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Phase Diagram of Horizontally Invariant Gibbs States for Lattice Models

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Abstract. We study interfaces between two coexisting stable phases for a general class of lattice models. In particular, we are dealing with the situation where several different interface configurations may enter the competition for the ideal interface between two fixed stable phases. A general method for constructing the phase diagram is presented. Namely, we give a prescription determining which of the phases and which of the interfaces are stable at a given temperature and for given values of parameters in the Hamiltonian. The stability here means that typical configurations of the limiting Gibbs state constructed with the corresponding interface boundary conditions differ only on a set consisting of finite components ("islands") from the corresponding ideal interface.

0 Introduction

Before stating our main result in its full generality in the next section, we shall explain the main idea for a particular model. Namely, we shall consider interfaces between two stable phases for the three-dimensional Blume-Capel model. To every site $i \in \mathbb{Z}^3$ a spin $x(i)$ is attached attaining the values $x(i) \in \{-1, 0, +1\}$. The Hamiltonian in a finite volume $\Lambda \subset \mathbb{Z}^3$ with boundary conditions z is given as

$$H_\Lambda(x|z) = J \sum_{\substack{\langle i,j \rangle \\ i,j \in \Lambda}} (x(i) - x(j))^2 + J \sum_{\substack{\langle i,j \rangle \\ i \in \Lambda, j \notin \Lambda}} (x(i) - z(j))^2 - \lambda \sum_{i \in \Lambda} x(i)^2 - h \sum_{i \in \Lambda} x(i).$$

First two sums are over pairs of nearest neighbours, $J > 0$ is fixed. It is easy to see that the phase diagram in the (λ, h)-plane and at vanishing temperature consists of three regions of ground states of constant spins, $x \equiv +1$, $x \equiv -1$, and $x \equiv 0$, separated by half-lines $h = 0, \lambda \geq 0$; $h = \lambda, \lambda \leq 0$; and $h = -\lambda, \lambda \leq 0$. With the help of Pirogov-Sinai theory [PS] one can show that the phase diagram at small temperatures is a smooth deformation of this zero temperature phase diagram [BS] as indicated in Fig. 1.

Notice that the region of the phase 0 is expanding as the temperature grows. This is easy to understand by observing that while the phases + and − have only one type of lowest energy excitation, namely, flipping a single spin to the value 0, the phase 0 allows two excitations of this order, flipping to +1 *or* −1, which gives the phase 0 an advantage at non-vanishing temperatures when excitations contribute to the free energy. The coexistence of all three phases occurs at the line ($\lambda = \lambda_0(T), h = 0$), where $\lambda_0(0) = 0$ and the function $\lambda_0(T)$ is growing in T.

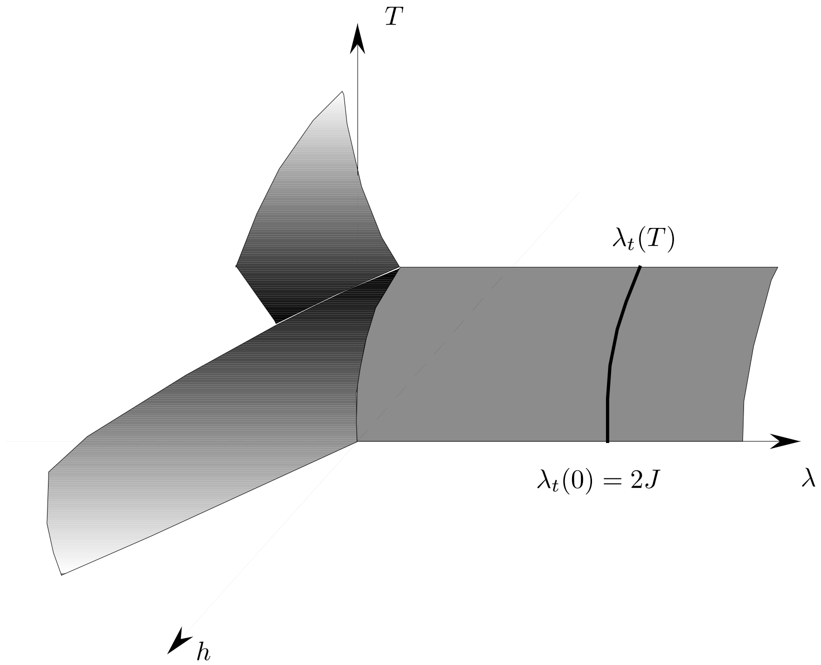

Fig. 1. Phase diagram of Blume-Capel model. The line $\lambda_t(T)$ corresponds to coexistence of two different interface patterns on the boundary between stable plus and minus phases.

Let us consider now, for $h = 0$, $\lambda > 0$, an interface between plus phase in the upper half-space and minus phase in the lower half-space. Depending on the value of λ, different arrangements of spins on the interface yield the minimal energy. Namely, if $\lambda \geq 2J$, the most convenient way is to switch directly from plus to minus spins — the configuration $y^{(I)}$, $y^{(I)}(i) = 1$ whenever $i_3 \geq 0$ and $y^{(I)}(i) = -1$ whenever $i_3 < 0$, is a ground state. Indeed, it is not difficult to see that the difference of the energy of any configuration x that differs from $y^{(I)}$ on finite number of sites and that of the configuration $y^{(I)}$ itself, is nonnegative, $H_\Lambda(x|y^{(I)}) \geq H_\Lambda(y^{(I)}|y^{(I)})$. On the other side, if $\lambda \leq 2J$, it is favourable to separate pluses and minuses by a layer of spins 0; the ground state is the configuration $y^{(II)}$, $y^{(II)}(i) = 1$ whenever $i_3 > 0$, $y^{(II)}(i) = 0$ for $i_3 = 0$, and $y^{(II)}(i) = -1$ whenever $i_3 < 0$.

We will see that this behaviour subsists at small temperatures. Namely, there exists a smooth transition function $\lambda_t(T)$ emanating from the point $\lambda_t(0) = 2J$ at $T = 0$ such that, if $\lambda \geq \lambda_t(T)$, the boundary conditions $y^{(I)}$ yield (at temperature T and in the thermodynamic limit) the Gibbs state whose typical configurations differ from $y^{(I)}$ only on small islands — the state corresponding to $y^{(I)}$ is stable. Similarly, for $\lambda_0(T) + \delta < \lambda \leq \lambda_t(T)$ the state corresponding to $y^{(II)}$ is stable. At the coexistence line $\lambda = \lambda_t(T)$ both states are stable. Notice that the transition is of the first order type — the variable that exhibits a discontinuity while passing through the transition line is, for example, the density of spin 0 at the interface.

We are actually concerned here with the phenomenon of "prewetting" of the microscopic $\pm$ interface by zero spins. Notice that we excluded a small neighbourhood of the coexistence line $\lambda_0(T)$. It is expected that for λ very close to λ_0 ($\lambda - \lambda_0 \sim O(e^{-\beta})$ as $\beta \to \infty$) the layer of zero spins spreads over several lattice sites, with its thickness growing due to "entropic repulsion" as $\lambda \searrow \lambda_0$ (it was this type of wetting that was discussed for $\lambda = \lambda_0$ in [BL]).

Our aim in this paper is to study this type of surface phenomena in a general case. Namely, we are interested in situations where, as in the example above, several different interface configurations may enter the competition for the ideal interface between two fixed stable phases. The simpler case with a single ground state interface in the considered region of parameters (as is the case, for example, for the standard Ising model) is well understood [HKZ] as a straightforward, though rather technically involved, generalization of the standard Dobrushin treatment [D 72]. Simplifying slightly, the main idea is to rewrite the finite volume Gibbs state with interface boundary conditions in terms of the probability distribution of the interface contour separating the regions of two coexisting stable phases. To prove the existence of the interface Gibbs state in the thermodynamic limit, one shows that typical interface contours differ only locally from the ideal ground state interface configuration. To this end one splits the interface contour into regions of ideal interface at different heights — the ceilings — separated by walls. The crucial observation that there is a one-to-one correspondence between interface contours and collections of compatible walls, and that the walls are distributed in an independent way, allows one to use a generalization of the Peierls argument, with walls playing the role of contours, to prove that the probability of walls is dumped and to conclude the existence of interface Gibbs state.

The most important fact that was skipped out in the simplified description above, is that the interface is actually surrounded, from above and below, by two different — possibly asymmetric — coexisting phases. Their influence is taken into account by rewriting the corresponding partition functions, with the help of the Pirogov-Sinai theory, in terms of cluster expansions. After separating a suitable normalizing factor, we expand the cluster terms intersecting the interface. As a final result, one has to deal with an interface decorated by clusters.

The problem in the general case is that presence of various competing interfaces leads to existence of different types of ceilings. To treat the probability distribution of collection of walls one has to cope with the matching conditions on families of walls. Namely, each wall is "labeled" on any connected component of its boundary, where it is attached to surrounding ceilings, by the type of the corresponding ceiling. For the collection of walls stemming from an interface, these labels on boundaries of different walls attached to the same ceiling must coincide. But this is the starting point of the standard Pirogov-Sinai analysis of probabilities of collections of labeled contours. One only has to be slightly more careful when applying these ideas.

One point to consider is the nonlinear dependence on the original Hamiltonian (and its parameters) of the decorating cluster terms originated from the coexisting phases surrounding the interface. These have to be attached to the considered walls and one thus has to be prepared to deal with a nontrivial dependence of wall weights on parameters of the original Hamiltonian.

Second point is that the decorations may actually stick out of the considered finite volume and as a result one is dealing with infinitely many wall-cluster aggregates intersecting the given finite volume. One thus has to be careful when applying standard cluster expansions to treat correctly these situations having also in mind that the weight factors of terms that are not entirely contained in a given volume may actually depend on it yielding a fixed limiting weight factor only when volume expands to infinity.

Last but not least, the weight factors of the wall-cluster aggregates might be actually negative, due to presence of cluster terms whose sign is not determined.

There are different strategies how to deal with these problems. Here, we have chosen the most conservative one. Namely, we rewrite the interface partition function (and the corresponding Gibbs state) in such a way that we can use directly (essentially) standard Pirogov-Sinai theory with the role of contours played by "shadows" of wall-cluster aggregates projected to the plane of ideal interface. To get positive weights allowing in the final account, for example, to estimate the probability of external shadows in the standard manner, we add a suitable cluster sum into the exponent contributions to the interface partition function, absorbing it in the same time into a small change of weight of interfaces. The cluster contributions of this added sum can be easily chosen in such a way that the positivity of combined cluster terms is assured.

There are at least two alternative approaches. First, one may base all the discussion on an extension of Pirogov-Sinai theory to complex parameters [BI]. The first steps in this direction (dealing only with one type of interface and generalizing thus [HKZ]) are done in [BCF 1]. The study in [BCF 1] is motivated by the investigation of interfaces in quantum lattice models [BCF 2]. Other approach, proposed recently by two of us [HZ], is to develop a new alternative to Pirogov-Sinai theory based on the idea of "expanding away", one by one, all contours and walls without ever passing through intermediate contour models with their cluster expansions as it is the case in the standard Pirogov-Sinai theory. This method is conceptually promising and we expect that it will allow a treatment of a great variety of models with different types of interface, wetting, and other "stratified" states (i.e. consisting of several interfaces). The paper [HZ] develops the theory in a very general situation and it has to be supplied by a detailed study of the dependence of stability of resulting interfaces on parameters of the original Hamiltonian to draw the phase diagram at nonvanishing temperatures. A great deal of the present paper treats this particular problem in the situation of an interface with different ceilings and has thus its importance even though it is based on a rather standard Pirogov-Sinai approach and does not evoke the approach from [HZ].

The paper is organized as follows. In Section 1, we set the problem, introduce the class of models to be treated with assumptions (Peierls conditions) that assure a dominance by a particular class of ideal interfaces (ceilings). The main result (Theorem 1) is presented in a general form stressing the smooth dependence of the resulting full phase diagram on the temperature (including the description of regions where particular interfaces become stable — cf. Fig. 1). It follows from Basic Lemma whose proof is postponed, after various preparatory steps in Sections 3 and 4, to Section 4.4.

Section 1 contains, in addition to Theorem 1 with its proof following from Basic Lemma, also the characterization of the ground state phase diagram including interface ground states (Proposition 1.1.3) as well as its completeness under the condition of removing of degeneracy (Corollary 1.1.4) with their proofs in Section 1.3.

Section 2 is devoted to a brief reformulation and a slight extension of the Pirogov-Sinai theory. First, we summarize the results concerning contour models in a form needed for our purposes. This part includes contour models with boundary dependence (models whose contour weights depend slightly on the boundary of the considered volume and are translation invariant only for contours far from the boundary). The corresponding results are of an independent interest, given the fact that one often obtains a reformulation in terms of such a contour model. Next, we introduce labeled contour models and summarize the Pirogov-Sinai theory in Theorem 2 (characterization of stable phases) proved as a consequence of Proposition 2.2.1 (properties of stable phases) that is proved in Section 2.3. Again, also the situation of labeled contour models with boundary dependence is considered (Corollary 2.2.2).

A standard application is the description of periodic Gibbs states in terms of contour models. This is needed in the further treatment and it is presented in Section 3 to set the notation.

Finally, in Section 4 we reformulate the Gibbs states with interfaces in terms of labeled contour models with role of contours played by the shadows of walls decorated by the clusters of the stable phases above and below the interface. This is done in a series of steps that yield an expression for the weights of shadows that are sufficiently dumped to allow once more the application of Theorem 2.

1 Setting and the main result

1.1 Setting; ground state phase diagram

We shall consider *classical lattice models* on a ν-dimensional lattice $\mathbb{Z}^\nu (\nu \geq 3)$ with a finite set S of spin values attached to each lattice site $i \in \mathbb{Z}^\nu$. The configuration space will be denoted by $X(= S^{\mathbb{Z}^\nu})$, the space of restrictions $x_\Lambda = (x(i) : i \in \Lambda)$ of configurations $x = (x(i) : i \in \mathbb{Z}^\nu) \in X$ to $\Lambda \subset \mathbb{Z}^\nu$ by $X_\Lambda(= S^\Lambda)$. We endow the lattice $\mathbb{Z}^\nu$ with the ℓ_∞-metric ($\rho(i,j) = \max_{k=1,\dots,\nu} |i_k - j_k|$). *Connected* (*R-connected*) set $A \subset \mathbb{Z}^\nu$ is then defined as a set whose any two sites $i, j \in A$ can be joined by a sequence of sites $i = i_1, i_2, \ldots, j = i_k$ from A such that $\rho(i_\ell, i_{\ell+1}) \le 1$ ($\rho(i_\ell, i_{\ell+1}) \le R$) for all $\ell = 1, 2, \ldots, k-1$.

Every translation invariant Hamiltonian $H = (U_A; A \subset \mathbb{Z}^\nu, \mathrm{diam} A < R)$ of range R with interactions $U_A : X_A \to \mathbb{R}$ can be identified with $H = (U_{[A]})$, where $[A]$ runs over all (finitely many) equivalence classes of subsets of $\mathbb{Z}^\nu$ defined by shifts of an $A \subset \mathbb{Z}^\nu$ with diamA < R. So the translation invariant Hamiltonians of range R form a finite-dimensional vector *space of Hamiltonians* denoted by $\mathcal{H}(R)$. More precisely, it can be identified with the space of all vectors $(U_A(x_A); i^{(A)} = 0, \mathrm{diamA} < \mathrm{R}, \mathrm{x_A} \in \mathrm{X_A})$, where x_A are ordered in a fixed way. Here $i^{(A)} \in A$ is a site chosen in A in a fixed canonical way (say, the first site in A in a fixed lexicographic order). Throughout we use $\|\cdot\|$ to denote the euclidean norm on $\mathcal{H}(R)$, $\|\cdot\|_\infty$ the maximum norm on it, and $\dim \mathcal{H}(R)$ to denote its dimension.

Using $H_\Lambda(x|z)$ to denote the Hamiltonian in Λ with boundary conditions $z \in X$,

$$H_\Lambda(x|z) = \sum_{A \cap \Lambda \neq \emptyset} U_A(x_A)$$

with $x = z$ in Λ^c, we introduce the *partition function*

$$Z(\Lambda|z; H) = \sum_{x = z \mathrm{in} \Lambda^c} \exp\{-H_\Lambda(x|z)\}.$$

It will be useful to define also

$$Z(Y, \Lambda|z; H) = \sum_{\substack{x_\Lambda \in Y \\ x = z \mathrm{in} \Lambda^c}} \exp\{-H_\Lambda(x|z)\}$$

for any $Y \subset X_\Lambda$.

The *Gibbs state in a finite volume* $\Lambda \subset \mathbb{Z}^\nu$ *under a boundary condition* $x \in X$ *with Hamiltonian* H is the probability $\mu(\cdot, \Lambda|z; H)$ on X defined by

$$\mu(\{x\}, \Lambda|z; H) = \frac{Z(\{x_\Lambda\}, \Lambda|z; H)}{Z(\Lambda|z; H)}$$

whenever $x = z$ in Λ^c.

The set of all *Gibbs states in* (possibly infinite) $V \subset \mathbb{Z}^\nu$ *under a boundary condition* $x \in X_{\mathbb{Z}^\nu \setminus V}$ *with Hamiltonian* H introduced by means of the DLR equations (see [HKZ], Section 2.1, for discussion of the situation with $V \subsetneq \mathbb{Z}^\nu$) will be denoted by $\mathcal{G}(V|x; H)$. For $V = \mathbb{Z}^\nu$ the boundary condition x is necessarily empty and we get the standard definition of Gibbs states ($\mathcal{G}(\mathbb{Z}^\nu \mid \emptyset, H) = \mathcal{G}(H)$). The inverse temperature β does not appear here as an independent parameter; it is incorporated into the constants of the Hamiltonian H. (See also Remark 4 in Section 1.2.)

It is well known that the set of all Gibbs states $\mathcal{G}(H)$ is the closed convex hull of all possible weak limits $\lim_{\Lambda_n \nearrow \mathbb{Z}^\nu} \mu(\cdot, \Lambda_n|z; H)$ of finite volume Gibbs states.

A configuration $x \in X$ is called a *ground configuration* of a Hamiltonian $H = (U_A)$ if "the energy of every finite perturbation $\tilde{x}$ of x ($\tilde{x}$ differs from x at a finite number of lattice sites) is not smaller than the energy of x":

$$H(\tilde{x}; x) = \sum_A (U_A(\tilde{x}_A) - U_A(x_A)) \geq 0. \tag{1.1}$$

If the above difference is positive for each finite perturbation $\tilde{x} \neq x$, we refer to x as to a *ground state* (isolated ground configuration). We use $g(H)$ (resp. $g^{\text{per}}(H)$) to denote the set of all ground configurations corresponding to H (resp. periodic ground configurations).

Our aim will be to discuss phase diagrams including a class of Gibbs states with an ("horizontal") interface. Before taking into account any excitations, we shall describe the phase diagram at vanishing temperature—the *ground state phase diagram.*

We thus suppose that a set G of configurations is given (to play the role of possible ground states). We shall restrict ourselves to horizontally periodic configurations, i.e. we suppose that every $x \in G$ is periodic with respect to translations in the first $(\nu - 1)$ coordinates. We use G^{per} to denote the subset of all $x \in G$ that are also vertically periodic, i.e. periodic with respect to the ν-th coordinate, and we put $G^{\text{hor}} = G \setminus G^{\text{per}}$. Further we assume that each $x \in G^{\text{hor}}$ is identical to two configurations $y_1, y_2 \in G^{\text{per}}$ above and below certain heights, respectively. Namely, for each $x \in G^{\text{hor}}$ there exist two states $y_1, y_2 \in G^{\text{per}}$ and a pair of constants $t_1(x), t_2(x)$ such that $x(i) = y_1(i)$ once $i_\nu \geq t_1(x)$ and $x(i) = y_2(i)$ once $i_\nu \leq t_2(x)$. We may assume that $t_1(x)$ and $t_2(x)$ are chosen as the minimal, resp. maximal, constant with this property. We suppose also that the set G is finite up to vertical translations, i.e. there exists a finite subset of G such that any configuration $x \in G$ is a vertical translation of a configuration from the considered finite set. In particular, G^{per} is finite, $G^{\text{per}} = \{x_1, \ldots, x_r\}$, and there exist a finite constant t (*maximal thickness* of interfaces) such that $t \geq t_1(x) - t_2(x)$ for all pairs $t_1(x), t_2(x)$ above[1].

To control the suppression of excitations with respect to configurations from G, we rely on an extended Peierls condition. To introduce it, let us first define "the specific energy at the site i" by

$$E_i^{(H)}(x) = \sum_{A \ni i} \frac{U_A(x_A)}{|A|} \tag{1.2}$$

for each configuration $x \in X$. Here $|A|$ refers to the number of sites in A. The notation $E_\Lambda^{(H)}(x) = \sum_{i \in \Lambda} E_i^{(H)}(x)$ will be also used. For every periodic configuration $x, x \in X^{\text{per}}$, we also define the *specific energy* $e_x(H)$ of x by

$$e_x(H) = \lim_{n \to \infty} \frac{1}{|V_n^\nu|} \sum_{i \in V_n^\nu} E_i^{(H)}(x) \tag{1.3}$$

[1] This fact implies that the interface introduced below is necessarily connected.

(with V_n^ν denoting a cube consisting of n^ν lattice sites). For any cube V_n^ν whose side is a multiple of the periodicity of $x \in X^{\text{per}}$, one clearly has $E^{(H)}_{V_n^\nu}(x) = |V_n^\nu| e_x(H)$, i.e. $e_x(H) = \frac{1}{|V_n^\nu|}\sum_{i\in V_n^\nu} E_i^{(H)}(x)$.

It is useful to introduce an averaged specific energy[2] at site i for any configuration x by the right hand side of the last equation. Namely, let p be a common multiple of periods of all configurations from G^{per} as well as horizontal periods of all configurations from G^{hor}, for any $i \in \mathbb{Z}^\nu$ let $V_p(i)$ be the cube

$$V_p(i) = \bigl\{ j \in \mathbb{Z}^\nu; j = i + k, k = (k_1, \dots, k_\nu) \in \{0, 1, \dots, p-1\}^\nu \bigr\},$$

and for any configuration $x \in X$ let

$$\overline{E}_i^{(H)}(x) = \frac{1}{|V_p(i)|} \sum_{j \in V_p(i)} E_j^{(H)}(x).$$

It is easy to verify that for any Λ, the sum $\sum_{i\in\Lambda} \overline{E}_i^{(H)}(x)$ differs from $E_\Lambda^{(H)}(x)$ by a local boundary term,

$$\sum_{i\in\Lambda} \overline{E}_i^{(H)}(x) = E_\Lambda^{(H)}(x) + \sum_{j\in\Lambda^c} E_j^{(H)}(x) \frac{|V_p^*(j) \cap \Lambda|}{|V_p^*(j)|} - \sum_{j\in\Lambda} E_j^{(H)}(x) \frac{|V_p^*(j)\cap\Lambda^c|}{|V_p^*(j)|}.$$

Here, $V_p^*(j) = \{ i \in \mathbb{Z}^\nu; j \in V_p(i)\} = \bigl\{ i \in \mathbb{Z}^\nu; i = j - k, k \in \{0, 1, \dots, p-1\}^\nu \bigr\}$. The explicit form of the boundary term is not very relevant; an important fact is, however, that whenever x and $\tilde{x}$ differ only on a finite set, then

$$E_\Lambda^{(H)}(\tilde{x}) - E_\Lambda^{(H)}(x) = \sum_{i\in\Lambda} \overline{E}_i^{(H)}(\tilde{x}) - \sum_{i\in\Lambda} \overline{E}_i^{(H)}(x) \tag{1.4}$$

once Λ is sufficiently large. More exactly, the equality holds if $x(i) \neq \tilde{x}(i)$ implies that $d(i, \Lambda^c) > R + p$. Clearly, $\overline{E}_i^{(H)}(x) = e_x(H)$ for any $x \in G^{\text{per}}$.

Let, now, an integer $d \geq R$ be chosen so that it is surpassing both periodicity p as well as interface maximal thickness t, $d > \max\{R-1, p, t\}$. Consider the set of all *elementary cubes* consisting of d^ν lattice sites. A *bad cube* of a configuration $x \in X$ is an elementary cube D for which x_D differs from y_D for every $y \in G^{\text{per}}$. Notice that the choice of d ensures that the only configurations with no bad cube are those from G^{per}. The *boundary* $B(x)$ of x is the union of all bad cubes of x. If $x \in G^{\text{per}}$ and $\tilde{x}$ is its finite perturbation (differing from x on a finite set of

[2] The reader who is ready to sacrifice the subtlety of the case of periodic but not necessarily translation invariant configurations in G^{per} can skip the present paragraph, suppose that all $x \in G^{\text{per}}$ are translation invariant, and replace everywhere $\overline{E}_i^{(H)}(x)$ by $E_i^{(H)}(x)$. Actually, as explained after the formulation of Basic Lemma in Section 1.3 below, by introducing "block spins" one can rewrite the model (possibly lowering the upper bound on allowed temperatures) in such a way that all configurations in G^{per} become translation invariant as well as all configurations in G^{hor} become horizontally translation invariant.

lattice sites), then, necessarily, $B(\tilde{x})$ is finite. Any connected component Γ of $B(\tilde{x})$ is called a *contour* (of $\tilde{x}$) and we use $\partial(\tilde{x})$ to denote the set of all contours of $\tilde{x}$, $B(\tilde{x}) = \bigcup_{\Gamma\in\partial(\tilde{x})} \Gamma$. Notice that the configuration $\tilde{x}$ coincides with one of the states $x \in G^{\mathrm{per}}$ on every component of $\mathbb{Z} \setminus B(\tilde{x})$. The only $x \in G^{\mathrm{per}}$ that coincides with $\tilde{x}$ on d-boxes that intersect both Γ and the only infinite component of $\mathbb{Z}^\nu \setminus \Gamma$, is called the external boundary condition of the contour Γ of x. Finally, we use $e_0(H)$ for $\min_{x\in G^{\mathrm{per}}} e_x(H)$ and denote $g_0^{\mathrm{per}}(H) = \{x \in G^{\mathrm{per}}; e_x(H) = e_0(H)\}$.

The standard Peierls condition[3] , with $\rho > 0$ and with respect to G^{per}, that is used in the Pirogov-Sinai theory, can be formulated as the bound

$$\sum_{i\in\Gamma} \big(\overline{E}_i^{(H)}(x_\Gamma) - e_x(H)\big) \geq \rho|\Gamma| \qquad (\overline{\mathbf{P}}^{(\mathrm{per})})$$

for any $x \in G^{\mathrm{per}}$, any contour Γ, and any configuration x_Γ such that Γ is its only contour with external boundary condition x. Notice that if all configurations in G^{per} are actually translation invariant, $\overline{E}_i$ can be simply replaced by E_i, yielding Peierls condition in the form

$$E_\Gamma^{(H)}(x_\Gamma) - e_x(H)|\Gamma| \geq \rho|\Gamma|. \qquad (\mathbf{P}^{(\mathrm{per})})$$

Let us remark that one can normalize the condition with respect to the minimum $e_0(H)$ instead of $e_x(H)$ [BK]. This is a useful trick that enables, with some additional care in relevant estimates, to get a uniform validity of the theory far away from lines of coexistence. However, we will restrict our considerations to a small neighbourhood of a fixed Hamiltonian anyway and will thus abstain from this extension.

Turning now to the configurations containing interfaces, we first introduce the notion of a *wall*. Namely, consider a configuration $x \in G^{\mathrm{hor}}$ and its excitation $\tilde{x}$ differing from x on a finite set of lattice sites. Let $I(\tilde{x})$ denote the infinite connected component of $B(\tilde{x})$ (notice that $B(\tilde{x})$ has only one infinite component (cf. footnote 1)) and let $\mathbf{I}(\tilde{x})$, an *interface*, be the pair $(I(\tilde{x}), \tilde{x}_{I(\tilde{x})})$. Notice that $I(\tilde{x})$ is splitting $\mathbb{Z}^\nu \setminus I(\tilde{x})$ into two infinite components.

Denoting, for any $i \in \mathbb{Z}^\nu$, by $C(i)$ the column $\{(i_1, \ldots, i_{\nu-1}, n); n \in \mathbb{Z}\}$ and by $C_d(i)$ its d-neighbourhood, we use $C(\tilde{x})$ to denote the set of those sites i of $I(\tilde{x})$ for which there exists a configuration $y \in G$ such that $I(\tilde{x}) \cap C_d(i) = I(y) \cap C_d(i)$ and $\tilde{x} = y$ on it. The set $C(i) \cap I(y)$, for such $i \in C(\tilde{x})$, is called a *y-ceiling column.*

Further, a pair $\mathbf{w} = (W, \tilde{x}_W)$, where W is a connected component of $I(\tilde{x}) \setminus C(\tilde{x})$, is a *wall* of $\mathbf{I}(\tilde{x})$. We denote by $\mathcal{W}(\mathbf{I}(\tilde{x}))$ the collection of all walls of $\mathbf{I}(\tilde{x})$. Whenever $\mathbf{w} = (W, \tilde{x}_W)$ is a wall, there exists a configuration $y_{\mathbf{w}} \in G$ and its perturbation $x_{\mathbf{w}}$ such that $\mathbf{w}$ is the only wall of $x_{\mathbf{w}}$, $\mathcal{W}(\mathbf{I}(x_{\mathbf{w}})) = \{\mathbf{w}\}$. For any wall

[3] The word *standard* here corresponds to the fact that here we are dealing only with "contours immersed in periodic configurations" in contrast with "walls of an interface" as will be the case below. On the other hand, we are dealing here with a Peierls condition valid uniformly on a neighbourhood of Hamiltonians — this type of the Peierls condition is sometimes [EFS] called "extended Peierls condition".

