ON THE CONTINUITY OF THE MAGNETIZATION AND THE ENERGY DENSITY FOR POTTS MODELS ON TWO-DIMENSIONAL GRAPHS

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Abstract. We consider the q-state Potts model on two-dimensional planar graphs. Our only assumptions concerning the graph and interaction are that the associated graphical representations satisfy the conclusion of the theorem of Gandolfi, Keane and Russo [GKR]. In addition to $\mathbb{Z}^2$, the class of graphs we consider contains, for example, the triangular, honeycomb, and Kagomé lattices. Under these conditions we show that the only possible point of discontinuity of the magnetization and the energy density is at the onset of the magnetic ordering transition (i.e., at the threshold for bond percolation in the random-cluster model). The result generalizes to any model with a natural dual, appropriate FKG monotonicity properties and a percolation characterization of the Gibbs uniqueness.

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

1. Introduction

In a recent paper [BC] coauthored by one of us, the continuity of the energy density in the Potts (and generalized Ashkin-Teller) ferromagnets on $\mathbb{Z}^2$ was investigated. As is well known, the Potts model on the square lattice is self-dual. Using this property it was established that if there is any discontinuity in the energy density, this must take place at the point where the model and its dual ostensibly coincide (i.e., at the self-dual point). Whenever such a discontinuity actually happens, the self-dual point was proved to coincide with the magnetic ordering transition. Thus, away from the transition point the energy is continuous. (The opposite case, where the energy density is presumed to be continuous even at the self-dual point was investigated in [CS], where partial information about the *critical* nature of the transition was obtained.)

Here we investigate the problem in a general two-dimensional context. We consider Potts models on two-dimensional planar graphs that satisfy certain hypotheses to be detailed below. For these systems, there is always a well defined ordering temperature, whose inverse we denote by β_t. The quantity β_t is characterized by "onset of spontaneous magnetization" and/or "onset of percolation" in the graphical representation. We show that for all β except possibly β_t the energy density is continuous. As a corollary, we obtain that for all $\beta > \beta_t$, there are exactly q pure phases (for $\beta < \beta_t$ there is a unique Gibbs state [ACCN]). Then, as our final result, we show that for all temperatures below β_t^{-1} the magnetization is also continuous in β.

Our setting is genuinely two-dimensional, no self-duality is required. Thus, above and beyond the square lattice, a large class of two-dimensional graphs can be treated. In order to ensure that both the magnetization and the energy density are well defined, we restrict ourselves to planar graphs that arise from Bravais lattices. Given an *elementary cell* $\mathbb{V}$ (i.e., a finite connected graph in $\mathbb{R}^2$), the latter arise as orbits of $\mathbb{V}$ under the action of an infinite group isomorphic to $\mathbb{Z}^2$. The only other (key) ingredient is the assumption is that the conclusion of Gandolfi-Keane-Russo's theorem [GKR] holds. Letting $\mathfrak{G}$ denote the set of graphs we consider, $\mathfrak{G}$ includes all graphs $\mathcal{G}$ such that

- $\mathcal{G}$ is a *planar Bravais lattice* with a basis (for a definition see e.g. [AM], page 75).

Furthermore, the interaction (Hamiltonian) must be such that

- The associated random cluster model on $\mathcal{G}$ excludes the simultaneous percolation of occupied bonds and dual percolation of vacant bonds.

As recently remarked by Georgii and Higuchi [GH], Zhang's argument streamlining the proof of the GKR-theorem (as described in [GHM] based on its reproduction in the forthcoming book [G1]) can be extended to rather general planar graphs. Thus, in particular, $\mathfrak{G}$ includes any planar Bravais lattice that is invariant under reflections through the horizontal and vertical axes; i.e., the triangular, honeycomb, Kagomé, dice and octagonal lattices, to name just a few. Moreover, models with asymmetric couplings can also be considered, provided the reflection symmetry is preserved.

The remaining sections of this paper are organized as follows: In the next section we collect the necessary facts about the Potts model and the associated random cluster models. In the third section we state and prove the theorem (Theorem A) on the continuity of the energy density on any graph in $\mathfrak{G}$ as well as some corollaries on the number of phases below the ordering transition and on the location of the transition line in the asymmetric model on $\mathbb{Z}^2$. The last section is devoted to the proof of continuity of the magnetization (Theorem B).

2. The Potts models

Consider a two-dimensional graph $\mathcal{G} = (\mathbb{S}, \mathbb{B})$, where $\mathbb{S}$ denotes the vertex and $\mathbb{B}$ the edge sets, respectively. The Potts Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle\in\mathbb{B}} J_{ij}\delta_{\sigma_i,\sigma_j}, \tag{2.1}$$

with $J_{ij} > 0$ and $\sigma_i \in \{1, \dots, q\}$ the usual Potts variables.

