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Coexistence of Partially Disordered/Ordered Phases in an Extended Potts Model

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We consider a generalization of the standard Potts model in which there are $q = r + s$ states with an interaction that distinguishes the two subspecies. We develop a graphical representation (of the FK type) for the system and show that this representation may be incorporated directly into reflection positivity arguments. Using combinations of these techniques, we establish detailed properties of the phase diagram including the existence of sharp triple points. Whenever relevant, the phases are characterized by the percolation properties of the underlying representation.

KEY WORDS: Reflection positivity; graphical representation; Potts model.

1. INTRODUCTION

In this paper we consider a variant of the Potts model. As usual, the spins σ can take on one of q values in a set Q, but here the set Q splits into disjoint subsets R and S containing, respectively, r and s elements, i.e., $Q = R \cup S$, $q = r + s$. The Hamiltonian is given by the expression:

$$H(\boldsymbol{\sigma}) = -J \sum_{\langle x, y \rangle} \delta_{\sigma_x \sigma_y} - \kappa \sum_{\langle x, y \rangle} (\delta^R_{\sigma_x} \delta^R_{\sigma_y} + \delta^S_{\sigma_x} \delta^S_{\sigma_y}) - h \sum_x (\delta^S_{\sigma_x} - \delta^R_{\sigma_x}) \tag{1.1}$$

with κ, $J > 0$, $h \in \mathbb{R}$, and the symbol $\langle x, y \rangle$ denoting a nearest-neighbor pair on $\mathbb{Z}^d$. Here $\delta_{\sigma\sigma'} = 1$ if $\sigma = \sigma'$ and zero otherwise, δ^R_σ is the indicator of the event $\sigma \in R$ (and similarly for δ^S_σ), implying that $\delta^R_\sigma \delta^R_{\sigma'} + \delta^S_\sigma \delta^S_{\sigma'}$ vanishes unless σ and σ' belong to the same family. Notice that the second

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term causes a *repulsion* for the neighboring pairs $\langle x, y\rangle$ with $\sigma_x \in S$ and $\sigma_y \in R$. The magnetic field h acts on the entire set R or S; hence as $h \to \infty(-\infty)$ we recover the usual s (r) state Potts model. Throughout, it will be assumed that $r \gg s \gg 1$.

It is plausible to expect that the Potts part of the interaction will govern the symmetry breaking *within* the families R and S. Thence, for β large and h increasing, the system will undergo a first-order transition between a regime dominated by the R set with r ordered states and a regime dominated by the S set with s ordered states. On the other hand, for h large and positive, the system will undergo an order-disorder transition reminiscent of the s-state Potts model as β varies, and similarly an r-state Potts transition for h large negative. Thus, as h increases, a turnover from R-dominated disordered state to S-dominated disordered state can be expected at higher temperatures, with a complete symmetry maintained within each group (R or S) in both states. This latter change may or may not be signaled by a phase transition, however, certainly not for $\beta \ll 1$. For r and s large, we prove that such a transition indeed occurs at intermediate temperatures.

In the symmetric case, i.e., for $r = s$, both of these "$R \to S$" transitions occur at $h = 0$, which pretty much completes the picture. An interesting feature of the asymmetric case is that it makes conceivable, for $r > s$, a direct transition from R-disordered to S-ordered phase. Indeed, for $r \gg s$, we establish this fact along with the existence of two (sharp) triple points suggested by the presence of such a phase boundary; i.e., essentially the entire phase diagram depicted in Fig. 1 is established. As the insert shows, in the symmetric case, these triple points degenerate to a single "quadruple" point with 4 coexisting phases representing altogether $q+2$ different equilibrium states. In fact, a generalization of the symmetric case has already been discussed in [LMR], for Q decomposing into q_1 different families with each containing q_2 elements (i.e., $q = q_1 q_2$), but that only for $h = 0$. Different aspects (e.g., quasilocality) of the latter—so called "fuzzy Potts"—model have been addressed in [MVdV].

In order to facilitate our analysis, we have developed a graphical representation that is a natural generalization of the random cluster model for the Potts system. Thus there are "strong" and "weak" bonds of both the R and S type. For example, a weak R-bond insists that both end-points are in the R-set while a strong R-bond induces a matching pair of spin states of R-type. This graphical representation enjoys an FKG monotonicity that is useful for various portions of our analysis. In particular, all of the phases can be described in terms of the percolation of the corresponding bonds. However, one can only go so far with graphical representations; estimates for the probabilities of contours are also needed. In this regard, reflection

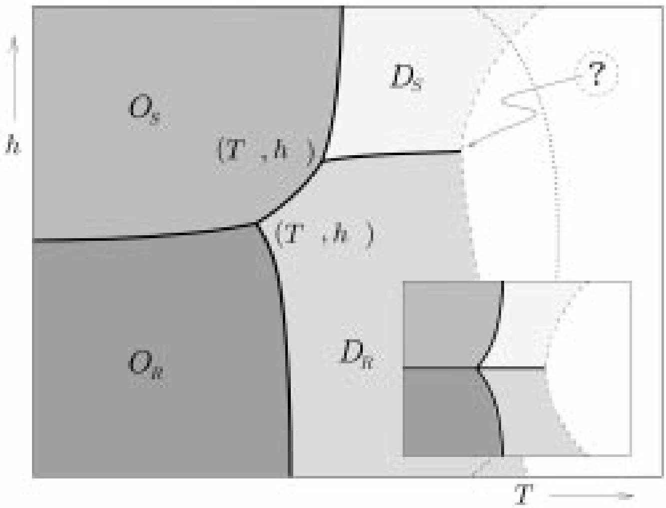

Fig. 1. Phase diagram for the model (1.1). Here $J, \kappa > 0$ are fixed, $r \gg s \gg 1$, and T is the temperature. The dashed line indicates the percolation thresholds of the "weak" bonds and the dotted line marks the area where Gibbs uniqueness follows from the high-temperature expansion. Insert shows the symmetric case $r = s \gg 1$. The question mark prompts that the suggested behavior at the endpoint of the transition line is hypothetical.

positivity (RP) methods combined with the chessboard estimate provide a rather effective service.

Somewhat to our surprise, we find that the graphical representations can be incorporated directly into the RP machinery. (*A priori*, one might imagine having to use RP on the spin system and processing this into a statement about the graphical representation.) This appears to be a promising technique—especially useful in higher dimensions—that is apparently generalizable. Thus, all in all, a rather seamless derivation is permitted. Unfortunately, there are certain limitations to this combination of RP and graphical representations. In particular, we can prove RP only for values of the parameters in the graphical representation that correspond to genuine spin-systems, i.e., integer values of r and s. In fact, this may represent a genuine "limitation:" It was recently shown by one of us [B] that the usual random cluster model with non-integer q is *not* RP in the desired sense. Here is our main result:

Main Theorem. Consider the system as described by the Hamiltonian (1.1) with $\kappa, J > 0$. Then for r, s and r/s large enough, there is a $\bar{\beta} < \infty$ and an ε small such that

(I) there are four closed and connected regions O_R, O_S, D_R, and D_S covering the set $\mathscr{J}=\{(\beta, h);\, \beta \geqslant \bar{\beta},\, h \in [-\infty, \infty]\}$, where the following translation-invariant states exist

(i) r "ordered" states $\langle\cdot\rangle^j_{O_R}$, $j=1,..., r$, in O_R

(ii) s "ordered" states $\langle\cdot\rangle^k_{O_S}$, $k=r+1,..., r+s$, in O_S

(iii) a single "disordered" state $\langle\cdot\rangle_{D_S}$ in D_S

(iv) a single "disordered" state $\langle\cdot\rangle_{D_R}$ in D_R

These states are characterized by the relations

$$\langle\delta_{\sigma_x j}\,\delta_{\sigma_x\sigma_y}\rangle^j_{O_R}\geqslant 1-\varepsilon, \qquad \langle\delta^S_{\sigma_x}\,\delta^S_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle_{D_S}\geqslant 1-\varepsilon$$

$$\langle\delta_{\sigma_x k}\,\delta_{\sigma_x\sigma_y}\rangle^k_{O_S}\geqslant 1-\varepsilon, \qquad \langle\delta^R_{\sigma_x}\,\delta^R_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle_{D_R}\geqslant 1-\varepsilon$$

valid for any pair x, y of neighbouring sites, any $1\leqslant j\leqslant r$ and any $r+1\leqslant k\leqslant r+s$.

(II) The intersections $(O_R\cup O_S)\cap(D_R\cup D_S)=\mathscr{C}_{OD}$ and $(O_R\cup D_R)\cap(O_S\cup D_R)=\mathscr{C}_{RS}$ constitute two continuous non-selfintersecting curves $\mathscr{C}_{OD}$ and $\mathscr{C}_{RS}$, where the order/disorder and the R/S states coexist, respectively. The curve $\mathscr{C}_{OD}$ admits a parametrization by βh, whereas $\mathscr{C}_{RS}$ is parametrizable by β. On the complement of these curves, the inequalities

$$\langle\delta_{\sigma_x\sigma_y}\,\delta^R_{\sigma_y}\rangle\geqslant 1-\varepsilon, \qquad \langle\delta^S_{\sigma_x}\,\delta^S_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle\geqslant 1-\varepsilon$$

$$\langle\delta_{\sigma_x\sigma_y}\,\delta^S_{\sigma_y}\rangle\geqslant 1-\varepsilon, \qquad \langle\delta^R_{\sigma_x}\,\delta^R_{\sigma_y}(1-\delta_{\sigma_x\sigma_y})\rangle\geqslant 1-\varepsilon$$

hold for every translation-invariant Gibbs state in the respective region and any pair x and y of nearest-neighbor sites.

(III) There are exactly two triple points, where $O_S+O_R+D_R$ and $O_S+D_S+D_R$ meet, respectively. These triple points are connected by a single line of coexistence between s-order and r-disorder.

(IV) In the high-temperature region (i.e., for $\beta\kappa$, $\beta J\ll 1$) the conditions of complete analyticity are met. In particular, the set of all Gibbs measures is a singleton in this region.

(V) The phases O_S and O_R can alternatively be characterized by spontaneous magnetization and/or percolation. In D_S and D_R, there is percolation of the corresponding disordered bonds.

The organization of the remainder of this paper is as follows. In Section 2 we introduce the graphical representation, establish some of its useful properties, and relate the percolation characterization of the respective phases to the non-vanishing of an order parameter. Section 3 is devoted to the proof of the Main Theorem. In this section, Lemma III.3 is of special interest for the asymetric case, because it directly rules out quadruple coexistence (whenever $r \gg s$). For reader's convenience, the computations based on chessboard estimates are performed here only in the two-dimensional case. We relegate the full (and somewhat clumsy) arguments to the Appendix.

2. THE GRAPHICAL REPRESENTATION

In this section we develop a graphical representation of the model and establish its various useful properties. To simplify our derivations, we will restrict ourselves to free (or periodic) boundary conditions in these preliminary discussions and defer the detailed analysis of boundary-condition issues to a later subsection.

Let us start with the identity

$$e^{\beta J\delta_{\sigma\sigma'}+\beta\kappa(\delta^S_\sigma\delta^S_{\sigma'}+\delta^R_\sigma\delta^R_{\sigma'})}$$
$$=1+(e^{\beta\kappa}-1)(\delta^S_\sigma\delta^S_{\sigma'}+\delta^R_\sigma\delta^R_{\sigma'})+e^{\beta\kappa}(e^{\beta J}-1)(\delta^S_{\sigma\sigma'}+\delta^R_{\sigma\sigma'}) \tag{2.1}$$

where $\delta^R_{\sigma\sigma'}$ ($\delta^S_{\sigma\sigma'}$) indicates that both spins coincide and belong to R (S). The five terms on the r.h.s. give rise to five different species of bonds—vacant, s-disorded, r-disorded, s-ordered, and r-ordered, with the prefactors representing the *a priori* weights of the corresponding bonds. For easier verbal reference, we will introduce a paralel notation in terms of colors: the five types of bonds above are called vacant, light-blue, light-red, dark-blue, and dark-red, in the order of their appearance.

We may label the five species by $\alpha\in I=\{v, s^{\mathrm{d}}, r^{\mathrm{d}}, s^{\mathrm{o}}, r^{\mathrm{o}}\}$ and define $w(\alpha)$ to be the corresponding coefficient in the identity (2.1). Thus, $w(v)=1$, $w(r^{\mathrm{d}})=w(s^{\mathrm{d}})=e^{\beta\kappa}-1$, and $w(r^{\mathrm{o}})=w(s^{\mathrm{o}})=e^{\beta\kappa}(e^{\beta J}-1)$. Let Λ denote a graph with sites $\mathbb{S}_\Lambda$ and bonds $\mathbb{B}_\Lambda$. Let $\Omega_\Lambda=I^{\mathbb{B}_\Lambda}$ denote the set of bond configurations in Λ. As will become clear, not all Ω_Λ will be used; configurations in which blue and red bonds share an endpoint need not be considered. With the above notations, $e^{-\beta H_\Lambda}$ may be written as

$$e^{-\beta H_\Lambda(\boldsymbol{\sigma})}=\sum_{\boldsymbol{\omega}\in\Omega_\Lambda}\mathbf{D}(\boldsymbol{\omega})\prod_{b\in\mathbb{B}_\Lambda}w(\omega_b)\,\chi_b(\boldsymbol{\omega},\boldsymbol{\sigma})\prod_{x\in\mathbb{S}_\Lambda}e^{\beta h(\delta^S_{\sigma_x}-\delta^R_{\sigma_x})} \tag{2.2}$$

where $\chi_b(\boldsymbol{\omega},\boldsymbol{\sigma})$ is one of the functions 1, $\delta^S_{\sigma_x}\delta^S_{\sigma_y}$, $\delta^R_{\sigma_x}\delta^R_{\sigma_y}$, $\delta^S_{\sigma_x\sigma_y}$, $\delta^R_{\sigma_x\sigma_y}$, according to the label ω_b of the bond $b=\langle x, y\rangle$, and $\mathbf{D}(\boldsymbol{\omega})$ is the indicator that $\boldsymbol{\omega}$ fulfils the aforementioned restriction.

For each such $\boldsymbol{\omega}$, we must now perform the trace (sum over spin configurations) to arrive at the weights, $W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\omega})$, for the graphical representation. The result is not particularly difficult, but only after the introduction of some additional notation. For each $\boldsymbol{\omega}$, let $\Lambda_R(\boldsymbol{\omega})$ denote the red portion of the graph: those bonds that are R-ordered or R-disordered, and all sites that are endpoints of such bonds. Let $N_R(\boldsymbol{\omega})$ denote the number of R-sites in $\Lambda_R(\boldsymbol{\omega})$. Some of the bonds in $\Lambda_R(\boldsymbol{\omega})$ are R-ordered bonds. These divide $\Lambda_R(\boldsymbol{\omega})$ into "R-connected" components; let $C_R(\boldsymbol{\omega})$ denote the number of such components. Similar notation applies to the blue portion of the configuration. Finally, let $N_\varnothing(\boldsymbol{\omega}) \equiv |\mathbb{S}_\Lambda| - [N_R(\boldsymbol{\omega}) + N_S(\boldsymbol{\omega})]$ denote the number of sites that do not fall into either category.

It is not hard to see that

$$W_{s,r;\Lambda}^{(\beta,h)}(\boldsymbol{\omega}) = \Bigl[\prod_{b\in\mathbb{B}_\Lambda} w(\omega_b)\Bigr] (se^{\beta h} + re^{-\beta h})^{N_\varnothing(\boldsymbol{\omega})}\, e^{\beta h N_S(\boldsymbol{\omega})} e^{-\beta h N_R(\boldsymbol{\omega})} s^{C_S(\boldsymbol{\omega})} r^{C_R(\boldsymbol{\omega})} \tag{2.3}$$

Indeed, each unmatched site just contributes the factor $se^{\beta h} + re^{-\beta h}$, each S-site a factor $e^{\beta h}$, and similarly for the R-sites. Finally, the ordered bonds dictate which fraction of the s^{N_S} (or r^{N_R}) spin states are actually allowed on each connected component, resulting thus in the factors $s^{C_S(\boldsymbol{\omega})}$ (and $r^{C_R(\boldsymbol{\omega})}$).