$\mathbf{w}$, we use $I_W = I(y_{\mathbf{w}}) \cap (\bigcup_{i \in W} C(i))$. Notice that this definition is consistent—the right hand side above is well defined only from geometry of W. Indeed, any wall has an "outside rim" from which the set $I(y_{\mathbf{w}})$ can be uniquely read of.

A Hamiltonian $H = (U_A)$ is said to fulfill the *Peierls condition* ($\overline{\mathbf{P}}$) *with respect to the set* G (and with constant $\rho > 0$) if it satisfies ($\overline{\mathbf{P}}^{(\mathrm{per})}$) as well as

$$\sum_{i \in W} \big(\overline{E}_i^{(H)}(x_{\mathbf{w}}) - e_0(H)\big) - \sum_{i \in I_W} \big(\overline{E}_i^{(H)}(y_{\mathbf{w}}) - e_0(H)\big) \geq \rho |W| \qquad (\overline{\mathbf{P}}^{(\mathrm{hor})})$$

for any wall $\mathbf{w}$ and the corresponding configuration $y_{\mathbf{w}} \in G^{\mathrm{hor}}$ and its excitation $x_{\mathbf{w}}$. Again, if all configurations in G^{per} are translation invariant, the condition takes simpler form

$$\big(E_W^{(H)}(x_{\mathbf{w}}) - e_0(H)|W|\big) - \big(E_{I_W}^{(H)}(y_{\mathbf{w}}) - e_0(H)|I_W|\big) \geq \rho |W|. \qquad (\mathbf{P}^{(\mathrm{hor})})$$

The symbol ($\mathbf{P}$) denotes that both ($\mathbf{P}^{(\mathrm{per})}$) and ($\mathbf{P}^{(\mathrm{hor})}$) are satisfied.

The following lemma is related to Lemma 2.1 from [S].

Lemma 1.1.1 *Let H satisfy the Peierls condition with respect to a non-empty finite set $G^{\mathrm{per}} \subset X^{\mathrm{per}}$. Then all periodic ground configurations of H are ground states, their set $g^{\mathrm{per}}(H)$ coincides with the set $g_0^{\mathrm{per}}(H)$, and $g_0^{\mathrm{per}}(H)$ minimizes specific energy over all X^{per}, $g_0^{\mathrm{per}}(H) = \{x \in X^{\mathrm{per}}; e_x(H) = e_0(H)\}$.*

Proof. 1. We first show that each element x of G^{per} with $e_x(H) = e_0(H)$, is a ground state of H. This, in particular, implies $g_0^{\mathrm{per}}(H) \subset g^{\mathrm{per}}(H)$.

Let y differ from x on a non-empty finite subset of $\mathbb{Z}^\nu$. Let $\Lambda \subset \mathbb{Z}^\nu$ be finite and such that $x(i) \neq y(i)$ implies that $\mathrm{dist}(i, \Lambda^c) > p + R$.

By (1.4), ($\overline{\mathbf{P}}^{(\mathrm{per})}$), and the choice of x and Λ, we get

$$\begin{aligned} H(y;x) = E_\Lambda^{(H)}(y) - E_\Lambda^{(H)}(x) = \\ \textstyle\sum_{i\in\Lambda}\big(\overline{E}_i^{(H)}(y) - \overline{E}_i^{(H)}(x)\big) = \sum_{i\in\Lambda}\big(\overline{E}_i^{(H)}(y) - e_x(H)\big) = \\ = \textstyle\sum_{i\in B(y)}\big(\overline{E}_i^{(H)}(y) - e_x(H)\big) + \sum_{i\in\Lambda\setminus B(y)}(\overline{E}_i^{(H)}(y) - e_x(H)) > \rho|B(y)| > 0. \end{aligned}$$

Notice that for $i \notin B(y)$ the values $\overline{E}_i^{(H)}(y)$ equal to some of $e_z(H)$ with $z \in G^{\mathrm{per}}$ and so they are greater or equal to $e_x(H) = e_0(H)$. Hence the claim of the first step is verified.

2. Let $e_y(H) > e_0(H)$ for some periodic configuration y. Then we may consider a configuration $\tilde{y}$ that is equal to y outside of some large cube $\Lambda \subset \mathbb{Z}^\nu$ and coincides with x in Λ, where $x \in g^{\mathrm{per}}(H)$. It is obvious that $H(\tilde{y}; y) < 0$ if Λ is large enough due to the inequality $e_y(H) > e_0(H) = e_x(H)$. So y is not a ground configuration.

3. Finally, let y be a periodic configuration and $y \notin G^{\mathrm{per}}$. Let us consider configurations y_Λ^x such that y_Λ^x equals to y on a sufficiently large cube $\Lambda \subset \mathbb{Z}^\nu$ and

to $x \in g_0^{\mathrm{per}}(H)$ on Λ^c. We may notice that by the finite range of the potential

$$e_y(H) - e_x(H) = \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{H(y^x_\Lambda; x)}{|\Lambda|},$$

where the limit is e.g. over any sequence of cubes Λ with diameters tending to infinity.

We get, similarly as in step 1, for sufficiently large neighbourhood V of Λ that

$$H(y^x_\Lambda; x) = E_V^{(H)}(y^x_\Lambda) - E_V^{(H)}(x) > \rho |B(y^x_\Lambda)|.$$

There is $c > 0$ independent of Λ such that $|B(y^x_\Lambda)| > c|\Lambda|$ as y is periodic.

Summing up what we observed, we get that $e_y(H) > e_x(H)$ implying the inclusion $\{x \in X^{\mathrm{per}}; e_x(H) = e_0(H)\} \subset G^{\mathrm{per}}$ and finishing thus the proof. □

Our next aim is to study all horizontally periodic ground configurations of H.

In analogy with $e_x(H)$ for $x \in X^{\mathrm{per}}$, we consider a configuration $x \in G \cap X^{\mathrm{hor}}_{y,z}$ with $y, z \in g_0^{\mathrm{per}}(H)$, $y \neq z$, where $X^{\mathrm{hor}}_{y,z}$ is defined as the set of all horizontally periodic configurations asymptotically coinciding with y and z (i.e. $\overline{x} \in X^{\mathrm{hor}}_{y,z}$ if $\overline{x}(i) = y(i)$ whenever $i_\nu \geq t_1$ and $\overline{x}(i) = z(i)$ whenever $i_\nu \leq t_2$ for some $t_1, t_2 \in \mathbb{Z}$), and define

$$e_x(H) = \lim_{n \to \infty} \frac{1}{|V_n^{\nu-1}|} \sum_{i:(i_1,\ldots,i_{\nu-1}) \in V_n^{\nu-1}} \big[E_i^{(H)}(x) - E_i^{(H)}(y)\big]. \tag{1.5}$$

Notice that only finite number of terms in the sum does not vanish and that the configurations y and z can be read off from x and we do not introduce them explicitly into the notation. Notice also that

$$e_x(H) = \sum_{i \in C(0)} \big[\overline{E}_i^{(H)}(x) - e_0(H)\big],$$

where the sum is again only formally infinite. Notice, that even though we keep the same notation as in (1.3), no confusion can arise since these two definitions concern disjoint classes of configurations x; periodic ones in (1.3) and horizontally periodic ones above.

Similar notion to $e_0(H)$, that according to Lemma 1.1.1 equals $\min\{e_x(H); x \in X^{\mathrm{per}}\}$, is

$$e_0^{y,z}(H) = \min\{e_x(H); x \in G \cap X^{\mathrm{hor}}_{y,z}\}.$$

We also use the notation

$$g_0^{y,z}(H) = \{x \in G^{\mathrm{hor}} \cap X^{\mathrm{hor}}_{y,z}; e_x(H) = e_0^{y,z}(H)\}.$$

Finally, we say that a set G is *admissible* if it is non-empty and, up to vertical translations, finite set of horizontally periodic configurations such that for each $x \in G^{\mathrm{hor}}$ it is $x \in X^{\mathrm{hor}}_{y,z}$ for some $y, z \in G^{\mathrm{per}}$, $y \neq z$, and that $G \cap X^{\mathrm{hor}}_{y,z} \neq \emptyset$ for each $y, z \in G^{\mathrm{per}}$, $y \neq z$.

Lemma 1.1.2 *Let G be admissible and let H satisfy the Peierls condition ($\overline{\mathbf{P}}$) with respect to G and some $\rho > 0$.*

Then a horizontally periodic configuration x is a ground state, equivalently a ground configuration, of H if and only if
either $x \in g_0^{\mathrm{per}}(H)$
or $x \in X_{y,z}^{\mathrm{hor}}$ for some $y, z \in g_0^{\mathrm{per}}(H)$, $y \neq z$, and

$$e_x(H) = e_0^{y,z}(H).$$

Moreover, such configuration x automatically belongs to G.

Proof. According to Lemma 1.1.1, it suffices to consider only non-periodic configurations.

1. We first show that each $x \in g_0^{y,z}(H)$ for some $y, z \in g^{\mathrm{per}}(H)$, $y \neq z$, is a ground state.

Let $\tilde{x}$ differ from x on a non-empty finite subset of $\mathbb{Z}^\nu$ and let $\Lambda \subset \mathbb{Z}^\nu$ be a finite cube such that $\tilde{x}(i) \neq x(i)$ implies that $\mathrm{dist}(i, \Lambda^c) > R + p$. Let x^I be the unique configuration with $B(x^I)$ consisting only of the infinite component of $B(\tilde{x})$.

Then
$$H(\tilde{x}; x) = H(\tilde{x}; x^I) + H(x^I; x).$$

Using ($\overline{\mathbf{P}}^{(\mathrm{per})}$), we get that $H(\tilde{x}; x^I) \geq 0$, or even that $H(\tilde{x}; x^I) > 0$ if $x^I = x$ (and thus $\tilde{x} \neq x^I$) when also $H(x^I; x) = 0$.

If $x^I \neq x$, we rewrite $H(x^I; x) = \sum_{i\in\Lambda} \big(\overline{E}_i^{(H)}(x^I) - \overline{E}_i^{(H)}(x)\big)$, by adding and subtracting the term $e_0(H)|\Lambda|$, as the sum over disjoint columns C of the terms of the form

$$\sum_{i\in C\cap\Lambda} \big(\overline{E}_i^{(H)}(x^I) - e_0(H)\big) - \sum_{i\in C\cap\Lambda} \big(\overline{E}_i^{(H)}(x) - e_0(H)\big).$$

For any $\bar{x}$-ceiling column C this term equals to $e_{\bar{x}}(H) - e_0^{y,z}(H)$ that is nonnegative by the definition of $e_0^{y,z}(H)$. Noticing also that for a column C that intersects a wall $\mathbf{w}$ of x_I every contribution $\overline{E}_i^{(H)}(x^I) - e_0(H)$ with $i \in C \cap W^c$ is nonnegative, and using the bound $e_{y_{\mathbf{w}}}(H) \geq e_x(H) = e_0^{y,z}(H)$, and the transcription of the definition of $e_x(H)$ after (1.5) above, we are left with

$$H(x^I; x) \geq \sum_{\mathbf{w}} \Big[\sum_{i\in W} \big(\overline{E}_i^{(H)}(x^I) - e_0(H)\big) - \sum_{i\in I_W} \big(\overline{E}_i^{(H)}(y_{\mathbf{w}}) - e_0(H)\big)\Big] > 0$$

by the Peierls condition ($\overline{\mathbf{P}}^{\mathrm{hor}}$).

2a. Let x be a horizontally periodic ground configuration that is not periodic. If, in the upper half space, there are only finitely many horizontal layers of width d with x equal to some element of $g^{\mathrm{per}}(H)$ on them, then we easily obtain a contradiction with ($\overline{\mathbf{P}}^{\mathrm{per}}$) by considering cubes Λ that are large enough and take x on Λ and some $x_0 \in g^{\mathrm{per}}(H)$ on Λ^c. Notice that the volume of bad cubes in Λ

is of the order of the volume of Λ. So there must be infinitely many such layers of a configuration $y \in g^{\mathrm{per}}(H)$ in the upper half-space and infinitely many layers of $z \in g^{\mathrm{per}}(H)$ (possibly equal to y) in the lower half-space .

Further, whenever there are two disjoint layers on which x equals to the same element y of $g^{\mathrm{per}}(H)$, it necessarily equals y also in-between of these two layers. Indeed, let us suppose that x is not equal to y in-between. Considering parallelepipeds Λ that have the upper mentioned layer just above its top and the lower mentioned layer just below its bottom and changing x on the d-neighbourhood of the boundary of Λ to y may increase the energy at most by a contribution of the order of the size of the side-wise boundary of Λ (excluding its top and bottom), whereas changing additionally x to y everywhere inside Λ causes a decrease of the energy by a contribution of the order of the size of the top of Λ (applying ($\overline{\mathbf{P}}^{\mathrm{per}}$ and the fact that between those two layers there must be a layer containing, due to horizontal periodicity of x, a periodic horizontal grid of bad cubes). Since the size of the top, or bottom, of Λ are asymptotically larger than that of the rest of the boundary of Λ, the energy for Λ large enough decreases and so x is not a ground configuration. This contradiction shows that $x \in X^{\mathrm{hor}}_{y,z}$ for some $y, z \in g^{\mathrm{per}}(H)$, $y \neq z$.

2b. If $e_x(H) > e_0^{y,z}(H)$ for $y, z \in g^{\mathrm{per}}(H)$, $y \neq z$, $x \in X^{\mathrm{hor}}_{y,z}$, then x is not a ground configuration because replacing it by some element $\bar{x}$ of $X^{\mathrm{hor}}_{y,z}$ for which $e_{\bar{x}}(H) = e_0^{y,z}$ on sufficiently large cube, with its top in the upper region for both configurations and its bottom in the lower region for both configurations x and $\bar{x}$, we increase the energy.

3. If $x \notin G$ and $x \in X^{\mathrm{hor}}_{y,z}$, we get that $e_x(H) > e_0^{y,z}(H)$ from ($\overline{\mathbf{P}}^{\mathrm{hor}}$) investigating the limit of $e_x(H) = \frac{1}{|V_n^{\nu-1}|} H(x^w_{V_n}; w)$ for some sequence of cubes V_n tending to $\mathbb{Z}^\nu$. Here $x^w_{V_n}$ is equal to w on V_n^c for some $w \in G \cap X^{\mathrm{hor}}_{y,z}$ with $e_w(H) = e_0^{y,z}(H)$ and to x on V_n. □

Our aim now is to describe "the phase diagram at zero temperature" — the *ground state phase diagram*. Consider a Hamiltonian H_0 with the set G_0 of all horizontally periodic ground states of H_0, $G_0 = g(H_0)$. To describe the ground state phase diagram in a (sufficiently small) neighbourhood of H_0 in $\mathcal{H}(R)$ means to specify, for every subset $G \subset G_0$, the set of all Hamiltonians H for which G is the set of all horizontally periodic ground configurations, $G = g(H)$. In fact we are describing the phase diagram for H from some cone containing a neighbourhood of H_0.

Proposition 1.1.3 (ground state phase diagram) *Suppose that H_0 satisfies the Peierls condition ($\overline{\mathbf{P}}$) with a constant $\rho_0 > 0$ and with respect to an admissible set G_0 of horizontally periodic ground states, $G_0 = g(H_0)$.*

Then there exists $\varepsilon > 0$ such that for each $H \in K_\varepsilon(H_0) = \{\beta \bar{H} : \|\bar{H} - H_0\| < \varepsilon \|H_0\|, \beta > 1\}$ the set $g(H)$ of all horizontally periodic ground configurations is contained in G_0, each $x \in g(H)$ is a ground state and either

- $$x \in G_0^{\mathrm{per}} \text{and } e_x(H) = \min\{e_{\bar{x}}(H) : \bar{x} \in G_0^{\mathrm{per}}\} \qquad (1.6)$$

- $x \in G_0 \cap X^{\rm hor}_{y,z}$ for some $y, z \in g^{\rm per}(H), y \neq z$, and
$$e_x(H) = \min\{e_{\bar{x}}(H) : \bar{x} \in G_0 \cap X^{\rm hor}_{y,z}\} \tag{1.7}$$

Moreover, for each $0 < \bar{\varepsilon} < \varepsilon$ *there exists a constant* $\rho_{\bar{\varepsilon}}$ *such that* $\lim_{\bar{\varepsilon}\to 0_+} \rho_{\bar{\varepsilon}} = \rho_0$ *and each* $H \in K_{\bar{\varepsilon}}(H_0)$ *satisfies the Peierls condition* $(\overline{\mathbf{P}})$ *with the constant* $\rho_{\bar{\varepsilon}} \frac{\|H\|}{\|H_0\|}$.

Proof. To show the validity of Peierls condition for the Hamiltonian H with respect to G_0 and a suitable $\rho_{\bar{\varepsilon}}$, it suffices to compare it with the condition for $H\frac{\|H_0\|}{\|H\|}$ close to H_0. Hence the characterization from Lemmas 1.1.1 and 1.1.2 is valid implying the claims. □

Remark. Notice that only non-empty admissible sets $g(H) \subset g(H_0)$ will appear.

To enable the realization of all admissible $G \subset G_0$ in a neighbourhood of H_0 ("full ground state phase diagram") it is necessary to assume that a condition of *removing of degeneracy* is fulfilled. To formulate it and to enable a global study of the phase diagram at a non-vanishing temperature it is convenient to extend the functional $H \to e_x(H)$ defined by (1.5) only for those Hamiltonians H for which $y, z \in g^{\rm per}(H)$. This can be done for example by defining

$$e_x(H) = \lim_{n\to\infty} \frac{1}{|V_n^{\nu-1}|} \sum_{\substack{i\in I(x)\\(i_1,\dots,i_{\nu-1})\in V_n^{\nu-1}}} [E_i(x) - \min_{y\in G_0^{\rm per}} e_y(H)] \tag{1.8}$$

for every horizontally periodic $x \in X^{\rm hor}_{y,z}$ with $y, z \in g^{\rm per}(H)$. Using $|G \cap X^{\rm hor}_{y,z}|_{\sim ver}$ to denote the number of elements of $G \cap X^{\rm hor}_{y,z}$ taken up to vertical translations, we formulate the condition of removing of degeneracy as the following assumption.

(RD) *An affine subspace* $\mathcal{H}_0 \subset \mathcal{H}(R)$ *removes degeneracy of* $G_0 = g(H_0)$*, an admissible set of horizontally periodic configurations, if the set of*

$$N_0 = \big(|g_0^{\rm per}(H_0)| - 1\big) + \sum_{\substack{y,z\in g_0^{\rm per}(H_0)\\ y\neq z}} (|g_0(H_0) \cap X^{\rm hor}_{y,z}|_{\sim ver} - 1)$$

linear functionals

$$\{e_x(H) - e_{x_0}(H), x \in g_0^{\rm per}(H_0), x \neq x_0\} \cup$$
$$\cup\{e_x(H) - e_{x_{y,z}}(H), x \in g_0(H_0) \cap X^{\rm hor}_{y,z}, x \neq x_{y,z}, y, z \in g_0^{\rm per}(H_0), y \neq z\}$$

with arbitrarily chosen $x_0 \in G^{\rm per}$ *and* $x_{y,z} \in G_0 \cap X^{\rm hor}_{y,z}$ *is a set of linearly independent functionals on* $\mathcal{H}_0$.

Notice that choosing $\mathcal{H}_0 = \mathcal{H}(R)$ with R large enough, degeneracy is always removed. (It is sufficient to choose R larger than the smallest periods of configurations in G_0 as well as "thickness" of $I(x)$ for all $x \in G_0 \cap X^{\rm hor}$.) If we considered G consisting of translation invariant configurations, already single site potentials ("external fields") would be enough to remove the degeneracy.

Let us also remark that one might consider a more general "manifold of parameters" $\mathcal{H}_0 \subset \mathcal{H}(R)$.

Corollary 1.1.4 (completeness of the ground state phase diagram) *Supposing the validity of (RD) (with $\mathcal{H}_0 = \mathcal{H}(R)$) in addition to the assumptions of Proposition 1.1.3 and denoting $\mathcal{H}_{\rm gr}(G) = \{H : G = g(H)\}$, one has $\mathcal{H}_{\rm gr}(G) \cap K_\varepsilon(H_0) \neq \emptyset$ for each admissible $G \subset g_0(H_0)$ and $\mathcal{H}_{\rm gr}(G_1) \cap K_\varepsilon(H_0) \subsetneq \mathcal{H}_{\rm gr}(G_2) \cap K_\varepsilon(H_0)$ whenever $G_1 \supsetneq G_2$. Actually the set $\mathcal{H}_{\rm gr}(G)$ is the intersection of the corresponding number of hyperplanes and open half-spaces in $\mathcal{H}(R)$ whose boundaries are yielded by equalities contained in (1.6) and (1.7).*

1.2 The main result

Our aim is to show that the phase diagram including all horizontally periodic states is a small distortion of the ground state phase diagram described in Proposition 1.1.3 and Corollary 1.1.4 above. A Gibbs state μ of a Hamiltonian $H \in \mathcal{H}(R)$ is said to be *a perturbation of a (ground) configuration* $x \in G_0$ if for μ-almost all configurations $\tilde{x}$ there exists a connected subset $M \subset \mathbb{Z}^\nu$ such that $\tilde{x}$ differs from x only outside M, $\tilde{x}_M = x_M$, the R-components of its complement $\mathbb{Z}^\nu \setminus M$ are finite and, if $x \in G_0 \cap X^{\rm hor}$, also the set $M_0 = \{i \in I(x) | M \supset I(x) \cap C(i)\}$ is connected.

Theorem 1 *Let H_0 be a translation invariant Hamiltonian, $H_0 \in \mathcal{H}(R)$, fulfilling the Peierls condition ($\overline{\mathbf{P}}$) with respect to an admissible set of ground states $G_0 = g(H_0)$ and a sufficiently large ρ_0. Let further an affine subspace $\mathcal{H}_0 \subset \mathcal{H}(R)$ containing H_0 remove degeneracy of G_0 (condition (**RD**)). Then there exist constants $\varepsilon > 0$, $c > 0$, and a one-to-one mapping $\mathcal{T}$ from $K_\varepsilon(H_0)$ onto a subset of $\mathcal{H}(R)$ so that*

(i) $\mathcal{T}(\mathcal{H}_0 \cap K_\varepsilon(H_0)) \subset \mathcal{H}_0$;

(ii) for any $x \in G$, with $G \subset g(H_0)$ admissible, there exists a Gibbs state that is a perturbation of x whenever $H \in \mathcal{T}(\mathcal{H}_{\rm gr}(G) \cap K_\varepsilon(H_0))(=: \mathcal{H}(G))$;

(iii) $\|\mathcal{T}(H) - H\| \leq e^{-(\rho_0 - c)\frac{\|H\|}{\|H_0\|}}$ *for each* $H \in K_\varepsilon(H_0)$;

(iv) $\|(\mathcal{T}(H_1) - H_1) - (\mathcal{T}(H_2) - H_2)\| \leq e^{-(\rho_0 - c)\min(\frac{\|H_1\|}{\|H_0\|}, \frac{\|H_2\|}{\|H_0\|})} \|H_1 - H_2\|$ *for all* $H_1, H_2 \in K_\varepsilon(H_0)$.

Remarks. 1. We expect that the phase diagram is complete in the sense that the only horizontally periodic Gibbs states for $H \in \mathcal{H}(G)$ are those corresponding to $x \in G$. Even though we have not a proof of this fact in a general situation, it should follow from the completeness of the ground state phase diagram by a method similar to that of [Z].

2. The Gibbs states from (ii) satisfy an exponential decay of correlations. Also, explicit integral formulas describing them can be written using the integration with respect to measures on families of external contours and on families of walls which describe the interface of the states corresponding to $x \in G^{\rm hor}$. We shall only describe the probabilities of external contours in Section 3 and of "shadows" in

Section 4 and we abstain from a discussion how to reconstruct all the probability using them. It could be done similarly like in [HKZ] Section 6.

3. The affine space $\mathcal{H}_0$ may be given in the form $\mathcal{H}_0 = \{H_\alpha = H_0 + \sum_{k=1}^N \alpha_k H_k\}$. It is customary to choose Hamiltonians $H_k \in \mathcal{H}(R)$ in a suitable way so that their number N may be taken as the minimal possible ($N = N_0$ from (**RD**)). One could replace $\mathcal{H}_0$ by an N-dimensional smooth manifold defined by a mapping $H(\alpha), \alpha \in \mathbb{R}^N$ (with $N \leq N_0$), from a neighbourhood V of a point $\alpha^{(0)} \in \mathbb{R}^N$ into $\mathcal{H}(R)$ with $H_0 = H(\alpha^{(0)})$. The condition of removing of degeneracy then would be stated as the condition of maximality of the rank of the $N_0 \times N$ matrix with the entries

$$\frac{\partial}{\partial \alpha_k}\big[e_x(H(\alpha)) - e_{x_0}(H(\alpha))\big]\Big|_{\alpha=\alpha^{(0)}}, x \in g^{\mathrm{per}}(H_0), x \neq x_0, \quad \text{and}$$

$$\frac{\partial}{\partial \alpha_k}\big[e_x(H(\alpha)) - e_{x_{y,z}}(H(\alpha))\big]\Big|_{\alpha=\alpha^{(0)}},$$

$$x \in g(H_0) \cap X^{\mathrm{hor}}_{y,z}, x \neq x_{y,z}, y, z \in g^{\mathrm{per}}(H_0), y \neq z.$$

We shall not pursue the case of a general manifold $\mathcal{H}_0$ any further.

4. Usually one considers pairs, a Hamiltonian H and a temperature T, and assigns them the Gibbs states defined by $\frac{1}{T}H$ (we put Boltzmann constant $k = 1$). In Theorem 1 we included the temperature into the Hamiltonian to avoid an overparametrization (multiplying both, the Hamiltonian and the temperature, by the same factor, we get exactly the same set of Gibbs states as originally). One can reformulate Theorem 1 exposing explicitly the temperature. Namely, one may introduce the temperature-depending mapping

$$\mathcal{T}_T(H) = T \cdot \mathcal{T}(\frac{H}{T}).$$

It describes the phase diagram at the slice of constant temperature T: the Gibbs states yielded by $\frac{1}{T}\mathcal{T}_T(H)$ correspond (by (ii)) to the ground states of $\frac{H}{T}$ (the same as the ground states of H). Thus, for a fixed temperature T, one actually considers $\mathcal{H}_0$ as a space of parameters and the mapping $\mathcal{T}_T$ shows how one should deform the ground state phase diagram to get the phase diagram at the temperature T. The condition (iii) shows that $\lim_{T\to 0} \mathcal{T}_T = \mathrm{id}$, the identical mapping, and that the limit is attained exponentially fast. Notice that iv) yields a Lipschitz condition for the mapping

$$(T, H) \to (T, \mathcal{T}_T(H)).$$

We derive Theorem 1 from the following lemma on the existence of a function characterizing the presence of stable states whose proof is the main content of the present paper and is presented in Sections 3, 4 and 5.

We use here and in what follows $\partial^+_{\bar\alpha}$ to denote the directional one-sided derivative in the direction $\bar\alpha$, i.e. $\partial^+_{\bar\alpha} f(\alpha) = \lim_{t\to 0+} \frac{f(\alpha+t\bar\alpha)-f(\alpha)}{t}$. Let us recall that the hamiltonian $H = (U_{[A]}; A \subset \mathbb{Z}^\nu, \mathrm{diam A} < \mathrm{R})$ is an element of the finite-dimensional space $\mathcal{H}(R)$.

Basic Lemma. *Let the assumptions of Theorem 1 be fulfilled (with ρ_0 sufficiently large). Then there exist $\varepsilon > 0$ and mappings $h_x : K_\varepsilon(H_0) \to \mathbb{R}$ $(H \mapsto h_x(H))$ for each $x \in G_0$ such that*

(i) a) *for $H \in K_\varepsilon(H_0)$ and $x \in G_0^{\text{per}}$ such that $h_x(H) = \min_{\tilde{x} \in G^{\text{per}}} h_{\tilde{x}}(H)$ (such x is then called stable), there exists an extremal Gibbs state $\mu \in \mathcal{G}(H)$ that is a perturbation of x;*

b) *for $H \in K_\varepsilon(H_0)$ and $x \in G_0 \cap X^{\text{hor}}_{y,z}$, where $y, z \in G_0^{\text{per}}$, $y \neq z$, are stable elements of G_0^{per} and $h_x(H) = \min_{\tilde{x} \in G_0 \cap X^{\text{hor}}_{y,z}} h_{\tilde{x}}(H)$ (x is stable), there exists an extremal Gibbs state $\mu \in \mathcal{G}(H)$ that is a perturbation of x;*

(ii) *there exists $c > 0$ (independent of ρ_0) such that*

a)
$$|h_x(H) - e_x(H)| \le e^{-(\rho_0 - c)\ \frac{\|H\|}{\|H_0\|}} \tag{1.9}$$

for each $H \in K_\varepsilon(H_0)$ and

b)
$$\|\partial^+_{\bar{H}} h_x(H) - \partial^+_{\bar{H}} e_x(H)\| \le e^{-(\rho_0 - c)\ \frac{\|H\|}{\|H_0\|}} \|\bar{H}\| \tag{1.10}$$

for any $H \in K_\varepsilon(H_0)$, $\bar{H} \in \mathcal{H}(R)$.

Remark (passing to horizontally translation invariant setting). Since we suppose that G_0 is finite up to vertical translations, we may and shall suppose that all elements of G_0^{per} are actually translation invariant and all elements of G_0^{hor} are horizontally translation invariant by considering a modified model. Namely, we can choose some partition of $\mathbb{Z}^d$ into a grid of cubes with edges of a length that is some (e.g. the smallest possible) common multiple of the periods of all concerned periods. The set of "spins" attained at such a cube B consist then of all configurations on B with values in S. This changes the number of "spins" in dependence on the periods only. In the proofs that follow in Sections 3, 4 and 5, we use this observation and suppose that G_0 consists of translation invariant and horizontally translation invariant configurations.