Let $\mathcal{L} \subset \mathcal{G}$ be a subgraph consisting of a collection of vertices $\mathbb{S}_{\mathcal{L}}$ together with the set $\mathbb{B}_{\mathcal{L}}$ of all edges connecting pairs of vertices in $\mathbb{S}_{\mathcal{L}}$. We use the notation $\mathcal{L} \Subset \mathcal{G}$ to indicate that $\mathcal{L} \subset \mathcal{G}$ is finite and connected. We define $\partial\mathbb{S}_{\mathcal{L}}$ to be the sites in $\mathbb{S}_{\mathcal{L}}^c$ with at least one neighbor in $\mathbb{S}_{\mathcal{L}}$:

$$\partial\mathbb{S}_{\mathcal{L}} = \{j \in \mathbb{S}_{\mathcal{L}}^c | \exists\langle i,j\rangle \in \mathbb{B} \text{ for some } i \in \mathbb{S}_{\mathcal{L}}\}. \tag{2.2}$$

The Gibbs measure on $\mathcal{L} \Subset \mathcal{G}$ with boundary conditions $\boldsymbol{\sigma}_{\partial\mathbb{S}_{\mathcal{L}}} = \{\sigma_i | i \in \partial\mathbb{S}_{\mathcal{L}}\}$ is defined by the usual Boltzmann weights in accord with the Hamiltonian $\mathcal{H}$. If # is a boundary condition, or a convex combination thereof, we denote the corresponding finite volume Gibbs measure by $\langle - \rangle^{\#}_{\beta\mathcal{H};\mathcal{L}}$. Of additional interest are the free boundary conditions (denoted by $\# = f$) which are obtained by setting the J_{ij}, $i \in \mathbb{S}_{\mathcal{L}}$, $j \in \mathbb{S}_{\mathcal{L}}^c$, to zero.

Let $R_{ij} = e^{\beta J_{ij}} - 1$ and let $\mathcal{L} \Subset \mathcal{G}$. For a given boundary condition # that is provided by a *fixed* spin configuration at $\partial\mathbb{S}_{\mathcal{L}}$, the associated random cluster measure in $\mathcal{L}$ is defined by the relation

$$\mu^{\#}_{\beta\mathcal{H};\mathcal{L}}(\boldsymbol{\omega}) \propto \Bigl(\prod_{\langle i,j\rangle\in\boldsymbol{\omega}} R_{ij}\Bigr) q^{\mathcal{C}^{\#}(\boldsymbol{\omega})} \chi^{\#}(\boldsymbol{\omega}). \tag{2.3}$$

Here $\boldsymbol{\omega}$ is a subset of the edges of $\mathcal{L}$ together with the boundary edges (namely, $\boldsymbol{\omega} \subset \{\langle i,j\rangle | i \in \mathbb{S}_{\mathcal{L}}, j \in \mathbb{S}\}$, while $\mathcal{C}^{\#}$ and $\chi^{\#}$ are defined as follows: Let $\partial\mathbb{S}_{\mathcal{L}}^{(1)}, \dots, \partial\mathbb{S}_{\mathcal{L}}^{(q)}$ denote the set of sites of $\partial\mathbb{S}_{\mathcal{L}}$ where the spins take on the values $1, \dots, q$, respectively. Then $\chi^{\#}(\boldsymbol{\omega})$ is zero if there is a connection between any of these components and it is one otherwise. Given that

$\chi^{\#}(\boldsymbol{\omega}) = 1$, $\mathcal{C}^{\#}(\boldsymbol{\omega})$ is the number of connected components of $\boldsymbol{\omega}$ (including the isolated sites in $\mathbb{S}_{\mathcal{L}}$), counting each connected component of each $\partial\mathbb{S}_{\mathcal{L}}^{(1)}, \dots, \partial\mathbb{S}_{\mathcal{L}}^{(q)}$ as a single component. In general, we may also consider any convex combination of such measures.

Every (infinite volume) spin Gibbs measure $\langle - \rangle_{\beta\mathcal{H}}^{\#}$ has a natural random cluster counterpart. Namely, this can be directly seen from the construction $\rho_{\beta\mathcal{H}|\mathcal{L}}^{\#}(-) = \langle \mu_{\beta\mathcal{H};\mathcal{L}}^{\ominus}(-) \rangle_{\beta\mathcal{H}}^{\#}$, where $\ominus$ denotes the *spin* boundary condition of the above type that is subject to the expectation under the state $\langle - \rangle_{\beta\mathcal{H}}^{\#}$. The random cluster measures $(\rho_{\beta\mathcal{H}|\mathcal{L}}^{\#})_{\mathcal{L}\Subset\mathcal{G}}$ form a consistent family (meaning that the projection of $\rho_{\beta\mathcal{H}|\mathcal{L}}^{\#}$ onto any $\mathcal{L}' \subset \mathcal{L}$ is precisely $\rho_{\beta\mathcal{H}|\mathcal{L}'}^{\#}$), hence they are finite-volume projections of a unique infinite-volume random cluster measure $\rho_{\beta\mathcal{H}}^{\#}$. The relation between $\langle - \rangle_{\beta\mathcal{H}}^{\#}$ and $\rho_{\beta\mathcal{H}}^{\#}$ can be characterized as follows: Every event of the type $\{\omega_b = 1 | b = b_1, \dots, b_n\}$ yields the same expectation under $\langle - \rangle_{\beta\mathcal{H}}^{\#}$ as does the product of functions $\frac{R_b}{1+R_b}\delta_{\sigma_i\sigma_j}$, $b = \langle i, j \rangle = b_1, \dots, b_n$, under $\rho_{\beta\mathcal{H}}^{\#}$. An alternative route to this relationship is by using the Edwards-Sokal Gibbs measures as worked out in [BBCK].