2.1. FKG Monotonicity

From the perspective of "S above R" (or blue above red), there is a natural ordering for the bond variables $s^{\mathrm{o}} \succ s^{\mathrm{d}} \succ v \succ r^{\mathrm{d}} \succ r^{\mathrm{o}}$, which induces a partial ordering on the configurations. We show that the graphical representation is monotone with respect to this ordering.

Proposition II.1. Let $\nu(\cdot) = \nu_{s,r;\Lambda}^{(\beta,h)}(\cdot)$ denote the random cluster measure on a finite graph Λ, defined according to the weights (2.3). Then ν is strong FKG w.r.t. the ordering $s^{\mathrm{o}} \succ s^{\mathrm{d}} \succ v \succ r^{\mathrm{d}} \succ r^{\mathrm{o}}$.

Proof. We must verify the FKG lattice condition. To facilitate matters, let us exchange "components for loops." Indeed, if $B_R(\boldsymbol{\omega})$ is the number of dark-red bonds and $\ell_R(\boldsymbol{\omega})$ is the number of independent loops formed by them, we may write

$$C_R(\boldsymbol{\omega}) = \ell_R(\boldsymbol{\omega}) - B_R(\boldsymbol{\omega}) + N_R(\boldsymbol{\omega}) \tag{2.4}$$

Thus, the weights may be written as

$$W_{s,\,r;\,\Lambda}^{(\beta,\,h)}(\boldsymbol{\omega}) = \left[\prod_{b\in\mathbb{B}_\Lambda} w'(\omega_b)\right](se^{\beta h}+re^{-\beta h})^{|\mathbb{S}_\Lambda|}$$
$$\times\left[\frac{se^{\beta h}}{se^{\beta h}+re^{-\beta h}}\right]^{N_S(\boldsymbol{\omega})}\left[\frac{re^{-\beta h}}{se^{\beta h}+re^{-\beta h}}\right]^{N_R(\boldsymbol{\omega})} s^{\ell_S(\boldsymbol{\omega})}r^{\ell_R(\boldsymbol{\omega})} \tag{2.5}$$

Here $w'(\omega_b)$ are changed *a priori* factors whose value will not play any role in the following.

As is well known, the verification of the FKG lattice condition may be done inductively by comparing configurations that disagree on at most two places. Thus, let $b_1, b_2 \in \mathbb{B}_\Lambda$, let $\boldsymbol{\theta}$ denote a bond configuration in $\mathbb{B}_\Lambda \backslash \{b_1, b_2\}$ and let η_1, η_2, ζ_1, and ζ_2 denote bond variables on b_1 and b_2 with $\eta_1 \succ \zeta_1$ and $\eta_2 \succ \zeta_2$. We use $(\boldsymbol{\theta}, \eta_1, \eta_2)$ to denote the configuration equal to η_1 (η_2) on b_1 (b_2) and $\boldsymbol{\theta}$ elsewhere on $\mathbb{B}_\Lambda$, and similarly for $(\boldsymbol{\theta}, \zeta_1, \zeta_2)$ etc. We must show

$$W_{s,\,r;\,\Lambda}^{(\beta,\,h)}(\boldsymbol{\theta}, \eta_1, \eta_2)\, W_{s,\,r;\,\Lambda}^{(\beta,\,h)}(\boldsymbol{\theta}, \zeta_1, \zeta_2)$$
$$\geqslant W_{s,\,r;\,\Lambda}^{(\beta,\,h)}(\boldsymbol{\theta}, \zeta_1, \eta_2)\, W_{s,\,r;\,\Lambda}^{(\beta,\,h)}(\boldsymbol{\theta}, \eta_1, \zeta_2) \tag{2.6}$$

It is clear that the *a priori* factors (i.e., the bond weights w') cancel exactly. Further, it is observed that $\mathbf{D}(\boldsymbol{\omega}_1), \mathbf{D}(\boldsymbol{\omega}_2) = 1$ imply $\mathbf{D}(\boldsymbol{\omega}_1 \vee \boldsymbol{\omega}_2)$, $\mathbf{D}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = 1$, where $\boldsymbol{\omega}_1 \vee \boldsymbol{\omega}_2$ denotes the maximum and $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2$ the minimum of the two configurations. Thus, we may omit any discussing of constraints.

We now claim that $N_S(\boldsymbol{\theta}, \eta_1, \eta_2) + N_S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \leqslant N_S(\boldsymbol{\theta}, \zeta_1, \eta_2) + N_S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Let $S(\boldsymbol{\omega}) \subset \mathbb{S}_\Lambda$ denote the set of sites that touch at least one blue bond (i.e., the site set of $\Lambda_S(\boldsymbol{\omega})$). It is not hard to see that $S(\boldsymbol{\theta}, \eta_1, \eta_2) = S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cup S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Indeed, $S(\boldsymbol{\theta}, \eta_1, \eta_2) \supset S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ (because the former contains more S-bonds), and similarly with $S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. So $S(\boldsymbol{\theta}, \eta_1, \eta_2)$ contains the union of both. Now suppose that $x \in S(\boldsymbol{\theta}, \eta_1, \eta_2)$. If this is caused by an S-bond in $\boldsymbol{\theta}$, then $x \in S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ *and* $x \in S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. If not, then x is either the endpoint of one (or both) of η_1 or η_2—say η_1, in which case x belongs to $S(\boldsymbol{\theta}, \eta_1, \zeta_2)$—or the end-point of ξ_1 or ξ_2, and then it also pertains to both sets. On the other hand, we only claim that $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cap S(\boldsymbol{\theta}, \eta_1, \zeta_2)$ (with the inclusion sometimes being strict). This is clear since $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \zeta_1, \eta_2)$ and $S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \subset S(\boldsymbol{\theta}, \eta_1, \zeta_2)$. Hence we have

$$
\begin{aligned}
N_S(\boldsymbol{\theta}, \eta_1, \eta_2) &+ N_S(\boldsymbol{\theta}, \zeta_1, \zeta_2) \\
&= |S(\boldsymbol{\theta}, \eta_1, \eta_2)| + |S(\boldsymbol{\theta}, \zeta_1, \zeta_2)| \\
&\leqslant |S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cup S(\boldsymbol{\theta}, \eta_1, \zeta_2)| + |S(\boldsymbol{\theta}, \zeta_1, \eta_2) \cap S(\boldsymbol{\theta}, \eta_1, \zeta_2)| \\
&= |S(\boldsymbol{\theta}, \zeta_1, \eta_2)| + |S(\boldsymbol{\theta}, \eta_1, \zeta_2)| = N_S(\boldsymbol{\theta}, \zeta_1, \eta_2) + N_S(\boldsymbol{\theta}, \eta_1, \zeta_2) \qquad (2.7)
\end{aligned}
$$

with the third line by inclusion-exclusion.

Thus we are down to showing the necessary inequalities among the loop counting functions. However, most of these are actually equalities; the only exception is when η_1, η_2, ζ_1, and ζ_2 are all bonds of the same family in which case the desired inequality reduces to the usual argument for the random cluster model. ▮

As an immediate consequence we obtain a domination comparison:

Corollary. If $h^{(1)} > h^{(2)}$, then

$$\nu_{s,\, r;\, \Lambda}^{(\beta,\, h^{(1)})}(\,\cdot\,) \geqslant_{\mathrm{FKG}} \nu_{s,\, r;\, \Lambda}^{(\beta,\, h^{(2)})}(\,\cdot\,)$$

Proof. It suffices to show that $W_{s,\, r;\, \Lambda}^{(\beta,\, h^{(1)})}(\boldsymbol{\omega})/W_{s,\, r;\, \Lambda}^{(\beta,\, h^{(2)})}(\boldsymbol{\omega})$ is an increasing function (assuming that both quantities are nonzero). Using formula (2.5) and defining

$$e^{-\alpha_S(h)} \equiv \frac{se^{\beta h}}{se^{\beta h} + re^{-\beta h}} \qquad \text{and} \qquad e^{-\alpha_R(h)} \equiv \frac{re^{-\beta h}}{se^{\beta h} + re^{-\beta h}} \tag{2.8}$$

we have

$$\frac{W_{s,\, r;\, \Lambda}^{(\beta,\, h^{(1)})}(\boldsymbol{\omega})}{W_{s,\, r;\, \Lambda}^{(\beta,\, h^{(2)})}(\boldsymbol{\omega})} = \Phi_\Lambda(h^{(1)}, h^{(2)}) \left[\frac{e^{-\alpha_R(h^{(1)})}}{e^{-\alpha_R(h^{(2)})}}\right]^{N_R(\boldsymbol{\omega})} \left[\frac{e^{-\alpha_S(h^{(1)})}}{e^{-\alpha_S(h^{(2)})}}\right]^{N_S(\boldsymbol{\omega})} \tag{2.9}$$

where $\Phi_\Lambda(h^{(1)}, h^{(2)})$ is a number independent of $\boldsymbol{\omega}$ (note that the modified weights w' are independent of the external field). It is thus sufficient to establish that $e^{-\alpha_R(h)}$ is monotone decreasing and $e^{-\alpha_S(h)}$ is monotone increasing as functions of h. This is easily checked. ▮

Remark. It is noted that the measures are perfectly well defined for non-integer r and s and that the above monotonicities hold for all r and s with $r, s \geqslant 1$. However, in this paper, we will need to make explicit use of the underlying spin-system; hence, we will not discuss these more general cases. Finally, we remark that all of the results of this subsection hold for an arbitrary (finite) graph with arbitrary fields h_i and arbitrary (non-negative) couplings $J_{x,y}$ and $\kappa_{x,y}$. In particular, FKG dominations are established under the condition that $h_x^{(1)} \geqslant h_x^{(2)}$ for all x.

2.2. Reflection Positivity

We begin with some preliminary notations. Let $\mathscr{T}_L$ denote a d-dimensional (lattice) torus, assumed for convenience to have an even number L of sites in all directions, and let P denote the intersection of $\mathscr{T}_L$ with a hyperplane orthogonal to one of the coordinate directions. (Thus P consists of two disconnected "planes of sites.") The set P divides $\mathscr{T}_L$ into left ($\mathscr{T}_L^{\mathscr{L}}$) and right ($\mathscr{T}_L^{\mathscr{R}}$) halves—*both* halves defined to include P.

Let $\boldsymbol{\omega} \in \Omega_{\mathscr{T}_\Lambda}$ denote bond configurations (such as the ones described above) on $\mathscr{T}_L$ and let $\mathscr{F}_{\mathscr{L}}$ ($\mathscr{F}_{\mathscr{R}}$) be the set of functions of bond configurations restricted to $\mathscr{T}_L^{\mathscr{L}}$ ($\mathscr{T}_L^{\mathscr{R}}$). Finally, let ϑ_P be the reflection operator that maps bond configuration on the left to configurations on the right and vice versa (i.e., $\boldsymbol{\omega}_{\mathscr{L}} \leftrightarrow \vartheta_P \boldsymbol{\omega}_{\mathscr{L}}$). For $f \in \mathscr{F}_{\mathscr{R}}$, define $\vartheta_P f \in \mathscr{F}_{\mathscr{L}}$ by $[\vartheta_P f](\boldsymbol{\omega}_{\mathscr{R}}) = f(\vartheta_P \boldsymbol{\omega}_{\mathscr{L}})$. Let $\mathbb{P}(\cdot)$ denote a probability measure on bond configurations and $\mathbb{E}$ the expectation with respect to this measure. Then $\mathbb{P}$ is said to be *reflection positive* if for every $g, f \in \mathscr{F}_{\mathscr{R}}$ one has

$$\mathbb{E}(f \vartheta_P g) = \mathbb{E}(g \vartheta_P f) \tag{2.10}$$

and

$$\mathbb{E}(f \vartheta_P f) \geqslant 0 \tag{2.11}$$

Proposition II.2. Let $\mathscr{T}$ denote the d-dimensional torus and let $\nu(\cdot) \equiv \nu_{s,r;\mathscr{T}}^{(\beta,h)}(\cdot)$ denote the random cluster measures as defined by the weights (2.3). Then ν is reflection positive with respect to reflections through all planes P containing sites.

Proof. The proof follows from the reflection positivity of a certain joint measure on bond-site configurations—the Edwards–Sokal measure—that contains the random cluster measure and the Gibbs measure of the considered spin system as marginals. This measure is essentially defined by the r.h.s. of (2.2). Namely, let us write

$$\begin{aligned} W_{\mathrm{ES},s,r;\mathscr{T}}^{(\beta,h)}(\boldsymbol{\omega},\boldsymbol{\sigma}) &= \left[\mathbf{D}(\boldsymbol{\omega}) \prod_{b \in \mathbb{B}_{\mathscr{T}}} \chi_b(\boldsymbol{\omega},\boldsymbol{\sigma})\right] \prod_{b \in \mathbb{B}_{\mathscr{T}}} w(\omega_b) \prod_{x \in \mathbb{S}_{\mathscr{T}}} e^{\beta h(\delta^S_{\sigma_x} - \delta^R_{\sigma_x})} \\ &\equiv \mathbf{D}(\boldsymbol{\omega},\boldsymbol{\sigma}) \prod_{b \in \mathbb{B}_{\mathscr{T}}} w(\omega_b) \prod_{x \in \mathbb{S}_{\mathscr{T}}} e^{\beta h(\delta^S_{\sigma_x} - \delta^R_{\sigma_x})} \end{aligned} \tag{2.12}$$

The weights $W^{(\beta,h)}_{\mathrm{ES},s,r;\mathscr{T}}$ define the Edwards–Sokal measure $\nu^{(\beta,h)}_{\mathrm{ES},s,r;\mathscr{T}}$ abbreviated hereafter as ν_{ES}. In the above language, Eqs. (2.2) and (2.3) read

$$e^{-\beta H_{\mathscr{T}}(\boldsymbol{\sigma})} = \sum_{\boldsymbol{\omega}} W^{(\beta,h)}_{\mathrm{ES},s,r;\mathscr{T}}(\boldsymbol{\omega},\boldsymbol{\sigma}) \tag{2.13a}$$

and

$$W^{(\beta,h)}_{\mathrm{ES},s,r;\mathscr{T}}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\sigma}} W^{(\beta,h)}_{\mathrm{ES},s,r;\mathscr{T}}(\boldsymbol{\omega},\boldsymbol{\sigma}) \tag{2.13b}$$

which verifies the preceding claim concerning the marginals. Our proof amounts to showing that the full ES-measure is reflection positive; here, of course, with respect to reflection operators acting on the larger space of bond-spin configurations and functions thereof. (Notwithstanding, we will make no notational distinctions.) Thus, let f denote a function that is determined by the bond-spin configurations on the "right" and consider $\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f)$, where $\mathbb{E}_{\mathrm{ES}}(\cdot)$ denotes expectation with respect to the measure ν_{ES}. Let $(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ denote a bond-spin configuration in the plane P; that is to say a spin value on each $x\in P$ and a bond value on each $\langle x,y\rangle$ with $x,y\in P$. We may write

$$\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f) = \sum_{(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)} \nu_{\mathrm{ES}}(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\,\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P) \tag{2.14}$$

The conclusion now follows from the observation that the restrictions of the conditional measures $\nu_{\mathrm{ES}}(\cdot\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ to the left and right half of the torus are independent and identical under reflections. Indeed, the "identical under reflections" property is an obvious consequence of the underlying symmetry of the model. Independence is established as follows:

Let us write $(\boldsymbol{\omega},\boldsymbol{\sigma}) = (\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P;\boldsymbol{\omega}_{\mathscr{R}},\boldsymbol{\sigma}_{\mathscr{R}};\boldsymbol{\omega}_{\mathscr{L}},\boldsymbol{\sigma}_{\mathscr{L}})$ corresponding to the configurations in P, $\mathscr{T}^{\mathscr{R}}_L$, and $\mathscr{T}^{\mathscr{L}}_L$, respectively. Let $\mathbf{D}_P(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ denote the analogue of the function $\mathbf{D}(\boldsymbol{\omega},\boldsymbol{\sigma})$, defined in (2.14), that here checks the consistency of the configuration only in P. Further, let $\mathbf{D}_{\mathscr{R}}(\boldsymbol{\omega}_{\mathscr{R}},\boldsymbol{\sigma}_{\mathscr{R}}\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ be the function that indicates consistency only for the right half of the configuration given $(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$ and similarly for $\mathbf{D}_{\mathscr{L}}(\boldsymbol{\omega}_{\mathscr{L}},\boldsymbol{\sigma}_{\mathscr{L}}\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)$. It is not hard to check that

$$\mathbf{D}(\boldsymbol{\omega},\boldsymbol{\sigma}) = \mathbf{D}_P(\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\,\mathbf{D}_{\mathscr{R}}(\boldsymbol{\omega}_{\mathscr{R}},\boldsymbol{\sigma}_{\mathscr{R}}\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P)\,\mathbf{D}_{\mathscr{L}}(\boldsymbol{\omega}_{\mathscr{L}},\boldsymbol{\sigma}_{\mathscr{L}}\mid\boldsymbol{\omega}_P,\boldsymbol{\sigma}_P) \tag{2.15}$$

for every $(\boldsymbol{\omega}, \boldsymbol{\sigma})$. By checking the terms in (2.6), we easily see that the weights factorize, which is equivalent to independence. Thus we have

$$\mathbb{E}_{\mathrm{ES}}(f\vartheta_P f) = \sum_{(\boldsymbol{\omega}_P, \boldsymbol{\sigma}_P)} \nu_{\mathrm{ES}}(\boldsymbol{\omega}_P, \boldsymbol{\sigma}_P)[\mathbb{E}_{\mathrm{ES}}(f \mid \boldsymbol{\omega}_P, \boldsymbol{\sigma}_P)]^2 \tag{2.16}$$

which is manifestly non-negative. Similarly one establishes

$$\mathbb{E}_{\mathrm{ES}}(f\vartheta_P g) = \mathbb{E}_{\mathrm{ES}}(g\vartheta_P f) \tag{2.17}$$

and the proof is complete. ▮

The use of RP for establishing the existence of discontinuous transitions is based on two standard Lemmas [FL, KS]. Here "behavioral pattern" refers to a particular set of configurations (typically on a box). Let $\mathscr{T}_L$ denote a torus of size L and let $\langle\cdot\rangle_L \equiv \langle\cdot\rangle^{(\beta,h)}_{s,r;\mathscr{T}_L}$ be a state on the configurations in $\mathbb{B}_{\mathscr{T}_L}$.