Proof of Theorem 1. Denoting the parameter ε of the cone from Basic Lemma by $\tilde{\varepsilon}$, the assertion (i) implies that it is enough to find a suitable mapping $\mathcal{T}(H)$ solving the equations

$$\begin{aligned} &h_x(\mathcal{T}(H)) - h_{x_0}(\mathcal{T}(H)) = e_x(H) - e_{x_0}(H) \text{ for } x, x_0 \in G_0^{\text{per}}, x \neq x_0, \text{ and} \\ &h_x(\mathcal{T}(H)) - h_{x_{y,z}}(\mathcal{T}(H)) = e_x(H) - e_{x_{y,z}}(H) \text{ for } y, z \in G_0^{\text{per}}, y \neq z \text{ and} \\ &\qquad\qquad x, x_{y,z} \in G_0 \cap X^{\text{hor}}_{y,z},\ x \neq x_{y,z} \end{aligned} \tag{1.11}$$

for all $H \in K_\varepsilon(H_0)$ for some $0 < \varepsilon < \tilde{\varepsilon}$. We see from Basic Lemma (i) immediately that, if x is a horizontally invariant ground configuration of H, then x is stable with respect to $\mathcal{T}(H)$.

The number of different equations in (1.11) is N_0 (see (**RD**) above). To get a unique solution $\mathcal{T}(H)$ satisfying (i) of the theorem we add the following equations. Namely, consider a decomposition $\mathcal{H}(R) = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$ into two orthogonal subspaces such that $\dim \mathcal{H}^{(0)} = N_0$, $\mathcal{H}^{(0)} + H_0 \subset \mathcal{H}_0$, and $\mathcal{H}^{(0)} + H_0$ removes degeneracy of G_0. Thus every $H \in \mathcal{H}(R)$ can be decomposed uniquely as $H = H_0 + H^{(0)} + H^{(1)}$, $H^{(0)} \in \mathcal{H}^{(0)}$, $H^{(1)} \in \mathcal{H}^{(1)}$. Denoting $H^{(1)} = \Pi^{(1)}(H)$ we consider the additional equations

$$\Pi^{(1)}(\mathcal{T}(H)) = \Pi^{(1)}(H). \tag{1.12}$$

The right hand sides of equations (1.11), (1.12) define an invertible linear mapping L from $\mathcal{H}(R)$ onto $\mathbb{R}^{N_0} \times \mathcal{H}^{(1)}$ while the left-hand sides define a mapping F that differs only slightly from L on $K_{\tilde{\varepsilon}}(H_0)$ according to Basic Lemma (ii).

More precisely, we put

$$L(H) = \Big((e_x(H) - e_{x_0}(H); x \in G_0^{\text{per}} \setminus \{x_0\}),$$
$$(e_x(H) - e_{x_{y,z}}(H); y \neq z, y, z \in G_0^{\text{per}}, x \in G_0 \cap X_{y,z}^{\text{per}} \setminus \{x_{y,z}\}), \Pi^{(1)}(H)\Big)$$

and

$$F(H) = \Big((h_x(H) - h_{x_0}(H); x \in G_0^{\text{per}} \setminus \{x_0\}),$$
$$(h_x(H) - h_{x_{y,z}}(H); y \neq z, y, z \in G_0^{\text{per}}, x \in G_0 \cap X_{y,z}^{\text{per}} \setminus \{x_{y,z}\}), \Pi^{(1)}(H)\Big).$$

If we express L in suitable orthogonal coordinates, we get by (**RD**) that the absolute value of the jacobian of L, $|j_L(H)|$, is strictly positive for all $H \in \mathcal{H}(R)$.

Our task thus is to find $\tilde{H} = \mathcal{T}(H)$ fulfilling (1.11) and (1.12), i.e. fulfilling the equation $F(\tilde{H}) = L(H)$ for all $H \in K_\varepsilon(H_0)$ for some ε. Equivalently, we want to solve the equation

$$\tilde{H} = H - L^{-1}P(\tilde{H}), \tag{1.13}$$

where $P(\tilde{H}) = F(\tilde{H}) - L(\tilde{H})$ is a mapping fulfilling the bound

$$\|P(\tilde{H})\| \leq \ e^{-(\rho_0 - c)\frac{\|\tilde{H}\|}{\|H_0\|}} \text{ for } \tilde{H} \in K_{\tilde{\varepsilon}}(H_0)$$

due to Basic Lemma (ii)a), and

$$\|P(\tilde{H}_1) - P(\tilde{H}_2)\| \leq \ e^{-(\rho_0 - c)\min(\frac{\|\tilde{H}_1\|}{\|H_0\|}, \frac{\|\tilde{H}_2\|}{\|H_0\|})} \|\tilde{H}_1 - \tilde{H}_2\| \text{ for } \tilde{H}_1, \tilde{H}_2 \in K_{\tilde{\varepsilon}}(H_0)$$

due to Basic Lemma (ii)b).

To find the fixed point $\tilde{H}$ of the mapping $Q : \tilde{H} \mapsto H - L^{-1}P(\tilde{H})$ in (1.13) for a fixed $H \in K_\varepsilon(H_0)$, we use the Banach contraction principle. We have

$$\|Q(\tilde{H}_1) - Q(\tilde{H}_2)\| \leq \|L^{-1}\| \ e^{-(\rho_0 - c)(1-\varepsilon)} \|\tilde{H}_1 - \tilde{H}_2\|$$

for $\tilde{H}_1, \tilde{H}_2 \in K_\varepsilon(H_0)$.

Since the constant $\|L^{-1}\|\ e^{(\rho_0-c)\ (1-\varepsilon)}$ is smaller than one if ρ_0 is sufficiently large (we may and shall suppose that $\varepsilon \leq \frac{1}{2}$), it suffices to show that Q maps $\overline{U_{\frac{\tilde{\varepsilon}}{2}}(H)} = \{\tilde{H} \mid \|\tilde{H}-H\| \leq \frac{\tilde{\varepsilon}}{2}\}$ into itself for $H \in K_{\frac{\tilde{\varepsilon}}{2}}(H_0)$. This allows us to take $\varepsilon = \frac{\tilde{\varepsilon}}{2}$.

Let $\|\tilde{H}-H\| \leq \frac{\tilde{\varepsilon}}{2}$. Then

$$\|Q(\tilde{H})-Q(H)\| \leq \|L^{-1}\| e^{-(\rho_0-c)(1-\tilde{\varepsilon})}\frac{\tilde{\varepsilon}}{2} \leq \frac{\tilde{\varepsilon}}{2}.$$

Hence, for sufficiently large ρ_0 (chosen independently of $H \in K_\varepsilon(H_0)$), we get that Q is a contraction on $\overline{U_\varepsilon(H)}$.

The unique solution $\mathcal{T}(H)$ in $\overline{U_\varepsilon(H)}$ fulfills the assertions (i) and (ii) of Theorem 1 according to (i) of Basic Lemma. Using (ii) of Basic Lemma, we get the asked properties (iii) and (iv) of Theorem 1 easily. □

2 Labeled contour models

We present here a brief reformulation of the essential part of Pirogov-Sinai theory and its slight extension to a form needed for our application. The task is to grasp some control over a description of "labeled contour models" that arise in the study of ("physical") Gibbs states. The characteristic feature of the Pirogov-Sinai theory is a reformulation in terms of "contour models". The following presentation follows essentially [Z], but it brings some improvements (see especially Theorem 2 and Proposition 2.2.1; compare also the paper [BK] using some ideas from a preliminary version of the present paper).

2.1 Contour models

We use here the word *contour* simply for any finite connected (in the sense of nearest neighbours) non-empty subset of $\mathbb{Z}^\nu$ ($\nu \geq 2$). Given any contour $\Gamma \subset \mathbb{Z}^\nu$, we define its *exterior*, ExtΓ, to be the only infinite connected component of $\mathbb{Z}^\nu \setminus \Gamma$. The *interior*, Int$\Gamma$, of Γ is the union of the other (finite) connected components of $\mathbb{Z} \setminus \Gamma$. We denote $V(\Gamma) = \Gamma \cup \text{Int}\Gamma$. A set ∂ of contours is *compatible* if any pair of distinct elements of ∂ has a disconnected union. A *contour model* is given by introducing a *contour functional*[4] ("contour weights") Ψ which maps the set of contours to $[0,\infty)$. Considering a set L of contours, we define the contour model *partition function* in L by

$$\mathcal{Z}(L;\Psi) = \sum_{\partial \subset L} \prod_{\Gamma \in \partial} \Psi(\Gamma)\,, \tag{2.1}$$

with the sum taken over all compatible families ∂ of contours from L. Notice that this definition makes sense not only for any finite set L, but supposing that the

[4]We restrict our attention here to the case of real-valued contour functionals that will arise in our context.

sum and products converge, also for some infinite L. In dealing with general sets L we follow the abstract setting from [KP]. Given a volume $\Lambda \subset \mathbb{Z}^\nu$, we shall be concerned with two particular cases of sets of contours, namely the set $L(\Lambda)$ of all *contours in* Λ, $L(\Lambda) = \{\Gamma : \mathrm{dist}(V(\Gamma), \Lambda^c) > 1\}$, and the set L_Λ of all *contours intersecting* Λ, $L_\Lambda = \{\Gamma : V(\Gamma) \cap \Lambda \neq \emptyset\}$. For a finite L we can also introduce the *contour model probability distribution* in L by

$$\mu(\{\partial\}, L; \Psi) = \frac{\prod_{\Gamma\in\partial} \Psi(\Gamma)}{\mathcal{Z}(L;\Psi)} \tag{2.2}$$

for any compatible ∂ in L. As usual, the empty product is put equal to one. For an infinite L we can introduce the compact space of families of contours from L as a closed subspace of $\{0,1\}^L$ with the product topology, and consider the weak limit of measures (2.2).

It is useful to introduce a special symbol, e.g. $\rho(\partial, L; \Psi)$, for the "*correlations*" $\mu(\{\bar\partial : \partial \subset \bar\partial\}, L; \Psi)$.

We may now summarize the main facts concerning contour models with a contour functional Ψ satisfying the inequality

$$\Psi(\Gamma) \le \exp(-\tau\,|\Gamma|) \text{ for any contour } \Gamma, \tag{2.3}$$

i.e. with Ψ being a τ-*functional*.

Most of the assertions of the following proposition as well as their proofs can be found, for example, in [S, Se, Br]. In view of an application to "volumes" of the form L_Λ we rely on an abstract version of contour models and cluster expansions as presented in [KP, D 96]. In particular, we define the distance $\mathrm{dist}(\partial, L)$ from a set ∂ of compatible contours to a set L of contours as $\min_{\boldsymbol{C}} \sum_{\Gamma\in\boldsymbol{C}} |\Gamma|$, where the minimum is taken over all clusters $\boldsymbol{C}$ (i.e. sets of contours whose union is connected) such that $\boldsymbol{C} \cap L \neq \emptyset$ and $\left(\bigcup_{\Gamma\in\partial}\Gamma\right) \bigcap \left(\bigcup_{\Gamma\in\boldsymbol{C}}\Gamma\right) \neq \emptyset$.

Let us introduce also the notion of external contours. Namely, a contour $\Gamma \in \partial$ is an *external* contour of ∂ if $\Gamma \subset \mathrm{Ext}\,\bar\Gamma$ for each $\bar\Gamma \in \partial$, $\bar\Gamma \neq \Gamma$. If ∂ is a family of contours such that every $\Gamma \in \partial$ is either external or $\Gamma \subset \mathrm{Int}\,\bar\Gamma$ for some external contour $\bar\Gamma$ of ∂, then we say that *external contours exist.* We use $\vartheta(\partial)$ to denote the set of all external contours.

Proposition 2.1.1 (contour models) *There exist constants* $\tau_{\mathrm{cl}} \equiv \tau_{\mathrm{cluster}}(\nu)$ *and* $c_{\mathrm{cl}} \equiv c_{\mathrm{cluster}}(\nu)$ *(both depending only on the dimension* ν *of the lattice) such that, whenever* L *is an arbitrary set of contours and* Ψ *is a* τ*-functional with* $\tau \ge \tau_{\mathrm{cl}}$*, then the weak limit*

$$\mu(\cdot, L; \Psi) = \lim_{K \nearrow L} \mu(\cdot, K; \Psi)$$

over the system of finite subsets ordered by inclusion exists, and :

a) For μ*-almost all families* ∂ *there exists the set* $\vartheta(\partial)$ *of external contours.*

b) Whenever ∂ *is a compatible family in* L*, we have*

$$\rho(\partial, L; \Psi) \le \exp(-(\tau - 1)\sum_{\Gamma\in\partial} |\Gamma|).$$

c) *Whenever ∂ is a compatible family of contours in the intersection of sets L_1, L_2, we have*

$$|\rho(\partial, L_1; \Psi) - \rho(\partial, L_2; \Psi)| \\ \leq \exp(-(\tau - 1) \sum_{\Gamma \in \partial} |\Gamma|) \exp(-(\tau - c_{\mathrm{cl}}) \operatorname{dist}(\partial, L_1 \triangle L_2)).$$

(Here $L_1 \triangle L_2$ is the symmetric difference $L_1 \triangle L_2 = (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$.) Hence also, for any finite non-empty set $L \subset L_1 \cap L_2$ and any mapping φ of families of contours to real numbers, living[5] *on L, with $|\varphi(\partial)| \leq \|\varphi\|$, one has*

$$\left| \int \varphi(\partial) \mu(d\partial, L_1; \Psi) - \int \varphi(\partial) \mu(d\partial, L_2; \Psi) \right| \\ \leq \|\varphi\| \exp(-(\tau - c_{\mathrm{cl}}) \operatorname{dist}(L, L_1 \triangle L_2)).$$

d) *Let us suppose further that Ψ is translation invariant. Then the limiting "pressure"*

$$p(\Psi) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mathcal{Z}(L(\Lambda); \Psi)$$

(with the limit in the van Hove sense) exists. The partition function $\mathcal{Z}(L(\Lambda); \Psi)$ satisfies the approximation

$$\log \mathcal{Z}(L(\Lambda); \Psi) = |\Lambda| \, p(\Psi) + \varepsilon \, |\partial \Lambda|$$

with $|\varepsilon| \leq \exp(-(\tau - c_{\mathrm{cl}}))$, and

$$|p(\Psi)| \leq \exp(-(\tau - c_{\mathrm{cl}})).$$

e) *Let us suppose now that a family of translation invariant $\tau^{(\alpha)}$-functionals $\Psi^{(\alpha)}$, $\tau^{(\alpha)} \geq \tau_{\mathrm{cl}}$, is given, depending on a parameter α from an open set $\Omega \subset \mathbb{R}^n$ and suppose that the one-sided derivative in the direction $\bar{\alpha}$ fulfills for some $\alpha \in \Omega$, $\bar{\alpha} \in \mathbb{R}^n$, the bound*

$$\left| \partial^+_{\bar{\alpha}} \Psi^{(\alpha)}(\Gamma) \right| \leq \exp(-\tau^{(\alpha)} |\Gamma|) ||\bar{\alpha}||$$

for every contour Γ. Then the derivative $\partial^+_{\bar{\alpha}} p(\Psi^{(\alpha)})$ satisfies the bound

$$\left| \partial^+_{\bar{\alpha}} p(\Psi^{(\alpha)}) \right| \leq \exp(-(\tau^{(\alpha)} - c_{\mathrm{cl}})) ||\bar{\alpha}||.$$

[5] We say that φ lives on L if $\varphi(\partial_1) = \varphi(\partial_2)$ whenever the set of all contours from ∂_1 that are contained in L coincides with the set of all contours from ∂_2 contained in L.

Proof. The assertions a)-c) are standard and their proof may be based on the cluster expansion as presented in the statement a) of Proposition 2.1.3. The statement d) follows from a rather straightforward extension of standard proofs of cluster expansion (Proposition 2.1.3 b)). □

The constant $c_{\text{cl}}(\nu)$ can be derived from the constant $c_{\#} = c_{\#}(\nu)$ determining the number of "contours of given length",

$$\big|\{\Gamma \mid \Gamma \ni 0, |\Gamma| = k\}\big| \leq e^{c_{\#}(\nu)k}. \tag{2.4}$$

Namely, it can be shown (see [KP]) that

$$c_{\text{cl}}(\nu) = c_{\#}(\nu) + a + \frac{\log(1+a)}{\log a} \sim c_{\#}(\nu) + 1.58 \tag{2.5}$$

with $a = \frac{\sqrt{5}-1}{2}$.

One often meets a situation where the contour functional $\Psi_\Lambda(\Gamma)$ depends on the volume Λ once the contour Γ crosses the boundary of Λ. Also in this case one has a full control of the limiting contour model.

Proposition 2.1.2 (contour models with dependence on the boundary) *Let $V \subset \mathbb{Z}^\nu$ be arbitrary and let a sequence V_n of sets converging to V be given, $V_n \nearrow V$. Let for any V_n a τ-functional Ψ_{V_n} be given so that $\Psi_{V_m}(\Gamma) = \Psi_{V_n}(\Gamma)$ whenever $\Gamma \notin L_{V_m \triangle V_n}$* [6] *and suppose that $\tau \geq \tau_{\text{cl}}$ with τ_{cl} from Proposition 2.1.1. Then the limit*

$$\Psi_V = \lim_{n\to\infty} \Psi_{V_n}$$

is uniquely determined and the limiting measure

$$\mu(\cdot, L_V; \Psi_V) = \lim_{n\to\infty} \mu(\cdot, L_{V_n}; \Psi_{V_n})$$

exists. It coincides with the measure

$$\mu(\cdot, L_V; \Psi_V) = \lim_{K \nearrow L_V} \mu(\cdot, K; \Psi_V)$$

from the preceding proposition. Further, for any (possibly infinite) V, V_1, $V_2 \subset \mathbb{Z}^\nu$, one has:

a) Whenever ∂ is a compatible family in L_V, we have

$$\rho(\partial, L_V; \Psi_V) \leq \exp(-(\tau - 1) \sum_{\Gamma \in \partial} |\Gamma|).$$

[6] It means also that $\Psi_\Lambda(\Gamma)$ coincides with $\Psi_{\mathbb{Z}^\nu}(\Gamma)$ for every contour Γ in Λ.

b) *Whenever ∂ is a compatible family of contours in the intersection of sets L_{V_1}, L_{V_2}, we have*

$$|\rho(\partial, L_{V_1}; \Psi_{V_1}) - \rho(\partial, L_{V_2}; \Psi_{V_2})| \\ \leq \exp(-(\tau-1)\sum_{\Gamma\in\partial}|\Gamma|)\exp(-(\tau-c_{\mathrm{cl}})\,\mathrm{dist}(\partial, L_{V_1}\triangle L_{V_2}))$$

with $c_{\mathrm{cl}} = c_{\mathrm{cl}}(\nu)$ from Proposition 2.1.1. For any finite non-empty set $\Lambda \subset V_1 \cap V_2$ and any mapping φ, living on L_Λ, of families of contours to real numbers, with $|\varphi(\partial)| \leq \|\varphi\|$, one has

$$\left|\int \varphi(\partial)\mu(d\partial, L_{V_1}; \Psi_{V_1}) - \int \varphi(\partial)\mu(d\partial, L_{V_2}; \Psi_{V_2})\right| \\ \leq \|\varphi\| \exp(-(\tau-c_{\mathrm{cl}})\,\mathrm{dist}(\Lambda, V_1\triangle V_2)).$$

Suppose, further, that $\Psi \equiv \Psi_{\mathbb{Z}^\nu}$ is translation invariant. Then

c) *The limiting "free energy" (or "pressure")*

$$p(\Psi) = \lim_{\Lambda\nearrow\mathbb{Z}^\nu}\frac{1}{|\Lambda|}\log\mathcal{Z}(L_\Lambda;\Psi_\Lambda)$$

(with the limit in the van Hove sense) exists with $p(\Psi)$ the same as in c) of the preceding proposition[7]*. The partition function $\mathcal{Z}(L_\Lambda;\Psi_\Lambda)$ satisfies also the approximation*

$$\log\mathcal{Z}(L_\Lambda;\Psi_\Lambda) = |\Lambda|\,p(\Psi) + \varepsilon\,|\partial\Lambda|$$

with $|\varepsilon| \leq \exp(-(\tau - c_{\mathrm{cl}}))$.

Proof. Due to the conditions on the functionals Ψ_{V_n}, the value $\Psi_{V_n}(\Gamma)$ stays constant, for every Γ, once n is sufficiently large. Taking into account the possibility to verify the convergence of measures by proving the convergence of correlations $\rho(\partial, L_{V_n}; \Psi_{V_n})$ for finite families ∂ and observing that it may be approximated, from a cluster expansion, by restricting the contour model on a finite number of contours, the statements of Proposition 2.1.2 follow from cluster expansions (Proposition 2.1.3 below) and Proposition 2.1.1 (cf. [HKZ] Proposition B.2). □

Let us also briefly summarize few standard facts about the cluster expansion in a form suitable for our purposes [KP, D 96]. A proof of b) in Proposition 2.1.3 appears in several papers [BK, DKS].

Propositions 2.1.1 and 2.1.2 follow easily from Proposition 2.1.3 below. The results do not serve for the proofs of the above propositions only; we use the explicit form of the expansion in an essential way in Section 4 when studying the probability of interfaces.

[7] Notice that it means in particular that the statement 1 c) is valid also with L_Λ replaced by sets $L^{(\Lambda)}$, where $L(\Lambda) \subset L^{(\Lambda)} \subset L_\Lambda$.

Proposition 2.1.3 (cluster expansion) *There exist constants $\tau_{\rm cl} \equiv \tau_{\rm cl}(\nu)$ and $c_{\rm cl} \equiv c_{\rm cl}(\nu)$ such that the following statements hold true.*

a) Whenever Ψ is a τ-functional with $\tau \geq \tau_{\rm cl}$, then there exists a ("cluster") functional Ψ^T that maps all non-empty clusters of contours (i.e. sets $\boldsymbol{C}$ of contours with connected support $C = \bigcup_{\Gamma \in \boldsymbol{C}} \Gamma$) to $\mathbb{R}$ such that

$$\log \mathcal{Z}(L; \Psi) = \sum_{\boldsymbol{C} \subset L} \Psi^T(\boldsymbol{C}) \tag{2.6}$$

for every finite set L of contours. The functional Ψ^T satisfies, for every $i \in \mathbb{Z}^\nu$, the bound

$$\sum_{\boldsymbol{C}: C \ni i} \left|\Psi^T(\boldsymbol{C})\right| \exp\bigl((\tau - c_{\rm cl}) \sum_{\Gamma \in \boldsymbol{C}} |\Gamma|\bigr) \leq 1, \tag{2.7}$$

with the sum taken over all clusters with support $C = \bigcup_{\Gamma \in \boldsymbol{C}} \Gamma$ containing a fixed site $i \in \mathbb{Z}^\nu$. The value $\Psi^T(\boldsymbol{C})$ depends only on values of $\Psi(\Gamma)$ with $\Gamma \in \boldsymbol{C}$, and it is translation invariant once the functional Ψ is translation invariant.

b) Suppose further that $\tau^{(\alpha)}$- functionals $\Psi^{(\alpha)}$, $\tau^{(\alpha)} \geq \tau_{\rm cl}$, depend on a parameter α from an open set $\Omega \subset \mathbb{R}^n$ and satisfy for every contour Γ, $\alpha \in \Omega$, and any $\bar{\alpha} \in \mathbb{R}^n$, the inequality

$$\left|\partial^+_{\bar{\alpha}} \Psi^{(\alpha)}(\Gamma)\right| \leq \exp(-\tau^{(\alpha)} |\Gamma|) ||\bar{\alpha}||. \tag{2.8}$$

Then, for every $i \in \mathbb{Z}^\nu$ and all $\alpha \in \Omega$, one has

$$\sum_{\boldsymbol{C}: C \ni i} \left|\partial^+_{\bar{\alpha}} \Psi^{(\alpha)T}(\boldsymbol{C})\right| \exp\bigl((\tau^{(\alpha)} - c_{\rm cl}) \sum_{\Gamma \in \boldsymbol{C}} |\Gamma|\bigr) \leq ||\bar{\alpha}||. \tag{2.9}$$

Remark. Assuming, instead of (2.8), the existence of the Fréchet derivative and a bound on its norm, we actually get the existence of the Fréchet derivative also for $\Psi^{(\alpha)T}$, with a corresponding bound.

Resuming over all clusters with coinciding support and denoting $\mathit{\Psi}^T(C) = \sum_{\boldsymbol{C}: \bigcup_{\Gamma \in \boldsymbol{C}} \Gamma = C} \Psi^T(\boldsymbol{C})$, we will get a formulation that is particularly suitable for our implementations.

Corollary 2.1.4 *Let $\tau_{\rm cl}$ and $c_{\rm cl}$ be the constants from Proposition 2.1.3. If Ψ is a τ-functional with $\tau \geq \tau_{\rm cl}$, then*

$$\log \mathcal{Z}(L(\Lambda); \Psi) = \sum_{C \subset \Lambda^{(0)}} \mathit{\Psi}^T(C) \tag{2.10}$$

for every finite $\Lambda \subset \mathbb{Z}^\nu$. The sum above runs over all connected subsets C of the set $\Lambda^{(0)} = \{i \in \Lambda \ \mathrm{dist}(i, V^c) > 1\}$. The functional $\Psi^T(C)$ satisfies the bound

$$\sum_{C \ni i} \left|\Psi^T(C)\right| \exp\bigl((\tau - c_{\mathrm{cl}})\,|C|\bigr) \le 1. \tag{2.11}$$

Let, moreover, Ψ be translation invariant. Then the limits

$$p(\Psi) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mathcal{Z}(L(\Lambda); \Psi) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mathcal{Z}(L_\Lambda; \Psi)$$

(the limit in the van Hove sense) exist and satisfy the bounds

$$\left|p(\Psi)\right| \le \exp\{-(\tau - c_{\mathrm{cl}})\} \tag{2.12}$$

and

$$\left|\partial^+_{\bar\alpha} p(\Psi)\right| \le \exp\{-(\tau - c_{\mathrm{cl}})\}||\bar\alpha||. \tag{2.13}$$

The function $p(\Psi)$ is explicitly given by

$$p(\Psi) = \sum_{C \ni 0} \frac{\Psi^T(C)}{|C|}. \tag{2.14}$$

Remark. Notice, that the assumption of the translation invariance of Ψ in Propositions 2.1.1 c), 2.1.2 c), and the above corollary is actually not necessary. It is enough to introduce

$$p_i(\Psi) = \sum_{\boldsymbol{C} : C \ni i} \frac{\Psi^T(\boldsymbol{C})}{|C|} = \sum_{C \ni i} \frac{\Psi^T(C)}{|C|}$$

and to replace the terms $|\Lambda|\, p(\Psi)$ by

$$\sum_{i \in \Lambda} p_i(\Psi).$$

For periodic Ψ the limit $p(\Psi)$ is obtained as a mean of $p_i(\Psi)$ over the cell of periodicity.

2.2 Labeled contour models

We consider a finite set $Q = \{1, \ldots, r\}$ of "*labels*" and call $\gamma = (\Gamma, \lambda)$ a *labeled contour* if its support $\Gamma \equiv \Gamma(\gamma) \subset \mathbb{Z}^\nu$ is a finite non-empty connected set (a contour), and $\lambda = \lambda(\gamma)$ assigns to each connected component of the boundary $\partial\Gamma$ of Γ some $q \in Q$. A labeled contour γ is called a *q-contour* if the label assigned to its *external boundary* (the boundary of $\Gamma \cup \mathrm{Int}\,\Gamma$) is q. A family x of labeled contours is said to be *compatible and matching* if their supports are compatible and their labels match (i.e., considering a connected component, say C, of $\mathbb{Z}^\nu \setminus \bigcup\{\Gamma(\gamma) : \gamma \in x\}$, all connected components of the boundaries of Γ's adjacent to C are labeled by the same label q; we say that C is a *q-component*). Compatible and matching families x of labeled contours play a role of configurations for labeled contour models [8]. Whenever x is such that for the family $\partial = \{\Gamma(\gamma) : \gamma \in x\}$ of its supports, there exists the set $\vartheta(\partial)$ of external contours (in the sense of the definition preceding Proposition 2.1.2), we introduce the set $\vartheta(x)$ ($\vartheta_q(x)$) of *external labeled contours* of the family x as the subset of those $\gamma = (\Gamma, \lambda) \in x$ whose support Γ is an external ($q-$)contour of ∂. We say that x is included in $V \subset \mathbb{Z}^\nu$ if $\operatorname{dist}(V(\Gamma(\gamma)), V^c) > 1$ for $\gamma \in x$. Whenever $\Lambda \subset \mathbb{Z}^\nu$, we consider the set $\mathfrak{X}(\Lambda)$ of compatible matching families x of labeled contours that are included in Λ. We introduce the set $\mathfrak{X}^q_\Lambda$ as the set of all compatible matching families of labeled contours that intersect Λ and are such that external contours exist and all external contours as well as all contours not belonging to $\mathfrak{X}(\Lambda)$ are q-contours. Let us also denote $\mathfrak{X}_\Lambda = \bigcup_q \mathfrak{X}^q_\Lambda$. If $x \in \mathfrak{X}_\Lambda$, the union of m-components of $\Lambda \setminus \bigcup\{\Gamma(\gamma) : \gamma \in x\}$ together with supports of all m-contours of x is denoted by $\Lambda_m(x)$.

Clearly, the number of labeled contours γ, for which $\operatorname{supp}\gamma = \Gamma$, is bounded by $|S|^{|\Gamma|}$.