Among all possible random cluster measures, of a particular interest are the ones generated by the free and wired boundary conditions ($\# = f, w$). The latter are defined by setting $\chi^f(\boldsymbol{\omega}) = \chi^w(\boldsymbol{\omega}) \equiv 1$, and interpreting the quantity $\mathcal{C}^f(\boldsymbol{\omega})$ as the usual number of components while $\mathcal{C}^w(\boldsymbol{\omega})$ is counting all clusters attached to the boundary as a single component. The free measure corresponds to the free boundary condition in the spin-system, whereas the wired measure corresponds to all boundary spins set to the same spin state r, i.e., $\partial\mathbb{S}_{\mathcal{L}}^{(s)} = \emptyset$ for $s \neq r$ and $\partial\mathbb{S}_{\mathcal{L}}^{(r)} = \partial\mathbb{S}_{\mathcal{L}}$.

We state without proof the following results that will be needed in subsequent developments. The proofs of these results can be found in (or easily extended from) [ACCN, G2] and the various other references stated.

(i) In the partial order defined by putting occupied bond above vacant one, the free and wired measures are (strong) FKG with

$$\mu_{\beta\mathcal{H};\mathcal{L}}^{w} \underset{\text{FKG}}{\geq} \mu_{\beta\mathcal{H};\mathcal{L}}^{f}. \tag{2.4}$$

Further, for any boundary condition $\#$ as described above,

$$\mu_{\beta\mathcal{H};\mathcal{L}}^{w} \underset{\text{FKG}}{\geq} \mu_{\beta\mathcal{H};\mathcal{L}}^{\#}, \tag{2.5}$$

even if the latter measure is not FKG.

(ii) For the free and wired cases, infinite volume (weak) limits exist: if $(\mathcal{L}_k)$ is a sequence of volumes with $\mathcal{L}_k \subset \mathcal{L}_{k+1} \Subset \mathcal{G}$ which eventually exhaust the entire graph $\mathcal{G}$, limiting measures emerge independent of the details of $(\mathcal{L}_k)$. The limiting objects will be denoted by $\mu_{\beta\mathcal{H}}^{w}$ and $\mu_{\beta\mathcal{H}}^{f}$. In the cases at hand, where the lattice can be described as a Bravis lattice with a basis and the couplings are invariant under translations by the primitive (generating) vectors, then the measures $\mu^w_{\beta\mathcal{H}}$ and $\mu^f_{\beta\mathcal{H}}$ are also invariant under these translations.

(iii) Percolation in these models is defined in the strongest possible sense. Supposing that a site x is contained in $\mathbb{S}_{\mathcal{L}_k}$, let $P^\beta_{\mathcal{L}_k}(x) = \mu^w_{\beta\mathcal{H};\mathcal{L}_k}(x \leftrightarrow \partial\mathbb{S}_{\mathcal{L}_k})$ denote the probability that the origin is connected to the boundary in the wired state. Then the limit

$$P^\beta_\infty(x) = \lim_{k\to\infty} P^\beta_{\mathcal{L}_k}(x) \tag{2.6}$$

exists and is independent of the sequence $(\mathcal{L}_k)$. Moreover, even though $P^\beta_\infty(x)$ in principle depends on x, the positivity of this quantity does not: Either $P^\beta_\infty(x) > 0$ for all $x \in \mathbb{S}$ or $P^\beta_\infty(x) = 0$ for all $x \in \mathbb{S}$. The necessary and sufficient condition for unicity of the Gibbs state is that $P^\beta_\infty(0) = 0$ (see [ACCN]). Under the condition that $P^\beta_\infty(0) > 0$, the spontaneous magnetization defined as

$$\mathfrak{m}(\beta) = \frac{\partial}{\partial\mathfrak{h}}\Big(\lim_{k\to\infty}\frac{1}{|\mathbb{S}_{\mathcal{L}_k}|}\log\sum_{\boldsymbol{\sigma}} e^{-\beta\mathcal{H}_{\mathcal{L}_k}(\boldsymbol{\sigma})+\mathfrak{h}\sum_{i\in\mathbb{S}_{\mathcal{L}_k}}\delta_{\sigma_i,1}}\Big)\Big|_{\mathfrak{h}=0^+} \tag{2.7}$$

is also strictly positive. (Here $\mathcal{H}_{\mathcal{L}_k}$ is the Hamiltonian restricted to $\mathcal{L}_k$ and $\boldsymbol{\sigma}$ denotes spin configurations in $\mathbb{S}_{\mathcal{L}_k}$.) In fact, $\mathfrak{m}(\beta) = P^\beta_\infty(0)$ for the homogeneous cases (i.e., $J_{ij} = J$ independent of i, j and $\mathcal{G}$ a homogeneous graph). In general, if $\mathcal{G} \in \mathfrak{G}$, then

$$\mathfrak{m}(\beta) = \frac{1}{|\mathbb{V}|}\sum_{x\in\mathbb{V}} P^\beta_\infty(x), \tag{2.8}$$

where $\mathbb{V}$ is the elementary lattice cell.