Lemma II.3. Let $\{c_\ell\}$ be a collection of (possibly overlapping) cubes of size one, and consider a behavioral pattern b_ℓ associated with each cube c_ℓ. Let $\chi_{b_\ell}(c_\ell)$ indicate the occurence of b_ℓ on c_ℓ. If $\langle\cdot\rangle_L$ is reflection positive and L is even, then

$$\left\langle \prod_\ell \chi_{b_\ell}(c_\ell)\right\rangle_L \leqslant \prod_\ell (\langle \chi_{b_\ell}(\mathscr{T}_L)\rangle_L)^{1/|L^d|}$$

where $\chi_{b_\ell}(\mathscr{T}_L)$ enforces the appropriate mirror image of the pattern b_ℓ to all translates of c_ℓ.

Lemma II.4. Let a and b denote two distinct patterns on a cube $c \in \mathscr{T}_L$. Let H be a Hamiltonian that depends on the parameter α that varies in the range $[\alpha_a, \alpha_b]$, and let $\langle\cdot\rangle_{L,\alpha}$ denote the Gibbs state on $\mathscr{T}_L$ induced by the Hamiltonian H at the parameter value α. Let $A \in (\frac{1}{2}, 1]$ and $B \in [0, \frac{1}{4}]$ be such that $B \leqslant [\frac{1}{2} - \sqrt{\frac{1}{2} - (A/2)}]^2$, and let $\varepsilon_a, \varepsilon_b \in (0, \frac{1}{2})$. Suppose that for all $\alpha \in [\alpha_a, \alpha_b]$, all $c, \tilde{c} \in \mathscr{T}_L$, and all L large enough

(0) $\chi_a(c)\,\chi_b(c) = 0$,

(i) $\langle \chi_a(c) + \chi_b(c)\rangle_{L,\alpha} \geqslant A$,

(ii) $\langle \chi_a(c)\,\chi_b(\tilde{c})\rangle_{L,\alpha} \leqslant B$,

and

(iiia) $\langle \chi_a(c)\rangle_{L,\alpha_a} > 1 - \varepsilon_a$

(iiib) $\langle \chi_b(c)\rangle_{L,\alpha_b} > 1 - \varepsilon_b$.

Then there is an $\alpha_c \in [\alpha_a, \alpha_b]$ and two distinct translation invariant Gibbs states $\langle \cdot \rangle_{\alpha_c}^{a}$ and $\langle \cdot \rangle_{\alpha_c}^{b}$ such that

$$\langle \chi_a(c) \rangle_{\alpha_c}^{a} \geqslant 1 - \bar{\varepsilon} \qquad \text{and} \qquad \langle \chi_b(c) \rangle_{\alpha_c}^{b} \geqslant 1 - \bar{\varepsilon}$$

where $\bar{\varepsilon} = \bar{\varepsilon}(A, B)$ is such that $\bar{\varepsilon} \to 0$ as $A \to 1$ and $B \to 0$.

Remark. We remark that in the above formulation it has been assumed that we are dealing with Gibbs distributions defined according to some particular Hamiltonian. In fact this is not really necessary in order to ensure that Lemma II.4 goes through. However, we may circumvent any possible difficulties by the observation that our system—bonds and all—can be obtained by considering the (annealed) bond-diluted version of the given Hamiltonian in the limit of zero temperature. When the bonds are integrated out, we recover the original model at some effective finite temperature. However, if we keep track of the bonds (and integrate out the spins) we find that these are distributed according to the desired random cluster measures.

2.3. Boundary Conditions

As already stated, graphical representation (2.3) will be our major tool of study of the Gibbs phases associated with the Hamiltonian (1.1). However, we first have to take properly into account the effect of boundary conditions. In particular, we have to clarify to what extent one can generalize the FKG domination arguments from the previous subsections. There are two ways that a random-cluster measure can be associated with a boundary condition. These lead eventually to two classes of random cluster measures: $\mathfrak{S}$-measures and $\mathfrak{G}$-measures, with the former defined by prescribing a *spin* boundary condition, whereas the latter is defined by prescribing a *graphical* boundary condition.

(1) $\mathfrak{S}$-measures. Given a finite set $\Lambda \subset \mathbb{Z}^d$, let $\partial\Lambda$ be its boundary, i.e., the set of sites in $\mathbb{Z}^d$ whose distance from Λ equals one. To implement the first possibility, we take a spin configuration $\tilde{\boldsymbol{\sigma}}$ and define the Edwards–Sokal measure on $\boldsymbol{\sigma}$'s in $\Lambda \cup \partial\Lambda$ and $\boldsymbol{\omega}$'s on the bonds thereof, however, with the spins at $\partial\Lambda$ fixed to $\tilde{\boldsymbol{\sigma}}$. Let $\nu_{\Lambda, \tilde{\boldsymbol{\sigma}}}$ denote the $\boldsymbol{\omega}$-marginal of this measure. Of particular interest are the limits of these measures as $\Lambda \nearrow \mathbb{Z}^d$: we denote by $\mathfrak{S}$ the set of all possible accumulation points, closed under convex combinations and closed in the weak topology on probability measures. We call these $\mathfrak{S}$-measures (they were called equilibrium random cluster measures in [ACCN]).

Our major object of interest is the original *spin* system (1.1). Even though all properties of the graphical marginals are derived starting from the spin system, not all information is retained by the marginals. This may actually cast doubts whether the representation (i.e., the $\mathfrak{S}$-measures) is still capable of capturing the important features of the spin system, e.g., the non-uniqueness of Gibbs states. This is (partially) answered in the following.

Lemma II.5. The existence of two distinct $\mathfrak{S}$-measures μ_1 and μ_2 implies the existence of two distinct Gibbs measures ν_1 and ν_2 for the Hamiltonian (1.1). Moreover, if μ_1 and μ_2 can be obtained for (expanding) sequences $\{\Lambda_n^{(1)}\}$, $\{\Lambda_n^{(2)}\}$, and boundary conditions $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, respectively, then ν_1 and ν_2 can be generated by $(\{\Lambda_n^{(1)}\}, \boldsymbol{\sigma}_1)$ and $(\{\Lambda_n^{(2)}\}, \boldsymbol{\sigma}_2)$, respectively, as (possibly subsequential) limits of finite-volume states.

Proof. See Appendix.

Let $\mathfrak{S}^\star$ be the translation invariant measures in $\mathfrak{S}$. Note that any accumulation point μ of the torus states is an $\mathfrak{S}^\star$-measure. Namely, if we go to a subsequence for which also the distribution functions for the *spins* at the exterior boundary of any finite volume converge, we see that the expectation of any cylinder function of bonds in the set A can be written as the combination of ($\mathfrak{S}$-class) states in any Λ encompassing A, where the spin boundary condition is now a subject of average. Moreover, the expectation is independent of Λ and thus only the tail of the boundary condition matters, but in that case the average runs over infinite-volume $\mathfrak{S}$-class measures, so μ is indeed an $\mathfrak{S}$-measure. The translation invariance of μ is trivial.

(2) $\mathfrak{G}$-measures. To implement the other possibility, i.e., to define boundary conditions directly in the graphical representation, one may explicitly consider a "graphical" boundary configuration $\tilde{\boldsymbol{\omega}}$, modify appropriately the weights in (2.3) for clusters that stick out of the finite volume so that a consistent family of measures is recovered, and then study the DLR measures associated therewith. While the techniques based on the DLR condition are well developed in the spin language (see, e.g., [Ge]), for the random-cluster measures the theory is still rather "weak in the knees," mainly for the lack of quasilocality (see [vEFS] Section 4.5.3 for a concrete problem of the latter kind, whose solution was given in [Gr] and [PVdV]). Therefore, we refrain from discussing it in its full generality.[5] On the other hand, we would like to have some way of incorporating explicit "graphical" boundary conditions into our considerations, because only then the full

[5] See [Gr] and [BC] for statements employing the DLR structure of the random cluster measures. An alternative is to rely on the DLR structure of Edwards–Sokal states—this is the approach used in [BBCK].

power of FKG-ordering can be employed. To this end, we will consider a restricted class of boundary conditions.

Let $\partial\mathbb{B}(\Lambda)$ to denote the (exterior) bond-boundary of Λ, i.e., the set of bonds with one end in $\partial\Lambda$ and the other in Λ. A boundary condition (i.e., a "graphical" configuration on $\partial\mathbb{B}(\Lambda)$) for the random cluster problem is said to be of the $\mathfrak{G}$-class[6] if

> $\partial\Lambda$ is divided into disjoint components $C_1^B,..., C_k^B$; $C_1^R,..., C_\ell^R$ and $C_1^f,..., C_t^f$. Each of the components $C_1^R,..., C_\ell^R$ act as a single site but always in a red state and similarly $C_1^B,..., C_k^B$ act as single blue sites and, finally, the components $C_1^f,..., C_t^f$ act as "free" sites with some *a priori* weights for red or blue.

Given a boundary condition from $\mathfrak{G}$-class, the weight (2.3) is modified to the effect that $C_S(\boldsymbol{\omega})$ and $C_R(\boldsymbol{\omega})$ are now counting the connected components *including* the boundary components. The infinite-volume states generated by a $\mathfrak{G}$-boundary condition from thus modified finite-volume measures will be denoted (with a slight abuse of notation) also by $\mathfrak{G}$.

It is clear from the definition of the $\mathfrak{G}$ measures that they can all be generated by convex combinations of measures with a spin boundary condition. Hence, $\mathfrak{G} \subset \mathfrak{S}$. On the other hand, while *constant* red (blue) boundary condition gives rise to red-(blue-)wired $\mathfrak{G}$-states $\nu_{\Lambda,\,\text{red-w}}^{(\beta,h)}$ ($\nu_{\Lambda,\,\text{blue-w}}^{(\beta,h)}$) (the "wiring" refers to connecting all boundary sites by bonds of the respective dark color to an auxiliary site), other spin boundary conditions do not necessarily yield a measure in the $\mathfrak{G}$-class. Indeed, if in the spin system one sets half of the boundary spins to one of the red-type spin-states and the other half to another type of red-state, the resulting random cluster measure is the red-wired measure conditioned on having no dark-red connection between the two halves. The conditioning here is crucial and one cannot specify this measure using only $\mathfrak{G}$-boundary conditions.

The fact that $\mathfrak{G}$ does not embody all Gibbs measures of interest is an unpleasant problem for techniques based on graphical representations. These difficulties (which have been encountered in other systems, c.f. [C]) can, to some extent, be circumvented but only after a certain amount of work. For our purposes, the key practical implication here is the lack of FKG (or at least of its proof) outside the $\mathfrak{G}$-class. (In the $\mathfrak{G}$-class, wherein each of the boundary conditions can be simulated on an extended graph, FKG follows from the observation that the FKG lattice condition is in power even after conditioning on a set of bonds taking a definite value.)

We close this section with the statement of two lemmas that concern ordering and uniqueness of the $\mathfrak{S}$-measures. In addition to the aesthetic

[6] A related notion in [BC] is the wiring diagram.

appeal, the following will be crucial in our subsequent analysis of the phase diagram. Both proofs are, for brevity of exposition, relegated to the Appendix. Note that the limits

$$\nu^{(\beta,h)}_{\text{blue-w}} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \nu^{(\beta,h)}_{\Lambda,\,\text{blue-w}} \qquad \text{and} \qquad \nu^{(\beta,h)}_{\text{red-w}} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \nu^{(\beta,h)}_{\Lambda,\,\text{red-w}} \tag{2.18}$$

exist by FKG and are measures both of $\mathfrak{G}$ and $\mathfrak{S}^{\star}$-class.

Lemma II.6. Consider the system defined by the Hamiltonian (1.1) with β fixed and h varying in $(-\infty, +\infty)$. Then for Lebesgue-a.e. h, both the sets of $\mathfrak{G}$-measures and $\mathfrak{S}^{\star}$-measures are singletons (and then necessarily $\mathfrak{G} = \mathfrak{S}^{\star}$).

Lemma II.7. Let $\nu^{(\beta,h)}$ be either a $\mathfrak{G}$-measure or a $\mathfrak{S}^{\star}$-measure. Then

$$\nu^{(\beta,h)}_{\text{blue-w}}(\cdot) \geqslant_{\text{FKG}} \nu^{(\beta,h)}(\cdot) \geqslant_{\text{FKG}} \nu^{(\beta,h)}_{\text{red-w}}(\cdot)$$

Moreover, for $h^{(1)} > h^{(2)}$, any $\mathfrak{S}^{\star}$-measure at $h^{(1)}$ FKG dominates every $\mathfrak{S}^{\star}$-measure at $h^{(2)}$, and similarly for $\mathfrak{G}$.

2.4. Percolation and Magnetization

The phases that we will study are all characterized by "percolation" of one sort or another. *A priori*, there are five distinct situations: Percolation of dark-blue bonds, percolation of blue bonds without percolation of dark-blue bonds, similarly for the reds, and no percolation at all. However, some caution is needed: the partially ordered phases (here defined by the intermediate sort of percolations) certainly do not represent genuine thermodynamic phases, except perhaps when they coexist. For this reason, it is mainly the dark-bond percolation that plays the role of an order parameter.

Given β and h, let $\Pi^{B}_{\infty}(\beta, h)$ ($\Pi^{R}_{\infty}(\beta, h)$) be the probabilities that dark-blue (dark-red) bonds percolate under the measures $\nu^{(\beta,h)}_{\text{blue-w}}$ ($\nu^{(\beta,h)}_{\text{red-w}}$, respectively). It is an easy corollary to Lemma II.7 that Π^{B}_{∞} is actually the maximal probability at which dark-blue bonds can percolate under any $\mathfrak{G} \cup \mathfrak{S}^{\star}$-measure, and similarly for Π^{R}_{∞}. As is standard, Π^{B}_{∞} and Π^{R}_{∞} are easily related to the spin order parameters, namely, to the blue and red magnetization. Here the red magnetization is defined by adding a symmetry breaking term of the type $-\mathfrak{h}^{R} \sum_x [\delta_{\sigma_x j} - (1/r)\, \delta^{R}_{\sigma_x}]$ (with $j \in R$) to the Hamiltonian and calculating the derivative of the free energy evaluated at $\mathfrak{h}^{R} = 0^{+}$. Similarly for the blue magnetization.