A *labeled contour functional Φ* ("an exponential of a contour Hamiltonian") maps the set of labeled contours into $[0, \infty)$. We also assume that a vector $\varphi = (\varphi_1, \ldots, \varphi_r) \in \mathbb{R}^r$ (of "specific energies of some translation invariant configurations") is given. We consider a *labeled contour model* with a boundary condition $q \in Q$ by introducing, for any $\Lambda \subset \mathbb{Z}^\nu$ and any $\mathfrak{X}^{(\Lambda)}$, $\mathfrak{X}(\Lambda) \subset \mathfrak{X}^{(\Lambda)} \subset \mathfrak{X}_\Lambda$, the (labeled contour model) *partition functions*

$$Z(\mathfrak{X}^{(\Lambda)}|q; \Phi, \varphi) = \sum_{x \in \mathfrak{X}^{(\Lambda)}} \exp\{-\sum_{m \in Q} \varphi_m\, |\Lambda_m(x)|\} \prod_{\gamma \in x} \Phi(\gamma). \tag{2.15}$$

Similarly we introduce the (labeled contour model) *probability distribution*

$$\mu(\{x\}, \mathfrak{X}^{(\Lambda)} \mid q; \Phi, \varphi) = Z^{-1}(\mathfrak{X}^{(\Lambda)}|q; \Phi, \varphi) \exp\{-\sum_{m \in Q} \varphi_m\, |\Lambda_m(x)|\} \prod_{\gamma \in x} \Phi(\gamma). \tag{2.16}$$

Again, the notation for partition functions and probability distributions above is distinguishing them from those for a lattice spin model (c.f. Section 1.1) only by using different variables. When stressing the fact that we are dealing with partition functions and probability distributions of a labeled contour model, we will use the notation Z^{cont} and μ^{cont}.

Remarks. 1. For any $\Lambda \subset \mathbb{Z}^\nu$, the sets $\mathfrak{X}(\Lambda)$ (resp. $\mathfrak{X}_\Lambda$) may be embedded into the product space $\{1, \ldots, r\}^{\mathbb{Z}^\nu}$ by assigning to every x that configuration from $\{1, \ldots, r\}^{\mathbb{Z}^\nu}$ which attains the value q at all lattice sites in $\Lambda_q(x)$. On the sets $\mathfrak{X}(\Lambda)$ (resp. $\mathfrak{X}_\Lambda$) we consider the topology inherited from the compact space $\{1, \ldots, r\}^{\mathbb{Z}^\nu}$.

[8] We use the same notation as for spin configurations of classical lattice models in view of the existence of a natural identification of a class of lattice configurations with a given compatible and matching family of labeled contours (see Section 3).

Let us notice that for an infinite Λ, even if $x = \lim x_n$ with x_n such that $\vartheta(x_n)$ consists of q-contours and in the same time $\partial(x_n)$ converges, in the topology of families of contours used in Section 2.1, to a compatible family of contours ∂ such that external contours $\vartheta(\partial)$ exists, it might be that $\vartheta(x) = \vartheta_{q'}(x)$ with $q' \neq q$.

2. Notice that there are small formal differences in our formulation and that of [S]. First, our definition of contours does not specify configurations on the support. We also found useful to consider directly the "weights of contours", thus the functional Φ above corresponds to $\exp(-\tilde{\Phi})$ in [S, Z].

In the standard Pirogov-Sinai theory, the functional $\tilde{\Phi}$ as well as the r-tuple $\varphi = (\varphi_1, \ldots, \varphi_r)$ are linear in the Hamiltonian H (they are given in terms of explicit formulae — see e.g. (1.4) and (1.6) from [Z]). However, in the present application, we shall meet a more general situation.

3. Readers accustomed to standard Pirogov-Sinai formulation may also notice that the partition function $Z(\mathfrak{X}(\Lambda)|q; \Phi, \varphi)$ corresponds, roughly speaking, to the (relative) diluted partition function multiplied by the factor $\exp(\varphi_q |\Lambda|)$ (since the standard approach is based on relative Hamiltonian with respect to the energy corresponding to the external boundary condition).

The crucial part of the Pirogov-Sinai theory is formulated in Theorem 2 below. First, we need a definition and some more notation.

Definition. A phase q is called c_s-*stable in* Λ (with respect to Φ and φ) if

$$\frac{Z(\mathfrak{X}(\Lambda)|\lambda; \Phi, \varphi)}{Z(\mathfrak{X}(\Lambda)|q; \Phi, \varphi)} \leq \exp(c_s |\partial\Lambda|),$$

for every labeling λ (i.e. a label $\lambda(b) \in Q$ for every connected component b of the boundary $\partial\Lambda$, chosen in a compatible way). A phase q is said to be c_s-*stable* (with respect to Φ and φ) if it is c_s-stable in Λ for every non-empty finite Λ.

The main aim of the Pirogov-Sinai theory is to provide a characterization of stable phases showing, in the same time, that any stable phase gives rise to a distinct Gibbs state that is a perturbation of the corresponding ground configuration. We split these claims into two statements. First, in Theorem 2, we characterize the stability (see (**S**)) in terms of certain functions $h_q(\Phi, \varphi)$ that are close to external fields φ_q. Proposition 2.2.1 then describes the properties of the stable phases.

Theorem 2 (characterization of stable phases). *Let, for every α from an open set $\Omega \subset \mathbb{R}^n$ of parameters and any $\Lambda \subset \mathbb{Z}^\nu$, a nonnegative labeled contour functional $\Phi_\Lambda^{(\alpha)}$ and a vector $\varphi^{(\alpha)}$ of "specific energies" be given (with labels from a finite set Q) such that $\Phi_{\Lambda_1}^{(\alpha)}(\gamma) = \Phi_{\Lambda_2}^{(\alpha)}(\gamma)$ whenever $\operatorname{supp}\gamma \subset \Lambda_1 \cap \Lambda_2$. We denote $\Phi^{(\alpha)} = \Phi_{\mathbb{Z}^\nu}^{(\alpha)}$. We suppose that a continuous function $\tau^{(\alpha)}$ is given so that for every labeled contour γ, and all $\alpha \in \Omega$, we have*

$$\Phi_\Lambda^{(\alpha)}(\gamma) \leq \exp(-\tau^{(\alpha)} |\Gamma(\gamma)|). \tag{1}$$

Then, there exist a constant $\tau_\ell = \tau_{\text{labeled}}(c_s, \nu, |S|)$ *and functions* $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ *characterizing* c_s*-stability*[9] *whenever* $\tau^{(\alpha)} \geq \tau_\ell$*. Namely,*

$$\begin{aligned}&\textit{the phase } q \textit{ is } c_s\textit{-stable with respect to } \Phi^{(\alpha)} \textit{ and } \varphi^{(\alpha)}\\ &\textit{if and only if } h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = \min_m (h_m(\Phi^{(\alpha)}, \varphi^{(\alpha)})).\end{aligned} \qquad (\mathbf{S})$$

The functions $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ *can be chosen in such a way that, denoting* $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = \min_m(h_m(\Phi^{(\alpha)}, \varphi^{(\alpha)}))$*, one has*

$$Z(\Lambda|q; \Phi^{(\alpha)}, \varphi^{(\alpha)}) \geq \exp\Big[-h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})\,|\Lambda| - \varepsilon\,|\partial\Lambda|\Big]$$

and

$$Z(\Lambda|q; \Phi^{(\alpha)}, \varphi^{(\alpha)}) \leq \exp\Big[-h(\Phi^{(\alpha)}, \varphi^{(\alpha)})\,|\Lambda| + \varepsilon\,|\partial\Lambda|\Big]$$

with $\varepsilon = e^{-(\tau^{(\alpha)} - c_{\text{cl}})} + e^{-(\tau^{(\alpha)} - c_{\text{cl}} - c_{\#} - \log|S| - 1)}$*.*

Moreover, there exists a constant $c_\ell = c_{\text{labeled}}(c_s, \nu, |S|)$ *such that*

$$\big|h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) - \varphi_q^{(\alpha)}\big| \leq \exp\big\{-(\tau^{(\alpha)} - c_\ell - c_{\text{cl}})\big\}$$

for $q = 1, \ldots, r$ *and* $\alpha \in \Omega$*.*

Supposing, moreover, for any $\alpha \in \Omega$ *and* $\bar{\alpha} \in \mathbb{R}^n$*, the bounds on the (one-sided) directional derivatives of* $\Phi_\Lambda^{(\alpha)}$ *and* $\varphi_q^{(\alpha)}$*,*

$$\big|\partial_{\bar{\alpha}}^+ \Phi_\Lambda^{(\alpha)}(\gamma)\big| \leq \exp\big\{-\tau^{(\alpha)}\,|\Gamma(\gamma)|\big\}||\bar{\alpha}|| \quad \textit{and} \tag{2}$$

$$\big|\partial_{\bar{\alpha}}^+ \varphi_q^{(\alpha)}\big| \leq M||\bar{\alpha}|| \quad \textit{for some } M > 0, \tag{3}$$

there exists constants $\tau_\ell = \tau_\ell(c_s, M, \nu, |S|)$ *and* $c_\ell = c_\ell(c_s, \nu, |S|)$ *(possibly larger than those above) such that for* $\tau^{(\alpha)} > \tau_\ell$ *we have:*

$$\big|\partial_{\bar{\alpha}}^+ \big[h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) - \varphi_q^{(\alpha)}\big]\big| \leq \exp\big\{-(\tau^{(\alpha)} - c_\ell - c_{\text{cl}})\big\}||\bar{\alpha}||$$

and, denoting $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = \min_m(h_m(\Phi^{(\alpha)}, \varphi^{(\alpha)}))$*, also*

$$\big|\partial_{\bar{\alpha}}^+ h(\Phi^{(\alpha)}, \varphi^{(\alpha)})\big| \leq \big(M + \exp\big\{-(\tau^{(\alpha)} - c_\ell - c_{\text{cl}})\big\}\big)||\bar{\alpha}||$$

for $\alpha \in \Omega$ *and* $\bar{\alpha} \in \mathbb{R}^n$*.*

Remark. Notice, in particular, that if Ω is convex, the functions $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) - \varphi_q^{(\alpha)}$ are, as functions of α, Lipschitz with the constant $\exp\big\{-(\tau^{(\alpha)} - c_\ell - c_{\text{cl}})\big\}$.

The theorem follows from an explicit construction of functions $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ in terms of contour functionals Ψ and $\bar{\Psi}$ introduced below. The expression in terms of those functionals also yields the properties of stable phases.

[9] Stability implies a good control of the corresponding states. These implications, as well as an explicit construction of the characterizing functions $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$, are presented in Proposition 2.2.1 and Corollary 2.2.2.

Proposition 2.2.1 (properties of stable phases) *Under assumption (1) of Theorem 2:*

a) Existence of contour functionals Ψ *For every finite Λ, $\alpha \in \Omega$ and $q = 1, \dots, r$, there exist uniquely determined nonnegative contour functionals $\Psi^{(\alpha)}_{q,\Lambda}$ such that*

$$Z(\mathfrak{X}^{(\Lambda)}|q; \Phi^{(\alpha)}, \varphi^{(\alpha)}) = \exp\{-\varphi^{(\alpha)}_q |\Lambda|\} \mathcal{Z}(L^{(\Lambda)}; \Psi^{(\alpha)}_q) \tag{2.17}$$

for any $L^{(\Lambda)}$, $L(\Lambda) \subset L^{(\Lambda)} \subset L_\Lambda$ with $\mathfrak{X}^{(\Lambda)} = \{x \in \mathfrak{X}_\Lambda; \gamma \in X$ implies $\Gamma(\gamma) \in L^{(\Lambda)}\}$, and

$$\mu(\{x : \vartheta(x) = \vartheta\}, \mathfrak{X}^{(\Lambda)} \mid q; \Phi^{(\alpha)}_\Lambda, \varphi^{(\alpha)}) = \mu(\{\partial : \vartheta(\partial) = \vartheta\}, L^{(\Lambda)}; \Psi^{(\alpha)}_{q,\Lambda}) \tag{2.18}$$

for any collection ϑ of external q-contours. Moreover, $\Psi^{(\alpha)}_{q,\Lambda_1}(\Gamma) = \Psi^{(\alpha)}_{q,\Lambda_2}(\Gamma)$ whenever[10] $V(\Gamma) \subset \Lambda_1 \cap \Lambda_2$. The functional $\Psi^{(\alpha)}_{q,\Lambda}$ satisfies, for any Λ, the bound

$$\left|\Psi^{(\alpha)}_{q,\Lambda}(\Gamma)\right| \le \exp\{-(\tau^{(\alpha)} - c_\ell)\,|\Gamma|\},$$

whenever Γ is such that q is c_s-stable in every component of $\operatorname{Int}\Gamma$ *(with respect to $\Phi^{(\alpha)}$ and $\varphi^{(\alpha)}$) and $c_\ell \ge c_\#(\nu) + c_s + \log|S|$.*

b) Existence of contour $\boldsymbol{\tau}$-functionals $\bar{\Psi}$ *Taking $\Psi^{(\alpha)}_q = \Psi^{(\alpha)}_{q,\mathbb{Z}^\nu}$, nonnegative contour functionals $\bar{\Psi}^{(\alpha)}_q$ exist, such that, for every contour Γ, we have*

- $\bar{\Psi}^{(\alpha)}_q(\Gamma) \le \Psi^{(\alpha)}_q(\Gamma)$,
- $\bar{\Psi}^{(\alpha)}_q(\Gamma) = \Psi^{(\alpha)}_q(\Gamma)$ *whenever q is c_s-stable in* $\operatorname{Int}\Gamma$,
- $\left|\bar{\Psi}^{(\alpha)}_q(\Gamma)\right| \le \exp\{-(\tau^{(\alpha)} - c_\ell)\,|\Gamma|\}$ *with $c_\ell \ge c_\# + c_s + \log|S|$.*

Supposing that the functional $\Phi^{(\alpha)}$ is translation invariant (in an obvious way with respect to shifts in $\mathbb{Z}^\nu$), the functional $\bar{\Psi}^{(\alpha)}_q$ can be chosen to be translation invariant. Further on, there exists $\tau_\ell = \tau_\ell(c_s, \nu, |S|)$ such that for $\tau \ge \tau_\ell$ we have

c) Description of $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ in terms of $\bar{\Psi}^{(\alpha)}$ *The functions $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})(\equiv h_q(\bar{\Psi}^{(\alpha)}, \varphi^{(\alpha)}))$ defined by*

$$h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = \varphi_q - p(\bar{\Psi}^{(\alpha)}_q),$$

with $p(\bar{\Psi}^{(\alpha)}_q)$ defined in Corollary 2.1.4, characterize the stability (i.e., satisfy the equivalence (**S**)*).*

Further on, whenever α is such that q is c_s-stable with respect to $\Phi^{(\alpha)}$, $\varphi^{(\alpha)}$, one has

$$h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = -\lim \frac{1}{|\Lambda|} \log Z(\mathfrak{X}(\Lambda)|q; \Phi^{(\alpha)}, \varphi^{(\alpha)}) = h(\Phi^{(\alpha)}, \varphi^{(\alpha)}).$$

---

[10] Supposing equality $\Phi_{\Lambda_1}(\gamma) = \Phi_{\Lambda_2}(\gamma)$ for $\operatorname{supp}\gamma \cap (\Lambda_1 \triangle \Lambda_2) = \emptyset$, we get $\Psi^{(\alpha)}_{q,\Lambda_1}(\Gamma) = \Psi^{(\alpha)}_{q,\Lambda_2}(\Gamma)$ whenever $\operatorname{dist}(V(\Gamma), \Lambda_1 \triangle \Lambda_2) > 1$.

Supposing, moreover, the smoothness of $\Phi_\Lambda^{(\alpha)}$ *and* $\varphi_q^{(\alpha)}$ *(the conditions (2) and (3) of Theorem 2), there exist a constant* $\tau_\ell = \tau_\ell(c_s, M, \nu, |S|)$ *(possibly larger then* $\tau_\ell(c_s, \nu, |S|)$ *above) such that for* $\tau^{(\alpha)} > \tau_\ell$ *we have :*

a′) Smoothness of contour functionals $\Psi^{(\alpha)}$ *For any* Λ*,*

$$\left|\partial_{\bar\alpha}^{+}\Psi_{q,\Lambda}^{(\alpha)}(\Gamma)\right| \le \exp\left\{-(\tau^{(\alpha)} - c_\ell)\,|\Gamma|\right\}||\bar\alpha||$$

with $c_\ell \ge c_s + c_\# + \log|S| + 1 + \ln 2$*, whenever* Γ *(and* α*) are such that* q *is* c_s*-stable in every component of* $\operatorname{Int}\Gamma$ *(with respect to* $\Phi^{(\alpha)}$ *and* $\varphi^{(\alpha)}$*).*

b′) Smoothness of contour functionals $\bar\Psi^{(\alpha)}$ *There exist functionals* $\bar\Psi_q^{(\alpha)}$ *satisfying b), with* $c_\ell \ge 2c_s + c_\# + \log|S| + 1$*, and*

$$\left|\partial_{\bar\alpha}^{+}\bar\Psi_{q}^{(\alpha)}(\Gamma)\right| \le \exp\left\{-(\tau^{(\alpha)} - c_\ell)\,|\Gamma|\right\}||\bar\alpha||$$

with $c_\ell \ge 4c_s + c_\# + 2 + 2\log(|S|+1)$*, for every* Γ *and every* $\alpha \in \Omega$*.*

Observation. Notice that, by b) and c), $h_q(\Phi,\varphi) = \min_m(h_m(\Phi,\varphi))$ iff $\bar\Psi_q^{(\alpha)} = \Psi_q^{(\alpha)}$. Further, the notion of c_s-stability of a phase q actually does not depend on c_s. Hence, in the following, we will just say that a phase q is stable. (However, changes of c_s may lead to changes of τ_ℓ.)

Remarks.

1. Clearly, there exists a choice of sufficiently large $c_\ell(c_s, \nu, |S|)$ such that it can be used simultaneously in the bounds on $\left|\Psi_q^{(\alpha)}(\Gamma)\right|$ and $\left|\bar\Psi_q^{(\alpha)}(\Gamma)\right|$ from a) and b) as well as in the bounds on $\left|\partial_{\bar\alpha}^{+}\Psi_q^{(\alpha)}(\Gamma)\right|$ and $\left|\partial_{\bar\alpha}^{+}\bar\Psi_q^{(\alpha)}(\Gamma)\right|$ from a′) and b′) .
2. The limit in c) is over finite volumes in the van Hove sense and defines the quantity corresponding to "pressure" in the physical model based on a Hamiltonian at a particular temperature.
3. Studying the contour model probabilities $\mu(\cdot, \Lambda; \Psi_q^{(\alpha)})$ for stable q and their limits for simply connected Λ tending to infinite simply connected volumes, we can describe labeled contour model probabilities in an infinite volume under boundary condition q by conditioning over external contours (cf. Proposition 2.1.2).
4. It suffices to suppose (1) (with sufficiently large $\tau^{(\alpha)}$), to get the existence of $\Psi^{(\alpha)} = (\Psi_q^{(\alpha)})$, $\bar\Psi^{(\alpha)} = (\bar\Psi_q^{(\alpha)})$ fulfilling a), b), and c). However, we were not able to follow exactly the method described in [Z] or [BI] to prove the smoothness b′) (and hence also a′)). Therefore, we extend the assertion of Theorem 1 from [Z] in Lemma 2.3.1 to get a), b), and c) for a wider class of functionals $\bar\Psi^{(\alpha)}$. We shall later find a functional $\bar\Psi^{(\alpha)}$ fulfilling b′) among them (Lemmas 2.3.1 and 2.3.2 and the final part of the proof of Theorem 2).
5. If we take $H \in \mathcal{H}(R)$ for the parameter α and $K_\varepsilon(H_0)$ for the set Ω, the function $\tau^{(H)}$ can be chosen as $\tau^{(H)} := \rho_\varepsilon \frac{\|H\|}{\|H_0\|}$ (see Proposition 1.1.3).

6. It is enough to suppose the estimates from (1) and (2) for $\sum_\gamma \Phi_\Lambda^{(\alpha)}(\gamma)$, where the sum is over q-contours $\gamma = (\Gamma, \lambda)$ with Γ fixed.
7. The first point in b) can be weakened to
$$\bar{\Psi}_q^{(\alpha)}(\Gamma) \leq e^{\text{const}|\Gamma|}\Psi_q^{(\alpha)}(\Gamma),$$
and the second point can be weakened by introducing a stronger notion of stability (see [BK]).

We postpone the proof of Proposition 2.2.1 to Section 2.3 below. Theorem 2 then easily follows.

Proof of Theorem 2. The only claims that are not directly included in Proposition 2.2.1 are the bounds on derivatives of $h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ and $h(\Phi^{(\alpha)}, \varphi^{(\alpha)})$. Using b′) of Proposition 2.2.1, the bound (2.13) on $p(\bar{\Psi}_q^{(\alpha)})$, and the assumption (3), we get
$$\left|\partial_{\bar{\alpha}}^{+} h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})\right| \leq \left(M + \exp\left\{-(\tau^{(\alpha)} - c_\ell - c_{\text{cl}})\right\}\right)||\bar{\alpha}||.$$
To estimate $\left|\partial_{\bar{\alpha}}^{+} h(\Phi^{(\alpha)}, \varphi^{(\alpha)})\right|$, we first notice that if, for a fixed α, there exists $q \in Q$ such that $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$ and $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) < h_m(\Phi^{(\alpha)}, \varphi^{(\alpha)})$, $m \neq q$, the claim immediately follows from the bound above. If $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$, $q \in \bar{Q} \subset Q$, and $h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) < h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)})$, $q \in Q \setminus \bar{Q}$, we get, for a fixed direction $\bar{\alpha}$,
$$\partial_{\bar{\alpha}}^{+} h(\Phi^{(\alpha)}, \varphi^{(\alpha)}) = \min_{q \in \bar{Q}} \partial_{\bar{\alpha}}^{+} h_q(\Phi^{(\alpha)}, \varphi^{(\alpha)}).$$
Indeed, if the right hand side is attained for several q's in $\bar{Q}$, the directional derivative on the left hand side equals any one of them, since they are equal anyway. □

We will use the results of Theorem 2 and Proposition 2.2.1 together with the results on contour models to describe the limit Gibbs states obtained under special boundary conditions as it is customary in the standard Pirogov-Sinai theory. However, in our application, the partition function of the "physical" model in Λ is described in terms of a labeled contour model with contours that may reach out of the volume Λ and with contour functional that (for such contours) depends on Λ. This is the reason why we formulate the following result on the description of labeled contour models under a stable boundary condition in a slightly more general situation than it is customary.

Let thus a translation invariant functional Φ and a vector φ be as above, and suppose that sequences of finite sets $\Lambda_n \nearrow V$ and functionals Φ_{Λ_n} are given so that

- $\Phi_{\Lambda_m}(\gamma) = \Phi_{\Lambda_n}(\gamma)$ whenever m, n, and γ with support Γ are such that $\{\gamma\} \in \mathfrak{X}_{\Lambda_n} \cap \mathfrak{X}_{\Lambda_m}$ and $\text{dist}(V(\Gamma), \Lambda_m \triangle \Lambda_n) > 1$,
- $0 \leq \Phi_{\Lambda_n}(\gamma) \leq \exp\{-\tau\,|\Gamma(\gamma)|\}$ for all n and all γ.

We study partition functions $Z(\mathfrak{X}^{(\Lambda)}|q;\Phi_\Lambda,\varphi)$ and the probabilities $\mu(\cdot,\mathfrak{X}^{(\Lambda)} \mid q;\Phi_\Lambda,\varphi)$ for $\mathfrak{X}(\Lambda)\subset\mathfrak{X}^{(\Lambda)}\subset\mathfrak{X}_\Lambda$.

Corollary 2.2.2 *For τ sufficiently large and a phase q stable with respect to Φ, φ (as in Theorem 2), there exists, for a (possibly infinite) simply connected set $V\subset\mathbb{Z}^\nu$ and any $\mathfrak{Y}\subset\mathfrak{X}_V$, a unique probability measure $\mu(\cdot,\mathfrak{Y}\mid q;\Phi_V,\varphi)$ defined as a limit of $\mu(\cdot,\mathfrak{Y}\cap\mathfrak{X}_\Lambda\mid q;\Phi_\Lambda,\varphi)$ with finite Λ converging to V. For almost all $x\in\mathfrak{X}_V$ there are external q-contours — the set $\vartheta(x)$ is defined and consists of q-contours. Moreover, for every bounded continuous function $f:\mathfrak{X}_V\to\mathbb{R}$, there exist $(\tau-c_\ell-c_s)$-functionals $\Psi_{q,V}$ such that*

$$\int f(x)\mu(dx,\mathfrak{X}_V|q;\Phi_V,\varphi)=\iint f(x)\mu(dx,\mathfrak{X}_V|q;\vartheta(x)=\vartheta(\partial))\mu(d\partial,L_V|\Psi_{q,V}). \tag{2.19}$$

Here $\mu(dx,\mathfrak{X}_V|q;\vartheta(x)=\vartheta(\partial))$ is naturally defined as the probability $\mu(dx,\mathfrak{X}_V|q;\Phi_V,\varphi)$ under the condition $\vartheta(x)=\vartheta(\partial)$.

The measures $\mu(d\partial,L_V|\Psi_{q,V})$ fulfill the assertions of Proposition 2.1.2.

Let $\mathcal{E}_i^{(a)}(V)$, $i\in V$, $a\in\mathbb{R}$, be the set $\mathcal{E}_i^{(a)}(V)=\{x\in\mathfrak{X}_V;\exists\gamma\in x$ such that $V(\Gamma)\ni i,|\Gamma|\geq a\}$. There exists a constant C such that $\mu(\mathcal{E}_i^{(a)}(V),\mathfrak{X}_V|q;\Phi_V,\varphi)\leq Ce^{-\tau a}$ for any $i\in V$ and $a\in\mathbb{R}$.

Proof. For any finite Λ we define $\Psi_{q,\Lambda}$ by

$$\Psi_{q,\Lambda}(\Gamma)=\sum_{\gamma:\substack{\{\gamma\}\in X^q_\Lambda\\ \operatorname{supp}\gamma=\Gamma}}\Phi_\Lambda(\gamma)\frac{Z(\mathfrak{X}(\operatorname{Int}_\Lambda\Gamma)|\lambda(\gamma);\Phi,\varphi)}{Z(\mathfrak{X}(\operatorname{Int}_\Lambda\Gamma)|q;\Phi,\varphi)}. \tag{2.20}$$

By $\lambda(\gamma)$ in the numerator we indicate that the partition function is considered with the boundary condition induced by $\lambda(\gamma)$. Here $\operatorname{Int}_\Lambda(\Gamma)$ is the union of those components of $\operatorname{Int}\Gamma$ whose distance from $\mathbb{Z}^\nu\setminus\Lambda$ is at least 1. In principle, also those components that are not fully contained in Λ are contributing, in a multiplicative way, to the numerator as well as denominator above. However, since for any γ in the sum above the label of the boundary of any such component is q, the concerned contributions to the numerator and denominator cancel. By induction, one can first verify that

$$Z(\mathfrak{X}^{(\Lambda)}|q;\Phi_V^{(\alpha)},\varphi^{(\alpha)})=\exp\{-\varphi_q^{(\alpha)}|\Lambda|\}\mathcal{Z}(L^{(\Lambda)};\Psi_{q,V}^{(\alpha)}).$$

The functionals $\Psi_{q,\Lambda}$ clearly satisfy the condition (about the independence on Λ for contours sufficiently far from the boundary) from Proposition 2.1.2 and the limiting functional $\Psi_{q,V}$ as well as the measure $\mu(d\partial,L_V;\Psi_{q,V})$ are well defined. To prove (2.19) we follow the proof of Proposition 3.4 from [HKZ]. The equality (2.19) holds for every finite Λ. Supposing that f is a cylindric function living on Λ, the right hand side of (2.19) can be approximated, up to a set of small measure, say ϵ, by a cylindrical continuous function living on a large finite $\Lambda(\epsilon)$.

The equality (2.19) for such a continuous bounded function then follows from Proposition 2.1.2 b).

The bound on probability of $\mathcal{E}_i^{(a)}(V)$ is a standard implication of Proposition 2.1.2 a) applied to the contour functional $\Psi_{q,V}$. □

2.3 Proof of Proposition 2.2.1

Since the assertion of Proposition 2.2.1 is an improvement of the results from [S] and [Z], we present here only the necessary changes or complements to the proofs from [Z].

In the same way as in Section 2.2, we often omit the superscript (α) if it cannot cause any misunderstanding. Sometimes we write $Z(\Lambda|q)$ instead of $Z(\mathfrak{X}(\Lambda)|q;\Phi^{(a)},\varphi^{(\alpha)})$ and $\mathcal{Z}(\Lambda;\Psi)$ instead of $\mathcal{Z}(L(\Lambda);\Psi)$.

1. Proof of a) Recall that the equality (2.18) is fulfilled iff Ψ is defined by (2.20) (cf. proof of Corollary 2.2.2). The estimate a) follows easily from the definition of stability of q in $\operatorname{Int}\Gamma$ and the bound $\exp\{(c_\# + \log|S|)\,|\Gamma|\}$ on the number of γ's in (2.20).