(iv) For any $\beta_1 < \beta_2$, we have

$$\mu^w_{\beta_1\mathcal{H}} \le \mu^f_{\beta_2\mathcal{H}}. \tag{2.9}$$

The proof is based on a "free energy" argument and can be found in [G2,BCK,BBCK]. In conjunction with the FKG domination stated in (i), this FKG bound implies that $\beta \mapsto P^\beta_\infty(0)$ and, consequently, $\beta \mapsto \mathfrak{m}(\beta)$ are monotone increasing (in fact $\beta \mapsto P^\beta_\infty(0)$ is right continuous). Hence $\beta_t = \inf\{\beta|\mathfrak{m}(\beta) > 0\}$ is a well-defined percolation threshold. In particular, we have that $\mu^f_{\beta_2\mathcal{H}}(x \leftrightarrow \infty) \ge \mu^w_{\beta_1\mathcal{H}}(x \leftrightarrow \infty)$, so the free measure at $\beta > \beta_t$ exhibits percolation almost surely.

(v) Using Strassen's theorem (see [St] or [L], page 75), the FKG domination bound from (i) implies that $\mu^f_{\beta\mathcal{H}} = \mu^w_{\beta\mathcal{H}}$ whenever $\mu^f_{\beta\mathcal{H}}(\omega_b = 1) = \mu^w_{\beta\mathcal{H}}(\omega_b = 1)$ for all bonds b. The same argument applies when $\mu^f_{\beta\mathcal{H}}$ is replaced by any (subsequential-)limiting state $\mu^\#_{\beta\mathcal{H}}$.

(vi) Let $\langle i,j\rangle$ be a nearest-neighbor bond. There is a one-to-one correspondence between the energy density $\mathfrak{e}_{ij}^{\#}(\beta)$ in the state $\langle - \rangle_{\beta\mathcal{H}}^{\#}$ and the bond density $\mathfrak{b}_{ij}^{\#}(\beta)$ in the corresponding random-cluster measure $\mu_{\beta\mathcal{H}}^{\#}$, with these quantities defined as

$$\begin{aligned}\mathfrak{e}_{ij}^{\#}(\beta) &= -J_{ij}\big\langle \delta_{\sigma_i,\sigma_j}\big\rangle_{\beta\mathcal{H}}^{\#}\\ \mathfrak{b}_{ij}^{\#}(\beta) &= \mu_{\beta\mathcal{H}}^{\#}(\omega_{ij}=1).\end{aligned} \tag{2.11}$$

The relation reads

$$\mathfrak{b}_{ij}^{\#}(\beta) = -\frac{1}{J_{ij}}\frac{R_{ij}}{1+R_{ij}}\mathfrak{e}_{ij}^{\#}(\beta) \tag{2.12}$$

(for a derivation see [MCLSC]). It follows by convexity of the free energy that both $\beta\mapsto\mathfrak{b}_{ij}^{\#}(\beta)$ and $\beta\mapsto\mathfrak{e}_{ij}^{\#}(\beta)$ are continuous except for countably many values of β. The overall energy density $\mathfrak{e}(\beta)$ is defined by averaging the values of $\mathfrak{e}_{ij}^{\#}(\beta)$ over the bonds in the elementary lattice cell $\mathbb{V}$ and taking the supremum over all boundary conditions #. By convexity arguments, $\mathfrak{e}(\beta)$ can alternatively be defined as the right-derivative of the free energy with respect to β. At the points of continuity of the latter, $\mathfrak{e}(\beta)$ is the energy density for all states.

(vii) Let $\mathcal{G}^*$ denote the dual graph of $\mathcal{G}$ (note that $\mathcal{G}^*$ exist because $\mathcal{G}$ is planar). We denote by $\boldsymbol{\omega}^*$ the complementary configuration of $\boldsymbol{\omega}$, where $\boldsymbol{\omega}^*$ is occupied at the dual bond whenever the direct bond is vacant in $\boldsymbol{\omega}$ and vice versa. Then, in the case of free and wired boundary conditions, the dual of a random cluster measure is again a random cluster measure with parameters

$$R_{ij}^* = \frac{q}{R_{ij}} \tag{2.13}$$

and with free and wired boundary conditions interchanged.

3. Continuity of the energy density

We begin by extending a theorem established in [BC] for the usual Potts model on $\mathbb{Z}^2$ that was proved using the *self*-duality of the lattice. It turns out that the reference to self-duality is not essential; it only matters that the model has a natural dual that can readily be analyzed.