Proposition II.8. Consider the system on $\mathbb{Z}^d$ described by the Hamiltonian (1.1) and let $M^R(\beta, h)$ and $M^B(\beta, h)$ denote the red and blue spontaneous magnetizations. Then

$$M^R(\beta, h) = \frac{r-1}{r}\, \Pi^R_\infty(\beta, h)$$

and

$$M^B(\beta, h) = \frac{s-1}{s}\, \Pi^B_\infty(\beta, h)$$

Proof. As follows from convexity of the free energy in the parameter $\mathfrak{h}^R$, the red magnetization can alternatively be defined by optimizing $\delta_{\sigma_0 j} - (1/r)\,\delta^R_{\sigma_0}$ over all possible translation-invariant Gibbs states (for an argument proving an analogous assertion see the proof of Lemma II.7 in the Appendix). Given β and h, all Gibbs states are parametrized by the spin boundary conditions. Fixing the boundary condition to $\tilde{\boldsymbol{\sigma}}$, it is straightforward to establish that the expected value of $\delta_{\sigma_0 j} - (1/r)\,\delta^R_{\sigma_0}$ under the Potts measure exactly equals $(1-(1/r))$-times the probability under the corresponding graphical marginal that x is connected by a path of dark-red bonds to an j component of $\tilde{\boldsymbol{\sigma}}$, i.e.,

$$\langle \delta_{\sigma_0 r_1} - (1/r)\,\delta^R_{\sigma_0} \rangle^{\tilde{\boldsymbol{\sigma}}}_{\Lambda} = \left(1 - \frac{1}{r}\right) \nu^{(\beta, h)}_{\tilde{\boldsymbol{\sigma}}, \Lambda}(\{0 \xrightarrow[\text{dark-red}]{} \partial\Lambda_j(\tilde{\boldsymbol{\sigma}})\}) \tag{2.19}$$

Here $\partial\Lambda_j(\tilde{\boldsymbol{\sigma}}) = \{y \in \partial\Lambda : \tilde{\sigma}_y = j\}$, $\langle\cdot\rangle^{\tilde{\boldsymbol{\sigma}}}_{\Lambda}$ is the spin state in Λ with boundary condition $\tilde{\boldsymbol{\sigma}}$ and $\nu^{(\beta, h)}_{\tilde{\boldsymbol{\sigma}}, \Lambda}$ is the corresponding $\mathfrak{S}$-measure.

The r.h.s. of (2.19) is dominated by the expectation of the local event that there is a dark-red path running from 0 outside a fixed volume $\Delta \subset \Lambda$. By taking expectation of both sides w.r.t. a translation invariant spin Gibbs measure and by taking the limit $\Lambda \nearrow \mathbb{Z}^d$ (possibly along a subsequence), we recover an $\mathfrak{S}^\star$-measure on the r.h.s. which is FKG dominated by the red-wired measure from Lemma II.7. Since $\{0 \xrightarrow[\text{dark-red}]{} \Delta^c\}$ is an increasing event, the limit $\Delta \nearrow \mathbb{Z}^d$ then shows that

$$\langle \delta_{\sigma_0 r_1} - (1/r)\,\delta^R_{\sigma_0} \rangle \leqslant \left(1 - \frac{1}{r}\right) \Pi^R_\infty(\beta, h) \tag{2.20}$$

for any translation invariant spin Gibbs state $\langle\cdot\rangle$. However, this inequality is clearly saturated for the state $\langle\cdot\rangle^j$ generated by a constant configuration

$\tilde{\boldsymbol{\sigma}} \equiv j$, which is translation invariant. Hence, $M^R(\beta, h) = ((r-1)/r)\, \Pi^R_\infty(\beta, h)$ as we were to prove. The case of blue magnetization is completely analogous. ∎

3. THE PROOF OF THE MAIN THEOREM

3.1. Preliminaries

The proof of the Main Theorem hinges on RP techniques as applied to graphical representations. In order to apply Lemma II.3 and II.4, we first need some definitions.

We will consider various behavioral patterns which are either *good* or *bad*. The good behavioral patterns will be denoted by $\mathscr{O}_{\mathscr{R}}$, $\mathscr{O}_{\mathscr{S}}$, $\mathscr{D}_{\mathscr{R}}$, $\mathscr{D}_{\mathscr{S}}$; these are defined by the property that every bond in a unit cube is of the corresponding type: r-ordered, s-ordered, r-disordered and s-disordered, respectively. Any other configuration on a cube is deemed to be bad. We recall that two bonds of differing color cannot share a vertex and hence the presence of two colors on a cube necessitates the intervention of a vacant bond. As a consequence, badness can occur for only one of two reasons, the occurrence of a vacant bond or a pair of adjacent bonds, one ordered, the other disordered, that are both of the *same* color. (In the latter case, we call such pairs *mismatched*.)

The good patterns are expected to dominate typical configurations in the low and intermediate temperature regimes. However, as requited by Lemma II.4(ii), simultaneous occurrence of different good patterns should be (sufficiently) improbable. The proof of the latter invokes the observation that a "barrier" of bad cubes necessarily separates any pair of different good cubes. Let us aggregate bad cubes that are joined through at least one edge into connected components that are called, traditionally, *contours*. Item (ii) of Lemma II.4 then boils down to showing that the probability that a contour "encircles" the origin is small. It is this step where the chessboard estimates (Lemma II.3) provide an effective service. In particular, all we need to show is that the *estimate* on the probability of the various contour elements is small—the rest is easily reduced to a counting argument which is identical to the one for the Ising contours.

As seen on the right hand side of the display in Lemma II.3, the relevant objects are the partition functions constrained so that the stated pattern repeats periodically. Of fundamental importance in the present analysis are the partition functions associated with the (purported) favored patterns. These will be denoted by $\mathscr{Z}_{\mathscr{O}_{\mathscr{R}}}$, $\mathscr{Z}_{\mathscr{O}_{\mathscr{S}}}$, $\mathscr{Z}_{\mathscr{D}_{\mathscr{R}}}$, and $\mathscr{Z}_{\mathscr{D}_{\mathscr{S}}}$, respectively (the dependence on the scale of the torus will always be understood from the context and will be supressed notationally). These objects are readily computed:

$$\mathscr{Z}_{\mathcal{O}_{\mathscr{R}}} = [e^{-\beta h} e^{d\beta\kappa} (e^{\beta J} - 1)^d]^{L^d} r \tag{3.1i}$$

$$\mathscr{Z}_{\mathcal{O}_{\mathscr{S}}} = [e^{\beta h} e^{d\beta\kappa} (e^{\beta J} - 1)^d]^{L^d} s \tag{3.1ii}$$

$$\mathscr{Z}_{\mathscr{D}_{\mathscr{R}}} = [re^{-\beta h} (e^{\beta\kappa} - 1)^d]^{L^d} \tag{3.1iii}$$

and

$$\mathscr{Z}_{\mathscr{D}_{\mathscr{S}}} = [se^{\beta h} (e^{\beta\kappa} - 1)^d]^{L^d} \tag{3.1iv}$$

Of further interest, there is the "vacant" partition function given by

$$\mathscr{Z}_v = [re^{-\beta h} + se^{\beta h}]^{L^d} \tag{3.1v}$$

The subject of our next lemma is that for all β sufficiently large (and for r and s large) the probability of any bad cube is small. Even though the general proof is not too involved, we relegate it to the Appendix. Here we provide the proof for $d=2$ that is particularly simple due to the employment of the diagonal torus (the SST) which reduces the problem to estimating single-bond events.

Lemma III.1. Let $\delta > 0$. Then there exist $\bar{r} = \bar{r}(d, \delta)$, $\bar{s} = \bar{s}(d, \delta)$, and $\bar{\beta} = \bar{\beta}(\kappa, d, \delta)$ so that for any $\kappa > 0$, $r \geqslant \bar{r}$, $s \geqslant \bar{s}$, and $\beta \geqslant \bar{\beta}$ such that the probability of a bad cube is less than δ. Namely, $\lim_{L\to\infty} \langle \chi_{\mathcal{O}_{\mathscr{R}}} + \chi_{\mathcal{O}_{\mathscr{S}}} + \chi_{\mathscr{D}_{\mathscr{R}}} + \chi_{\mathscr{D}_{\mathscr{S}}} \rangle_L > 1 - \delta$.

Proof (d=2). By the discussion in the second paragraph of the present section, it is only necessary to show that the probability of a vacant bond or a mismatched pair is small. Let us start with a vacant bond. We use standard reflection positivity arguments for the case of two-dimensional diagonal torus (see [S] for a full discussion or [CM] Lemma 4.3 for a proof along these lines complete with pictures). Let $\langle i, j \rangle$ denote any bond in $\mathscr{T}_L$ (the diagonal torus of size L). Reflecting the bond $n+1$ times where $n = \lfloor 2 \log_2 L \rfloor$ we obtain the estimate for the vacant bond:

$$\langle \mathbf{D}(\omega_{\langle i, j \rangle} = v) \rangle_L \leqslant \left(\frac{\mathscr{Z}_v}{\mathscr{Z}} \right)^{1/2L^2} = \frac{(se^{\beta h} + re^{-\beta h})^{1/2}}{\mathscr{Z}^{1/2L^2}} \tag{3.2}$$

An estimate for $\mathscr{Z}$ is provided by the r and/or s disordered partition functions, $\mathscr{Z} \geqslant \mathscr{Z}_{\mathscr{D}_{\mathscr{R}}} + \mathscr{Z}_{\mathscr{D}_{\mathscr{S}}}$. Since $[(se^h + re^{-h})^{L^2}]/[(se^h)^{L^2} + (re^{-h})^{L^2}] \leqslant 2^{L^2}$, we have

$$\langle \mathbf{D}(\omega_{\langle i, j \rangle} = v) \rangle_L \leqslant \frac{\sqrt{2}}{e^{\beta\kappa} - 1} \tag{3.3}$$

Next we consider the mismatched pairs. Focusing attention, say, on the s-type, let $\langle i, j\rangle$ and $\langle i, j'\rangle$ denote an adjacent pair of bonds. We will consider the event $\{\omega_{\langle i, j\rangle} = s^{\mathrm{d}}, \omega_{\langle i, j'\rangle} = s^{\mathrm{o}}\}$. If we agree to use only $\mathscr{Z}_{\mathscr{D}_{\mathscr{S}}}$ and $\mathscr{Z}_{\mathscr{O}_{\mathscr{S}}}$ in the lower bound of the partition function, a moments thought shows that the calculation is identical to the s-state Potts model. Thus we get

$$\lim_{L\to\infty} \langle \mathbf{D}(\omega_{\langle i, j\rangle} = s^{\mathrm{d}}, \omega_{\langle i, j'\rangle} = s^{\mathrm{o}})\rangle_L \leqslant s^{-1/4} \tag{3.4}$$

The argument is similar for the r-mismatched pairs. ∎

A second ingredient needed to set Lemma II.4 in motion is the bound (ii).

Lemma III.2. There exists a function $\varepsilon(d, \delta)$, such that $\varepsilon(d, \delta) \to 0$ as $\delta \to 0$, and a constant $\bar{\delta}(d)$ such that for any two distinct patterns $\mathscr{P}_1, \mathscr{P}_2 \in \{\mathscr{O}_{\mathscr{R}}, \mathscr{O}_{\mathscr{S}}, \mathscr{D}_{\mathscr{R}}, \mathscr{D}_{\mathscr{S}}\}$ one has

$$\langle \chi_{\mathscr{P}_1}(c)\, \chi_{\mathscr{P}_2}(\tilde{c})\rangle_L \leqslant \varepsilon(d, \delta)$$

whenever $0 < \delta < \bar{\delta}$, $\kappa > 0$, $r \geqslant \bar{r}$, $s \geqslant \bar{s}$, and $\beta \geqslant \bar{\beta}$ (with $\bar{r}$, $\bar{s}$, and $\bar{\beta}$ as in Lemma III.1).

Proof. As already noted before, by going along any connected path from c to $\tilde{c}$, one eventually bumps into a bad cube. Hence, on torus of size L, if the two patterns are to occur simultaneously, the cubes c and $\tilde{c}$ have to be separated by a contour consisting of bad cubes (that either surrounds one of the cubes or winds around the torus). The probability of such a contour is directly estimated by chessboard estimates: for a contour composed of $|\gamma|$ bad cubes we get the bound $\delta^{|\gamma|}$.

To get an explicit expression of the function $\varepsilon(d, \delta)$, it is actually convenient to consider just the surface of the above union of bad cubes, which is an Ising contour attached to the faces of the bad cubes in γ. Since there are at most $2d$ plaquettes per each cube in γ, each plaquette carries at most the weight $\delta^{1/2d}$. By employing the recent Lebowitz–Mazel [LM] estimate on the number of Ising contours (which is asymptotically optimal as $d \to \infty$), we arrive at the expression

$$\varepsilon(d, \delta) = (2 + o(C(d)\, \delta^{1/2d}))\, \delta^{2d} + (1 + o(C(d)\, \delta^{1/2d}))\, \tfrac{1}{2} d(d-1)\, L^d\, \delta^{L^{d-1}} \tag{3.5}$$

Here $C(d) = \exp(O(\log d/d))$ is the connectivity constant for the Ising contours.

Indeed, by Corollary 1.2 of [LM], the contribution to $\langle \chi_{\mathscr{P}_1}(c)\, \chi_{\mathscr{P}_2}(\tilde{c}) \rangle_L$ corresponding to a contour surrounding either c or $\tilde{c}$ is dominated by the lowest-order term, provided $C(d)\, \delta^{1/2d} \ll 1$. Hence, one gets $2\delta^{2d}$, where the "2" accounts for the uncertainty whether the contour runs around c or $\tilde{c}$. The case of a contour wrapped around the torus is fairly analogous; one only needs to observe that the Lebowitz–Mazel counting argument requires the contours neither to be closed nor to be encircling a given point —it is only required that the contour contains a given plaquette. Hence, also in this case the lowest-order term dominates, yielding $\frac{1}{2}d(d-1)\, L^d\, \delta^{L^{d-1}}$, where the prefactor counts the possible positions of the plaquette. ▌

3.2. Proof of (I) and (II)

We will start by setting the parameters. It will be assumed that κ and J are fixed and strictly positive, whereas r and s are to be adjusted such that the technical ingredients (i.e., Lemmas III.1 and III.2) are in power. The quantity δ will be our generic "small parameter," i.e., a number chosen small enough so that Lemma II.4 yields, in conjuction with Lemmas III.1 and III.2, the needed bound. In particular, $\delta \ll C(d)^{-2d}$ must be assumed.

For a fixed δ, let the numbers $r \geqslant s$ be such that $r \geqslant \bar{r}(d, \delta)$ and $s \geqslant \bar{s}(d, \delta)$, respectively, and the assumptions

(1) $\bar{\beta}(\kappa, d, \delta) \ll (1/Jd) \log s$

(2) $\bar{\varepsilon}(1-\delta, 3\delta^{2d}) \max\{1, \kappa/J\} \ll 1$

hold (with $\bar{\beta}(\kappa, d, \delta)$ from Lemma III.1 and $\bar{\varepsilon}(1-\delta, 3\delta^{2d})$ from Lemma II.4).

Remark. Since we are clearly about to use Lemma II.4 in the context of graphical representations, the reader may have doubts whether this result is really applicable, when the Gibbsianness (i.e., the property *defined* by stipulating the DLR condition) of the limiting states is questionable. But this is only due to the wording of the statement of Lemma II.4—as the proof shows, actually the existence of two distinct *limiting* states is established, obtained by conditioning from torus states. In particular, the states emerging from Lemma II.4 are of the $\mathfrak{S}^\star$-class, because they are limits of the torus states conditioned on densities of the respective patterns having large enough value. Since any finite volume can be omitted while evaluating the latter densities, the same argument we used to prove that torus states are $\mathfrak{S}^\star$-class applies also to these conditional measures.

The proof now comes in two stages.