2. Implication b) ⇒ c) Our strategy is not to define $\bar\Psi$ explicitly as in [Z], but to isolate those properties of $\bar\Psi$ that ensure, in particular, the conclusion c) of Proposition 2.2.1. Then we look for a suitably smooth $\bar\Psi^{(\alpha)}$ satisfying the assumptions of the following lemma whose assertion (iv) is essentially identical to c). A particular choice of $\bar\Psi^{(\alpha)}$ is given in Lemma 2.3.2. To satisfy the assumptions of the following Lemma 2.3.1, one has to assume that $\tau - c_s - c_\# - \log|S| \geq \tilde\tau_0(c_s,\nu,|S|)$ with $\tilde\tau_0(c_s,\nu,|S|)$ determined in course of the proof of Lemma 2.3.1. Thus τ_ℓ needed for validity of Proposition 2.2.1 c) can be taken as $\tau_\ell = \tilde\tau_0(c_s,\nu,|S|) + c_s + c_\# + \log|S|$.

If $\tilde\Psi_q$ are translation-invariant (non-negative) τ-functionals for $q = 1,\dots,r$, and $\tilde\Psi = (\tilde\Psi_1,\dots,\tilde\Psi_r)$, we define the following quantities:

$$\begin{aligned} h_q(\tilde\Psi,\varphi) &= \varphi_q - p(\tilde\Psi_q),\\ h(\tilde\Psi,\varphi) &= \min_q h_q(\tilde\Psi,\varphi),\\ a_q(\tilde\Psi,\varphi) &= h_q(\tilde\Psi,\varphi) - h(\tilde\Psi,\varphi). \end{aligned}$$

Lemma 2.3.1 *There exists $\tilde\tau_0 \equiv \tilde\tau_0(c_s,\nu,|S|)$ such that if $\tilde\Psi = \{\tilde\Psi_q\}$ is a contour functional satisfying*

$\tilde{\text{b}}$) • $\tilde\Psi_q(\Gamma) \leq \Psi_q(\Gamma)$,

• $\tilde\Psi_q(\Gamma) = \Psi_q(\Gamma)$ *whenever q is c_s-stable in* $\operatorname{Int}\Gamma$,

• $\left|\tilde\Psi_q(\Gamma)\right| \leq \exp\{-\tilde\tau\,|\Gamma|\}$

and if $\tilde\tau \geq \tilde\tau_0$, then, denoting $\tilde\varepsilon_0 \equiv \tilde\varepsilon_0(\tilde\tau) := \varepsilon(\tilde\tau) + \varepsilon(\tilde\tau - c_\# - \log|S| - 1)$ with $\varepsilon(\tau) = e^{-(\tau - c_{\rm cl})}$, the following holds:

(i) If q is not c_s-stable in Λ, then

$$a_q(\tilde{\Psi},\varphi)\,|\Lambda| > (c_s - 2\tilde{\varepsilon}_0)\,|\partial\Lambda|\,.$$

(ii) $Z(\Lambda|q;\Phi,\varphi) \geq \exp\Big[-h_q(\tilde{\Psi},\varphi)\,|\Lambda| - \tilde{\varepsilon}_0\,|\partial\Lambda|\Big].$

(iii) $Z(\Lambda|q;\Phi,\varphi) \leq \exp\Big[-h(\tilde{\Psi},\varphi)\,|\Lambda| + \tilde{\varepsilon}_0\,|\partial\Lambda|\Big].$

(iv) q is c_s-stable iff $a_q(\tilde{\Psi},\varphi)=0$. Whenever q is c_s-stable,

$$h_q(\tilde{\Psi},\varphi) = -\lim \frac{1}{|\Lambda|}\log Z(\Lambda|q;\Phi,\varphi)$$
$$= \min_{\lambda} \overline{\lim}\Big[-\frac{1}{|\Lambda|}\log Z(\Lambda|\lambda;\Phi,\varphi)\Big] (= h(\Phi,\varphi)),$$

with the minimum taken over all multi-indices λ.

Proof. Assuming (i)–(iii) for all proper subsets of Λ, one can prove (ii) and (iii) following the inductive proof of Theorem 1.7 in [Z].

To be more precise we give some commentary to it. The proof of (ii) remains unchanged (it suffices to take $\tilde{\varepsilon}_0 \sim \varepsilon(\tilde{\tau}) = e^{-\tilde{\tau}+c_{\rm cl}}$ here). In the proof of (iii) we get the formula [Z, (1.44)][11] and we intend to apply [Z, Main Lemma, (2.13)] to the functional

$$\Xi(\Gamma) = (\tilde{\tau} - c_{\#} - \log|S| - 2\tilde{\varepsilon}_0(\tilde{\tau}) - h + \varphi_q)\,|\Gamma| \equiv \bar{\tau}\,|\Gamma|$$

for a fixed q.

Observe now that the equation

$$\tau^* = \bar{\tau} - \varepsilon(\tau^*)$$

has, for every $\bar{\tau} \geq \tau_{\rm cl}$, a solution τ^* such that $\varepsilon(\tau^*) \to 0$ for $\bar{\tau} \to \infty$. For the auxiliary functional

$$\Xi^*(\Gamma) \equiv \tau^*|\Gamma| := \Xi(\Gamma) - \varepsilon(\tau^*)\,|\Gamma|$$

of [Z, (2.10)] we need $\tau^* \geq \tau_{\rm cl}$, i.e.,

$$\bar{\tau} = \tilde{\tau} - c_{\#} - \log|S| - 2\tilde{\varepsilon}_0(\tilde{\tau}) - h + \varphi_q \geq \tau_{\rm cl} + \varepsilon(\tau^*)$$

to be able to use the results of Section 2.1.

Therefore we choose $\tilde{\tau}_0$ large enough to ensure that for some $\delta > 0$ we have

$$2\varepsilon(\tilde{\tau}_0 - c_{\#} - \log|S| - 2\delta) < \delta, \quad \varphi_q \geq h(\tilde{\tau}_0) - \delta,$$

and

$$\tilde{\tau}_0 \geq \tau_{\rm cl} + c_{\#} + \log|S| + 2\delta + \varepsilon(\tilde{\tau}_0 - c_\ell - 2\delta), \varepsilon(\tau^*) < \delta.$$

[11] We only will have a factor $2\tilde{\varepsilon}_0$ instead of $3c_{\rm cl}$ in (1.44). This can be traced back to the fact that, in the induction hypothesis used for (1.38), the role of $2c_{\rm cl}$ is played by $\tilde{\varepsilon}_0$.

Then, for $\tilde\tau > \tilde\tau_0$, one has

$$\tilde\tau - c_{\#} - \log|S| - 2\varepsilon(\tilde\tau) - h + \varphi_q \geq \tilde\tau - c_{\#} - \log|S| - 2\delta$$
$$\geq \tau_{\mathrm{cl}} + \varepsilon(\tilde\tau - c_{\#} - \log|S| - 2\delta) \geq \tau_{\mathrm{cl}} + \varepsilon(\tau^*)$$

because $\tau^* \geq \tilde\tau - c_\ell - 2\delta$. With a particular choice of δ, say $\delta = 1/3$, the choice of $\tilde\tau_0$ depends on ν and $|S|$ only. Hence, we bound the right hand side of [Z, (1.44)] by

$$\exp\{-h\,|\Lambda| + \varepsilon(\tilde\tau)\,|\partial\Lambda|\}\boldsymbol{Z}_q(\Lambda, \Xi, a_q(\tilde\Psi, \varphi))$$

with $\boldsymbol{Z}_q(\Lambda, \Xi, a_q(\tilde\Psi, \varphi))$ the partition function from [Z] Main Lemma, (2.12) (denoted by $\boldsymbol{Z}(\Lambda, H)$ there) defined with the functional Ξ above and the parameter $a_q(\tilde\Psi, \varphi)$ defined before the present lemma. This yields [Z] (2.13) in the form

$$\boldsymbol{Z}_q(\Lambda, \Xi, a_q(\tilde\Psi, \varphi)) \leq e^{\varepsilon(\tau^*)|\partial\Lambda|},$$

or, equivalently,

$$\mathcal{Z}(\Lambda; \Xi^*) \leq e^{p(\Xi^*)|\Lambda| + \varepsilon(\tau^*)|\partial\Lambda|}.$$

The bound (iii) follows using [Z, Main Lemma] and the inequality $\varepsilon(\tau^*) \leq \varepsilon(\tilde\tau - c_{\#} - \log|S| - 1)$. The only difference between [Z, Theorem 1] and our Lemma 2.3.1 (i) – (iii) concerns now the derivation of (i) and stems from our definition of c_s-stable sets.

Indeed, if q is not c_s-stable in Λ, there exists a multi-index λ such that

$$\frac{Z(\Lambda|\lambda)}{Z(\Lambda|q)} > \exp\bigl\{c_s\,|\partial\Lambda|\bigr\}.$$

From (ii) and (iii) we have

$$Z(\Lambda|q) \geq \exp\bigl\{-h_q(\tilde\Psi, \varphi)|\Lambda| - \tilde\varepsilon_0\,|\partial\Lambda|\bigr\}, \text{ and } Z(\Lambda|\tilde q) \leq \exp\bigl\{-h(\tilde\Psi, \varphi)|\Lambda| + \tilde\varepsilon_0\,|\partial\Lambda|\bigr\}$$

and thus (i) follows.

To prove (iv), suppose first that $a_q(\tilde\Psi, \varphi) = 0$. Then q is stable by (i) once $\tilde\tau$ is so large that $c_s - 2\tilde\varepsilon_0 \geq 0$.

Let, on the other side, q be c_s-stable. Then $\Psi_q(\Gamma) = \tilde\Psi_q(\Gamma)$ for every Γ by the condition $\tilde{\mathrm{b}}$). Thus

$$\mathcal{Z}(\Lambda; \Psi_q) = \mathcal{Z}(\Lambda; \tilde\Psi_q) = \exp\bigl\{\varphi_q\,|\Lambda|\bigr\}Z(\Lambda|q; \varPhi, \varphi)$$

and

$$\varphi_q - p(\tilde\Psi_q) = h_q(\tilde\Psi, \varphi) = -\lim \frac{\log Z(\Lambda|q)}{|\Lambda|}. \tag{2.21}$$

Taking now any q_0 such that $a_{q_0}(\tilde\Psi, \varphi) = 0$, we know already that q_0 is stable and thus

$$\exp(-c_s\,|\partial\Lambda|) \leq \frac{Z(\Lambda|q_0)}{Z(\Lambda|q)} \leq \exp\bigl\{c_s\,|\partial\Lambda|\bigr\}.$$

Hence $h_{q_0}(\tilde\Psi, \varphi) = h_q(\tilde\Psi, \varphi)$ and $a_q(\tilde\Psi, \varphi) = 0$.

Further, if q is c_s-stable, we have

$$\frac{Z(\Lambda|\lambda)}{Z(\Lambda|q)} \le \exp\big\{c_s\,|\partial\Lambda|\big\}$$

for any λ. Thus, taking into account (2.21), we get

$$\overline{\lim}\left[-\frac{\log Z(\Lambda|\lambda)}{|\Lambda|}\right] \ge \lim\left[-\frac{\log Z(\Lambda|q)}{|\Lambda|}\right] = h(\Phi,\varphi).$$

Hence

$$h(\Phi,\varphi) = \min_{\lambda}\overline{\lim}\left[-\frac{1}{|\Lambda|}\log Z(\Lambda|\lambda;\Phi,\varphi)\right]. \qquad \square$$

3. The choice of $\tilde\Psi$ *satisfying* $\tilde{\text{b}})$ *of Lemma 2.3.1.*

Lemma 2.3.2 *Let* $\tilde c \ge c_s$*. The functionals*

$$\tilde\Psi_q(\Gamma) = \sum_{\gamma}\Phi(\gamma)\min\left[\frac{Z(\operatorname{Int}\Gamma|\lambda(\gamma))}{Z(\operatorname{Int}\Gamma|q)}, \frac{Z(\operatorname{Int}\Gamma|\lambda(\gamma))\exp\big\{\tilde c|\partial\operatorname{Int}\Gamma|\big\}}{\max_{\tilde\lambda} Z(\operatorname{Int}\Gamma|\tilde\lambda)}\right], \qquad (2.22)$$

with the sum taken over all labeled q*-contours with* $\Gamma = \Gamma(\gamma)$ *and with labeling* $\lambda(\gamma)$*, and with the maximum in the second term taken over all labelings* $\tilde\lambda$*, satisfy the assumptions* $\tilde{\text{b}})$ *of Lemma 2.3.1 above with* $\tilde\tau = \tau - \tilde c - c_{\#} - \log|S|$*.*

Proof. It follows clearly from (2.22) and the definition of stability. $\square$

Remark. 1. In particular, combining Lemma 2.3.2 with Lemma 2.3.1 (for $\tilde\tau = \tau - \tilde c - c_{\#} - \log|S| \ge \tilde\tau_0$), we can conclude that

- there exists a c_s-stable q_0 for every α;
- for any c_s-stable q one has

$$-\lim\frac{\log Z(\Lambda|q)}{|\Lambda|} = \min_{\lambda}\left[-\overline{\lim}\frac{\log Z(\Lambda|\lambda)}{|\Lambda|}\right];$$

- $Z(\Lambda|q) \le \exp\big\{-h(\Phi,\varphi)\,|\Lambda| + \tilde\varepsilon_0\,|\partial\Lambda|\big\}$ for every q;
- $Z(\Lambda|q) \ge \exp\big\{-h(\Phi,\varphi)\,|\Lambda| - \tilde\varepsilon_0\,|\partial\Lambda|\big\}$ for every c_s-stable q.

2. Once the existence of a stable q_0 is established, we notice that replacing $\tilde\lambda$ by a constant $\tilde q$ in (2.22) leads to a $\tilde\Psi_q(\Gamma)$ satisfying assumptions of Lemma 2.3.1 with $\tilde\tau = \tau - 2c_s - c_{\#} - \log|S|$ (taking $\tilde c$ and thus $\tilde\tau$ sufficiently large, we have $\tilde\epsilon_0 \le c_s$).

3. Taking $\tilde\Psi$ for $\bar\Psi$, we proved b) and c) from Proposition 2.2.1. The task now is, assuming conditions (2) and (3) from Theorem 2, to choose $\bar\Psi$ so that it also satisfies b′) (cf. the step 5 below).

4. Estimates of derivatives of partition functions and proof of a′).

Lemma 2.3.3 *For sufficiently large $\tau_\ell \equiv \tau_\ell(c_s, M, \nu, |S|)$, $\tau^{(\alpha)} > \tau_\ell$, and taking $\tilde\varepsilon_0 \equiv \tilde\varepsilon_0(\tau^{(\alpha)})$ as in Lemma 1, one has*[12]

$$\big|\partial^+_{\bar\alpha} Z^{(\alpha)}(\Lambda \mid \lambda)\big| \le M\ |\Lambda|(1+\tilde\varepsilon_0)\exp\big\{-h^{(\alpha)}|\Lambda| + \tilde\varepsilon_0|\partial\Lambda|\big\}||\bar\alpha||, \tag{2.23}$$

whenever Λ is a union of simply connected finite sets and λ is an arbitrary labeling. Moreover,

$$\Big|\partial^+_{\bar\alpha}\Big(\frac{Z^{(\alpha)}(\Lambda \mid \lambda)}{Z^{(\alpha)}(\Lambda \mid q)}\Big)\Big| \le \exp\big\{|\partial\Lambda|\big\}||\bar\alpha||, \tag{2.24}$$

if q is c_s-stable in Λ with respect to $\Phi^{(\alpha)}$, $\varphi^{(\alpha)}$.

Proof. The first inequality can be proven in a rather straightforward way, by induction in $|\Lambda|$ (cf. also [BK] (Lemma 2.2)).

Namely, considering the derivative of

$$Z^{(\alpha)}(\Lambda \mid q) = \sum_\vartheta e^{-\varphi_q^{(\alpha)}|\Lambda\setminus\bigcup \operatorname{Int}\gamma|}\prod_{\gamma\in\vartheta}\Phi^{(\alpha)}(\gamma)Z^{(\alpha)}(\operatorname{Int}\gamma \mid \lambda(\gamma)),$$

with the sum over collections ϑ of external q-contours in Λ (including the empty one), we get (taking, without loss of generality, $||\bar\alpha|| = 1$)

$$\begin{aligned}
&\big|\partial^+_{\bar\alpha} Z^{(\alpha)}(\Lambda \mid q)\big| \le \\
&\le M|\Lambda|Z^{(\alpha)}(\Lambda \mid q) + \sum_\gamma\big|\partial^+_{\bar\alpha}\big(\Phi^{(\alpha)}(\gamma)Z^{(\alpha)}(\operatorname{Int}\gamma \mid \lambda(\gamma))\big)\big|e^{-\varphi_q^{(\alpha)}|\gamma|}Z^{(\alpha)}(\operatorname{Ext}\gamma \mid q) \le \\
&\le \exp\{-h^{(\alpha)}|\Lambda| + \tilde\varepsilon_0|\partial\Lambda|\}\Big(M|\Lambda| + \sum_\gamma\big(1 + M|\operatorname{Int}\gamma|(1+\tilde\varepsilon_0)\big)e^{-\tau^{(\alpha)}|\gamma|+3\tilde\varepsilon_0|\gamma|}\Big) \le \\
&\qquad\qquad \le M|\Lambda|(1+\tilde\varepsilon_0)\exp\{-h^{(\alpha)}|\Lambda| + \tilde\varepsilon_0|\partial\Lambda|\}.
\end{aligned}$$

The first term corresponds to the derivative of $\varphi_q^{(\alpha)}$ using assumption (3) of Theorem 2. In the second term, we summed over contours not affected by the derivative, yielding $Z^{(\alpha)}(\operatorname{Ext}\gamma \mid q)$ with $\operatorname{Ext}\gamma = \Lambda \setminus (\operatorname{Int}\gamma \cup \gamma)$. To get the second inequality we use Lemma 1 (iii) (with the help of $\tilde\Psi$ from Lemma 2) to bound $Z^{(\alpha)}(\Lambda \mid q)$, $Z^{(\alpha)}(\operatorname{Int}\gamma \mid \lambda(\gamma))$, and $Z^{(\alpha)}(\operatorname{Ext}\gamma \mid q)$, the assumption (2) of Theorem 2 to bound $|\partial^+_{\bar\alpha}\Phi^{(\alpha)}(\gamma)|$, induction hypothesis (2.23) to bound $|\partial^+_{\bar\alpha}Z^{(\alpha)}(\operatorname{Int}\gamma \mid \lambda(\gamma))|$, and the fact that $\varphi_q^{(\alpha)} \ge h^{(\alpha)} - \tilde\varepsilon_0$. To get the last inequality, we use the bound

$$\sum_{\gamma:\gamma\cup\operatorname{Int}\gamma\ni i}\big(1 + M|\operatorname{Int}\gamma|(1+\tilde\varepsilon_0)\big)e^{-\tau^{(\alpha)}|\gamma|+3\tilde\varepsilon_0|\gamma|} \le \tilde\varepsilon_0,$$

where the sum is taken over all γ encircling a fixed site i.

[12] We explicitly indicate here the dependence on α.

To get (2.24), we use (2.23) and Lemma 1 (ii) and (iii) (with existence of needed $\tilde{\Psi}$ assured by Lemma 2) and get

$$\Big|\partial^+_{\bar{\alpha}} \frac{Z^{(\alpha)}(\Lambda \mid \lambda)}{Z^{(\alpha)}(\Lambda \mid q)}\Big| \le M|\Lambda|(1+\tilde{\varepsilon}_0)\exp\big\{2\tilde{\varepsilon}_0|\partial\Lambda|\big\}$$

$$+M|\Lambda|(1+\tilde{\varepsilon}_0)\exp\big\{4\tilde{\varepsilon}_0|\partial\Lambda|\big\} \le \exp\big\{|\partial\Lambda|\big\}$$

for $\tau^{(\alpha)}$ sufficiently large (and thus $\tilde{\varepsilon}_0$ sufficiently small). □

Proof of a′) of Proposition 2.2.1. We differentiate the right hand side of (2.20). Evaluating the derivative of the product $\Phi^{(\alpha)}_\Lambda(\gamma)\frac{Z^{(\alpha)}(X(\mathrm{Int}_\Lambda \Gamma)|\lambda(\gamma);\Phi,\varphi)}{Z^{(\alpha)}(X(\mathrm{Int}_\Lambda \Gamma)|q;\Phi,\varphi)}$, we use the assumption (2) of Theorem 2, the c_s-stability of q in $\mathrm{Int}_\Lambda \Gamma$, assumption (1) of Theorem 2, and the bound (2.24) for the set $\mathrm{Int}_\Lambda \Gamma$, to get the bound (again, $||\bar{\alpha}|| = 1$)

$$\Big|\partial^+_{\bar{\alpha}}\Big(\Phi^{(\alpha)}_\Lambda(\gamma)\frac{Z^{(\alpha)}(X(\mathrm{Int}_\Lambda \Gamma)|\lambda(\gamma);\Phi,\varphi)}{Z^{(\alpha)}(X(\mathrm{Int}_\Lambda \Gamma)|q;\Phi,\varphi)}\Big)\Big| \le$$

$$\le e^{-\tau^{(\alpha)}|\Gamma|}e^{c_s|\Gamma|} + e^{-\tau^{(\alpha)}|\Gamma|}e^{|\Gamma|} \le e^{-(\tau^{(\alpha)}-c_s-1-\ln 2)|\Gamma|}$$

that yields (a′) with $c_\ell = c_s + 1 + \ln 2 + c_\# + \log|S|$. □

5. Definition of $\overline{\Psi}$ satisfying b), and simultaneously b'), of Proposition 2.2.1

According to 3. and 4., the only problem with $\tilde{\Psi}_q$ from (2.22) is the use of the min in its definition which makes $\tilde{\Psi}$ non-smooth. Therefore we introduce a "smooth version of min" first and then we apply it to define $\overline{\Psi}_q$ by a natural modification of (2.22).

Lemma 2.3.4 *For any $\eta > 0$, there is a function* $\min_\eta : \mathbb{R}^r \to \mathbb{R}, r \in \mathbb{N}$, *such that*

(i) $\min_\eta(u) \le \min(u)$ *for* $u = (u_1, \dots, u_r) \in \mathbb{R}^r$;

(ii) $\min_\eta(u) = u_i$ *whenever* $u_i \le \min\{u_j \mid j \ne i\} - \eta$;

(iii) $\min_\eta \in C^\infty(\mathbb{R}^r)$ *and it is 1-Lipschitz;*

(iv) $\frac{\partial}{\partial u_i}\min_\eta(u) = 0$ *if* $u_i \ge \min(u) + 2\eta$.

Proof. We may define $\min_\eta$ as the convolution of min and a nonnegative function $\varphi_\eta \in C^\infty(\mathbb{R}^r)$ that is symmetric (i.e. of the form $\psi_\eta(\|u\|)$), fulfills $\int_{\mathbb{R}^r} \varphi_\eta(u)du = 1$ and $\varphi_\eta(u) = 0$ if $\|u\| \ge \frac{\eta}{2}$.

Namely, we put $\min_\eta = \varphi_\eta * \min$.

Then (i) holds because

$$\varphi_\eta * \min(u) = \int_{\mathbb{R}^r} \varphi_\eta(u-v)\min(v)dv \le \int_{\mathbb{R}^r} \varphi_\eta(u-v)v_i dv = \int_{\mathbb{R}^r} \varphi_\eta(v)(u_i - v_i)dv =$$

$$= u_i \int_{\mathbb{R}^r} \varphi_\eta(v)dv - \int_{\mathbb{R}^r} \varphi_\eta(v)v_i dv = u_i.$$

The last integral is zero because of the symmetry of φ_η.

To verify (ii), notice that

$$\varphi_\eta * \min(u) = \int\limits_{\|v\|\le\frac{\eta}{2}} \varphi_\eta(v)\min(u-v)dv = \int\limits_{\|v\|\le\frac{\eta}{2}} \varphi_\eta(v)(u_i - v_i)dv = u_i.$$

The condition (iii) follows immediately from the fact that min is 1-Lipschitz and $\int_{\mathbb{R}^r} \varphi_\eta(v)dv = 1$ with $\varphi_\eta(v) \ge 0$.

Finally, we verify (iv). Let $|\tilde u_i - u_i| < \eta$ and $\tilde u_j = u_j$ for $j \ne i$. Then $\min_\eta(\tilde u) = \int_{\|v\|<\frac{\eta}{2}} \varphi_\eta(v)\min(\tilde u - v)dv = \int_{\|v\|<\frac{\eta}{2}} \varphi_\eta(v)\min(u-v)dv = \min_\eta(u)$. Here we used that $\tilde u_i \ge \min(u) + \eta$ and $\tilde u_i - v_i \ge \min(u) + \frac{\eta}{2} \ge \min(u-v)$. So (iv) holds and Lemma 4 is proved. □

Proof of b) and b′) of Proposition 2.2.1. For $\eta > 0$ and any contour γ, we introduce the shorthand

$$\mathcal{M}(\gamma,\eta) = \min_{\eta|\partial\Lambda|}\Big[\log\frac{Z(\Lambda|\lambda(\gamma))}{Z(\Lambda|q)}, \log\frac{Z(\Lambda|\lambda(\gamma))\exp\big\{(c_s+\eta)|\partial\Lambda|\big\}}{Z(\Lambda|\tilde q)}; \tilde q \ne q\Big],$$

where $\Lambda = \operatorname{Int}\Gamma$; $q, \tilde q \in \{1,\dots,r\}$. Put now

$$\overline{\Psi}_q^{(\alpha)}(\Gamma) = \sum_\gamma \Phi^{(\alpha)}(\gamma)\exp\big\{\mathcal{M}(\gamma,\eta)\big\}, \tag{2.25}$$

where the sum is over q-contours with support Γ.

First we show that the functional $\overline{\Psi}_q^{(\alpha)}$ fulfills the three assumptions of $\tilde b$) of Lemma 1 (with $\tilde\tau = \tau - c_\ell$, $c_\ell = c_\# + \log|S| + c_s + \tilde\varepsilon_0 + \eta + 1$) and thus b) of Proposition 2.2.1 (with $c_\ell = c_\# + \log|S| + 2c_s + 1$) .

Namely, the first point ($\overline{\Psi}_q^{(\alpha)} \le \Psi_q^{(\alpha)}$) follows from (2.20), (2.25) and the property (i) of $\min_\eta$ (cf. Lemma 4).

The second point ($\overline{\Psi}_q^{(\alpha)}(\Gamma) = \Psi_q^{(\alpha)}(\Gamma)$ if q is c_s-stable in Λ) needs moreover the definition of c_s-stability of q in Λ and the property (ii) of $\min_\eta$ (cf. Lemma 4).

To show the third point ($|\overline{\Psi}_q^{(\alpha)}(\Gamma)| \le \exp\big\{-(\tau^{(\alpha)} - c_\ell)|\Gamma|\big\}$), we use the fact that, by Lemma 2, there is a c_s-stable q_0. So $|\overline{\Psi}_q^{(\alpha)}(\Gamma)| \le \exp\big\{-(\tau^{(\alpha)} - c_\# - \log|S| - c_s - \tilde\varepsilon_0 - \eta))|\Gamma|\big\}$ by the definition of $\overline{\Psi}_q^{(\alpha)}$, the assumption (1) of Theorem 2, Lemma 4 (i), and the definition of c_s-stability of q in Λ. Using the choice $\eta = 1$, we get b) with $c_\ell = c_\# + \log|S| + 2c_s + 1$.

Now, we shall show that $\overline{\Psi}_q^{(\alpha)}$ fulfills the estimate (b′).

First, notice that the following auxiliary estimate holds due to Lemma 4 (i),

$$\Big|\frac{\partial}{\partial t_i}\exp\big\{\min_\eta(\log t_1,\cdots,\log t_r)\big\}\Big| \le \Big|\exp\big\{\min_\eta(\log t_1,\cdots,\log t_r)\big\}\cdot\frac{\partial}{\partial u_i}\min_\eta(u)\cdot\frac{1}{t_i}\Big| \le 1$$

for $t_1,\ldots,t_r > 0$. We differentiate

$$\partial^+_{\bar\alpha} \sum_\gamma \Phi^{(\alpha)}(\gamma)\exp\{\mathcal{M}(\gamma,\eta)\} = \sum_\gamma \partial^+_{\bar\alpha}\Phi^{(\alpha)}(\gamma)\exp\{\mathcal{M}(\gamma,\eta)\} + \sum_\gamma \Phi^{(\alpha)}(\gamma)\partial^+_{\bar\alpha}\big(\exp\{\mathcal{M}(\gamma,\eta)\}\big). \quad (2.26)$$

We are going to use (2) of Theorem 2 to estimate $\partial^+_{\bar\alpha}\Phi^{(\alpha)}(\gamma)$ and (1) of Theorem 2 to estimate $\Phi^{(\alpha)}(\gamma)$. Using the existence of a c_s-stable q_0 we get further that

$$\exp\{\mathcal{M}(\gamma,\eta)\} \le \le \exp\Big\{\min\Big[\log\frac{Z(\Lambda|\lambda(\gamma))}{Z(\Lambda|q)}, \log\frac{Z(\Lambda|\lambda(\gamma))\exp\{(c_s+\eta)|\partial\Lambda|\}}{Z(\Lambda|\tilde q)}; \tilde q \ne q\Big]\Big\} \le \le \exp\Big\{(2c_s+\eta)|\partial\Lambda|\Big\}.$$

To establish the estimate (b′) it remains to estimate the derivative

$$\Big|\partial^+_{\bar\alpha}\exp\{\mathcal{M}(\gamma,\eta)\}\Big| \le \exp\{\mathcal{M}(\gamma,\eta)\}\Big|\partial^+_{\bar\alpha}\mathcal{M}(\gamma,\eta)\Big| \le \le \exp\{\mathcal{M}(\gamma,\eta)\}\Big(\sum_j \frac{\partial}{\partial t_j}\min{}_{\eta|\partial\Lambda|}(\log t_1,\ldots,\log t_r)_{t_i=t_{q_i}(\alpha)} \cdot \frac{\partial}{\partial^+_{\bar\alpha}}(t_{q_j}(\alpha))\Big),$$

where we use the notation $t_q(\alpha) = \frac{Z(\Lambda|\lambda(\gamma))}{Z(\Lambda|q)}$ and $t_{\tilde q}(\alpha) = \frac{Z(\Lambda|\lambda(\gamma))\exp\{(c_s+\eta)|\partial\Lambda|\}}{Z(\Lambda|\tilde q)}$ for $\tilde q \ne q$.

