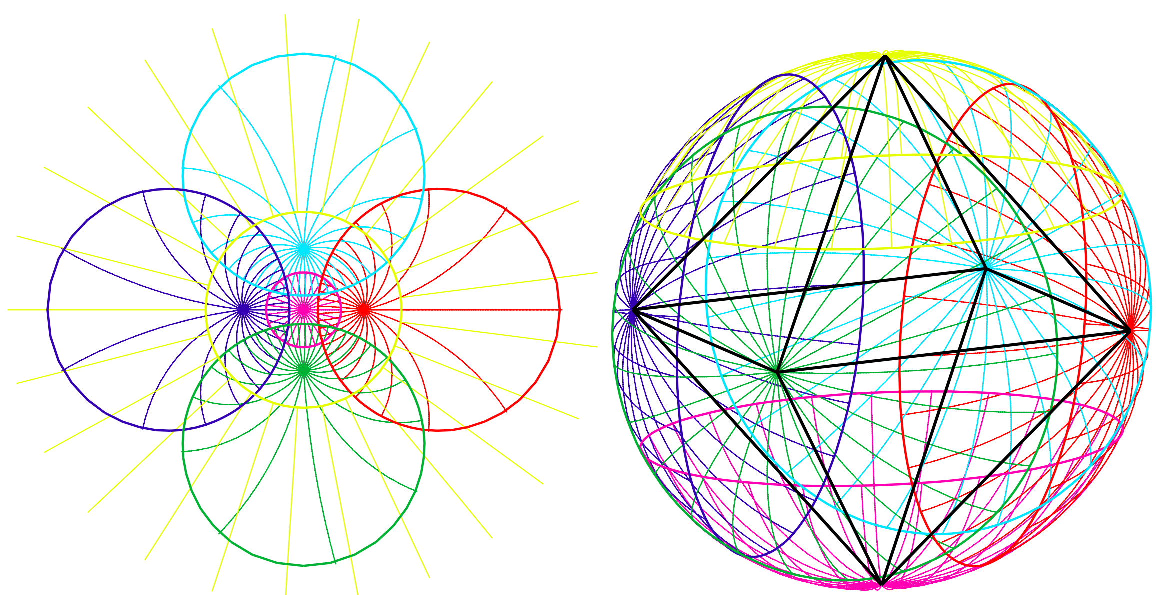


Figure 3: Expansion disks V_i of the octahedron action

Theorem 12 *Let $F : \mathcal{L}(\mathcal{O}) \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the $\mathcal{L}(\mathcal{O})$ action of contractions with quotient $q_4 = (3-\sqrt{5})/2$. There exists a factor map $\Theta : \mathcal{O} \to \overline{\mathbb{C}}$ such that for each $u \in \mathcal{O}$,*

$$\lim_{k\to\infty} F_{u_{[0,k)}}\kappa = \delta_{\Theta(u)}, \quad \bigcap_{k>0} V_{u_{[0,k)}} = \{\Theta(u)\}.$$

Proof: For $ij \in L = \mathcal{L}(\mathcal{O})$ and $jk \in L$ set

$$W_{ij} = F_j(V_i) \cap V_j \cap \left(\bigcap_{k:ijk\in L} U_k \right), \quad Z_{jk} = V_j \cap U_k \cap \left(\bigcup_{l:jkl\in L} F_k^{-1}(U_l) \right).$$

Then $\{\mathrm{int}(Z_{jk}) : jk \in L\}$ is an open cover of $\overline{\mathbb{C}}$, and for any $ijk \in L$ we have

$$F_k(W_{ij}) \subseteq W_{jk}, \quad F_j^{-1}(Z_{jk}) \subseteq \bigcup_{i:ijk\in L} Z_{ij}.$$

Some of these relations are displayed in Figures 4 and 5. The sets of the left-hand sides are in blue, and the complements of the sets in the right-hand sides are in red. Thus the two regions should not intersect. To prove convergence pick any $u \in \mathcal{O}$. Then we have $F_{u_{[2,k)}}(W_{u_{[0,1]}}) \subseteq W_{u_{[k-2,k)}}$ and F_k is contracting on $W_{u_{[k-2,k)}}$, so $F_{u_{[0,n)}}\kappa$ converges to a point measure. Conversely, to prove surjectivity, take any $x_0 \in \overline{\mathbb{C}}$. Since $\mathrm{int}(Z_{ij})$ cover $\overline{\mathbb{C}}$, there exists $u_{[0,1]} \in L$ such that $x_0 \in \mathrm{int}(Z_{u_{[0,1]}})$. There exists u_2 such that $u_{[0,2]} \in L$ and $x_1 = F_{u_0}(x_0) \in Z_{u_{[1,2]}}$. Since F_j^{-1} are expansive on Z_{jk}, we get the result.