(1) $\mathscr{R}$–$\mathscr{S}$ Transition. Let $\chi_{\mathscr{R}} = \chi_{\mathscr{D}_{\mathscr{R}}} + \chi_{\mathscr{O}_{\mathscr{R}}}$ be the indicator for a red cube and similarly for $\chi_{\mathscr{S}}$. Since $\beta \geqslant \bar{\beta}$, we have by Lemma III.1 that

$\langle \chi_{\mathscr{R}} + \chi_{\mathscr{S}} \rangle_L^{\beta, h}$ is close to one for any h. Clearly, as $h \to \infty$, the mean value $\langle \chi_{\mathscr{R}} \rangle_L^{\beta, h}$ is supressed and as $h \to -\infty$, the mean value $\langle \chi_{\mathscr{S}} \rangle_L^{\beta, h}$ vanishes—both of these uniformly in L. The conditions of Lemma II.4 are met; for each $\beta \geqslant \bar{\beta}$, there is an h_β with $|h_\beta| < \infty$ at which R and S-type phases coexist: there exist two $\mathfrak{S}^\star$-states $\langle \cdot \rangle_{\mathscr{R}}^{\beta, h_\beta}$ and $\langle \cdot \rangle_{\mathscr{S}}^{\beta, h_\beta}$ satisfying

$$\langle \chi_{\mathscr{R}} \rangle_{\mathscr{R}}^{\beta, h_\beta} \geqslant 1 - \varepsilon' \tag{3.6r}$$

and

$$\langle \chi_{\mathscr{S}} \rangle_{\mathscr{S}}^{\beta, h_\beta} \geqslant 1 - \varepsilon' \tag{3.6s}$$

with $\varepsilon' = \bar{\varepsilon}(1 - \delta, 3\delta^{2d})$.

Although the conditions of Lemma II.4 do not rule out the existence, for a given β, of several such points (i.e., "reentrance"), the FKG monotonicity (as proved in Lemma II.7) resoundingly does. Indeed, any $\mathfrak{S}^\star$-state at parameters (β, h) with $h > h_\beta$ will FKG dominate the state $\langle \cdot \rangle_{\mathscr{S}}^{\beta, h_\beta}$ and similarly for $h < h_\beta$.

We now claim that the function h_β is continuous. Indeed let $\beta_* > \bar{\beta}$ and consider a sequence $\{\beta_k\}$ with $\beta_k \to \beta_*$. Crude estimates show that the $|h_{\beta_k}|$ are uniformly bounded so let $-\infty < h_* < \infty$ denote an accumulation point of $\{h_{\beta_k}\}$. Now at the points (β_k, h_{β_k}), we have the $\mathscr{S}$-state where the inequality (3.6s) holds. It follows by a compactness argument that there exists a $\mathfrak{S}^\star$-state at (β_*, h_*), where the same inequality holds. Thus $h_* \geqslant h_{\beta_*}$. Using the same argument with the r's, we conclude that $h_* \leqslant h_{\beta_*}$ and continuity is established. The curve given by the function h_β will constitute our $\mathscr{C}_{RS}$.

(2) Order-Disorder Transition. The situation with the order-disorder transition is similar; the disadvantage is that we do not have an FKG monotonicity as the temperature varies. This is remedied by convexity arguments, based on the fact that β couples almost directly to the relevant observables.

It will be useful to consider the events $\mathfrak{b}_{\mathscr{O}}$ and $\mathfrak{b}_{\mathscr{D}}$ that a given bond is ordered/disordered respectively: $\mathfrak{b}_{\mathscr{O}} = \mathfrak{b}_{\mathscr{O}_{\mathscr{S}}} \cup \mathfrak{b}_{\mathscr{O}_{\mathscr{R}}}$ and similarly for $\mathfrak{b}_{\mathscr{D}}$. As above, we may also define the indicators $\chi_{\mathscr{O}}$ and $\chi_{\mathscr{D}}$ for events on cubes. However we claim that for $\beta \geqslant \bar{\beta}$ the quantities $\langle \chi_{\mathscr{O}} \rangle_L^{\beta, h}$ and $\langle \chi_{\mathfrak{b}_{\mathscr{O}}} \rangle_L^{\beta, h}$ are essentially interchangable. Indeed, first $\langle \chi_{\mathscr{O}} \rangle_L^{\beta, h} \leqslant \langle \chi_{\mathfrak{b}_{\mathscr{O}}} \rangle_L^{\beta, h}$. On the other hand, writing $\langle \chi_{\mathscr{O}} \rangle_L^{\beta, h} = \langle \chi_{\mathfrak{b}_{\mathscr{O}}} \rangle_L^{\beta, h} \langle \chi_{\mathscr{O}} \mid \mathfrak{b}_{\mathscr{O}} \rangle_L^{\beta, h}$ and using FKG and Lemma III.2 we get $\langle \chi_{\mathscr{O}} \rangle_L^{\beta, h} \geqslant \langle \chi_{\mathfrak{b}_{\mathscr{O}}} \rangle_L^{\beta, h} (1 - \delta)$.

For this portion of the proof, we will keep $\tilde{h} = \beta h$ fixed and allow β to vary in $[\bar{\beta}, \infty)$. It follows by inspection of (3.1i–v) that, provided the assumption (1) above holds, the variables $\chi_{\mathfrak{b}_{\mathscr{O}}}$ and $\chi_{\mathfrak{b}_{\mathscr{D}}}$ satisfy the conditions of Lemma II.4 and thus there is a $\beta_{\tilde{h}}$ at which two $\mathfrak{S}^\star$-states $\langle\cdot\rangle_{\mathscr{D}}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}$ and $\langle\cdot\rangle_{\mathscr{O}}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}$ coexist satisfying $\langle\chi_{\mathfrak{b}_{\mathscr{O}}}\rangle_{\mathscr{O}}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}\geqslant 1-\varepsilon'$ and $\langle\chi_{\mathfrak{b}_{\mathscr{D}}}\rangle_{\mathscr{D}}^{\beta_{\tilde{h}},\tilde{h}/\beta_{\tilde{h}}}\geqslant 1-\varepsilon'$.

Consider the "bond densities" $\rho_{\mathscr{O}}(\beta,h)$ and $\rho_{\mathscr{D}}(\beta,h)$ representing the thermodynamic density of bonds of the two types. These objects are well defined in most states—certainly in translation invariant states where they equal to the expectations of $\mathfrak{b}_{\mathscr{O}}$ and $\mathfrak{b}_{\mathscr{D}}$—but their value may depend on the state. To account notationally for such a case, we will indicate different states by a superscript, e.g., $\rho^*_{\mathscr{O}}(\beta,h)$. However, the densities for different states are not completely uncorrelated: an examination of the weights in (2.1) shows that the quantity

$$\mathscr{Q}(\beta,h)=\left(k+J\frac{e^{\beta J}}{e^{\beta J}-1}\right)\rho_{\mathscr{O}}(\beta,h)+\kappa\frac{e^{\beta\kappa}}{e^{\beta\kappa}-1}\rho_{\mathscr{D}}(\beta,h)$$
$$\equiv\mathscr{A}_{\mathscr{O}}(\beta)\,\rho_{\mathscr{O}}(\beta,h)+\mathscr{A}_{\mathscr{D}}(\beta)\,\rho_{\mathscr{D}} \tag{3.7}$$

has the property that if $\beta_1>\beta_2$, then for any translation-invariant states $*$ and $\#$,

$$\mathscr{Q}^*(\beta_1,\tilde{h}/\beta_1)\geqslant\mathscr{Q}^\#(\beta_2,\tilde{h}/\beta_2) \tag{3.8}$$

This follows because $\mathscr{Q}$ represents the derivative of the free energy with respect to β with $\tilde{h}=\beta h$ held fixed. Note that $*,\#\in\mathfrak{S}$ is not required.

In what follows, the treatment is slightly simplified by assuming that $\kappa\leqslant J$; we will proceed under this assumption and, at the end, discuss briefly the complementary case. Let $\beta>\beta_{\tilde{h}}$ with $\tilde{h}=\beta h$ and let $*$ denote the translation-invariant state at (β,h) designed to minimize $\rho_{\mathscr{O}}$. Then

$$\mathscr{A}_{\mathscr{O}}(\beta)\,\rho^*_{\mathscr{O}}(\beta)+\mathscr{A}_{\mathscr{D}}(\beta)\,\rho^*_{\mathscr{D}}(\beta)\geqslant\mathscr{Q}^{\mathscr{O}}(\beta_{\tilde{h}})\geqslant\mathscr{A}_{\mathscr{O}}(\beta_{\tilde{h}})(1-\varepsilon') \tag{3.9}$$

where the suppressed h's in all arguments are evaluated at $h=\tilde{h}/\beta_{\tilde{h}}$. Since both $\mathscr{A}_{\mathscr{O}}$ and $\mathscr{A}_{\mathscr{D}}$ decrease with β we may replace on the left side β by $\beta_{\tilde{h}}$:

$$[\mathscr{A}_{\mathscr{O}}(\beta_{\tilde{h}})-\mathscr{A}_{\mathscr{D}}(\beta_{\tilde{h}})]\,\rho^*_{\mathscr{O}}(\beta)+\mathscr{A}_{\mathscr{D}}(\beta_{\tilde{h}})\geqslant\mathscr{A}_{\mathscr{O}}(\beta_{\tilde{h}})(1-\varepsilon') \tag{3.10}$$

where we have further used that $\rho^*_{\mathscr{O}}(\beta)+\rho^*_{\mathscr{D}}(\beta)\leqslant 1$. If $J\geqslant\kappa$ then all the quantities $(e^{\beta J}-1)^{-1}$, $(e^{\beta\kappa}-1)^{-1}$ are uniformly small by the condition that all inverse temperatures are larger than $\bar{\beta}$. Since $\mathscr{A}_{\mathscr{O}}(\beta)-\mathscr{A}_{\mathscr{D}}(\beta)=J(1-e^{-\beta J})^{-1}-\kappa(e^{\beta\kappa}-1)^{-1}\approx J$, we arrive at $\rho^*_{\mathscr{O}}(\beta)\geqslant 1-\varepsilon$ with $\varepsilon\approx\varepsilon'[J+\kappa]/J$.

The argument for $\beta<\beta_{\tilde{h}}$ is similar but requires the additional ingredient that for $\beta>\bar{\beta}$, vacant bonds are rare in *any* $\mathfrak{S}^\star$-state. (Of course we already know this for the limit of any torus state.) We will be content to prove this statement away from the coexistence line $\mathscr{C}_{RS}$. Consider the point $(\beta, \tilde{h}/\beta)$ with $\beta < \tilde{h}/h_\beta$ (i.e., "above" $\mathscr{C}_{RS}$). Now, at the point (β, h_β) the $\mathscr{S}$-state has a blue bond density in excess of $1-\varepsilon'$. Since the (blue-ordered $\cup$ blue-disordered)-bond event is clearly increasing, this probability is larger in any $\mathfrak{S}^\star$-state at $(\beta, \tilde{h}/\beta)$, by the assumption upon β. Thus the vacant bond density is less than ε'. A similar argument using in turn red-bond density shows that vacant bonds are uniformly rare in all $\mathfrak{S}^\star$-states at $(\beta, \tilde{h}/\beta)$ with $\beta > \tilde{h}/h_\beta$ (i.e., "below" $\mathscr{C}_{RS}$).

Now consider $\beta < \beta_{\tilde{h}}$ and assume that $(\beta, \tilde{h}/\beta) \notin \mathscr{C}_{RS}$. Let # denote the state at $(\beta, \tilde{h}/\beta)$ designed to maximize $\rho_{\mathscr{O}}$ within the $\mathfrak{S}^\star$-class. We have

$$\begin{aligned}\varepsilon' \mathscr{A}_{\mathscr{O}}(\beta_{\tilde{h}}) + \mathscr{A}_{\mathscr{D}}(\beta_{\tilde{h}}) \geqslant \mathscr{Q}^{\mathscr{D}}(\beta_{\tilde{h}}) &\geqslant \mathscr{A}_{\mathscr{O}}(\beta)\, \rho^{\#}_{\mathscr{O}}(\beta) + \mathscr{A}_{\mathscr{D}}(\beta)\, \rho^{\#}_{\mathscr{D}}(\beta) \\ &= (\mathscr{A}_{\mathscr{O}}(\beta) - \mathscr{A}_{\mathscr{D}}(\beta))\, \rho^{\#}_{\mathscr{O}}(\beta) + (\rho^{\#}_{\mathscr{O}}(\beta) + \rho^{\#}_{\mathscr{D}}(\beta))\, \mathscr{A}_{\mathscr{D}}(\beta)\end{aligned} \tag{3.11}$$

where we have again supressed the h dependence in our arguments. Using the fact that the density of vacants is always less than ε', we have $\rho^{\#}_{\mathscr{O}}(\beta) + \rho^{\#}_{\mathscr{D}}(\beta) \geqslant 1 - \varepsilon'$ and thus

$$\varepsilon'(\mathscr{A}_{\mathscr{O}}(\beta_{\tilde{h}}) + \mathscr{A}_{\mathscr{D}}(\beta)) + \mathscr{A}_{\mathscr{D}}(\beta_{\tilde{h}}) - \mathscr{A}_{\mathscr{D}}(\beta) \geqslant (\mathscr{A}_{\mathscr{O}}(\beta) - \mathscr{A}_{\mathscr{D}}(\beta))\, \rho^{\#}_{\mathscr{O}}(\beta) \tag{3.12}$$

Notice that $\mathscr{A}_{\mathscr{D}}(\beta_{\tilde{h}}) - \mathscr{A}_{\mathscr{D}}(\beta)$ is negative (and anyway small). We obtain $\rho^{\#}_{\mathscr{O}}(\beta) \leqslant \varepsilon$, with $\varepsilon \approx \varepsilon'(J + 2\kappa)/J$.

If $\kappa/J > 1$ the argument is pretty much the same only we cannot immediately discard terms like $e^{-\beta J}$ on the grounds that $\beta > \bar{\beta}$. Two ingredients are required: First, the δ parameter must be made small enough (i.e., s must be large enough) so that when the ε' term emerges from the Lemma II.4, the quantity $\varepsilon'\kappa/J$ is still small. Second, the value of $\bar{\beta}$ must be trimmed so that it can be stipulated that for $\beta > \bar{\beta}$ the quantity $(e^{\beta J} - 1)$ is no smaller than, say unity. Under these conditions, the proof follows *mutatis mutandis*.

The proof of the continuity of $\beta_{\tilde{h}}$ is the same as for h_β; thus we have our curve $\mathscr{C}_{OD}$. These two curves define our four regions: O_S, O_R, D_R and D_S. In the interior of these regions, the characterizations corresponding to the bounds in (II) of the Main Theorem are clearly satisfied. Namely, if $\beta < \beta_{\tilde{h}}$ and $h > h_\beta$, then just about all of the bonds are disordered and blue in any $\mathfrak{S}^\star$-state. This means that in all translation invariant Gibbs spin states, each of them being paired with a translation-invariant $\mathfrak{S}$-state in an Edwards–Sokal measure, satisfy $\langle \delta_{\sigma_x \sigma_y} \delta^S_{\sigma_x} \rangle \geqslant 1 - \varepsilon$. The proof of the other inequalities is similar. The existence of r and s separate magnetized states in their respective ordered regions is obvious: All of these emerge from the torus states. (These will be discussed further below in the *Proof of (IV) and (V).*). ▌

3.3. Proof of (III)

On the basis of crude estimates, both functions $\tilde{h} \mapsto \beta_{\tilde{h}}$ and $\beta \mapsto h_\beta$ are bounded away from the boundary of $\mathcal{J}$, hence, it is clear that they must coincide at least at one point. When $r = s$, both $\mathscr{C}_{OD}$ and $\mathscr{C}_{RS}$ enjoy the flip symmetry $h \to -h$, which entails that $\mathscr{C}_{RS}$ lies on $h = 0$ and is intersected by $\mathscr{C}_{OD}$ at *exactly* one (quadruple-coexistence) point—this establishes the phase diagram in the symmetric case. Insofar we have not yet made use of the condition $r/s \gg 1$. It is this condition that forces non-trivial coincidence of the curves and, consequently, the two triple points. The existence of the triple points is a consequence of the following claim.