We consider the product of the two terms of the last bound. The first one is bounded by $\exp\{(2c_s+\eta)|\Gamma|\}$ as we already noticed. The other one may be reduced to the sum over j's such that $t_{q_j}(\alpha) < \min(t)\cdot\exp\{2\eta|\partial\Lambda|\}$ by Lemma 4 (iv). Now we may use our auxiliary estimate of the partial derivatives over t_j's. Since we confined ourselves to those special j's, we may use Lemma 3 to estimate the partial derivatives of $t_q(\alpha)$ and $t_{\tilde q}(\alpha)$ for those q and $\tilde q$ for which $t_q(\alpha) \le \exp\{c_s|\Gamma|\}$ and $t_{\tilde q}(\alpha) \le \exp\{(2c_s+\eta)|\Gamma|\}$.

Inserting the above bounds into (2.26), we finally get

$$\Big|\partial^+_{\bar\alpha}\overline{\Psi}^{(\alpha)}_q(\Gamma)\Big| \le e^{c_\#|\Gamma|}e^{-\tau^{(\alpha)}}\big(\exp\{(2c_s+\eta)|\Gamma|\} +|S|\exp\{(2c_s+\eta)|\Gamma|\}\exp\{(2c_s+\eta)|\Gamma|\}\big)||\bar\alpha||.$$

Hence, taking again $\eta = 1$, we get (b′) with $c_\ell = c_\# + 4c_s + 2 + 2\log(|S|+1)$. □

3 Periodic Gibbs states

In this short section we recall how the standard Pirogov-Sinai theory leads to a description of Gibbs states of classical lattice models in terms of contour models. Indeed, after reformulating a model with Hamiltonian H in terms of a labeled contour model, we can apply results of Section 2.2.

Recall that we suppose that all periodic configurations from $G_0^{\mathrm{per}} = \{x_1, \ldots, x_r\}$ are actually translation invariant. This does not mean a loss of generality as explained in the remark following Basic Lemma 1.2.

Considering, in the standard way (in the present context, cf. [[HKZ, Lemma 3.1]), the "diluted" partition functions

$$Z^d(\Lambda|x_q; H) = \sum_{x = x_q \text{ in } (\Lambda^c)_R} \exp\{-E_\Lambda^{(H)}(x)\},$$

with $(\Lambda^c)_R = \{i \in \mathbb{Z}^\nu; \mathrm{dist}(i, \Lambda^c) \leq R+1\}$, we introduce $\varphi^{(H)} = (\varphi_q^{(H)})$ and $\Phi^{(H)}$ in such a way that we can replace the diluted sums $Z^d(\Lambda|x_q; H)$ by the corresponding labeled contour model partition functions $Z^{\mathrm{cont}}(\mathfrak{X}(\Lambda)|q; \Phi^{(H)}, \varphi^{(H)})$ discussed in Section 2.2 (cf(2.15)); we use here the superscript "cont" to stress that we are concerned with partition functions (resp., probability distributions) of a labeled contour model.

Namely, let us introduce the labeled contour model (with labels $Q = \{1, \ldots, r\}$, where we denote $G_0^{\mathrm{per}} = \{x_1, x_2, \ldots, x_r\}$) with

$$\varphi_q^{(H)} = e_{x_q}(H) \tag{3.1}$$

and

$$\Phi^{(H)}(\gamma) = \sum_{x \sim \gamma} \exp\left[-E_\Gamma^{(H)}(x) + E_\Gamma^{(H)}(x_q)\right], \tag{3.2}$$

whenever γ is a q-contour. The sum is over configurations on Γ that can be extended to a configuration having γ as its contour (for definitions of $E_A^{(H)}(x)$ and $e_x(H)$ see (1.2) and (1.3)).

Then, one can easily verify that

$$Z^d(\Lambda|x_q; H) = Z^{\mathrm{cont}}(\mathfrak{X}(\Lambda)|q; \Phi^{(H)}, \varphi^{(H)}) \tag{3.3}$$

for every finite Λ. Moreover, the state $\mu(\{x\}, \Lambda|x_q; H)$ can be linked with the labeled contour model probability distribution (2.16),

$$\sum_{\substack{x \in X_\Lambda \\ x \sim z}} \mu(\{x\}, \Lambda|x_q; H) = \mu^{\mathrm{cont}}(\{z\}, \mathfrak{X}^{(\Lambda)} \mid q; \Phi^{(H)}, \varphi^{(H)}).$$

Here the sum is over all spin configurations $x \in X_\Lambda$ such that the set of their labeled contours is just z (it means that one is summing only over configurations on $\bigcup_{\gamma \in z} \mathrm{supp}\, \Gamma$ with boundaries fixed by labeling of contours $\gamma \in z$).

Next, we verify that the assumptions of Theorem 2 for the labeled model described by $\Phi^{(H)}$ and $\varphi^{(H)}$ are satisfied. Recall that the Hamiltonian H_0 satisfies the Peierls condition ($\mathbf{P}^{(\mathrm{per})}$) with respect to the set G_0 and constant ρ_0. We can choose ε so that each $H \in K_\varepsilon(H_0)$ satisfies, according to Proposition 1.1.3, the Peierls condition ($\mathbf{P}^{(\mathrm{per})}$) with a sufficiently large constant $\rho_\varepsilon \frac{\|H\|}{\|H_0\|}$, where $\lim_{\varepsilon\to 0} \rho_\varepsilon = \rho_0$, if ρ_0 is large enough. Now, we consider the open set $K_\varepsilon(H_0) \subset \mathcal{H}(R)$ to play the role of the set Ω from Theorem 2 (thence the Hamiltonians H stand for the parameters α of Theorem 2).

The functional $\Phi^{(H)}(\gamma)$ defined by (3.2) satisfies, for any q-contour γ, the condition (1) of Theorem 2. Indeed, using ($\mathbf{P}^{(\mathrm{per})}$) and the estimate $|S|^{|\Gamma|}$ on the cardinality of the set $\{x; x \sim \gamma\}$ we get $\Phi^{(H)}(\gamma) \le e^{-(\rho_\varepsilon \frac{\|H\|}{\|H_0\|} - \log|S|)|\Gamma|}$.

To verify the condition (2), we consider the derivative

$$|\partial^+_{\bar H} \Phi^{(H)}(\gamma)| = \big|\partial^+_{\bar H} \sum_{x\sim\gamma} \exp\big(-\sum_{i\in\Gamma}(E_i^{(H)}(x) - E_i^{(H)}(x_q)\big)\big|. \tag{3.4}$$

Referring to (1.2), notice that $\big|\frac{\partial}{\partial U_{A_0}(z_{A_0})}(\sum_{i\in A} \frac{U_A(x_A)}{|A|})\big| \le \sum_{i\in A,[A]=[A_0]} \frac{1}{|A|} \le 1$, whenever A_0 is such that the canonically fixed site $i^{(A_0)}$ equals 0 (cf. the discussion at the beginning of Section 1.1) and a configuration $z_{A_0} \in S^{A_0}$ is fixed. Therefore, taking again into account that the cardinality of $\{x; x \sim \gamma\}$ is at most $|S|^{|\Gamma|}$ and that $e^{-(E_\Gamma^{(H)}(x) - E_\Gamma^{(H)}(x_q))} \le e^{-\rho_\varepsilon \frac{\|H\|}{\|H_0\|}|\Gamma|}$, we get, for $\bar H = (\bar U_{A_0}(z_{A_0}) | i^{(A_0)} = 0)$, the bound

$$\begin{aligned}
|\partial^+_{\bar H} \Phi^{(H)}(\gamma)| &\le e^{|\Gamma|\log|S|} e^{-\rho_\varepsilon \frac{\|H\|}{\|H_0\|}|\Gamma|} \sum_{i\in\Gamma} (|\partial^+_{\bar H} E_i^{(H)}(x)| + |\partial^+_{\bar H} E_i^{(H)}(x_q)|) = \\
&= e^{|\Gamma|(\log|S| - \rho_\varepsilon \frac{\|H\|}{\|H_0\|})} \times \sum_{i\in\Gamma} \sum_{A_0, z_{A_0}} \Big(\Big| \frac{\partial}{\partial U_{A_0}(z_{A_0})} \Big(\sum_{i\in A} \frac{U_A(x_A)}{|A|} \Big) \overline{U}_{A_0}(z_{A_0}) \Big| + \\
&\quad + \Big| \frac{\partial}{\partial U_{A_0}(z_{A_0})} \Big(\sum_{i\in A} \frac{U_A((x_q)_A)}{|A|} \Big) \overline{U}_{A_0}(z_{A_0}) \Big| \Big) \le \\
&\le e^{|\Gamma|(-\rho_\varepsilon \frac{\|H\|}{\|H_0\|} + \log|S|)} |\Gamma| 2 \dim \mathcal{H}(R) \|\bar H\| \\
&\le e^{|\Gamma|(-\rho_\varepsilon \frac{\|H\|}{\|H_0\|} + \log|S| + 1 + \log 2 + \log\dim\mathcal{H}(R))} \|\bar H\|.
\end{aligned}$$

Hence,

$$|\partial^+_{\bar H} \Phi^{(H)}(\gamma)| \le e^{-\tau_\Phi^{(H)}|\Gamma|} \|\bar H\|, \tag{3.5}$$

with

$$\tau_\Phi^{(H)} = \rho_\varepsilon \frac{\|H\|}{\|H_0\|} - \log|S| - 1 - \log 2 - \log\dim\mathcal{H}(R). \tag{3.6}$$

Thus $\Phi^{(H)}$ satisfies the conditions (1) and (2) from Theorem 2 and Proposition 2.2.1 with the constant $\tau_\Phi^{(H)}$ playing the role of $\tau^{(\alpha)}$. Further, $\varphi_q^{(H)}$, in the place of $\varphi_q^{(\alpha)}$, satisfies the condition (3) from Theorem 2 with $M = 1$. Recalling the notation $\tilde{\varepsilon}_0$, c_{cl}, and c_ℓ (cf. Lemma 2.3.1, Proposition 2.1.3, and Theorem 2), and applying Theorem 2, the equality (3.3), and Lemma 2.3.1, we obtain

Proposition 3.1 *Under assumptions of Theorem 1:*

a) For every $H \in K_\varepsilon(H_0)$ as above, defining $\Phi^{(H)}$ and $\varphi^{(H)}$ by (3.1) and (3.2), we obtain a labeled contour model that satisfies (1) – (3) with the respective constants $\tau^{(H)} = \tau_\Phi^{(H)}$ and $M = 1$. Moreover,

$$Z^d(\Lambda|x_q; H) = Z^{\mathrm{cont}}(\mathfrak{X}(\Lambda)|q; \Phi^{(H)}, \varphi^{(H)})$$

for every finite Λ.

b) For every $x \in G_0^{\mathrm{per}}$, there exists a function $h_x(\cdot)$ on $K_\varepsilon(H_0)$ such that there exists an extremal Gibbs state $\mu \in \mathcal{G}(H)$ that is a perturbation of x whenever $h_x(H) = h(H) = \min_{\tilde{x} \in G_0^{\mathrm{per}}} h_{\tilde{x}}(H)$. Moreover,

$$|h_x(H) - \varphi_x^{(H)}| \le e^{-\tau_{\mathrm{h}}^{(H)}}, \tag{3.7}$$

$$|\partial_{\bar{H}}^+ (h_x(H) - \varphi_x^{(H)})| \le e^{-\tau_{\mathrm{h}}^{(H)}}, \tag{3.8}$$

and, finally,

$$|\partial_{\bar{H}}^+ h(H)| \le (1 + e^{-\tau_{\mathrm{h}}^{(H)}}) \|\bar{H}\| \tag{3.9}$$

for every $x \in G_0^{\mathrm{per}}$ and any $H \in K_\varepsilon(H_0)$. Here, and in what follows,

$$\tau_{\mathrm{h}}^{(H)} = \tau_\Phi^{(H)} - c_{\mathrm{cl}} - c_\ell. \tag{3.10}$$

Further,

$$Z^d(\Lambda|x; H) \ge \exp\bigl[-h_x|\Lambda| - \tilde{\varepsilon}_0|\partial\Lambda|\bigr] \tag{3.11}$$

for every $x \in G_0^{\mathrm{per}}$. If x_q is stable, then

$$Z^d(\Lambda|x_q; H) \le \exp\bigl[-h|\Lambda| + \tilde{\varepsilon}_0|\partial\Lambda|\bigr] \tag{3.12}$$

and there exists a $(\tau_\Phi^{(H)} - c_\ell)$-functional Ψ_q such that

$$Z^d(\Lambda|x_q; H) = e^{-e_{x_q}(H)|\Lambda|} \mathcal{Z}(L(\Lambda); \Psi_q) \tag{3.13}$$

for every finite Λ.

4 Gibbs states with interfaces

The aim of the present section is to start from the situation of Theorem 1 and to rewrite the partition functions $Z(Y, \Lambda|y; H)$, $y \in G_0^{\mathrm{hor}}$, in terms of labeled contour models that can be treated by the methods of Section 2.2.1. In fact, we are doing this only for a class of sets Y of configurations with the aim to estimate the probability of a particular interface. Throughout this section we suppose that the assumptions of Theorem 1 (i.e. also of Basic Lemma) are satisfied. In particular,

ρ_0 is assumed to be sufficiently large, with the exact bounds specified in the course of the exposition in the present section.

We shall proceed in several steps that culminate in Proposition 4.4 below that yields an expression of probabilities of some sets of configurations from X in terms of corresponding interfaces which, in their turn, are related to certain labeled contour model. The probability measure that we have in mind is constructed, following Dobrushin [D 72], by a suitable limit starting from Gibbs states in finite volumes with boundary conditions $y \in G_0^{\text{hor}}$.

4.1 Interfaces in finite volumes

To consider Gibbs states in Basic Lemma for non-periodic elements of G_0 we fix a particular $H \in K_\varepsilon(H_0)$, two different configurations from G_0^{per}, say x_p, x_q with $p, q \in \{1, \dots, r\}$, and a configuration $y \in G_0 \cap X^{\text{hor}}_{x_p,x_q}$ to play the role of boundary conditions. The Hamiltonian H being fixed, we shall often omit a reference to it from the notation.

Following [HKZ] we say that a configuration x has a y-interface if its boundary $B(x)$ has a unique infinite component $I(x)$, $I(x) \setminus I(y)$ has only finite components, and $\mathbb{Z}^\nu \setminus I(x)$ has exactly two infinite R-components. We say that the sites in one of them are lying above and those in the other one below the interface. Further, we use $\boldsymbol{I}(x)$ to denote the pair $(I(x), x_{I(x)})$ and say that $\boldsymbol{I}$ is a y-interface if there exists a configuration x that has a y-interface $\boldsymbol{I} = \boldsymbol{I}(x)$.

We begin with a study of the partition function $Z(\Lambda|y; H)$ in a fixed finite volume $\Lambda \subset \mathbb{Z}^\nu$. If x is any configuration (i.e. $x \in S^{\mathbb{Z}^\nu}, \nu \geq 2$) that equals y in Λ^c, then x has an interface ([HKZ], Lemma 4.1). We use $\mathcal{J}(y, \Lambda)$ to denote the set of interfaces of configurations x considered above.

Our aim is to study the probability of interfaces and thus we begin with rewriting, for a fixed $\boldsymbol{I} \in \mathcal{J}(y, \Lambda)$, the partition function $Z(\boldsymbol{I}, \Lambda|y; H)$ defined as $Z(\boldsymbol{I}, \Lambda|y; H) = Z(\{x : \boldsymbol{I}(x) = \boldsymbol{I}\}, \Lambda|y; H)$ in five steps.

Step 1 (sum over interfaces) Notice first that the energy $E^{(H)}_{I\cap\Lambda}(x)$ does not depend on x_{I^c} and will be denoted by $E^{(H)}_{I\cap\Lambda}(\boldsymbol{I})$.

The volume $\Lambda_R (= \{i \in \mathbb{Z}^\nu : d(i, \Lambda) \leq R + 1\})$ is, by means of an interface $\boldsymbol{I} \in \mathcal{J}(y, \Lambda)$ with the support I, split up into several parts: $I \cap \Lambda_R$, the R-components $\Lambda_m(I), m \in \{p, q\}$, of $\Lambda_R \setminus I$ containing the sites lying above or below I, respectively, and, finally, the remaining finitely many R-components of $\Lambda_R \setminus I$ denoted by $\text{Int}_j I, j = 1, 2, \dots$. The configuration x equals to some $x_{q(j)} \in G_0^{\text{per}} = \{x_1, x_2, \dots, x_r\}$, in $I_R \cap \text{Int}_j I$ if $\boldsymbol{I}(x) = \boldsymbol{I}$.

We recall that, given the Hamiltonian H, the "physical" partition function in Λ under the boundary condition $z \in X$ is expressed by

$$Z(\Lambda|z; H) = \sum_{x=z \text{ in } \Lambda^c} \exp\{-H_\Lambda(x|z)\},$$

where $H_\Lambda(x|z) = \sum_{A\cap\Lambda\neq\emptyset} U_A(x_A)$ with $x = z$ in Λ^c.

Using now directly concerned definitions, we get

$$Z(\Lambda|y;H) = \sum_{\boldsymbol{I}\in\mathcal{J}(y,\Lambda)} Z(\boldsymbol{I},\Lambda|y;H),$$

where

$$\begin{aligned} &Z(\boldsymbol{I},\Lambda|y;H)\exp\{-\sum_{A\subset\Lambda^c} U_A(y)\frac{|A\cap\Lambda_R|}{|A|}\} = \\ &\quad = \exp\{-E^{(H)}_{I\cap\Lambda_R}(\boldsymbol{I})\}\prod_{m\in\{p,q\}} Z^d(\Lambda_m(I)|x_m;H)\prod_j Z^d(\mathrm{Int}_j\, I|x_{q(j)};H). \end{aligned} \tag{4.1}$$

The equalities above correspond to Lemma 4.2 from [HKZ] that differs only by the usage of the relative diluted partition function Θ instead of Z^d.

Step 2 (cluster expansion) From now on we suppose that x_p *and* x_q *are stable with respect to the considered Hamiltonian* H and so we may apply the Pirogov-Sinai theory as formulated in Section 2. We express the diluted partition functions with boundary conditions x_p, x_q in terms of contour functionals $\Psi_p^{(H)}$ and $\Psi_q^{(H)}$ and thus, referring to (3.13) and (2.10), in terms of cluster functionals $\Psi_p^{(H)T}$ and $\Psi_q^{(H)T}$. Namely,

$$\begin{aligned} &Z(\boldsymbol{I},\Lambda|y;H)\exp\{-\sum_{A\subset\Lambda^c} U_A(y)\frac{|A\cap\Lambda_R|}{|A|}\} = \prod_j Z^d(\mathrm{Int}_j\, I|x_{q(j)};H)\times \\ &\quad\times\exp\{-E^{(H)}_{I\cap\Lambda_R}(\boldsymbol{I}) + \sum_{m\in\{p,q\}}\big(\sum_{C\subset\Lambda_m(I)} \Psi_m^{(H)T}(C) - e_{x_m}(H)\,|\Lambda_m(I)|\big)\}. \end{aligned} \tag{4.2}$$

Here, the contour functionals $\Psi_p^{(H)}$ and $\Psi_q^{(H)}$ satisfy, according to Proposition 2.2.1, the bounds a) and a′) with the sufficiently large constant $\tau_\Phi^{(H)} - c_\ell$. Further, the cluster functionals $\Psi_p^{(H)T}$ and $\Psi_q^{(H)T}$ satisfy, in view of Proposition 2.1.3 (if $\tau_\Phi^{(H)} - c_\ell \geq \tau_{\mathrm{cl}}$), the bounds (2.7) and (2.9) with the constant $\tau_{\mathrm{h}}^{(H)}$ defined in Proposition 3.1 above.

Step 3 (extraction of bulk terms) We are going to extract a bulk term independently of $\boldsymbol{I}$ using the fact that (see (**S**) in Theorem 2, Proposition 2.2.1 c), (2.14) of Corollary 2.1.4 and (3.1))

$$\varphi_p^{(H)} - \sum_{C\ni i}\frac{\Psi_p^{(H)T}(C)}{|C|} = \varphi_q^{(H)} - \sum_{C\ni i}\frac{\Psi_q^{(H)T}(C)}{|C|}.$$

Denoting this expression by $h(H)$ (notice that, since the phases x_p and x_q are stable, one has

$$h(H) = -\lim_{\Lambda\nearrow\mathbb{Z}^\nu}\frac{1}{|\Lambda|}\log Z^d(\Lambda|x_{\bar q};H) = -\lim_{\Lambda\nearrow\mathbb{Z}^\nu}\frac{1}{|\Lambda|}\log Z(\Lambda|z;H)$$

for any $\bar{q} \in \{1, \ldots, r\}$ and $z \in X$ (the first equality, for stable $\bar{q}$, follows by (3.13), (3.1), (2.10) and Proposition 2.2.1 c); the other one is standard), we get

$$Z(\boldsymbol{I}, \Lambda|y; H) \exp\Big\{h(H)\,|\Lambda_R|\} \exp\{-\sum_{A\subset\Lambda^c} U_A(y)\frac{|A\cap\Lambda_R|}{|A|}\Big\} =$$

$$= \exp\Big\{-\Big[E_{I\cap\Lambda_R}(\boldsymbol{I}) - h(H)\,|I\cap\Lambda_R|\Big]\Big\}$$

$$\times \exp\Big\{\sum_j \log Z^d(\mathrm{Int}_j\, I|x_{q(j)}; H) + h(H)\,|\mathrm{Int}_j\, I|\Big\}$$

$$\times \exp\Big\{-\sum_{m\in\{p,q\}}\sum_{C\not\subseteq\Lambda_m(I)} \Psi_m^{(H)T}(C)\frac{|C\cap\Lambda_m(I)|}{|C|}\Big\}. \quad (4.3)$$

Step 4 (extraction of surface terms) The next step is to extract a surface term that does not depend on $\boldsymbol{I}$. If $C \subset \mathbb{Z}^\nu$ and $m \in \{p,q\}$, we write $C * m$ whenever there exist $i, j \in C$ such that $|i-j| = 1$, $i \in \partial\Lambda_R \cap \partial\Lambda_m(I(y))$, $j \in (\Lambda_R)^c$. We put $\chi_m^\Lambda(C) = 1$ if $C * m$ and $\chi_m^\Lambda(C) = 0$ otherwise. Using this notation, we have

$$Z(\boldsymbol{I},\Lambda|y;H)\exp\Big\{h(H)|\Lambda_R| - \sum_{A\subset\Lambda^c} U_A(y)\frac{|A\cap\Lambda_R|}{|A|} + \sum_{m\in\{p,q\}}\sum_{C*m}\Psi_m^{(H)T}(C)\frac{|C\cap\Lambda_R|}{|C|}\Big\}$$

$$= \exp\Big\{-\big(E_{I\cap\Lambda_R}(\boldsymbol{I}) - h(H)\,|I\cap\Lambda_R|\big) + \big(\sum_j \log Z^d(\mathrm{Int}_j\, I|x_{q(j)}; H) + h(H)\,|\mathrm{Int}_j\, I|\big)\Big\}$$

$$\times \exp\Big\{\sum_{m\in\{p,q\}}\sum_{C\cap I\neq\emptyset} \Psi_m^{(H)T}(C)\Big(\chi_m^\Lambda(C)\frac{|C\cap\Lambda_R|}{|C|} - \frac{|C\cap\Lambda_m(I)|}{|C|}\Big)\Big\}. \quad (4.4)$$

The above equality corresponds to Lemma 4.3 in [HKZ]. Notice that the factor on the left-hand side of (4.4) does not depend on $\boldsymbol{I}$.

Step 5 (positivity of cluster terms) The terms $\Psi_m^{(H)T}(C)(\chi_m^\Lambda(C)\frac{|C\cap\Lambda_R|}{|C|} - \frac{|C\cap\Lambda_m(I)|}{|C|})$ are not necessarily positive, a feature that would be useful in further application of our version of the Pirogov-Sinai theory. However, the terms depending on C may be turned into explicitly positive by adding a suitable sum in the exponent and absorbing it into a small change of weight of interfaces in the same time. For a reasonable choice of the added sum (to secure the positivity of cluster terms and to allow the proof of Lemma 4.3 below) we shall use the bounds $|\Psi_m^{(H)}(\Gamma)| \leq e^{-\tau_{\mathrm{h}}^{(H)}|\Gamma|}$ and $\Big|\Psi_m^{(H)T}(C)\Big| \leq e^{-\tau_{\mathrm{h}}^{(H)}|C|}$ with $\tau_{\mathrm{h}}^{(H)}$ from Proposition 3.1 (see also Step 2). Hence

$$Z(\boldsymbol{I}, \Lambda|y; H) \exp\Big\{h(H)\,|\Lambda_R| - \sum_{A\subset\Lambda^c} U_A(y)\frac{|A\cap\Lambda_R|}{|A|}\Big\}$$

$$\times \exp\Big\{\sum_{m\in\{p,q\}}\sum_{C*m}\Psi_m^{(H)T}(C)\frac{|C\cap\Lambda_R|}{|C|} + 2\sum_{\substack{C*p\\ C*q}} e^{-\tau_{\mathrm{h}}^{(H)}|C|}\Big\} =$$

$$= \exp\Big\{-\big[E^{(H)}_{I\cap\Lambda_R}(\boldsymbol{I})-h(H)\,|I\cap\Lambda_R|\big]+\Big(\sum_j \log Z^d(\mathrm{Int}_j\, I|x_{q(j)};H)+h(H)\,|\mathrm{Int}_j\, I|\Big)\Big\}$$

$$\times \exp\Big\{\sum_{m\in\{p,q\}}\sum_{C\cap I\neq\emptyset}\Big[\Psi^{(H)T}_m(C)(\chi^\Lambda_m(C)\frac{|C\cap\Lambda_R|}{|C|}-\frac{|C\cap\Lambda_m(I)|}{|C|})\Big]\Big\}$$

$$\times \exp\Big\{2\sum_{C\cap I\neq\emptyset} e^{-\tau^{(H)}_{\mathrm{h}}|C|}(|C\cap I\cap\Lambda_R|+\chi^\Lambda_p(C)\cdot\chi^\Lambda_q(C))\Big\}$$

$$\times \exp\Big\{-2\sum_{C\cap I\neq\emptyset} e^{-\tau^{(H)}_{\mathrm{h}}|C|}\,|C\cap I\cap\Lambda_R|\Big\}. \quad (4.5)$$

We added the terms $2\sum_{\substack{C*q\\C*p}} e^{-\tau^{(H)}_{\mathrm{h}}|C|}$ and added and subtracted $2\sum_{C\cap I\neq\emptyset} e^{-\tau^{(H)}_{\mathrm{h}}|C|}$ $|C\cap I\cap\Lambda_R|$. The idea is that the latter equals $\varkappa(\tau^{(H)}_{\mathrm{h}})\,|I\cap\Lambda_R|$ with

$$\varkappa(\tau^{(H)}_{\mathrm{h}}) = 2\sum_{C:0\in C} e^{-\tau^{(H)}_{\mathrm{h}}|C|} \quad (4.6)$$

and may be absorbed into a controllable change of energy, while the former does not depend on I and may be extracted as a "border of surface term". In the same time, whenever the term $\chi^\Lambda_m(C)\frac{|C\cap\Lambda_R|}{|C|}-\frac{|C\cap\Lambda_m(I)|}{|C|}$ is non-vanishing (in any case, its absolute value is bounded by 1), the term $|C\cap I\cap\Lambda_R|+\chi^\Lambda_p(C)\cdot\chi^\Lambda_q(C)$ is at least 1. Indeed, if $\chi^\Lambda_m(C)=0$ and in the same time $C\cap\Lambda_m(I)\neq\emptyset$, then necessarily $C\cap I\cap\Lambda_R\neq\emptyset$ (we took into account that $C\cap I\neq\emptyset$). If, say, $\chi^\Lambda_p(C)=1$ and in the same time $\chi^\Lambda_q(C)=0$ and $C\cap\Lambda_R\neq C\cap\Lambda_p(I)$, then $C\cap I\cap\Lambda_R\neq\emptyset$. Finally, the claim is trivial if $\chi^\Lambda_p(C)=\chi^\Lambda_q(C)=1$. As a result,

$$\Psi^{(H)T}_m(C)\Big(\chi^\Lambda_m(C)\frac{|C\cap\Lambda_R|}{|C|}-\frac{|C\cap\Lambda_m(I)|}{|C|}\Big)+ \\ + e^{-\tau^{(H)}_{\mathrm{h}}|C|}\Big(|C\cap I\cap\Lambda_R|+\chi^\Lambda_p(C)\cdot\chi^\Lambda_q(C)\Big)\geq 0 \quad (4.7)$$

since $\big|\Psi^{(H)T}_m(C)\big|\leq e^{-\tau^{(H)}_{\mathrm{h}}|C|}$.

Using $\tilde{Z}(\boldsymbol{I},\Lambda|y,H)$ to denote the left-hand side of (4.5), we stress that the ratio $\tilde{Z}/Z$ does not depend on $\boldsymbol{I}$ and thus the probabilities of interfaces defined by $\tilde{Z}$ or Z do not differ.