Theorem A. *For the two-dimensional Potts model on any graph $\mathcal{G}\in\mathfrak{G}$, the overall energy density $\mathfrak{e}(\beta)$ is continuous at every $\beta\neq\beta_t$.*

Proof. First note that, whenever a discontinuity in the energy density occurs, there are (at least) two coexisting pure spin states, one with the higher and the other with the lower value of the energy density, as follows by a limiting argument for which we refer e.g. to [BC] or any standard textbook on statistical mechanics. If $\beta < \beta_t$, which means no percolation, the claim in (iii) implies uniqueness of the spin Gibbs state. This rules out discontinuity in this region.

For $\beta > \beta_t$, let us consider the dual system. This is a similar Potts spin system, however, now at dual values of the parameters, which correspond to high temperatures. Indeed, for any $\beta > \beta_t$, there is percolation in the free measure (as was observed in statement (iv) above). Invoking the [GKR] result, which asserts that percolation implies finiteness of all connected components of "dual-to-vacant" bonds, we have that there is no percolation in the wired state of the dual model. The latter rules out percolation in all states, hence, the dual model has a unique Gibbs state and thus a continuous energy/bond density. Since the free energies of the model and its dual are equal up to an analytic factor, the desired result is established. $\square$

Corollary I. *For the two-dimensional Potts model on any $\mathcal{G} \in \mathfrak{G}$, at every $\beta > \beta_t$ there are exactly q pure (i.e., translation invariant extremal) states.*

Remark. The result that the continuity of the energy at $\beta > \beta_t$ implies the existence of exactly q pure phases in the Potts model was established in [Pf]. Additional results along these lines for other spin systems were also established there, however, the arguments were restricted to systems on $\mathbb{Z}^d$. Presumably, this is not essential but, in any case, certain modifications would have to be implemented.

With the help of Pirogov-Sinai theory, even more can be proven in the case of very large values of q. Namely, the class of all translation invariant states is exhausted by q phases for $\beta > \beta_t$, a single phase at $\beta < \beta_t$, and $q + 1$ phases at $\beta = \beta_t$ [M]. Again, even though these results are explicitly proven only for $\mathbb{Z}^d$ they are directly extendable to other periodic lattices, provided q stays large.

Here we use an alternative method, which is perhaps less generalizable (e.g., in the direction of non-Potts type systems and for $d > 2$), but is more in accord with the percolation spirit of the present work.

Proof of Corollary I. Let $\langle - \rangle^{\odot}_{\beta\mathcal{H}}$ be a translation-invariant Gibbs measure. As argued previously, there is a unique (translation-invariant) random cluster measure $\rho^{\odot}_{\beta\mathcal{H}}$ associated with this measure. In particular, the corresponding bond and energy densities $\mathfrak{b}^{\odot}_{ij}(\beta)$ and $\mathfrak{e}^{\odot}_{ij}(\beta)$ satisfy the same relationship as stated in (vi). Since at $\beta > \beta_t$ the energy density is equal for all states, we conclude that $\rho^{\odot}_{\beta\mathcal{H}}$ has the same bond density as the wired state $\mu^{w}_{\beta\mathcal{H}}$ (namely, a jump in any $\mathfrak{b}_{ij}$, with $\langle i, j \rangle$ being a bond in $\mathbb{V}$, implies a jump in the overall bond and hence also energy density). Moreover, by the FKG domination

$$\mu^{w}_{\beta\mathcal{H}} \underset{\text{FKG}}{\geq} \rho^{\odot}_{\beta\mathcal{H}}, \tag{3.1}$$

so Strassen's theorem implies that $\mu^{w}_{\beta\mathcal{H}} = \rho^{\odot}_{\beta\mathcal{H}}$. We conclude, in particular, that $\rho^{\odot}_{\beta\mathcal{H}}$ has a unique infinite cluster and only finite components of the dual bonds to vacant bonds.

Given $\epsilon > 0$ and ℓ an integer, with probability at least $1-\epsilon$ under $\rho^{\odot}_{\beta\mathcal{H}}$ there is a circuit of occupied bonds enclosing a box Λ_ℓ of size ℓ such that it is contained in a large-enough box Λ, both centered at the origin (otherwise there is a dual connection between the boundaries of the boxes). However, this means that with probability $\ge 1-\epsilon$ under $\langle - \rangle^{\odot}_{\beta\mathcal{H}}$, there is a circuit of sites in the ring $\Lambda \setminus \Lambda_\ell$ whereupon the spin is constant. Hence, $\langle - \rangle^{\odot}_{\beta\mathcal{H}}$ can be approximated by a convex combination of finite-volume measures with constant boundary conditions: If $\lambda_{\Gamma;k}$ is the probability that the circuit Γ occurs and the spin thereupon is equal to k, then

$$\sum_{\Gamma,k} \lambda_{\Gamma;k} \langle \mathcal{O} \rangle^{[k]}_{\beta\mathcal{H};\mathcal{L}(\Gamma)} - \epsilon \le \langle \mathcal{O} \rangle^{\odot}_{\beta\mathcal{H}} \le \epsilon + \sum_{\Gamma,k} \lambda_{\Gamma;k} \langle \mathcal{O} \rangle^{[k]}_{\beta\mathcal{H};\mathcal{L}(\Gamma)}, \tag{3.2}$$