Lemma III.3. Suppose that $r \gg s$ and consider the quantities $\langle \chi_{\mathcal{O}_\mathscr{R}} \rangle_L$, $\langle \chi_{\mathcal{O}_\mathscr{S}} \rangle_L$, $\langle \chi_{\mathscr{D}_\mathscr{R}} \rangle_L$ and $\langle \chi_{\mathscr{D}_\mathscr{S}} \rangle_L$. Then for all β and h, at least one of these objects is small. In particular, let Θ_L denote the chessboard estimate for $\min\{\langle \chi_{\mathcal{O}_\mathscr{S}} \rangle_L, \dots, \langle \chi_{\mathscr{D}_\mathscr{S}} \rangle_L\}$:

$$\Theta_L = \left[\frac{\min\{\mathscr{Z}_{\mathcal{O}_\mathscr{R}}, \mathscr{Z}_{\mathcal{O}_\mathscr{S}}, \mathscr{Z}_{\mathscr{D}_\mathscr{R}}, \mathscr{Z}_{\mathscr{D}_\mathscr{S}}\}}{\mathscr{Z}_{\mathcal{O}_\mathscr{R}} + \mathscr{Z}_{\mathcal{O}_\mathscr{S}} + \mathscr{Z}_{\mathscr{D}_\mathscr{R}} + \mathscr{Z}_{\mathscr{D}_\mathscr{S}}} \right]^{1/L^d}$$

Then there is $L_0 = L_0(r/s)$ such that

$$\Theta_L \leqslant 2\sqrt{\frac{s}{r}}$$

for all $L \geqslant L_0$.

Proof. We first invite the reader to reexamine the weights for $\mathscr{Z}_{\mathcal{O}_\mathscr{S}}, \dots, \mathscr{Z}_{\mathscr{D}_\mathscr{S}}$ in (3.1). If $h \leqslant 0$, then $\mathscr{Z}_{\mathscr{D}_\mathscr{S}}/\mathscr{Z}_{\mathscr{D}_\mathscr{R}} \leqslant (s/r)^{L^d}$ and we are done. If $h > 0$, we may have $se^{\beta h} \geqslant re^{-\beta h}$. But then $\mathscr{Z}_{\mathcal{O}_\mathscr{R}}/\mathscr{Z}_{\mathcal{O}_\mathscr{S}} = r(e^{-2\beta h})^{L^d}/s \leqslant (s/r)^{L^d - 1}$ and we are again done, provided L is large enough. Thus suppose $re^{-\beta h} > se^{\beta h}$. We have

$$\begin{aligned} \Theta_L^{L^d} &\leqslant \frac{[\mathscr{Z}_{\mathcal{O}_\mathscr{R}} \mathscr{Z}_{\mathscr{D}_\mathscr{S}}]^{1/2}}{\mathscr{Z}_{\mathcal{O}_\mathscr{S}} + \mathscr{Z}_{\mathscr{D}_\mathscr{R}}} = \frac{r^{1/2}([ABs]^{1/2})^{L^d}}{s[Ae^{\beta h}]^{L^d} + [Bre^{-\beta h}]^{L^d}} \\ &= \left(\frac{s}{r}\right)^{1/2(L^d - 1)} \frac{r^{1/2}([ABr]^{1/2})^{L^d}}{s^{1/2}[Ae^{\beta h}]^{L^d} + s^{-1/2}[Bre^{-\beta h}]^{L^d}} \end{aligned} \tag{3.13}$$

where we have abbreviated $A = [e^{\beta\kappa}(e^{\beta J} - 1)]^d$ and $B = [e^{\beta\kappa} - 1]^d$. By noting that the second ratio at the extreme right is less than a half, the proof is over once L is large enough. ∎

For the duration of this portion of the proof, it is more convenient to use the parametrization by β and $\tilde{h}$, where $\tilde{h} = \beta h$. The argument comes again in two steps.

(1) Lower Triple Point. In the region $\beta \gg 1$ the curve $\mathscr{C}_{RS}$ separates O_R from O_S and in the region $\tilde{h} \ll -1$, the curve $\mathscr{C}_{OD}$ separates O_R from D_R. Let us follow the curves from these extreme ranges of parameters until they first touch. This point will be denoted by $\odot \equiv (\beta^{\odot}, \tilde{h}^{\odot})$. At $\odot$, we have coexistence of O_R, O_S and D_R phases. Note, as is thus clear by Lemma III.3, that in a neighborhood of $\odot$, the *chessboard estimate* for $\langle \chi_{\mathscr{D}_{\mathscr{S}}} \rangle_L$ is small. (In fact, chessboard estimates are continuous in all parameters. The existence of nearby regions of states of the other types forces the small one to be D_S by Lemma III.3) We will successively establish three claims:

(A) The "unused" portions of $\mathscr{C}_{RS}$ and $\mathscr{C}_{OD}$ lie in the quadrant $Q = \{\tilde{h} > \tilde{h}^{\odot}, \beta < \beta^{\odot}\}$.

(B) In this quadrant, in a neighborhood of $\odot$, the two curves coincide.

(C) The point $\odot$ is the unique triple point where O_R, O_S and D_R coexist.

Let us start on (A). Consider the line segment $\{\tilde{h} = \tilde{h}^{\odot}, \bar{\beta} \leqslant \beta < \beta^{\odot}\}$. This is a line where *only* disordered states can exist because one exists at $\odot$ and hence for all higher temperatures with the same value of $h^{\odot}$. Similarly, consider the line segment $\{\beta = \beta^{\odot}, \tilde{h} > \tilde{h}^{\odot}\}$. Here only O_S states can exist by FKG-domination from the previous subsection (note that $h > \tilde{h}^{\odot}/\beta^{\odot}$ on this segment). Consequently, the unused parts of both $\mathscr{C}_{OD}$ and $\mathscr{C}_{RS}$ must avoid these segments. Since $\mathscr{C}_{OD}$ is parametrizable by $\tilde{h}$ and $\mathscr{C}_{RS}$ by β, these parts have to lie inside the quadrant defined by these segments.

We now claim that in this quadrant, in a neighborhood of $\odot$, the curve $\mathscr{C}_{OD}$ cannot rise above $\mathscr{C}_{RS}$. Indeed, let

$$Q^{+} = \{(\beta, \tilde{h}) \in Q : \tilde{h} > \beta h_{\beta}\} \tag{3.14}$$

By definition, everything in Q^{+} is $\mathscr{S}$-type. If $\mathscr{C}_{OD}$ has a point inside Q^{+} then in a neighborhood of this point, there is a region of (exclusive) D_S states. Here we have invoked the continuity of $\tilde{h} \mapsto \beta_{\tilde{h}}$, i.e., the function that determines $\mathscr{C}_{OD}$, and the fact that Q^{+} is an open set. However, if we are sufficiently close to $\odot$, continuity of the chessboard estimate forbids at least the torus state from being a D_S state. By exactly the same argument as the rareness of vacant bonds in any $\mathfrak{S}^\star$-state was established in the paragraph right after (3.9), this implies that the probability of $\mathscr{D}_{\mathscr{S}}$ patterns is uniformly low near $\odot$ in Q^+. Hence, $\mathscr{C}_{OD}$ stays below $\mathscr{C}_{RS}$ in the vicinity of $\odot$ in Q.

Finally we will show that in Q, the curve $\mathscr{C}_{OD}$ is *never* below $\mathscr{C}_{RS}$. This will finish off (B) and prove (C) as well; however this time the result is global. Indeed, setting

$$Q^- = \{(\beta, \tilde{h}) \in Q : \tilde{h} < \beta h_\beta\} \tag{3.15}$$

let us suppose there is a point of $\mathscr{C}_{OD}$ in Q^-. It follows that there is a whole *open* region of O_R-states in Q. By FKG, this implies that all $\mathfrak{S}^\star$-states below this region are O_R and hence ordered. On the other hand, the lower boundary of the quadrant has already been determined to consist exclusively of disordered $\mathfrak{S}^\star$-states. Thus the two curves coincide for a while and, when they eventually split, they must do so in such a way that $\mathscr{C}_{OD}$ does not dip below $\mathscr{C}_{RS}$ ever more. In other words, the O_R states are gone for good.

(2) Upper Triple Point. The argument for the other triple point is similar but hindered by the absence of FKG-monotonicity. As the previous discussion was to rule out the "bubbles" *below* $\mathscr{C}_{RS}$, now the key point will be to deal with the "bubbles" that can appear *above* $\mathscr{C}_{RS}$.

Following the curves from the high-temperature/high-field side, let $\otimes = (\beta^\otimes, \tilde{h}^\otimes)$ denote the first point that these curves coincide. The quadrant below and to the left of this point will be denoted by K, i.e., $K = \{(\beta, \tilde{h}) : \beta > \beta^\otimes, \tilde{h} < \tilde{h}^\otimes\}$. Again, "beyond" the point $\otimes$, both curves are locked into the quadrant. Indeed, $\{\beta > \beta^\otimes, \tilde{h} = \tilde{h}^\otimes\}$ is a line that consists exclusively of ordered states and $\{\beta = \beta^\otimes, \tilde{h} < \tilde{h}^\otimes\}$ is a line that consists exclusively of $\mathscr{R}$-states. (As a matter of fact, D_R-states.) We already know that $\mathscr{C}_{OD}$ lies above or on $\mathscr{C}_{RS}$ all the way down to $\odot$. We must rule out the possibility that it lies strictly above. Explicitly, we will rule out the existence of D_S-states in K. To that end, we note that K can be reparametrized as

$$K = \{(\beta, \tilde{h}) : \beta = \beta^\otimes + \alpha, h = \tilde{h}^\otimes - \alpha\Delta, 0 < \alpha \leqslant \infty, \Delta > 0\} \tag{3.16}$$

The argument we will use is similar to the one in (3.7)–(3.11), which is just the case $\Delta = 0$.

Let $\eta_{\mathscr{R}}$ denote the density of R-sites. As differentiating of the free energy with respect to α (with Δ fixed) reveals, in any translation-invariant state at $(\beta, \tilde{h}) = (\beta^\otimes + \alpha, \tilde{h}^\otimes - \alpha\Delta)$, the quantity $\mathscr{A}_{\mathscr{O}}(\beta)\, \rho_{\mathscr{O}} + \mathscr{A}_{\mathscr{D}}(\beta)\, \rho_{\mathscr{D}} + 2\Delta\eta_{\mathscr{R}}$ must be larger than its value in any translation-invariant state at

$(\beta^{\otimes}, \tilde{h}^{\otimes})$. Let Δ, α be fixed and let $\bigstar$ denote any such state. By comparison to the O_S-state at $\otimes$ we have

$$\mathscr{A}_{\mathscr{O}}(\beta)\,\rho^{\bigstar}_{\mathscr{O}} + \mathscr{A}_{\mathscr{D}}(\beta)\,\rho^{\bigstar}_{\mathscr{D}} + 2\Delta\eta^{\bigstar}_{\mathscr{R}} \geqslant \mathscr{A}_{\mathscr{O}}(\beta^{\otimes})(1-\varepsilon') \tag{3.17i}$$

and similarly, by comparison with the D_R state at $\otimes$,

$$\mathscr{A}_{\mathscr{O}}(\beta)\,\rho^{\bigstar}_{\mathscr{O}} + \mathscr{A}_{\mathscr{D}}(\beta)\,\rho^{\bigstar}_{\mathscr{D}} + 2\Delta\eta^{\bigstar}_{\mathscr{R}} \geqslant \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})(1-\varepsilon') + 2\Delta(1-\varepsilon') \tag{3.17ii}$$

This easily finishes the proof. Namely, let $2\Delta \leqslant \mathscr{A}_{\mathscr{O}}(\beta^{\otimes}) - \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})$. Then by the same arguments as in (3.8)–(3.9) we get

$$[\mathscr{A}_{\mathscr{O}}(\beta^{\otimes}) - \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})](\rho^{\bigstar}_{\mathscr{O}} + \eta^{\bigstar}_{\mathscr{R}}) + \mathscr{A}_{\mathscr{D}}(\beta^{\otimes}) \geqslant \mathscr{A}_{\mathscr{O}}(\beta^{\otimes})(1-\varepsilon') \tag{3.18i}$$

However, this entails that the bigger of the quantities $\rho^{\bigstar}_{\mathscr{O}}$, $\eta^{\bigstar}_{\mathscr{R}}$ is close to a half, which rules out that $\bigstar$ is any D_S state. If $2\Delta > \mathscr{A}_{\mathscr{O}}(\beta^{\otimes}) - \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})$, then (assuming without loss of generality $\eta^{\bigstar}_{\mathscr{R}} \leqslant 1-\varepsilon'$) we are led to

$$\begin{aligned}
&[\mathscr{A}_{\mathscr{O}}(\beta^{\otimes}) - \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})]\,\rho^{\bigstar}_{\mathscr{O}} + \varepsilon'\mathscr{A}_{\mathscr{D}}(\beta^{\otimes}) \\
&\qquad \geqslant [\mathscr{A}_{\mathscr{O}}(\beta^{\otimes}) - \mathscr{A}_{\mathscr{D}}(\beta^{\otimes})](1-\varepsilon'-\eta^{\bigstar}_{\mathscr{R}})
\end{aligned} \tag{3.18ii}$$

which again cannot be satisfied if both $\rho^{\bigstar}_{\mathscr{O}}$ and $\eta^{\bigstar}_{\mathscr{R}}$ are small.

Since Δ was arbitrary, there is no alternative to the scenario that the curves coincide down to $\odot$, where the "new" O_R states enter the play. But this proves that there are just two triple points and that they are indeed connected by a single line of $O_S + D_R$ coexistence. ∎

3.4. Proof of (IV) and (V)

Item (IV) is a standard high temperature result that is particularly easy to justify in the graphical representation. Consider any infinite-volume spin Gibbs measure ν and its (arbitrary) Edwards–Sokal coupling ν^{ES}. Then we claim that under the $\boldsymbol{\omega}$ marginal of ν^{ES} (and β small), most of the configuration is actually in the vacant state. Namely, if two neighboring sites are of the same spin-type but not the same spin-state, then the only other possibility is a disordered bond-state, whose relative weight is then $1-e^{-\beta\kappa}$. Similarly, if two neighboring sites are in the same state, the relative weight of an ordered bond is $1-e^{-\beta J}$ and of a disordered bond is $e^{-\beta J}(1-e^{-\beta\kappa})$.

Let us call non-vacant bonds "open" and the vacant ones "closed." Since the bond states are independent under the measure $\nu^{\mathrm{ES}}(\cdot \mid \boldsymbol{\sigma})$, the "open/closed" marginal of $\nu^{\mathrm{ES}}(\cdot \mid \boldsymbol{\sigma})$ is FKG-dominated by the ordinary bond percolation on $\mathbb{Z}^d$ with parameter $p = 1 - e^{-\beta(J+\kappa)}$, where $J, \kappa > 0$ is used to bound the other possibilities. Let now $p < p_c(d)$, where $p_c(d)$ marks the onset of bond percolation on $\mathbb{Z}^d$. Then a classical result [MMS, AB] yields that the size of the maximal cluster intersecting a fixed finite volume Λ has an exponentially small tail, with a Λ-independent rate (if we are content to restrict p and, consequently, β only to small values, then this follows already from a Peierls argument). The FKG domination says the same applies to the graphical marginal of $\nu^{\mathrm{ES}}(\cdot \mid \boldsymbol{\sigma})$ for ν-a.s. $\boldsymbol{\sigma}$.