To evaluate those probabilities, we recall the notion of walls in order to rewrite (4.5) in terms of them. A pair $\mathbf{w}=(W,x_W)$, where W is a connected component of $I(x)\setminus C(x)$, is a *wall* of $\boldsymbol{I}(x)$. We denote by $\mathcal{W}(\boldsymbol{I}(x))$ the collection of all walls of $\boldsymbol{I}(x)$. For any $\mathbf{w}\in\mathcal{W}(\boldsymbol{I}(x))$ we put

$$E^{(H)}(\mathbf{w}) = E^{(H)}_W(x). \quad (4.8)$$

Let us recall also that given a wall $\mathbf{w}$, we use $x_{\mathbf{w}}\in X$ to denote the configuration for which $\boldsymbol{I}(x_{\mathbf{w}})=B(x_{\mathbf{w}})$ and $\mathbf{w}$ is the only wall of $\boldsymbol{I}(x_{\mathbf{w}})$, and $y_{\mathbf{w}}$ to denote the unique element of G_0 which differs from $x_{\mathbf{w}}$ in at most finite number of sites. Recall that the set $G_0 \cap X^{\mathrm{hor}}_{x_p,x_q}$ consists of configurations that are vertical shifts of finitely many "representatives" $x_{p,q;1}, \dots, x_{p,q;n_{p,q}}$. Let $s(\mathbf{w})$ be such that $x_{p,q;s(\mathbf{w})}$ is a vertical shift of $y_{\mathbf{w}}$.

We use π to denote the projection of $\mathbb{Z}^\nu$ onto $\mathbb{Z}^{\nu-1}$ defined by

$$\pi(i_1, \dots, i_{n-1}, i_n) = (i_1, \dots, i_{n-1}),$$

and recall that $I_W = I(y_{\mathbf{w}}) \cap \pi^{-1}(W)$.

Notice further that each set $\mathrm{Int}_j\, I$ is surrounded by the support W of a single wall $\mathbf{w}$ of $\boldsymbol{I}$ and use $j \circ W$ to denote that this is the case.

Let $I_{p,q;s}$ be the support of the interface of $x_{p,q;s}$, $s = 1, 2, \dots, n_{p,q}$ and $B_{p,q;s}(\boldsymbol{I})$ be the set

$$\{i \in B_R = \pi(\Lambda_R) : \pi^{-1}(i) \cap I \text{ is a vertical shift of the "}x_{p,q;s}\text{-ceiling column" }\} \cup \bigcup \{\pi(W); s(\mathbf{w}) = s\}$$

for $\boldsymbol{I} \in \mathcal{J}(y, \Lambda)$. Let $T_{p,q;s}$ be the number of sites in $\pi^{-1}(0) \cap I_{p,q;s}$, i.e. the "thickness" of $I_{p,q;s}$, and T be the maximum of all $T_{p,q;s}$'s over $p, q \in \{1, \dots, r\}, s \in \{1, \dots, n_{p,q}\}$. Let us recall that $2 \sum_{C \cap I \neq \emptyset} e^{-\tau_{\mathrm{h}}^{(H)}|C|}\, |C \cap I \cap \Lambda_R| = \varkappa(\tau_{\mathrm{h}}^{(H)})|I \cap \Lambda_R|$.

Further, we use $\tilde{E}^{(H)}(\mathbf{w})$ to denote the "modified energy of the wall $\mathbf{w}$",

$$\tilde{E}^{(H)}(\mathbf{w}) = E^{(H)}(\mathbf{w}) - h(H)\,|W| + \varkappa(\tau_{\mathrm{h}}^{(H)})\,|W| - \\ - \sum_{j \circ W} (\log Z^d(\mathrm{Int}_j\, I | x_{q(j)}; H) + h(H)\,|\mathrm{Int}_j\, I|), \quad (4.9)$$

and $\varphi^{(H)}_{p,q;s}$ to denote the "modified specific energy of an s-ceiling between x_p above and x_q below",

$$\varphi^{(H)}_{p,q;s} = E^{(H)}_{\pi^{-1}(0) \cap I_{p,q;s}}(x_{p,q;s}) + \big[\varkappa(\tau_{\mathrm{h}}^{(H)}) - h(H)\big] T_{p,q;s}. \quad (4.10)$$

We also use $\Phi^{(H)}_\Lambda(\mathbf{w}; p, q)$ to denote the "modified weight corresponding to a wall $\mathbf{w}$ of an interface separating x_p and x_q", the *wall functional*

$$\Phi^{(H)}_\Lambda(\mathbf{w}; p, q) = \exp\big\{-\big[\tilde{E}^{(H)}(\mathbf{w}) - \varphi^{(H)}_{p,q;s(\mathbf{w})}\,|\pi(W)|\big]\big\} \quad (4.11)$$

and, finally, we define the *cluster functional*

$$\Phi^{(H)}_{\Lambda,I}(C; p, q) = \exp\Big\{ \sum_{m \in \{p,q\}} \Big[\Psi^{(H)T}_m(C) \Big(\chi^\Lambda_m(C) \frac{|C \cap \Lambda_R|}{|C|} - \frac{|C \cap \Lambda_m(I)|}{|C|} \Big) + \\ + e^{-\tau_{\mathrm{h}}^{(H)}|C|} \Big(|C \cap I \cap \Lambda_R| + \chi^\Lambda_p(C) \chi^\Lambda_q(C) \Big) \Big] \Big\} - 1. \quad (4.12)$$

Using this notation, we rewrite (4.5) as

$$\tilde{Z}(\boldsymbol{I},\Lambda|y;H)=\prod_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})}\Phi_\Lambda^{(H)}(\mathbf{w};p,q)\prod_{s=1}^{n_{p,q}}\exp\{-\varphi_{p,q;s}^{(H)}\,|B_{p,q;s}(\boldsymbol{I})|\}\times$$
$$\times\prod_{C:C\cap I\neq\emptyset,C\cap B_R\neq\emptyset}\Big(1+\Phi_{\Lambda,I}^{(H)}(C;p,q)\Big),\quad(4.13)$$

and notice that by (4.7) we have

$$\Phi_{\Lambda,I}^{(H)}(C;p,q)\geq 0.\qquad(4.14)$$

4.2 Interfaces in cylinders with finite base

Our next aim is to show that $\tilde{Z}(V|y;H)=\sum_{\boldsymbol{I}\in\mathcal{J}(y;V)}\tilde{Z}(\boldsymbol{I},V|y;H)$ can be defined for an infinite cylinder $V=\pi^{-1}(B)$ with a finite base $B(=\pi(V))\subset\mathbb{Z}^{\nu-1}$ and to study $\tilde{Z}(I,V|y;H)$.

Step 6 (a wall bound) The main aim of this step is to prove that, for any finite volume $\Lambda\subset\mathbb{Z}^\nu$ and any interface $\boldsymbol{I}$ of a configuration which equals y in Λ^c,

$$\tilde{Z}(\boldsymbol{I},\Lambda|y;H)\leq e^{c_{\mathbb{I}}|B_R|(\|H\|+1)}\exp\big\{-(\rho_\varepsilon\frac{\|H\|}{\|H_0\|}-c_{\mathbb{I}})\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})}|W|\big\},\qquad(4.15)$$

for some "interface" constant $c_{\mathbb{I}}$, once ρ_ε is sufficiently large.

To estimate the wall functional $\Phi_\Lambda^{(H)}(\mathbf{w};p,q)$ defined by (4.11), we bound first (cf. (4.9) and (4.10))

$$\begin{aligned}E^{(H)}(\mathbf{w})-h(H)\,|W|+\varkappa(\tau_{\mathrm{h}}^{(H)})\,|W|-\varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)|=\\=\Big(\big[E_W^{(H)}(x_{\mathbf{w}})-e_0(H)|W|\big]-\big[E_{I_W}^{(H)}(y_{\mathbf{w}})-e_0(H)|I_W|\big]\Big)+\\+(e_0(H)-h(H))(|W|-|I_W|)+\varkappa(\tau_{\mathrm{h}}^{(H)})(|W|-|I_W|).\quad(4.16)\end{aligned}$$

First of all we use Peierls condition ($\mathbf{P^{hor}}$) with the constant $\rho_\varepsilon\frac{\|H\|}{\|H_0\|}$ (see Proposition 1.1.3) to the first parenthesised part on the right-hand side and we get

$$\big[E_W^{(H)}(x_{\mathbf{w}})-e_0(H)|W|\big]-\big[E_{I_W}^{(H)}(y_{\mathbf{w}})-e_0(H)|I_W|\big]>\rho_\varepsilon\frac{\|H\|}{\|H_0\|}|W|.\qquad(4.17)$$

By (3.7) we get

$$\big|(e_0(H)-h(H))(|W|-|I_W|)\big|\leq e^{-\tau_{\mathrm{h}}^{(H)}}|W|(1+T).\qquad(4.18)$$

We estimate the artificial term $\varkappa(\tau_{\mathrm{h}}^{(H)})$, used to achieve the positivity of $\Phi_{\Lambda,I}^{(H)}(C;p,q)$, using its definition (4.6) and the bound (2.4), by

$$0 \le \varkappa(\tau_{\mathrm{h}}^{(H)}) \le 2 \sum_{k=1}^{\infty} e^{-(\tau_{\mathrm{h}}^{(H)}-c_{\#})k} \le e^{-(\tau_h^{(H)}-c_{\#}-\log 3)} \tag{4.19}$$

once $\tau_{\mathrm{h}}^{(H)}$ is sufficiently large. Hence

$$\varkappa(\tau_{\mathrm{h}}^{(H)})(|W|-|I_W|) \ge -e^{-(\tau_h^{(H)}-c_{\#}-\log 3)}|W|T. \tag{4.20}$$

Further, due to (3.12) we have the bound

$$\sum_{j\circ W}{}' \bigl(\log Z^d(\mathrm{Int}_j\, I|x_{q(j)};H)+h(H)|\,\mathrm{Int}_j\, I|\bigr) \le \sum_{j\circ W} \tilde\varepsilon_0(\tau_{\mathrm{h}}^{(H)})|\partial\, \mathrm{Int}_j\, I| \le$$
$$\le 2\nu\tilde\varepsilon_0(\tau_{\mathrm{h}}^{(H)})|W| \le e^{-(\tau_h^{(H)}-c_{\mathrm{Z}})}|W| \tag{4.21}$$

for a constant c_{Z}. Here we used that

$$\tilde\varepsilon_0(\tau_{\mathrm{h}}^{(H)}) \le 2e^{-(\tau_{\mathrm{h}}^{(H)}-c_{\mathrm{cl}}-c_{\#}-\log|S|-1)} \le e^{-(\tau_h^{(H)}-c_\varepsilon)},$$

for a suitable constant c_ε, by Lemma 2.3.1. Here $\sum'_{j\circ W}$ means any sum over a subset of $\{j; j\circ W\}$.

Substituting the just derived inequalities (4.17), (4.18), (4.20) into (4.16), we get

$$E^{(H)}(\mathbf{w}) - h(H)\,|W| + \varkappa(\tau_{\mathrm{h}}^{(H)})\,|W| - \varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)| \ge$$
$$\rho_\varepsilon\frac{\|H\|}{\|H_0\|}|W| - e^{-\tau_{\mathrm{h}}^{(H)}}|W|(1+T) - e^{-(\tau_{\mathrm{h}}^{(H)}-c_{\#}-\log 3)}|W|T. \tag{4.22}$$

Using further (4.21) and having in mind (4.11) with (4.9) and (4.10), we conclude that

$$\Phi_\Lambda^{(H)}(\mathbf{w};p,q) \le$$
$$\le \exp\Bigl(-\rho_\varepsilon\frac{\|H\|}{\|H_0\|}|W|+e^{-\tau_{\mathrm{h}}^{(H)}}|W|(1+T)+e^{-(\tau_{\mathrm{h}}^{(H)}-c_{\#}-\log 3)}|W|T+e^{-(\tau_h^{(H)}-c_{\mathrm{Z}})}|W|\Bigr)$$
$$\le \exp\bigl(-(\rho_\varepsilon\frac{\|H\|}{\|H_0\|}-c_w)|W|\bigr), \tag{4.23}$$

where c_w is a positive constant which can be chosen arbitrarily small if taking $\tau_{\mathrm{h}}^{(H)}$, i.e. ρ_ε, sufficiently large in the same time.

Now we estimate the contribution of the specific ceiling energies to the logarithm of $\tilde Z(\boldsymbol{I},\Lambda|y;H)$. Due to (4.10), (4.19), and the inequality $|h(H)| \le \|H\| + e^{-\tau_h^{(H)}}$ following from (3.7), we get

$$|\varphi_{p,q;s}^{(H)}| \le t(\|H\| + e^{-(\tau_h^{(H)}-c_{\#}-\log 3)} + \|H\| + e^{-\tau_h^{(H)}}) \le c_\varphi(\|H\|+1), \tag{4.24}$$

for some real constant c_φ. It follows that

$$\Big|\sum_{s=1}^{n_{p,q}} \varphi^{(H)}_{p,q;s} |B_{p,q;s}(\boldsymbol{I})|\Big| \le \max_{s=1,\dots,n_{p,q}} \big\{|\varphi^{(H)}_{p,q;s}|\big\} \big|\pi(\Lambda_R)\big| \le c_\varphi(\|H\|+1)\big|B_R\big|. \quad (4.25)$$

Finally, we get bounds for the nonnegative cluster functionals $\Phi^{(H)}_{\Lambda,I}(C;p,q)$.

Using (4.14) from Step 5, (4.12), the fact that $|\Psi^{(H)T}_m(C)| \le e^{-\tau^{(H)}_{\mathrm{h}}|C|}$, and the inequalities $0 \le e^x - 1 \le 2x$ for x nonnegative and small enough, we get

$$0 \le \Phi^{(H)}_{\Lambda,I}(C;p,q) \le 4(|C|+2)e^{-\tau^{(H)}_{\mathrm{h}}|C|} \le e^{-(\tau^{(H)}_{\mathrm{h}}-c_{\mathrm{C}})|C|}, \quad (4.26)$$

where c_{C} is a real constant.

So we can estimate the third product from (4.13) by

$$\prod_{C: C\cap I\neq\emptyset, C\cap B_R\neq\emptyset} \big(1+\Phi^{(H)}_{\Lambda,I}(C;p,q)\big) \le \prod_{i\in I\cap\Lambda_R}\ \prod_{C:\mathrm{dist}(C,i)\le|C|} \big(1+e^{-(\tau^{(H)}_{\mathrm{h}}-c_{\mathrm{C}})|C|}\big) \le$$
$$\le \exp\big\{c_\pi(|I\cap\Lambda_R|)\big\} \le \exp\big\{c_\pi(|B_R| + \sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} |W|)\big\} \quad (4.27)$$

for some constant c_π. Here, we used

$$\prod_{C:\mathrm{dist}(C,i)\le|C|} \big(1+e^{-(\tau^{(H)}_{\mathrm{h}}-c_{\mathrm{C}})|C|}\big) \le \exp\big\{\sum_{C:\mathrm{dist}(C,i)\le|C|} e^{-(\tau^{(H)}_{\mathrm{h}}-c_{\mathrm{C}})|C|}\big\}$$
$$\le \exp\big\{\sum_{C: i\in C} |C|^\nu e^{-(\tau^{(H)}_{\mathrm{h}}-c_{\mathrm{C}})|C|}\big\}. \quad (4.28)$$

Applying the estimates (4.23), (4.25), and (4.27) to (4.13), we get

$$\tilde{Z}(\boldsymbol{I},\Lambda|y;H) \le$$
$$\le \exp\{-\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} (\rho_\varepsilon \frac{\|H\|}{\|H_0\|} - c_{\mathrm{w}})|W|\} \exp\{c_\varphi(\|H\|+1)|B_R|\} \exp\{c_\pi|B_R|$$
$$+ \sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} c_\pi|W|\} \le e^{c_{\mathbb{I}}|B_R|(\|H\|+1)} \exp\big\{-(\rho_\varepsilon \frac{\|H\|}{\|H_0\|} - c_{\mathbb{I}}) \sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} |W|\big\}$$

with a suitable constant $c_{\mathbb{I}}$, getting thus (4.15).

Step 7 (partition functions in infinite cylinder sets) Let us use χ^V to denote, for $V = \pi^{-1}(B)$, the analogue of χ^Λ. Further, we introduce the analogs of the wall and cluster functionals $\Phi^{(H)}_\Lambda(\mathbf{w};p,q)$ and $\Phi^{(H)}_{\Lambda,I}(C;p,q)$, defined by (4.11) and (4.12). We denote the corresponding functionals, obtained by substituting V for Λ, by $\Phi^{(H)}_V(\mathbf{w};p,q)$ and $\Phi^{(H)}_{V,I}(C;p,q)$.

To avoid overloading the functionals with an additional label, we slightly abuse the notation observing that the functionals defined above can be easily distinguished by the number of their indices and arguments from general labeled contour functional of Section 2.2 as well as from the functionals $\Phi_B^{(H)}(A;p,q)$ and $\mathit{\Phi}_B^{(H)}(\sigma;p,q)$, introduced in the next section, that yield the labeled aggregate and shadow models.

In correspondence to Lemma 4.7 in [HKZ], and due to the estimate (4.26), we may use the fact that

$$\prod_{C:C\cap I\neq\emptyset, C\cap V_R\neq\emptyset}\bigl(1+\Phi_{V,I}^{(H)}(C;p,q)\bigr)=\sum_{\substack{\{C_k\}\text{ finite}\\ C_k\cap I\neq\emptyset, C_k\cap V_R\neq\emptyset}}\prod_k \Phi_{V,I}^{(H)}(C_k;p,q)$$

to obtain the following expression for the limit partition function

$$\begin{aligned}\tilde Z(\boldsymbol{I},V|y;H)=\lim_{\Lambda\nearrow V}\tilde Z(\boldsymbol{I},\Lambda|y;H)=\sum_{\substack{\{C_k\}\text{ finite}\\ C_k\cap I\neq\emptyset, C_k\cap V_R\neq\emptyset}}\prod_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})}\Phi_V^{(H)}(\mathbf{w};p,q)\times\\ \times\prod_{s=1}^{n_{p,q}}\exp\{-\varphi_{p,q;s}^{(H)}\,|B_{p,q;s}(\boldsymbol{I})|\}\prod_k\Phi_{V,I}^{(H)}(C_k;p,q).\end{aligned}\qquad(4.29)$$

Thus the probabilities of any interface $\boldsymbol{I}\in\mathcal{J}(y,\Lambda)$ defined by

$$\mu\bigl(\{x\in X:\boldsymbol{I}(x)=\boldsymbol{I}\},\Lambda|y;H\bigr)=\frac{\tilde Z(\boldsymbol{I},\Lambda|y;H)}{\sum_{\bar{\boldsymbol{I}}\in\mathcal{J}(y,\Lambda)}\tilde Z(\bar{\boldsymbol{I}},\Lambda|y;H)}$$

converge to the probability of $\boldsymbol{I}\in\mathcal{J}(y,V)=\bigcup_{\Lambda\subset V}\mathcal{J}(y,\Lambda)$ defined by

$$\mu\bigl(\{x\in X:\boldsymbol{I}(x)=\boldsymbol{I}\},V|y;H\bigr)=\frac{\tilde Z(\boldsymbol{I},V|y;H)}{\sum_{\bar{\boldsymbol{I}}\in\mathcal{J}(y,V)}\tilde Z(\bar{\boldsymbol{I}},V|y;H)}.\qquad(4.30)$$

The sum in the denominator above converges since, according to (4.15), it can be bounded by $e^{c_{\mathbb{I}}|B_R|(\|H\|+1)}\prod_{i\in B}\bigl(\sum_{i\in W}e^{-(\rho_\varepsilon\frac{\|H\|}{\|H_0\|}-c_{\mathbb{I}})|W|}\bigr)$, where W stands for connected subsets of $\mathbb{Z}^\nu$ (possible supports of walls shifted vertically to intersect B).

Here we have used the important fact that the interfaces from $\mathcal{J}(y,V)$ are uniquely determined by their walls even if we "forget" their vertical position. This observation goes back to "(y,V)-admissible families of standard walls ([HKZ] Lemma 2.2)" and originates in [D 72].

Now we shall recall or derive some estimates on the functionals $\Phi_V^{(H)}(\mathbf{w};p,q)$ and $\Phi_{V,I}^{(H)}(C;p,q)$ and their derivatives as well as a bound on the derivative of $\varphi_{p,q;s}^{(H)}$ needed later.

Lemma 4.3 *Let $x_p,x_q\in G_0$ be two stable translation invariant configurations at $H\in K_\varepsilon(H_0)$. Let $s\in\{1,\dots,n_{p,q}\}$ be arbitrary.*

Then $\Phi_V^{(H)}(\mathbf{w};p,q)$ is translation invariant, $\Phi_{V,I}^{(H)}(C;p,q)$ is translation invariant "inside V", and the following bounds are fulfilled.

(a) $0<\Phi_V^{(H)}(\mathbf{w};p,q)\le e^{-\tau_{\mathbb{I}}^{(H)}|W|}$;

(b) $0\le\Phi_{V,I}^{(H)}(C;p,q)\le e^{-\tau_{\mathbb{I}}^{(H)}|C|}$;

(a′) $|\partial_{\overline{H}}^{\pm}\Phi_V^{(H)}(\mathbf{w};p,q)|\le\|\overline{H}\|\exp\{-\tau_{\mathbb{I}}^{(H)}|W|\}$;

(b′) $|\partial_{\overline{H}}^{\pm}\Phi_{V,I}^{(H)}(C;p,q)|\le\|\overline{H}\|\exp\{-\tau_{\mathbb{I}}^{(H)}|C|\}$;

(c) $|\partial_{\overline{H}}^{\pm}\varphi_{p,q;s}^{(H)}|\le\|\overline{H}\|M_{\mathrm{s}}$.

Here $M_{\mathrm{s}}(\equiv M_{shadow})$ and $c'_{\mathbb{I}}$ are suitable constants, and $\tau_{\mathbb{I}}^{(H)}=\tau_{\mathrm{h}}^{(H)}-\log\bigl(\frac{\rho_\varepsilon}{\|H_0\|}\bigr)-c'_{\mathbb{I}}\|H\|$.

Proof. (a) Taking the limit $\Lambda\nearrow V$ in (4.23), we get

$$0\le\Phi_V^{(H)}(\mathbf{w};p,q)\le\exp\bigl(-(\rho_\varepsilon\frac{\|H\|}{\|H_0\|}-c_w)|W|\bigr)\le e^{-(\tau_h^{(H)}-c_w)|W|}$$

by (3.6) and (3.10).

(b) Taking the limit $\Lambda\nearrow V$ in (4.26) we get

$$0\le\Phi_{V,I}^{(H)}(C;p,q)\le\exp\bigl\{-(\tau_{\mathrm{h}}^{(H)}-c_{\mathrm{C}})|C|\bigr\}.$$

(a′) By the definitions (4.9) of $\tilde{E}^{(H)}(\mathbf{w})$ and (4.11) of $\Phi_\Lambda^{(H)}(\mathbf{w};p,q)$, with Λ replaced by V, we have

$$\begin{aligned}
&\partial_{\overline{H}}^{\pm}\Phi_V^{(H)}(\mathbf{w};p,q)=\\
&\quad=\partial_{\overline{H}}^{\pm}\Bigl[\exp\bigl\{-E^{(H)}(\mathbf{w})+(h(H)-\varkappa(\tau_{\mathrm{h}}^{(H)}))|W|+\varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)|\bigr\}\times\\
&\qquad\qquad\prod_{j\circ W}\frac{Z^d(\mathrm{Int}_j\,I|x_{q(j)};H)}{\exp\{-h(H)|\,\mathrm{Int}_j\,I|\}}\Bigr]=\\
&\quad=\Phi_V^{(H)}(\mathbf{w};p,q)\partial_{\overline{H}}^{\pm}\bigl\{-E^{(H)}(\mathbf{w})+(h(H)-\varkappa(\tau_{\mathrm{h}}^{(H)}))|W|+\varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)|\bigr\}+\\
&\qquad+\exp\bigl\{-E^{(H)}(\mathbf{w})+(h(H)-\varkappa(\tau_{\mathrm{h}}^{(H)}))|W|+\varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)|\bigr\}\times\\
&\qquad\times\sum_{j\circ W}\prod_{\substack{j'\neq j\\ j'\circ W}}\frac{Z^d(\mathrm{Int}_{j'}\,I|x_{q(j)};H)}{\exp\{-h(H)|\,\mathrm{Int}_{j'}\,I|\}}\partial_{\overline{H}}^{\pm}\Bigl(\frac{Z^d(\mathrm{Int}_j\,I|x_{q(j)};H)}{\exp\{-h(H)|\,\mathrm{Int}_j\,I|\}}\Bigr).
\end{aligned}\tag{4.31}$$

To get the sought estimate (a′), we begin by estimating the first summand in (4.31). Due to (a), it suffices to show that

$$|\partial_{\overline{H}}^{\pm}\{-E^{(H)}(\mathbf{w})+(h(H)-\varkappa(\tau_{\mathrm{h}}^{(H)}))|W|+\varphi_{p,q;s(\mathbf{w})}^{(H)}|\pi(W)|\}|\le C^{(1)}|W|\|\overline{H}\|$$

with a suitable constant $C^{(1)}$ that can be chosen independent of $\tau_{\mathrm{h}}^{(H)}$ for large $\tau_{\mathrm{h}}^{(H)}$.

Indeed, this follows from (3.9), and the following bounds:

$$\big|\partial^+_{\bar H} E^{(H)}(\mathbf{w})\big| \le \big|\partial^+_{\bar H}\big(\sum_{i\in W} E^{(H)}_i(x)\big)\big| \le \dim \mathcal{H}(R)\big\|\overline{H}\big\|\big|W\big| \tag{4.32}$$

by linearity of $E^{(H)}_i(x)$ in H (see (4.8) to understand the first inequality);

$$\partial^+_{\bar H}\tau^{(H)}_{\mathrm{h}} = \rho_\varepsilon \frac{\|\bar H\|}{\|H_0\|} \tag{4.33}$$

which follows from (3.10) and (3.6);

$$|\partial^+_{\bar H}\varkappa(\tau^{(H)}_{\mathrm{h}})| \le 2\sum_{0\in C} e^{-\tau^{(H)}_h|C|}|C|\rho_\varepsilon\frac{\|\bar H\|}{\|H_0\|}$$

$$\le \|\bar H\| e^{-(\tau^{(H)}_{\mathrm{h}} - \log\left(\frac{\rho_\varepsilon}{\|H_0\|}\right) - c_\kappa)} \le \|\bar H\|\,\mathrm{const}, \tag{4.34}$$

with a suitable constant c_κ, follows from (4.6) and (4.33);

$$|\partial^+_{\bar H}\varphi^{(H)}_{p,q;s}| \le M_{\mathrm{s}}\|\bar H\| \tag{4.35}$$

follows from (4.10) using an analogy with (4.32), (3.9), and (4.34).

To estimate the other summand in (4.31), we use (4.22), (4.21) and, moreover,

$$\Big|\partial^+_{\bar H}\Big(\frac{Z^d(\mathrm{Int}_j\, I|x_{q(j)};H)}{\exp\{-h(H)|\,\mathrm{Int}_j\, I|\}}\Big)\Big| \le \|\bar H\|\exp\{c_{\mathrm{Z}}\|W\|\}. \tag{4.36}$$

To show the last bound, we use (2.23) of Lemma 2.3.3, (3.12), (3.9), and the bound $|\,\mathrm{Int}_j\, I| \le |\partial\,\mathrm{Int}_j\, I|^{\frac{\nu}{\nu-1}} \le e^{|\partial\,\mathrm{Int}_j\, I|}$ to get

$$\begin{aligned}\Big|\partial^+_{\bar H}\Big(\frac{Z^d(\mathrm{Int}_j\, I|x_{q(j)};H)}{\exp\{-h(H)|\,\mathrm{Int}_j\, I|\}}\Big)\Big| \le \frac{|(\partial^+_{\bar H}Z^d(\mathrm{Int}_j\, I|x_{q(j)};H))\ \exp\{-h(H)|\,\mathrm{Int}_j\, I|\}|}{\exp\{-2h(H)|\,\mathrm{Int}_j\, I|\}} +\\ + \frac{|Z^d(\mathrm{Int}_j\, I|x_{q(j)};H))\ \exp\{-h(H)|\,\mathrm{Int}_j\, I|\}\ |\,\mathrm{Int}_j\, I|\ \partial^+_{\bar H}h(H)|}{\exp\{-2h(H)|\,\mathrm{Int}_j\, I|\}} \le\\ \le \|\bar H\||\,\mathrm{Int}_j\, I|\exp\big\{\tilde\varepsilon_0|\partial(\mathrm{Int}_j\, I)|\big\}\big(M_{\mathrm{s}}(1+\tilde\varepsilon_0) + (1+e^{-\tau^{(H)}_{\mathrm{h}}})\big) \le \|\bar H\| e^{c_{\mathrm{Z}}|W|},\end{aligned} \tag{4.37}$$

where c_{Z} is a suitable constant. We used the obvious inequality $|\,\mathrm{Int}_j\, I| \le |\partial(\mathrm{Int}_j\, I)|^\nu$.

(b′) From the definition (4.12) we have

$$|\partial^+_{\bar H}\Phi^{(H)}_{V,I}(C;p,q)| \le \Big|\exp\Big\{\sum_{m\in\{p,q\}}\Big[\Psi^{(H)T}_m(C)(\chi^V_m(C)\frac{|C\cap V_R|}{|C|}-\frac{|C\cap V_m(I)|}{|C|})+ \\ + e^{-\tau^{(H)}_{\mathrm{h}}|C|}(|C\cap I\cap V_R|+\chi^V_p(C)\chi^V_q(C))\Big]\Big\}\Big|\times \\ \times\Big(\sum_{m\in\{p,q\}}|\partial^+_{\bar H}\Psi^{(H)T}_m(C)|+(|C|+1)\exp\{-\tau^{(H)}_{\mathrm{h}}|C|\}|\partial^+_{\bar H}\tau^{(H)}_{\mathrm{h}}||C|\Big)\le \\ \le \|\bar H\|\exp\{-(\tau^{(H)}_{\mathrm{h}}-\log\big(\frac{\rho_\varepsilon}{\|H_0\|}\big)-c'_{\mathrm{C}})|C|\} \quad (4.38)$$

with some constant c'_{C}.
Here we used the bound (2.9) from Proposition 2.1.3 to estimate $|\partial^+_{\bar H}\Psi^{(H)T}_m(C)|$, as well as the bound (4.33) above.