for any observable $\mathcal{O}$ in Λ_ℓ with $\|\mathcal{O}\|_\infty \le 1$. Here $\langle - \rangle^{[k]}_{\beta\mathcal{H};\mathcal{L}(\Gamma)}$ denotes the spin state in $\operatorname{Int}\Gamma$ with boundary condition $\sigma_i = k$ at $i \in \Gamma$. As $\mathcal{L}(\Gamma) \nearrow \mathcal{G}$, each such state has a unique thermodynamic limit [C, BBCK], which altogether give rise to q distinct states for $\beta > \beta_t$. By taking a subsequential limit of the coefficients $\lambda_k = \sum_\Gamma \lambda_{\Gamma,k}$ (and noting that $\sum_k \lambda_k$ tends to one as $\ell \to \infty$) we see that $\langle - \rangle^{\odot}_{\beta\mathcal{H}}$ is indeed a mixture of these q states. $\square$

Corollary II. *Consider the asymmetric Potts models on $\mathbb{Z}^2$ with couplings $J_{ij} = K$ in the vertical direction and $J_{ij} = L$ in the horizontal direction (with $0 \le K, L \le \infty$). If this model has a discontinuous transition (which is the case for large q), then it occurs exactly at $\beta_t(K,L)$ determined by the equation $(e^{\beta_t K} - 1)(e^{\beta_t L} - 1) = q$.*

Proof. Abbreviating $\mathbb{K} = e^{\beta K} - 1$ and $\mathbb{L} = e^{\beta L} - 1$, let us define $\mathbb{K}^*$ and $\mathbb{L}^*$ by the formula

$$\begin{aligned} \mathbb{K}\,\mathbb{L}^* &= q \\ \mathbb{K}^*\mathbb{L} &= q. \end{aligned} \tag{3.3}$$

Since the original model is on $\mathbb{Z}^2$, it is easy to verify that its dual is the same model with $(\mathbb{K},\mathbb{L})$ replaced by $(\mathbb{K}^*,\mathbb{L}^*)$ (and the wired and free boundary measures interchanged). The $(\mathbb{K},\mathbb{L})$-parameter space $\varSigma = \{(\mathbb{K},\mathbb{L}) | \mathbb{K} \ge 0,\ \mathbb{L} \ge 0\}$ splits into three disjoint parts: $\varSigma_0$ and $\varSigma_\infty$, with the former containing the point $(0,0)$ and the latter containing (∞,∞), and the self-dual line $\mathcal{C} = \{(\mathbb{K},\mathbb{L}) | \mathbb{K}\mathbb{L} = q\}$.

On the other hand, the free and wired random cluster measures are both increasing in $\mathbb{K}$ and $\mathbb{L}$, hence we can define a unique transition line $\widetilde{\mathcal{C}} = \{(\mathbb{K}_\alpha,\mathbb{L}_\alpha) | 0 < \alpha < \infty\}$, where $\mathbb{L}_\alpha = \alpha\mathbb{K}_\alpha$ and where $\mathbb{K}_\alpha$ is the infimum of all $\mathbb{K}$ such that there is percolation in the model with parameters $(\mathbb{K},\alpha\mathbb{K})$. Similarly, $\widetilde{\mathcal{C}}$ induces a trichotomy: $\varSigma$ splits into $\widetilde{\mathcal{C}}$, the high-temperature part $\widetilde{\varSigma}_0$, and the low-temperature part $\widetilde{\varSigma}_\infty$. It is of importance that $\widetilde{\mathcal{C}}$ is parametrizable as a function either of $\mathbb{K}$ or of $\mathbb{L}$ as follows from monotonicity analogous to claim (iv) above. As a consequence, $\widetilde{\varSigma}_\infty$ is mapped into $\widetilde{\varSigma}_0$ under the duality map, and thus $\widetilde{\mathcal{C}} \subset \mathcal{C} \cup \widetilde{\varSigma}_\infty$.

Consider now the curve γ that $(\mathbb{K},\mathbb{L})$ sweeps out as β increases from 0 to ∞ and suppose that there is a discontinuity in the energy density at some purported $(\mathbb{K}^\dagger,\mathbb{L}^\dagger)\in\gamma$. As before, a limiting argument ensures the existence of two Gibbs measures at $(\mathbb{K}^\dagger,\mathbb{L}^\dagger)$ exhibiting the two values of $\mathfrak{e}$. However, if $(\mathbb{K}^\dagger,\mathbb{L}^\dagger)\in\widetilde{\Sigma}_0$, then this contradicts the no-percolation uniqueness theorem whereas if $(\mathbb{K}^\dagger,\mathbb{L}^\dagger)\in\Sigma_\infty$ (note the absence of "tilde") the same is applies to the dual model. Consequently, $(\mathbb{K}^\dagger,\mathbb{L}^\dagger)\in\Sigma\setminus(\widetilde{\Sigma}_0\cup\Sigma_\infty)\subset\mathcal{C}$. $\square$

4. Continuity of the magnetization

Here we state and prove the following claim:

Theorem B. *For the Potts model on any graph $\mathcal{G}\in\mathfrak{G}$, the spontaneous magnetization $\mathfrak{m}(\beta)$ is continuous for all $\beta>\beta_t$.*

Remark. We need only a slight variant of the argument in the proof of Theorem A. Namely, such a discontinuity implies, it would seem, two different translation invariant states at the purported point of discontinuity (distinguished by the value of the magnetization), which in turn implies different energy densities. However, this time the argument is not quite so straightforward because there is no guarantee that the lower state will be pure—conceivably, it can be a particular mixture of the various upper states craftily tuned to reduce the magnetization. Indeed this phenomenon presumably occurs in the one-dimensional $1/r^2$ models: There is a discontinuity in the magnetization at the critical point (the Thouless effect) and yet at the critical point, there is no extremal state with zero magnetization. Thus we must proceed with caution.

Proof of Theorem B. Suppose the magnetization is discontinuous at a $\beta=\beta^\circledast>\beta_t$. In the inhomogeneous case, at least one sublattice magnetization (defined by restricting the effect of the field $\mathfrak{h}$ in (2.7) to the sublattice)—say the one of the vertex-type of the origin—necessary undergoes a jump at $\beta=\beta^\circledast$, because the magnetization of *each* sublattice increases. The whole problem is thus converted to the sublattice containing the origin to which we now restrict our attention (and to which we will not make any further explicit reference).

Obviously, the wired state at $\beta=\beta^\circledast$ gives rise to the upper value of the magnetization. Let us use $\mathfrak{m}_+$ to denote this value and let $\mathfrak{m}_-$ be the lower value, i.e., $\mathfrak{m}_-=\lim_{\beta\uparrow\beta^\circledast}\mathfrak{m}(\beta)$. Define $\Delta=(\mathfrak{m}_+)^2-(\mathfrak{m}_-)^2$. Then, for any x and y,

$$\mu^w_{\beta^\circledast\mathcal{H}}(x\leftrightarrow y)=\tfrac{q}{q-1}\bigl\langle\delta_{\sigma_x\sigma_y}-\tfrac{1}{q}\bigr\rangle^{[k]}_{\beta^\circledast\mathcal{H}}\ge(\mathfrak{m}_+)^2,\tag{4.1}$$

where $[k]$, $k=1,\dots,q$, denotes any of the q ordered states obtained as the limit of finite volume states with all the boundary spins set to the k-th spin state.

Let $\epsilon > 0$ be a small number, in particular, we demand $\epsilon < \Delta/6$. We shall show that for L large enough there is a Gibbs state $\langle - \rangle^{\#}_{\beta^{\circledast}\mathcal{H}}$ such that

$$\tfrac{q}{q-1}\bigl\langle \delta_{\sigma_0\sigma_{2L}} - \tfrac{1}{q}\bigr\rangle^{\#}_{\beta^{\circledast}\mathcal{H}} \le (\mathfrak{m}_+)^2 - \Delta + 6\epsilon, \tag{4.2}$$

where σ_{2L} is a shorthand for the spin at the site $(2L,0)$.

To show this, notice first that at any inverse temperature β, the quantity $\frac{q}{q-1}\langle \delta_{\sigma_x\sigma_y} - \frac{1}{q}\rangle^{f}_{\beta\mathcal{H}} = \mu^{f}_{\beta\mathcal{H}}(x \leftrightarrow y)$ is decomposed into two events. Namely, the event $\{[x \leftrightarrow y]_F\}$ that x and y are in the same finite cluster and the event $\Pi_\infty(x) \cap \Pi_\infty(y)$ that both are in an infinite cluster. Let $\beta_0 \in (\beta_t, \beta^{\circledast})$ and let $L \gg \ell \gg 1$ be two length scales that satisfy the following two conditions for the measure $\mu^{f}_{\beta_0\mathcal{H}}$:

- (a) With probability larger than $1-\epsilon$, a site at the boundary of the square vessel Λ_ℓ centered at the origin is connected to infinity.
- (b) With probability larger than $1-\epsilon$, there is a circuit of occupied bonds in the region $\Lambda_L \setminus (\Lambda_\ell \cup \partial\Lambda_\ell)$.

We denote the events described in these conditions by $\mathcal{A}$, $\mathcal{B}$, respectively. Clearly, (a) can be achieved because there is percolation whereas (b) holds because there is no dual percolation.

Let $\mathcal{C}$ be the event that there is no dual circuit surrounding the origin containing any site a distance greater than $2L$ away. Since $\mathcal{C} \supset \mathcal{A} \cap \mathcal{B}$, it occurs with probability that is, assuming (a) and (b), larger than $1-2\epsilon$. Moreover, it is obvious that all three events are increasing. Hence, if $\mu^{f}_{\beta_0\mathcal{H}}(\mathcal{D}) > 1-\epsilon$, then $\mu^{f}_{\beta\mathcal{H}}(\mathcal{D}) > 1-\epsilon$ for all $\beta > \beta_0$ and $\mathcal{D} = \mathcal{A}, \mathcal{B}$, while for $\mathcal{D} = \mathcal{C}$ the same holds with ϵ replaced by 2ϵ.