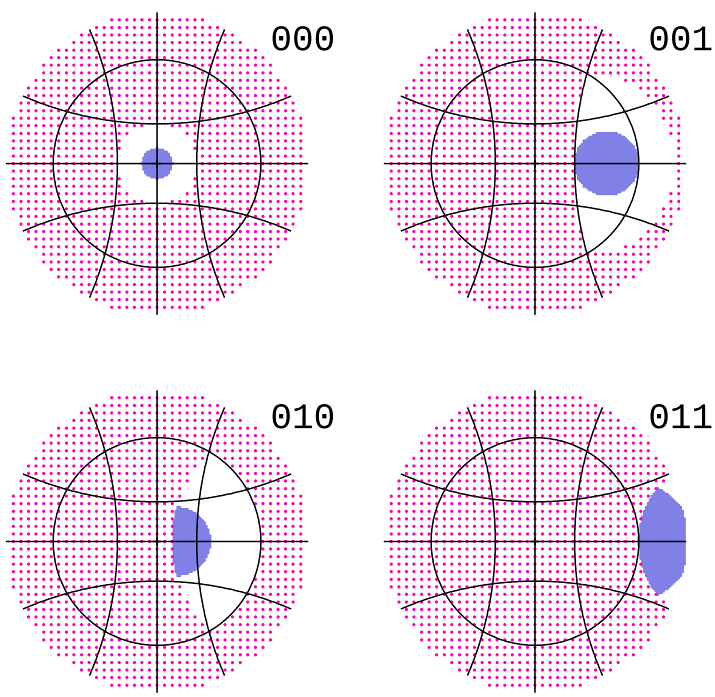


Figure 4: Forward relations for the octahedron action

8 Conclusion

Besides the cases treated in the present paper, other examples of holomorphic arithmetics might be of interest. In the one-dimensional case, the triangle seems to require even slower walk subshift of higher order. The charactreristic angle should be obtained as a solution of a higher order algebraic equation. (For the walk subshift of order 2 the equation is linear, for the slow walk subshift of

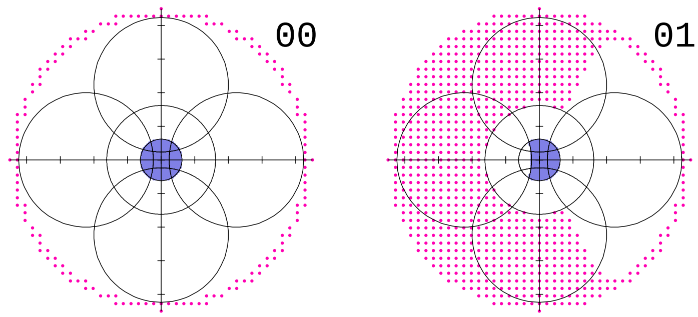


Figure 5: Backward relations for the octahedron action

order three, the equation is quadratic.) A general question is to characterize those subshifts of $\mathbb{Z}_n^{\mathbb{N}}$ which admit a holomorphic arithmetics with a factor map. In the two-dimensional case, other Platonic solids than octahedron are obvious candidates for nice holomorphic arithmetics.

A quite interesting use of the octahedron representation would be the construction of efficient algorithms for arithmetical operations. Holomorphic arithmetics could avoid the notorious overflow problems of computer arithmetics, since ∞ is a legitimate number in the system. At least two arithmetical operations are trivial in the octahedron system. To obtain the negation of a number, just interchange 1 with $7 = -1$, and $\mathrm{I} = i$ with $\mathrm{J} = -i$. To obtain the inverse element, interchange 0 with $8 = \infty$ and I with J. Other arithmetical operations are less obvious, but it is clear that there exist algorithms (recursive functions) for them. For any Möbius transformation M whose parameters are algorithmic real or complex numbers there exists an algorithm which computes a continuous function $F_M : \mathcal{O} \to \mathcal{O}$ such that $\Theta \circ F_M = M \circ \Theta$. The algorithm works on infinite words $u \in \mathcal{O}$ in the sense that for each $k \in \mathbb{N}$ it determines $m_k > 0$, and then computes $F_M(u)_k$ from $u_{[0,m_k)}$ (see Weihrauch [6]). The operations of addition and multiplication are realizable by partial recursive functions, since these operations cannot be continuously extended to whole product $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ (addition is not defined at (∞, ∞) and the multiplication is not defined at $(0, \infty)$). All these arithmetic algorithms can be described and implemented by passing through standard real or complex arithmetics. Of course, their direct combinatorial description would be much more interesting.

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