It is not difficult to observe that this actually implies the exponential decay of all truncated correlations with a uniform decay rate, because disconnected regions act independently of each other. Namely, let $(A_i)_{i=1}^N$ be disjoint finite sets of sites, ψ_i be a cylindric function in A_i, let $\chi = \chi(A_1,..., A_N)$ indicate that all A_i's are connected and let $\ell(A_1,..., A_N)$ denote the minimal number of bonds needed for this connection. Let $\langle \cdot \rangle_\Lambda$ be any state in $\Lambda \supset \bigcup_i A_i$ with some fixed boundary condition. Then the logarithmic generating function

$$\mathscr{H}_\Lambda(z_1,..., z_N) = \log \left[\frac{\langle e^{\sum_{i=1}^N z_i \psi_i} \rangle_\Lambda}{\langle e^{\sum_{i=1}^N z_i \psi_i} \mid \{\chi = 0\} \rangle_\Lambda} \right] \tag{3.19}$$

of the functions f_i (note that the denominator plays no role for truncated correlators involving *all* functions f_i) exists for any $z_i \in \mathbb{C}$ such that $|z_i|$ is small enough, and it satisfies the inequality

$$|\mathscr{H}_\Lambda(z_1,..., z_N)| \leqslant C \left(\prod_{i=1}^N e^{2\,|z_i|\,\|\psi_i\|} \right) e^{-\delta \ell(A_1,..., A_N)} \tag{3.20}$$

as is verified by splitting the numerator in (3.18) depending whether $\chi = 0$ or 1 and then using that $\log(1-x) \leqslant x_0^{-1} \log(1-x_0)\, x$ for $0 \leqslant x \leqslant x_0 < 1$ and that $\langle \chi \rangle_\Lambda \leqslant \mathcal{O}(1)\, e^{-\delta \ell(A_1,..., A_N)}$, where δ is the rate of the connectivity function in the percolation model above. Note that C is finite and independent of Λ whenever $\max_i\{|z_i|\,\|\psi_i\|\}$ is small enough. The multidimensional Cauchy theorem then implies that condition IIc stating that the truncated correlator obeys the bound

$$|\langle \psi_1^{k_1};...; \psi_N^{k_N} \rangle_\Lambda| \leqslant k_1! \cdots k_N!\, \tilde{C}^{k_1 + \cdots + k_N} e^{-\delta \ell(A_1,..., A_N)} \tag{3.21}$$

and consequently all the other conditions in [DS] (or, alternatively, the condition in [vdBM]) for complete analyticity are satisfied.

In order to prove item (V), note that in the interior of O_R, O_S, D_R and D_S there is percolation of the appropriate type. Indeed, in any

$\mathfrak{S}^{\star}$-state, we know that the probability of the dominant type of bond is close to one in any $\mathfrak{S}^{\star}$-state and, on the torus, we know that contours are rare. Thence the event that two bonds at the opposite ends of the torus are connected by a-path of relevant bonds is close to one uniformly in the size of the torus. Thus, in any torus state, percolation of the appropriate bond-type is inevitable.

We will finish by showing that, for the interior points, in every $\mathfrak{S}^{\star}$-state there is no percolation of any of the sub-dominant types. First note that, throughout the interior of the regions, one can produce $\mathfrak{S}^{\star}$-states where no other than the relevant bonds percolate. Namely, in the torus states, the probability that the non-appropriate bonds form a contour running around the torus tends to 0 in the thermodynamic limit, hence, conditioning on the complement event yields an $\mathfrak{S}^{\star}$-state in this limit, assigning a uniformly positive probability to the event that any two sites are connected by the relevant bonds. With this in the hand, percolation of non-relevant bonds can be immediately ruled out inside the O-regions, using FKG domination from Lemma II.7, because "non-appropriate percolation" is a monotone event in this case. In the D-regions, one first rules on the percolation of O-bonds of the respective color (again by FKG) and then the percolation of the bonds of the complementary color. ▮

APPENDIX

A1. Proof of Lemma II.5

The measures μ_1 and μ_2 are distinct, hence they are distinguished by the expectation of a local function g. Since both have been obtained essentially as limits of finite-volume states, it is enough to show that the expectation of the bond configuration function g under the corresponding Edwards–Sokal measure can be expressed as an expectation, under the same measure, of a local function f depending only on spin configuration $\boldsymbol{\sigma}$. The compactness of the space of all measures (in the weak-* topology) then proves the existence of the two desired distinct spin-states in the respective sets of cluster points.

The function g can be rewritten as $g(\cdot)=\sum_{\boldsymbol{\omega}} g(\boldsymbol{\omega})\,\delta_{\{\boldsymbol{\omega}\}}(\cdot)$. Thus, relying also on the inclusion-exclusion principle, without loss of generality it suffices to construct the function f only for g that indicates a fixed configuration $\bar{\boldsymbol{\omega}}$ on a finite set of bonds Σ, with *no* vacant bonds. Then, Σ decomposes into the disjoint union of the sets $\Sigma_{\mathcal{O}_{\mathcal{R}}}$, $\Sigma_{\mathcal{D}_{\mathcal{R}}}$, $\Sigma_{\mathcal{O}_{\mathcal{S}}}$, and $\Sigma_{\mathcal{D}_{\mathcal{S}}}$ of those bonds where $\bar{\boldsymbol{\omega}}$ is dark-red, light-red, dark-blue, and light-blue, respectively. Now, the corresponding $f_{\bar{\boldsymbol{\omega}}}$ will be the product $f_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\sigma})=\prod_{b=\langle x,\,y\rangle\in\Sigma} f_{\bar{\omega}_b}(\sigma_x,\sigma_y)$, where

$$f_{\bar{\omega}_b}(\sigma_x, \sigma_y) = \begin{cases} \dfrac{e^{\beta J}-1}{e^{\beta J}}\,\delta^S_{\sigma_x\sigma_y} & b=\langle x, y\rangle \in \Sigma_{\mathcal{O}_{\mathcal{S}}} \\[2ex] \dfrac{e^{\beta J}-1}{e^{\beta J}}\,\delta^R_{\sigma_x\sigma_y} & b=\langle x, y\rangle \in \Sigma_{\mathcal{O}_{\mathcal{R}}} \\[2ex] \dfrac{e^{\beta\kappa}-1}{e^{\beta\kappa}}\left(1-\dfrac{e^{\beta J}-1}{e^{\beta J}}\,\delta^S_{\sigma_x\sigma_y}\right)\delta^S_{\sigma_x}\delta^S_{\sigma_y} & b=\langle x, y\rangle \in \Sigma_{\mathcal{D}_{\mathcal{R}}} \\[2ex] \dfrac{e^{\beta\kappa}-1}{e^{\beta\kappa}}\left(1-\dfrac{e^{\beta J}-1}{e^{\beta J}}\,\delta^R_{\sigma_x\sigma_y}\right)\delta^R_{\sigma_x}\delta^R_{\sigma_y} & b=\langle x, y\rangle \in \Sigma_{\mathcal{D}_{\mathcal{S}}} \end{cases} \tag{A.1}$$

Observe that for every $b \in \Sigma$ we have (note that $\bar{\omega}_b \neq v$)

$$f_{\bar{\omega}_b}(\sigma_x, \sigma_y) \sum_{\omega_b} w(\omega_b)\,\chi_b(\omega_b, \sigma_x, \sigma_y) = w(\bar{\omega}_b)\,\chi_b(\bar{\omega}_b, \sigma_x, \sigma_y) \tag{A.2}$$

(see (2.2) to recall the meaning of w's and χ's), where in the arguments of χ we have retained only the relevant terms. By noting that the external-field terms have not been tampered with at all during this operation we can easily convince ourselves that

$$\sum_{\boldsymbol{\omega},\,\boldsymbol{\sigma}} f_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\sigma})\, W^{\beta,h}_{\mathrm{ES},\,s,\,r;\,\Lambda}(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \sum_{\boldsymbol{\omega},\,\boldsymbol{\sigma}} \delta_{\bar{\boldsymbol{\omega}}}(\boldsymbol{\omega})\, W^{\beta,h}_{\mathrm{ES},\,s,\,r;\,\Lambda}(\boldsymbol{\omega}, \boldsymbol{\sigma}) \tag{A.3}$$

where $W^{\beta,h}_{\mathrm{ES},\,s,\,r;\,\Lambda}$ is the corresponding Edwards–Sokal weight in Λ (see (2.14)), with the dependence on the boundary condition $\tilde{\boldsymbol{\sigma}}$ being only implicit. Namely, fix a configuration $\boldsymbol{\sigma}$ and carry out the summation over $\boldsymbol{\omega}_\Sigma$ on the l.h.s. Then (A.2) asserts that this is replacable by putting the indicator $\delta_{\{\bar{\boldsymbol{\omega}}\}}$, independently of $\boldsymbol{\sigma}$ to have been chosen. The equality then follows by summing over the rest of the variables. ∎

A2. Proof of Lemmas II.6 and II.7

We first prove Lemma II.6 for any $\mathfrak{G}$-measure. The claim for $\mathfrak{S}^{\star}$-measures will be then an easy corollary of the FKG domination in Lemma II.7. At several points in the forthcoming derivation, we will have occasion to use the Strassen theorem [S]—or more precisely the corollary to Strassen's theorem. This result may be stated as follows:

Theorem A.1. Let $X_1,\dots, X_N$ denote a collection of real-valued random variables (for simplicity each assumed to take on only a finite number of values) and let μ_1 and μ_2 denote measures on the configurations of these variables. Suppose that *a priori* $\mu_1 \geqslant_{\text{FKG}} \mu_2$. If the marginal distributions of X_i are equal for all i, then $\mu_1 = \mu_2$.

Proof. See, e.g., [L] p. 75.

Given a spin configuration $\boldsymbol{\sigma}$, let $\boldsymbol{\xi}$ denote its red/blue marginal, i.e., $\xi_x = R$ if $\sigma_x \in R$ and $\xi_x = B$ if $\sigma_x \in B$. We start with a somewhat weaker version of Lemma II.6.

Lemma A.2. Let $\tilde{\mu}^{(\beta, h), *}_{r, s, \Lambda}(\cdot)$ denote the $\boldsymbol{\xi}$-marginal of the Edwards–Sokal measure with a $\mathfrak{G}$ boundary condition. Then for Lebesgue-a.e. h, there is a unique infinite-volume measure for this marginal.

Proof. We first claim that $\tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\cdot)$ is FKG w.r.t. the order $R \prec B$. Indeed, first let us observe that cylinder functions of these site variables may be evaluated via conditional expectations given a bond configuration $\boldsymbol{\omega}$. In particular, each site that is the endpoint of a blue bond *is* blue, similarly for the reds, and vacant sites are independently red or blue with probability $re^{-\beta h}/(se^{\beta h} + re^{-\beta h})$ and $se^{\beta h}/(se^{\beta h} + re^{-\beta h})$, respectively. Thus, if $\mathfrak{A}$ is an increasing event determined by the site variables we may write

$$\tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\mathfrak{A}) = \sum_{\boldsymbol{\omega}} \nu^{(\beta, h)}_{s, r; \Lambda}(\boldsymbol{\omega})\, A(\boldsymbol{\omega}) \tag{A.4}$$

Here $A(\boldsymbol{\omega}) = \mathbb{E}_{\text{ES}}(\mathbb{I}_{\mathfrak{A}} \mid \boldsymbol{\omega})$, with $\mathbb{I}_{\mathfrak{A}}$ being the indicator function, and $\nu^{(\beta, h)}_{s, r; \Lambda}$ is the random-cluster measure in Λ (with the boundary-condition dependence being only implicit).

It is not hard to see that $A(\boldsymbol{\omega})$ is an increasing function. Furthermore, if $\mathfrak{A}$ and $\mathfrak{B}$ are both increasing events, it is easy to see that

$$\mathbb{E}_{\text{ES}}(\mathbb{I}_{\mathfrak{A}} \mathbb{I}_{\mathfrak{B}} \mid \boldsymbol{\omega}) \geqslant \mathbb{E}_{\text{ES}}(\mathbb{I}_{\mathfrak{A}} \mid \boldsymbol{\omega})\, \mathbb{E}_{\text{ES}}(\mathbb{I}_{\mathfrak{B}} \mid \boldsymbol{\omega}) \equiv A(\boldsymbol{\omega})\, B(\boldsymbol{\omega}) \tag{A.5}$$

Indeed the only randomness in the above conditional expectations come from the "vacant" sites which are independently assigned their colors; thus we use the FKG property for Bernoulli measures. Hence we have

$$\tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\mathfrak{A} \cap \mathfrak{B}) \geqslant \sum_{\boldsymbol{\omega}} \nu^{(\beta, h)}_{s, r; \Lambda}(\boldsymbol{\omega})\, A(\boldsymbol{\omega})\, B(\boldsymbol{\omega}) \geqslant \tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\mathfrak{A})\, \tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\mathfrak{B}) \tag{A.6}$$

where the last step follows from the FKG property of $\nu^{(\beta, h)}_{s, r; \Lambda}(\cdot)$ and the identity in (A.6). It is also evident that, among the $\mathfrak{G}$-type boundary conditions, the measure $\tilde{\mu}^{(\beta, h), *}_{s, r; \Lambda}(\cdot)$ enjoys the same FKG hierarchy as the corresponding $\nu^{(\beta, h)}_{s, r; \Lambda}(\cdot)$, e.g., the blue-wired is highest, red-wired is lowest, etc.

We now claim that for a.e. h, there is a single limiting $\tilde{\mu}$ measure in the $\mathfrak{G}$-class. Indeed, the free energy has a.e. a continuous derivative w.r.t. h and at the points of continuity, the fraction of blue sites (which is coupled to h in the Hamiltonian) exists and is independent of the state. Moreover, in both the red-wired and blue-wired states (which are both translation invariant) this fraction is exactly the probability of a blue at any fixed site. Hence, by the corollary to Strassen's theorem, these are the *same* state. Since all $\mathfrak{G}$-states lie in between these extremes, there is just one such state. ∎

Corollary. Let $\tilde{\mu}_{s, r; \Lambda}^{(\beta, h), \dagger}(\cdot)$ denote the marginal of the random cluster measure that counts only whether each bond is red, blue or vacant. Then for Lebesgue-a.e. h, the limiting red- and blue-wired measures coincide.

Proof. Let x and y denote a neighboring pair of sites. Since by Lemma A.2 the limiting ξ-marginals $\tilde{\mu}_{r, s, \text{blue-w}}^{(\beta, h), *}(\cdot)$ and $\tilde{\mu}_{r, s, \text{red-w}}^{(\beta, h), *}(\cdot)$ agree, it follows that the probability that both x and y are blue is the same in both systems. Let us denote this probability by $g_{x, y}^{b}$ and further let b_B and b_R denote the probabilities that the bond $\langle x, y\rangle$ is blue in the blue-wired and red-wired measures, and similarly v_B and v_R for the probabilities that the bonds is vacant. Finally, let $\lambda_{v, B}^{b}$ denote the conditional probability, in the limiting blue-wired measure, that both endpoints of the bond $\langle x, y\rangle$ are blue given that this bond is vacant. Let $\lambda_{v, R}^{b}$ be the similar quantity for the red-wired boundary condition.

Since the underlying random clutter measures are *strong* FKG, it is observed that

$$\lambda_{v, B}^{b} \geqslant \lambda_{v, R}^{b} \tag{A.7}$$

It is also observed that $\lambda_{v, R}^{b} < 1$. Clearly

$$b_B + v_B \lambda_{v, B}^{b} = g_{x, y}^{b} b_R + v_R \lambda_{v, R}^{b} \tag{A.8}$$

i.e.,

$$0 = (b_B - b_R)(1 - \lambda_{v, R}^{b}) + v_B(\lambda_{v, B}^{b} - \lambda_{v, R}^{b}) + ([b_B + v_B] - [b_R + v_R])\, \lambda_{v, R}^{b} \tag{A.9}$$

But *a priori* each term on the right hand side is non-negative so it follows that all three are zero. In particular, $b_B = b_R$ (implying that the blue-bond densities are equal) and thus also $v_B = v_R$. Using the corollary to Strassen's theorem, the desired conclusion is obtained. ∎

Proof of Lemma II.6—$\mathfrak{G}$-Measures. Let x and y denote a neighboring pair of sites and let $\alpha_{x, y}^{B}$ denote the conditional probability in the blue-wired measure that the sites x and y are connected in the complement of the bond $\langle x, y\rangle$ given that both x and y are blue. Let $\alpha^{R}_{x,y}$ denote the corresponding probability in the red-wired measure. Now given that x and y are both blue, the only possibilities for the bond $\langle x, y\rangle$ are to be vacant, light blue or dark blue.