For the part $\{[0 \leftrightarrow 2L]_F\}$ of the event $0 \leftrightarrow 2L$, let us first note that $\{[0 \leftrightarrow 2L]_F\} \subset \mathcal{C}^c$. We thus have, for $\beta \ge \beta_0$, the bound

$$\mu^{f}_{\beta\mathcal{H}}\bigl([0 \leftrightarrow 2L]_F\bigr) \le \mu^{f}_{\beta\mathcal{H}}(\mathcal{C}^c) < 2\epsilon. \tag{4.3}$$

Concerning the event $\Pi_\infty(x) \cap \Pi_\infty(y)$, define Λ_L to be a box of size $2L-1$ centered at 0 and let $P^{\beta}_{L}(0)$ be the probability that the origin is connected to $\partial\Lambda_L$ in the finite volume system with wired boundary conditions. Clearly,

$$\mu^{f}_{\beta\mathcal{H}}\bigl(\Pi_\infty(0) \cap \Pi_\infty(2L)\bigr) \le \bigl[P^{\beta}_{L}(0)\bigr]^2, \tag{4.4}$$

which means that we just need to bound the quantity $P^{\beta}_{L}(0)$ in terms of $P^{\beta}_{\infty}(0) = \mathfrak{m}(\beta)$.

Let $\mathcal{F}$ denote the event that there is a circuit of occupied bonds in the region $\Lambda_L \setminus (\Lambda_\ell \cup \partial\Lambda_\ell)$ that is connected to infinity. Clearly $\mathcal{A} \cap \mathcal{B} \subset \mathcal{F}$. Under the condition $\mathcal{F}$, it is not hard to see that the probability of $\Pi_\infty(0)$ exceeds $P^{\beta}_{L}(0)$. Indeed, conditioning further on the outermost circuit in Λ_L that satisfies the requirements for $\mathcal{F}$, all that is required is a connection from the origin to this circuit. Let $\mathbb{1}_\Gamma$ indicate that the outermost such circuit is precisely Γ. Then we have

$$\begin{aligned}\mu^w_{\beta\mathcal{H}}\bigl(\Pi_\infty(0)\cap\mathcal{F}\bigr) &= \sum_\Gamma \mu^w_{\beta\mathcal{H}}\bigl(\,\mathbb{1}_\Gamma\,\mu^w_{\beta\mathcal{H};\mathrm{Int}\,\Gamma}(0\leftrightarrow\Gamma)\bigr)\\ &\ge \mu^w_{\beta\mathcal{H};\Lambda_L}(0\leftrightarrow\partial\Lambda_L)\mu^w_{\beta\mathcal{H}}(\mathcal{F}),\end{aligned} \tag{4.5}$$

where we used that conditioning on Γ yields the wired measure for the interior of Γ and that $\Lambda\mapsto\mu^w_{\beta\mathcal{H},\Lambda}(0\leftrightarrow\partial\Lambda)$ is monotone decreasing in Λ. Finally, we summed over Γ to get $\mu^w_{\beta\mathcal{H}}(\mathcal{F})$ back. Invoking the bound $\mu^w_{\beta\mathcal{H}}(\mathcal{F})>1-2\epsilon$, we have

$$P^\beta_\infty(0)\ge\mu^w_{\beta\mathcal{H}}(\mathcal{F})P^\beta_L(0)>P^\beta_L(0)(1-2\epsilon). \tag{4.6}$$

whenever $\beta>\beta_0$. Now let $(\beta_k)_{k\ge0}$ denote a sequence of inverse temperatures increasing to $\beta^\circledast$ and let # denote the Gibbs state defined as a $k\to\infty$ limit of the free Gibbs states at β_k. By putting (4.3), (4.4) and (4.6) together, we have for all k that

$$\tfrac{q}{q-1}\bigl\langle\delta_{\sigma_0\sigma_{2L}}-\tfrac1q\bigr\rangle^f_{\beta_k\mathcal{H}}<\mathfrak{m}(\beta_k)^2+6\epsilon, \tag{4.7}$$

where we remind the reader that ϵ is uniform in k. Hence

$$\tfrac{q}{q-1}\bigl\langle\delta_{\sigma_0\sigma_{2L}}-\tfrac1q\bigr\rangle^\#_{\beta^\circledast\mathcal{H}}\le(\mathfrak{m}_-)^2+6\epsilon=(\mathfrak{m}_+)^2-\Delta+6\epsilon<(\mathfrak{m}_+)^2. \tag{4.8}$$

A comparison with (4.1) reveals that the state $\langle-\rangle^\#_{\beta^\circledast\mathcal{H}}$ as well as the random cluster measure $\mu^\#_{\beta^\circledast\mathcal{H}}$ associated with this state are evidently different from the wired state. However, by the last display, $\mu^\#_{\beta^\circledast\mathcal{H}}$ is lower than the wired state, which implies that the bond density is strictly below the value in the wired state. Thence, the existence of a discontinuity of the magnetization at $\beta^\circledast>\beta_t$ implies a discontinuity in the energy density, which contradicts the result of Theorem A. □

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