When these sites are externally connected by dark-blue bonds, the ratio of these probabilities is $1 : e^{\beta\kappa} - 1 : e^{\beta\kappa}(e^{\beta J} - 1)$. For notational clarity, let us temporarily denote these quantities by $1 : C : D$. On the other hand, if the two sites are disconnected, the ratios read $1 : C : s^{-1}D$. Now in the red- and blue-wired states, we have determined that the probability of a blue bond is the same and the probability of a neighboring pair of blue sites is the same. The ratio of these probabilities is thus equal which gives us

$$\alpha^{B}_{x,y} \frac{C+D}{1+C+D} + (1-\alpha^{B}_{x,y}) \frac{C+s^{-1}D}{1+C+s^{-1}D}$$
$$= \alpha^{R}_{x,y} \frac{C+D}{1+C+D} + (1-\alpha^{R}_{x,y}) \frac{C+s^{-1}D}{1+C+s^{-1}D} \tag{A.10}$$

The above is only possible if $\alpha^{B}_{x,y} = \alpha^{R}_{x,y}$ and from this it follows that the dark-blue bond density is the same in both measures. Thence, all bond densities are the same and, again using the corollary to the Strassen theorem, the measures coincide. ▮

As a simple Corollary, we obtain a domination bound for the extreme $\mathfrak{G}$-measures:

Corollary. For any $h^{(1)} > h^{(2)}$,

$$\nu^{(\beta, h^{(1)})}_{\text{red-w}}(\cdot) \geqslant_{\text{FKG}} \nu^{(\beta, h^{(2)})}_{\text{blue-w}}(\cdot)$$

Proof. Let g be a monotone increasing cylinder function. Let

$$\bar{h} = \inf\{h\colon \nu^{(\beta, h)}_{\text{blue-w}}(g) > \nu^{(\beta, h^{(1)})}_{\text{red-w}}(g)\} \tag{A.11}$$

and suppose $\bar{h} < h^{(1)}$. Since both $h \mapsto \nu^{(\beta,h)}_{\text{blue-w}}$ and $h \mapsto \nu^{(\beta,h)}_{\text{red-w}}$ are increasing, this makes inevitable that $\nu^{(\beta,h)}_{\text{blue-w}}(g) > \nu^{(\beta,h)}_{\text{red-w}}(g)$ for all $h \in (\bar{h}, h^{(1)})$. However, this is in contradiction with $\nu^{(\beta,h)}_{\text{blue-w}} = \nu^{(\beta,h)}_{\text{red-w}}$ for Lebesgue almost all h, hence $\bar{h} \geqslant h^{(1)}$ (in fact, the equality holds). Since g was arbitrary, the proof is over. ▮

Proof of Lemma II.7. Let β, h be fixed and let us suppress them from the notation. For $\mathfrak{G}$-measures the claim is a trivial corollary of Proposition II.1 and Corollary above, thus we shall concentrate on $\mathfrak{S}^{\star}$-masures.

We shall show that $\nu_{\Lambda,\,\text{blue-w}}(g) \geqslant \nu(g)$ for any increasing cylinder function g and any $\mathfrak{S}^\star$-measure ν.

Let thus g be a cylinder function with support on a finite set A of bonds. Consider its "spread" over Λ, i.e., $\mathfrak{g}_\Lambda = \sum_{x\,:\,\tau^x(A) \subset \mathbb{B}_\Lambda} g \circ \tau^x$, with τ^x denoting the "shift by x." Let now $\langle \cdot \rangle^\alpha_{\Lambda,\,\tilde{\boldsymbol{\sigma}}}$ denote the $\boldsymbol{\omega}$-marginal of the Edwards–Sokal measure with the weight (2.12) (considering the volume Λ with the boundary condition $\tilde{\boldsymbol{\sigma}}$ instead of the torus $\mathscr{T}$) modified by the factor $e^{\alpha \mathfrak{g}_\Lambda(\boldsymbol{\omega})}$. The respective normalizing constant $Z^\alpha_{\Lambda,\,\tilde{\boldsymbol{\sigma}}}$ of this measure is then also α-dependent and is logarithmically convex in α. Observing that $F(\alpha) = \lim_{\Lambda \uparrow \mathbb{Z}^d} (1/|\Lambda|) \log Z^\alpha_{\Lambda,\,\tilde{\boldsymbol{\sigma}}}$ does not depend on the boundary condition and that it is a convex function in α, we can employ the standard convexity argument to infer that

$$\left\langle \frac{\mathfrak{g}_\Lambda}{|\Lambda|} \right\rangle^0_{\Lambda,\,\tilde{\boldsymbol{\sigma}}_1} - \varepsilon_\Lambda \leqslant \left.\frac{\mathrm{d}F}{\mathrm{d}\alpha^+}\right|_{\alpha_1} \leqslant \left.\frac{\mathrm{d}F}{\mathrm{d}\alpha^-}\right|_{\alpha_2} \leqslant \left\langle \frac{\mathfrak{g}_\Lambda}{|\Lambda|} \right\rangle^\alpha_{\Lambda,\,\tilde{\boldsymbol{\sigma}}_2} + \varepsilon_\Lambda \tag{A.12}$$

for any $\alpha > \alpha_2 > \alpha_1 > 0$ and any two spin boundary conditions $\tilde{\boldsymbol{\sigma}}_1, \tilde{\boldsymbol{\sigma}}_2$. Here $\varepsilon_\Lambda = O(|\partial\Lambda|/|\Lambda|)$ uniformly in the boundary condition.

For the interpretation of the l.h.s. it is important that any $\mathfrak{S}$ measure ν is the $\boldsymbol{\omega}$-marginal of some Edwards–Sokal measure whose $\boldsymbol{\sigma}$ marginal is a (conventional) Gibbs measure. By using the fact that every local cylinder function g of bonds can be interchanged, under expectation w.r.t. the Edwards–Sokal measure, into a spin function f (as follows from the proof of Lemma II.5), we can view the l.h.s. of (A.12) as a spin Gibbs specification. By averaging over a translation-invariant spin Gibbs measure we arrive at $\langle \mathfrak{f}_\Lambda/|\Lambda| \rangle = \langle f \rangle = \nu(g)$, where we denoted by $\mathfrak{f}_\Lambda$ the "spread" of f.

It remains to work out the r.h.s. of (A.12) into the desired form. Let us consider an auxiliary measure $\langle \cdot \rangle^\alpha_{\Lambda,\,*,\,\text{blue-w}}$, derived from $\nu_{\Lambda,\,\text{blue-w}}$ by modifying the *a priori* weights (see (2.1)) in the following manner: dark-red, light-red, light-blue, dark-blue will pick up additional factors $e^{-2\alpha\,\text{var}(g)}$, $e^{-\alpha\,\text{var}(g)}$, $e^{\alpha\,\text{var}(g)}$, $e^{2\alpha\,\text{var}(g)}$, respectively, where the variance $\text{var}(g)$ of the function g is defined as

$$\text{var}(g) = \sup_b \sup_{\substack{\boldsymbol{\omega},\,\tilde{\boldsymbol{\omega}}\,:\,\omega_{b'} = \tilde{\omega}_{b'} \\ \forall b' \neq b}} |g(\boldsymbol{\omega}) - g(\tilde{\boldsymbol{\omega}})| \tag{A.13}$$

By observing the proof of Proposition II.1, it is easily checked that $\langle \cdot \rangle^\alpha_{\Lambda,\,*,\,\text{blue-w}}$ satisfies the FKG lattice condition and, for $\alpha \geqslant 0$, a similar argument we used in (2.9) proves that it actually dominates the wired measure $\langle \cdot \rangle^\alpha_{\Lambda,\,\text{blue-w}}$.

Now we can use the fact that any constant blue boundary condition generates the blue-wired measure, so by choosing $\boldsymbol{\sigma}_2$ in (A.12) to be such *prior* to averaging over the state $\langle\cdot\rangle$, we arrive at the inequality

$$\nu(g)-2\varepsilon_{\Lambda}\leqslant\langle g\rangle^{\alpha}_{\Lambda,\,\text{blue-w}}\leqslant\langle g\rangle^{\alpha}_{\Lambda,\,*,\,\text{blue-w}} \tag{A.14}$$

for all $\alpha>0$ and all finite Λ. By passing to the limits $\alpha\downarrow 0$ and $\Lambda\nearrow\mathbb{Z}^d$ we get the desired bound. Since g was arbitrary the upper bound in the display in Lemma II.7 is proved. The lower bound is completely analogous.

The FKG-domination for the $\mathfrak{S}^{\star}$-measures is then the result of the following estimate

$$\nu^{(\beta,\,h^{(1)})}(\cdot)\geqslant_{\text{FKG}}\nu^{(\beta,\,h^{(1)})}_{\text{red-w}}(\cdot)\geqslant_{\text{FKG}}\nu^{(\beta,\,h^{(2)})}_{\text{blue-w}}(\cdot)\geqslant_{\text{FKG}}\nu^{(\beta,\,h^{(2)})}(\cdot) \tag{A.15}$$

for any $h^{(1)}>h^{(2)}$, and any $\mathfrak{S}^{\star}$-measures $\nu^{(\beta,\,h^{(1)})}$ and $\nu^{(\beta,\,h^{(2)})}$. The middle inequality follows by Corollary above. ▮

A3. Proof of Lemma III.1—General Case

In dimensions $d>2$, no diagonal torus is available. Hence we have to proceed by brute force in deriving the chessboard estimate. In particular, since the events we shall be studying (i.e., the various ways that a given cube is bad) do not, after dissemination, result in the probability of a *definite* configuration, no direct use of the formulas (3.1i)–(3.v) can be made for the numerator estimate (c.f. the proof of Lemma III.1 in the case of $d=2$).

The way we estimate the r.h.s. of the formula in Lemma II.3 in the case of $d>2$ is by redistributing the weights of the graphical representation: we define new *a priori* weights

$$\tilde{w}(s^{\mathrm{o}})=w(s^{\mathrm{o}})\,e^{\beta h/d}\qquad\qquad \tilde{w}(r^{\mathrm{o}})=w(r^{\mathrm{o}})\,e^{-\beta h/d}$$

$$\tilde{w}(s^{\mathrm{d}})=w(s^{\mathrm{d}})(se^{\beta h})^{1/d}\qquad\qquad \tilde{w}(r^{\mathrm{d}})=w(r^{\mathrm{d}})(re^{-\beta h})^{1/d}$$

$$\tilde{w}(v)=w(v)(se^{\beta h}+re^{-\beta h})^{1/d}$$

The weights $\tilde{w}$ have one significant advantage over w. Namely, the weight W (see (2.3)) corresponding to constant configurations (which appear, standardly, in the denominator estimate) is given *exactly* by taking the product of the respective $\tilde{w}$'s. For non-constant configurations we obtain the following estimate.

Lemma A.3. Given a configuration ω, let $\tilde{C}_R(\omega)$ and $\tilde{C}_S(\omega)$ denote the number of connected r- and s-ordered components containing at least one bond. Further, let $\mathscr{P}_{\mathscr{R}}(\omega)$ and $\mathscr{P}_{\mathscr{S}}(\omega)$ denote the number of mismatched r- and s-pairs in ω. Then

$$W^{(\beta,h)}_{s,r;\Lambda}(\omega) \leqslant r^{\tilde{C}_R(\omega)} s^{\tilde{C}_S(\omega)} \left[\prod_{b\in\mathbb{B}(\mathscr{T}_L)} \tilde{w}(\omega) \right] r^{-\mathscr{P}_{\mathscr{R}}(\omega)/2d} s^{-\mathscr{P}_{\mathscr{S}}(\omega)/2d}$$

Proof. Since the *a priori* factors of W (i.e., the square bracket in (2.3)) are trivially reproduced, we just have to show that neither the site-terms nor the terms counting the connected components have decreased.

To see the former observe that the vertex terms $e^{\pm\beta h}$ and $se^{\beta h}+re^{-\beta h}$ can be split equally over the neighbouring bonds, giving rise to powers $1/d$ in (3.1), where for the vacant bond adjacent to a non-vacant one we used $se^{\beta h}, re^{-\beta h} \leqslant se^{\beta h}+re^{-\beta h}$. The same holds for numbers s, r inherent to isolated vertices of the s- (r-)disorder (note that the additional negative powers $r^{-\mathscr{P}_{\mathscr{R}}(\omega)/2d}$ and $s^{-\mathscr{P}_{\mathscr{R}}(\omega)/2d}$ partly compensate for the mismatched pairs where neither s nor r are needed). Finally, the non-trivial connected components are dominated by $r^{\tilde{C}_R(\omega)}$ and $s^{\tilde{C}_S(\omega)}$, respectively. ∎

Proof of Lemma III.1. As follows from the chessboard estimates (Lemma II.3), it suffices to prove that a each particular bad pattern a (i.e., a "graphical" configuration on a cube c) has small probability. We use $\mathscr{Z}_a$ for the partition function constrained on the disseminated pattern a, i.e., $\mathscr{Z}_a = \sum_\omega W^{(\beta,h)}_{s,r;\mathscr{T}}(\omega)[\chi_a(\mathscr{T}_L)](\omega)$.

It will be important to know, for counting the connecting components under the indicator $\chi_a(\mathscr{T}_L)$, how many connected components are there within the pattern a itself. Let us use $\tilde{C}_R(a)$ and $\tilde{C}_S(a)$ to denote the number of nontrivial (i.e., containing at least one bond) r and s-ordered components of the configuration a on the cube c. It is an elementary observation that each such component gives rise to at most $(L/2)^{d-1}$ connected components in the disseminated configuration. Indeed, a single bond is disseminated just into $(L/2)^{d-1}$ parallel lines through the torus. Similarly, each mismatched pair yields in total $L^2(L/2)^{d-2}=4(L/2)^d$ clones in the dissemination; there are L^2 clones in the plane containing the initial pair and the plane itself is replicated into $(L/2)^{d-2}$ parallel planes.

Note that every bond is shared by a total of $d2^{d-1}$ elementary cubes. Then, under the condition that $s\leqslant r$, Lemma A.3 implies

$$\mathscr{Z}_a \leqslant r^{[\tilde{C}_R(a)+\tilde{C}_S(a)](L/2)^{d-1}} \left[\prod_{b\in c} \tilde{w}(a_b) \right]^{L^d/2^{d-1}} s^{-2/d[\mathscr{P}_{\mathscr{S}}(a)+\mathscr{P}_{\mathscr{R}}(a)](L/2)^d} \qquad \text{(A.17)}$$

with $\mathscr{P}_{\mathscr{R}}(a)$ and $\mathscr{P}_{\mathscr{S}}(a)$ denoting the number of mismatched pairs in a. Since, trivially, the full partition sum $\mathscr{Z} \geqslant \sum_{\alpha \in I} \tilde{w}(\alpha)^{dL^d}$, by Lemma II.3 we get

$$\langle \chi_a(c) \rangle_L \leqslant \left(\frac{\mathscr{Z}_a}{\mathscr{Z}} \right)^{1/|\mathscr{T}_L|} \leqslant r^{1/L} \left[\prod_{b \in c} \frac{\tilde{w}(a_b)}{\max_{\alpha \in I} \tilde{w}(\alpha)} \right]^{1/2^{d-1}} s^{-1/d2^{d-1}[\mathscr{P}_{\mathscr{S}}(a) + \mathscr{P}_{\mathscr{R}}(a)]} \tag{A.18}$$

where we used that $\tilde{C}_R + \tilde{C}_S \leqslant 2^{d-1}$, with the r.h.s. corresponding to a "dimer" covering of the elementary cube. Now, the pattern a is bad and thus it contains either a vacant bond or a mismatched pair. Since

$$\frac{\tilde{w}(v)}{\max_\alpha \tilde{w}(\alpha)} \leqslant \frac{2^{1/d}}{e^{\beta\kappa} - 1} \tag{A.19}$$

(compare (3.3)) the total probability that a bad pattern occurs is simply computed by enumerating all possible arrangements of the pattern a. In this way we get the bound bounded as

$$\sum_{\substack{a \\ \text{bad pattern}}} \langle \chi_a(c) \rangle_L \leqslant r^{1/L} \left(\frac{d2^d}{2} \left(\frac{2^{1/d}}{e^{\beta\kappa} - 1} \right)^{-1/2^{d-1}} + 2^d d(d-1)\, s^{-1/d2^{d-1}} \right) \tag{A.20}$$

once L is large enough. To count the possible configurations, we used that there are $d(d-1)\, 2^{d-1}$ places to put a mismatched pair on the cube, and that each has two possible colors. By setting the quantity on the r.h.s. equal to δ the proof is finished. ∎

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