(c) The needed estimate was already proved in (4.35) above. □

4.3 Aggregate and labeled shadow models

We shall now project the walls, as well as clusters decorating the interface, to $\mathbb{Z}^{\nu-1}(\subset\mathbb{Z}^\nu)$, by applying the projection π. In that way, using the equalities (4.29) and (4.30), we introduce a shadow model, in terms of which we shall grasp the characteristic features of interfaces. The series of preceding steps will be concluded by Proposition 4.4 that describes the properties of the labeled shadow model as well as the relations to the probabilities of interfaces via an intermediary aggregate model.

Given an interface $\boldsymbol{I}=(I,x_I)\in\mathcal{J}(y,V)$ (where $V=\pi^{-1}(B)$ as in Section 4.2 and $y\in X^{\mathrm{hor}}_{x_p,x_q}$) and a finite family $\{C_k\}$ of clusters $C_k, C_k\cap I\ne\emptyset, C_k\cap V_R\ne\emptyset$ (we say that "$\{C_k\}$ decorates $\boldsymbol{I}$ at V"), and using $\mathcal{W}(\boldsymbol{I})$ to denote the collection of walls of $\boldsymbol{I}$, we first consider connected components, to be called *shadows of* $(\boldsymbol{I},\{C_k\})$, of the projection $\pi(\bigcup_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})}W\cup\bigcup_k C_k)$. If $(\mathcal{W},\mathcal{C})$ is a collection of walls of $\boldsymbol{I}$, $\mathcal{W}\subset\mathcal{W}(\boldsymbol{I})$, and clusters from $\{C_k\}$, $\mathcal{C}\subset\{C_k\}$, such that $\Sigma=\pi(\bigcup_{\mathbf{w}\in\mathcal{W}}W\cup\bigcup_{C\in\mathcal{C}}C)$ is a shadow, we introduce an *aggregate* A *of* $(\boldsymbol{I},\{C_k\})$ as the "piece of interface above the shadow" together with the corresponding clusters. Namely, $A=(\boldsymbol{I}_A,\mathcal{C})$, where $\boldsymbol{I}_A=(I_A,x_{I_A})$ with $I_A=\pi^{-1}(\Sigma)\cap I$. The shadow Σ is called the *support* of A. We identify aggregates that can be moved one into another by a vertical translation (by shifting both, the corresponding walls and clusters). Notice, that the relative position of concerned clusters with respect to the set I_A is, in a given aggregate, always fixed.

An aggregate $A=(\boldsymbol{I}_A,\mathcal{C})$ with support Σ is naturally labeled since for any $i\in\partial\Sigma$, the column $\pi^{-1}(i)$ is an "s-column" of x_I with some $s\in\{1,\dots,n_{p,q}\}$. It is clear that this s is identical for all i's from each component of $\partial\Sigma$. We denote this labeling by $\lambda(A)$.

As a result, the shadow Σ corresponding to an aggregate A may be labeled by $\lambda(A)$ as described above. A shadow Σ endowed with such a labeling $\lambda=\lambda(A)$

will be called a *labeled shadow* (of $(\boldsymbol{I}, \{C_k\})$). We write $\sigma = (\Sigma, \lambda)$, $\lambda(\sigma) = \lambda$, and $\operatorname{supp} \sigma = \Sigma$.

Notice that the labeled shadows of a pair $(\boldsymbol{I}, \{C_k\})$, as above, form compatible matching families in the sense of Section 2.2, whereas, strictly speaking, labeled aggregates do not fit into that scheme, neither as a system of contours in $\mathbb{Z}^\nu$ nor in $\mathbb{Z}^{\nu-1}$. To include them would need a slight generalization of the setting from Section 2.2.

Labeled shadows are "labeled contours" in the sense of Section 2.2. Considering now generally a configuration $x_{p,q;s} \in G_0 \cap X^{\mathrm{hor}}_{x_p,x_q}$ with fixed $p, q \in Q, p \neq q$, such that x_p, x_q are stable, and $s \in (1, \ldots, n_{p,q})$ (in the role of the triplet y, x_p, x_q from the preceding subsection), we introduce the weight of an aggregate $A = (\boldsymbol{I}_A, \mathcal{C})$ compatible with V and y by

$$\Phi^{(H)}_B(A; p, q) = \prod_{\mathbf{w} \in \mathcal{W}(\boldsymbol{I}_A)} \Phi^{(H)}_V(\mathbf{w}; p, q) \prod_{C \in \mathcal{C}} \Phi^{(H)}_{V,I}(C; p, q). \tag{4.39}$$

Summing over all aggregates A corresponding to a fixed labeled shadow σ (the same support and the labeling $\lambda(\sigma) = \lambda(A)$ corresponding to x_A as above), we get the shadow weight

$$\mathit{\Phi}^{(H)}_B(\sigma; p, q) = \sum_{\substack{A: \operatorname{supp} A = \operatorname{supp} \sigma \\ \lambda(A) = \lambda(\sigma)}} \Phi^{(H)}_B(A; p, q). \tag{4.40}$$

Finally, we use $\varphi^{(H)}_{p,q}$ to denote the vector $(\varphi^{(H)}_{p,q;s})_{s=1}^{n_{p,q}}$. It is defined as in (4.10) with stable x_p, x_q.

Preparing for a direct application of the Pirogov-Sinai theory from Section 2 to the model defined by labeled shadows, notice first that, even though we took with a volume $B \subset \mathbb{Z}^{\nu-1}$ only interfaces from $\mathcal{J}(x_{p,q;s}, V)$ with the walls inside of $V = \pi^{-1}(B)$, the clusters C_k may stick out of V and, correspondingly, the aggregates may not be contained in V and the shadows may not be contained in B. Anticipating this feature, we actually considered, in Section 2, the generalization allowing for volume depending contour weights.

Thus, for any finite $B \subset \mathbb{Z}^{\nu-1}$, we introduce the generalized ensemble $\mathfrak{X}^{\mathrm{aggr}}_B(p, q)$ of all compatible and matching families of labeled aggregates with walls in $V = \pi^{-1}(B)$ and with clusters intersecting V and the ensemble $\mathfrak{X}^{\mathrm{shad}}_B(p, q)$ of all families of shadows that correspond to families from $\mathfrak{X}^{\mathrm{aggr}}_B(p, q)$. The set $\mathfrak{X}^{\mathrm{aggr}}_B(p, q)$ ($\mathfrak{X}^{\mathrm{shad}}_B(p, q)$) is actually the union, over $s \in (1, \ldots, n_{p,q})$, of all sets characterized by the external labels of all external aggregates (shadows) being fixed to equal s — i.e. families of aggregates (shadows) consistent with boundary conditions $x_{p,q;s}$. The set $\mathfrak{X}^{\mathrm{shad}}_B(p, q)$ will play the role of the abstract set $\mathfrak{X}_\Lambda$ from Section 2.2.

Recall that, in accordance with the notation from the first paragraph of Section 2.2, for any $\mathcal{A} \in \mathfrak{X}^{\mathrm{aggr}}_B(p, q)$ ($\mathcal{S} \in \mathfrak{X}^{\mathrm{shad}}_B(p, q)$), we use $B_s(\mathcal{A})$ ($B_s(\mathcal{S})$) to denote the set of sites in B that are outside of the supports of all aggregates $A \in \mathcal{A}$

(shadows $\sigma \in \mathcal{S}$) and are labeled by the label s of the corresponding ceiling. Now, we define

$$Z^{\text{aggr}}\big(\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\big) = \sum_{\substack{\mathcal{A}\in\mathfrak{X}_B^{\text{aggr}}(p,q)\\ \text{ext. label } s_0}} Z^{\text{aggr}}\big(\{\mathcal{A}\},\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\big),$$

where

$$Z^{\text{aggr}}\big(\{\mathcal{A}\},\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\big) = \prod_{A\in\mathcal{A}} \Phi_B^{(H)}(A;p,q)\exp\big[-\sum_{s=1}^{n_{p,q}}\varphi_{p,q;s}^{(H)}\big|B_s(\mathcal{A})\big|\big] \tag{4.41}$$

if $\mathcal{A} \in \mathfrak{X}_B^{\text{aggr}}(p,q)$ with external label s_0; otherwise it is not defined (or, rather, it is put to equal 0). Similarly,

$$Z^{\text{shad}}\big(\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\big) = \sum_{\substack{\mathcal{S}\in\mathfrak{X}_B^{\text{shad}}(p,q)\\ \text{ext. label } s_0}} Z^{\text{shad}}\big(\{\mathcal{S}\},\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\big)$$

with

$$Z^{\text{shad}}\big(\{\mathcal{S}\},\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\big) = \prod_{\sigma\in\mathcal{S}} \mathit{\Phi}_B^{(H)}(\sigma;p,q)\exp\big[-\sum_{s=1}^{n_{p,q}}\varphi_{p,q;s}^{(H)}\big|B_s(\mathcal{S})\big|\big], \tag{4.42}$$

whenever $\mathcal{S} \in \mathfrak{X}_B^{\text{shad}}(p,q)$ with external label s_0; otherwise it is put to equal 0.

Notice that $\mathfrak{X}_B^{\text{aggr}}(p,q)$ (respectively, $\mathfrak{X}_B^{\text{shad}}(p,q)$) contains an infinite number of configurations of aggregates (shadows). Nevertheless, the convergence of (4.41) and (4.42) is an easy consequence of the bounds (a) and (b) from Lemma 4.3.

The probability of a compatible matching family of labeled aggregates $\mathcal{A} \in \mathfrak{X}_B^{\text{aggr}}(p,q)$ with external label s_0 is, correspondingly, defined by

$$\mu^{\text{aggr}}\big(\{\mathcal{A}\},\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\big) = \frac{Z^{\text{aggr}}\big(\{\mathcal{A}\},\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\big)}{Z^{\text{aggr}}\Big(\mathfrak{X}_B^{\text{aggr}}(p,q)|s_0;H\Big)}. \tag{4.43}$$

Similarly, for $\mathcal{S} \in \mathfrak{X}_B^{\text{shad}}(p,q)$ with external label s_0 we introduce

$$\mu^{\text{shad}}\big(\{\mathcal{S}\},\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\big) = \frac{Z^{\text{shad}}\big(\{\mathcal{S}\},\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\big)}{Z^{\text{shad}}\Big(\mathfrak{X}_B^{\text{shad}}(p,q)|s_0;H\Big)}. \tag{4.44}$$

These measures can be used to evaluate the probability of perturbations of the interface of $x_{p,q;s_0}$, once a sufficient decay of functional $\mathit{\Phi}_B^{(H)}(\sigma;p,q)$ in dependence on the size of $|\operatorname{supp}\sigma|$ is guaranteed as we shall see in Section 4.4.

Proposition 4.4 (interfaces in terms of aggregates and shadows) *Let x_p, x_q be two translation invariant ground states of H_0 that are stable for some $H \in K_\varepsilon(H_0)$ for ε small enough, ρ_0 large enough, and let $B \subset \mathbb{Z}^{d-1}$ be finite. Then the following claims hold.*

(i) *For any* $s \in \{1, \dots, n_{p,q}\}$*:*

$$Z^{\text{shad}}\big(\mathfrak{X}_B^{\text{shad}}(p,q)|s;H\big) = Z^{\text{aggr}}\big(\mathfrak{X}_B^{\text{aggr}}(p,q)|s;H\big) = \sum_{\boldsymbol{I}} \tilde{Z}(\boldsymbol{I}, \pi^{-1}(B)|x_{p,q;s};H).$$

Further,

$$\mu^{\text{shad}}\big(\{\mathcal{S}\}, \mathfrak{X}_B^{\text{shad}}(p,q)|s;H\big) = \sum_{\mathcal{A}} \mu^{\text{aggr}}\big(\{\mathcal{A}\}, \mathfrak{X}_B^{\text{aggr}}(p,q)|s;H\big),$$

where the sum runs through $\mathcal{A} \in \mathfrak{X}_B^{\text{aggr}}(p,q)$ *such that* $\operatorname{supp} A = \operatorname{supp} \sigma$ *for all* $A \in \mathcal{A}$ *and the labels corresponding to*
mathcalA and $\mathcal{A}$ *coincide,* $\lambda(\mathcal{A}) = \lambda(\mathcal{S})$*.*

Finally, using $\mathcal{W}(\mathcal{A})$ *to denote the set of all walls corresponding to the collection of aggregates* $\mathcal{A}$*, we have*

$$\mu\big(\{x \in X : \mathcal{W}(\boldsymbol{I}(x)) = \mathcal{W}\}, \pi^{-1}(B)|x_{p,q;s};H\big) = \sum_{\mathcal{A}:\mathcal{W}(\mathcal{A})=\mathcal{W}} \mu^{\text{aggr}}\big(\{\mathcal{A}\}, \mathfrak{X}_B^{\text{aggr}}(p,q)|s;H\big)$$

for any admissible collection of walls $\mathcal{W}$*.*

(ii) *The functional* $\Phi_B^{(H)}(\sigma;p,q)$ *and the vector* $\varphi_{p,q}^{(H)} = (\varphi_{p,q;s}^{(H)})_{s=1}^{n_{p,q}}$ *define a labeled contour model that fulfills all the assumptions of Theorem 2 and Proposition 2.2.1. Namely,*

- $\Phi_B^{(H)}(\sigma;p,q)$ *is nonnegative,*
- $\Phi_B^{(H)}(\sigma;p,q) = \Phi_B^{(H)}(\sigma';p,q) = \Phi_{\mathbb{Z}^{\nu-1}}^{(H)}(\sigma;p,q)$ *whenever* σ *and* σ' *are such that* $\operatorname{supp}\sigma, \operatorname{supp}\sigma' \subset B$ *and* σ' *is a translation of* σ *("the functional* $\Phi_B^{(H)}(\cdot;p,q)$ *is, inside* B*, translation invariant and independent of* B*"). Also, there exists a constant* $\tau_{\text{s}}^{(H)}$ *differing from* $\tau_{\mathbb{I}}^{(H)}$ *by a fixed constant depending only on* ν *and* $|S|$ *and a constant* M_{s}*, such that the following bounds hold:*

(1) $\Phi_B^{(H)}(\sigma;p,q) \le \exp\{-\tau_{\text{s}}^{(H)}|\operatorname{supp}\sigma|\}$ *for any labeled shadow* σ *consistent with boundary conditions* $x_{p,q,s}$ *in* B*;*

(2) $|\partial_{\bar{H}}^{+}\Phi_B^{(H)}(\sigma;p,q)\| \le \|\bar{H}\| \exp\{-\tau_{\text{s}}^{(H)}|\operatorname{supp}\sigma|\}$*;*

(3) $|\partial_{\bar{H}}^{+}\varphi_{p,q;s}^{(H)}| \le M_{\text{s}}\|\bar{H}\|$*.*

Proof. (i) It follows from (4.39) – (4.44) with the use of (4.29) and (4.30).

(ii) The functional $\Phi_B^{(H)}(A;p,q)$ is nonnegative by its definition (4.39) and by (4.14).

The translation invariance inside B follows from the definitions (4.39), (4.11), (4.12), from the translation invariance of the functionals involved and the invariance inside B of $\chi_m^V(C)$, and by inspecting cardinalities of the sets involved in (4.12) for sets C fully contained in V.

The corresponding claims for $\Phi_B^{(H)}(\sigma;p,q)$ follow from (4.40).

(1) We use first Lemma 4.3(a) and (b) to get, by (4.39) and (4.40), that

$$0 \le \Phi_B^{(H)}(\sigma;p,q) \le \sum_{(\boldsymbol{I},\mathcal{C})} \exp\{-\tau_{\mathbb{I}}^{(H)} \sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} |W| - \tau_{\mathbb{I}}^{(H)} \sum_{C\in\mathcal{C}} |C|\},$$

where the sum is over aggregates that correspond each to a pair of an interface $\boldsymbol{I} \in \mathcal{I}(x_{p,q;s}, V)$ and of a finite family $\mathcal{C}$ such that $C \cap V_R \neq \emptyset$ and $C \cap I \neq \emptyset$, for all $C \in \mathcal{C}$, and $\pi(\bigcup_{C\in\mathcal{C}} C \cup \bigcup_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} W) = \Sigma = \operatorname{supp}\sigma$ as above (i.e. over $(\boldsymbol{I},\mathcal{C})$ with the only aggregate $A = (\boldsymbol{I}_A, \mathcal{C})$).

Since, obviously, $\|A\| = \|(\mathcal{W}(\boldsymbol{I}),\mathcal{C})\| \equiv \sum_{C\in\mathcal{C}} |C| + \sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} |W| \ge |\operatorname{supp}\sigma|$, we have

$$\Phi_B^{(H)}(\sigma;p,q) \le e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta)|\operatorname{supp}\sigma|} \sum_{(\boldsymbol{I},\mathcal{C})} \exp\{-\zeta(\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} |W| + \sum_{C\in\mathcal{C}} |C|)\},$$

for any $\zeta > 0$, $\zeta < \tau_{\mathbb{I}}^{(H)}$.

We shall show that, for some sufficiently large ζ, in dependence on ν and $|S|$ only, the last sum is at most $e^{c|\operatorname{supp}\sigma|}$ for suitable c which also depends on $|S|$ and ν only. We begin with an observation.

Namely, introducing the notion of an *extent* of an aggregate A as $\operatorname{ext} A = I_A \cup \bigcup_{C\in\mathcal{C}} C$, we notice that it is a connected set of cardinality at most $\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I}_A)} |W| + (1+T) \sum_{C\in\mathcal{C}} |C| \le (1+T)\|A\|$. In the same time we may suppose, referring to the identification of aggregates under vertical translations, that each $\operatorname{ext} A$ contains an element i of $\operatorname{supp}\sigma$.

As a consequence, there are at most $e^{c_s\|A\|}$ aggregates A with $i \in \operatorname{ext} A$ and given $\|A\|$. Namely, every such A can be identified with a connected path over $\operatorname{ext} A$ of the length at most $2\nu(T+1)\|A\|$, each point of which is moreover equipped with a label saying whether this point of the path belongs to a wall, to a cluster, or to the remaining part

$$\operatorname{ext} A \setminus \bigcup_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I}_A)} W \cup \bigcup_{C\in\mathcal{C}} C$$

of the interface $\boldsymbol{I}_A$. The label will also say what spin at this point is attained if it is a point of a wall. The number of needed labels is thus depending on ν and $|S|$

only. As a result, we get

$$\Phi_B^{(H)}(\sigma;p,q) \le e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta)|\operatorname{supp}\sigma|} \sum_{A=(\boldsymbol{I},\mathcal{C})} \exp\{-\zeta(\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})}|W| + \sum_{C\in\mathcal{C}}|C|)\} \le$$
$$\le e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta)|\operatorname{supp}\sigma|} \sum_{i\in\operatorname{supp}\sigma}\sum_{A:i\in\operatorname{ext}A} e^{-\zeta\|A\|} \le$$
$$\le e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta)|\operatorname{supp}\sigma|} \sum_{i\in\operatorname{supp}\sigma}\sum_{k\ge|\operatorname{supp}\sigma|} e^{-(\zeta-c_{\mathrm{s}})k} \le e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta-1+\log(1-e^{-c_{\mathrm{s}}}))|\operatorname{supp}\sigma|}$$
$$\le e^{-\tau_{\mathrm{s}}^{(H)}|\operatorname{supp}\sigma|},$$

where $\zeta \ge 2c_{\mathrm{s}}$.

(2) By differentiating (4.40) and using (4.39) as well as the estimates (a), (b), (a′), (b′) of Lemma 4.3, we obtain

$$|\partial_{\bar H}^{+}\Phi_B^{(H)}(\sigma;p,q)| \le \sum_{A=(\boldsymbol{I},\mathcal{C})}\sum_{\mathbf{w}\in\mathcal{W}(\boldsymbol{I})} \|\bar H\| \exp\{-\tau_{\mathbb{I}}^{(H)} \sum_{\bar{\mathbf{w}}\in\mathcal{W}(\boldsymbol{I})}|\bar{\mathbf{w}}| - \tau_{\mathbb{I}}^{(H)}\sum_{\bar C\in\mathcal{C}}|\bar C|\}+$$
$$+ \sum_{A=(\boldsymbol{I},\mathcal{C})}\sum_{C\in\mathcal{C}} \|\bar H\| \exp\{-\tau_{\mathbb{I}}^{(H)} \sum_{\bar{\mathbf{w}}\in\mathcal{W}(\boldsymbol{I})}|\bar{\mathbf{w}}| - \tau_{\mathbb{I}}^{(H)}\sum_{\bar C\in\mathcal{C}}|\bar C|\} \le$$
$$\le \|\bar H\| e^{-(\tau_{\mathbb{I}}^{(H)}-\zeta-1)|\operatorname{supp}\sigma|} \sum_{A=(\boldsymbol{I},\mathcal{C})} \exp\{(-\zeta+1)\|A\|\}.$$

Here the sum $\sum_{A=(\boldsymbol{I},\mathcal{C})}$ above is taken over all aggregates A with $\operatorname{supp} A = \operatorname{supp}\sigma$ and $\lambda(A) = \lambda(\sigma)$. Now, we use the same argument as in (1) with $\zeta - 1$ instead of ζ, so that for $\zeta - 1 \ge 2c_{\mathrm{s}}$ we get

$$|\partial_{\bar H}^{+}\Phi_B^{(H)}(\sigma;p,q)| \le e^{-\tau_{\mathrm{s}}^{(H)}|\operatorname{supp}\sigma|}.$$

(3) This is the estimate of Lemma 4.3(c) above. □

4.4 Gibbs states with interfaces. Proof of Basic Lemma

To conclude the proof of Basic Lemma, it remains to define, for any $y \in G_0^{\mathrm{hor}}$, the functions $h_y : K_\varepsilon(H_0) \to \mathbb{R}$ such that the statements (i) (b) and (ii) of Basic Lemma hold.

As anticipated in Section 4.1, we fix a particular $H \in K_\varepsilon(H_0)$, two different configurations from G_0^{per}, say x_p, x_q with $p,q \in \{1,\dots,r\}$, and a configuration $y \in G_0 \cap X^{\mathrm{hor}}_{x_p,x_q}$ to be identified with a triplet (p,q,s), $s \in \{1,\dots,n_{p,q}\}$. The corresponding functionals $\Phi_B^{(H)}(A;p,q)$, $\mathit{\Phi}_B^{(H)}(\sigma;p,q)$, and vectors $\varphi_{p,q}^{(H)}$ describing the aggregate and shadow models in the preceding Section 4.4 were defined (or considered) only if x_p and x_q were stable and distinct. We may however notice that $\varphi_{p,q;s}^{(H)}$ are by (4.10) actually well-defined for general pairs of (distinct) x_p and x_q

from $G_0^{\rm per}$. To define $\Phi_B^{(H)}(A;p,q)$ and $\mathit{\Phi}_B^{(H)}(\sigma;p,q)$, we then use formulas (4.39) and (4.40) referring to (4.11) and (4.12). While $\Phi_V^{(H)}(\mathbf{w};p,q)$ again is well-defined even if x_p, x_q are arbitrary elements of $G_0^{\rm per}$, we need to define $\Phi_{V,I}^{(H)}(C;p,q)$ so that the bounds (1), (2), and (3) of Proposition 4.4 (ii) hold true. This can be easily achieved if we simply replace Ψ_p and Ψ_q in (4.12), and further on, by $\overline{\Psi}_p$ and $\overline{\Psi}_q$, respectively, where $\overline{\Psi}_p$ and $\overline{\Psi}_q$ are the auxiliary functionals from Proposition 2.2.1 (b$'$), so that we may apply all the cluster expansion technique to them to get (ii) of Proposition 4.4 for all p,q's.

This is the starting point for a second use of the Pirogov-Sinai strategy. This time with different $y \in G_0 \cap X^{\rm hor}_{x_p,x_q}$ (and fixed p,q) playing the role of reference states and thus the label in Theorem 2 and Proposition 2.2.1 taking values s with $s \in \{1,\ldots,n_{p,q}\}$. The function $h_y(H)$ from Basic Lemma is, for y corresponding to the triplet (p,q,s), defined to be equal to the function $h_s(\Phi^{(H)}(\cdot;p,q),\varphi^{(H)}_{p,q})$ whose existence is assured by Theorem 2. The claim (ii) of Basic Lemma is then an immediate consequence of Theorem 2 (approximating $h_y(H)$ (and its derivative) in terms of $\varphi^{(H)}_{p,q;s}$) and the equation (4.10) relating $\varphi^{(H)}_{p,q;s}$ to $e_y(H)$ (cf. (3.7), (3.9), and (4.34)). (Of course, if either x_p or x_q is not stable, we lose the equality (4.13) and as a result we cannot claim anything like (i) of Proposition 4.4.)

It remains to prove the statement (i) (b) of Basic Lemma. Therefore we assume that both x_p and x_q are stable, and that $y = x_{p,q;s}$ is stable for the shadow model determined by $\mathit{\Phi}_B^{(H)}(\sigma;p,q)$ and $\varphi^{(H)}_{p,q}$.

We are going to derive the properties of the Gibbs measure $\mu(\cdot|y,H)$ obtained from measures $\mu(\cdot, V_n = \pi^{-1}(B_n)|y,H)$ as a weak limit with B_n growing up to $\mathbb{Z}^{\nu-1}$, which were investigated in Step 7 above. Such measures are Gibbs states as they are weak limits of $\mu(\cdot,\Lambda_n|y,H)$ for volumes $\Lambda_n = B_n \times [-k_n,k_n]$, with k_n growing to infinity sufficiently quickly (cf. [D 72, HKZ (3.2)]). To show that almost every configuration is a perturbation of y, it suffices to verify that for every ε and any $\Lambda \subset \mathbb{Z}^\nu$, there exist constants $a,b,d>0$ so that, defining

$$X(\Lambda,a,b) = \Big\{x \in X; \big[\mathbf{w} \in \mathcal{W}(\boldsymbol{I}(x)), V(W)\cap\Lambda \neq \emptyset \Rightarrow \mathrm{diamW} \le \mathrm{a}\big] \text{ and } \big[\gamma \text{ a contour of } x, V(\Gamma)\cap\Lambda\neq\emptyset \Rightarrow \mathrm{diam}\Gamma \le \mathrm{b}\big]\Big\},$$

with $V(W)$ denoting the union of W and all finite components of $\mathbb{Z}^\nu \setminus W$, we have

$$\mu\Big(X(\Lambda,a,b),\pi^{-1}(B)|y;H\Big) > 1-\varepsilon \tag{4.45}$$

whenever $B \subset \mathbb{Z}^{\nu-1}$ is such that $\mathrm{dist}(\pi(\Lambda),B^c) \ge d$.

To this end we first show that it is unprobable that shadows intersecting a given finite volume in $\mathbb{Z}^{\nu-1}$ are large and similarly for corresponding aggregates in $\mathbb{Z}^\nu$. Finally we estimate the probability of existence of large contours intersecting a fixed finite volume in $\mathbb{Z}^\nu$.

Let thus $\mathcal{E}_M^{(c)}$ be the set of all shadow configurations containing a large shadow intersecting or surrounding a fixed finite set $M \subset \mathbb{Z}^{\nu-1}$. Namely, for any such M and $c > 0$ we introduce

$$\mathcal{E}_M^{(c)} = \big\{\mathcal{S} \in \mathfrak{X}_B^{\text{shad}}(p,q);\ \text{there exists } \sigma \in \mathcal{S} \text{ such that}$$
$$V(\Sigma) \cap M \neq \emptyset \text{ and } \operatorname{diam}\Sigma > \mathrm{c}\big\},$$

with $\Sigma = \operatorname{supp}\sigma$ and $V(\Sigma)$ (for any finite set $\Sigma \subset \mathbb{Z}^{\nu-1}$) denoting the union of Σ with all finite components of $\mathbb{Z}^{\nu-1} \setminus \Sigma$. Using the main results of Section 2 (Theorem 2, Proposition 2.2.1, and Corollary 2.2.2) to the shadow model, we may show, in a standard way, that for a given finite $M \subset \mathbb{Z}^{\nu-1}$ and a positive ε, constants $d' = d'(M,\varepsilon)$ and $c = c(M,\varepsilon)$ may be found such that

$$\mu^{\text{shad}}\big(\mathcal{E}_M^{(c)}, \mathfrak{X}_B^{\text{shad}}(p,q)|s;H\big) \leq \frac{1}{2}\varepsilon \tag{4.46}$$

for all $B \subset \mathbb{Z}^{\nu-1}$ such that $\operatorname{dist}(M,B^c) \geq d'$.

For a fixed configuration $\mathcal{S}$ of shadows of diameter at most c and with $V(\Sigma) \cap M \neq \emptyset$ for every $\sigma \in \mathcal{S}$, consider the event consisting of those configurations of walls and clusters (configurations of aggregates) whose set of shadows intersecting M is fixed and equals $\mathcal{S}$. The conditional probability of each of these events, given such a fixed $\mathcal{S}$, is independent of B for B sufficiently large. Since there are countably many such events, and due to (4.46), we may find $a = a(M,\varepsilon)$ such that, supposing $\operatorname{dist}(M,B^c) > d'$, we get

$$\mu^{\text{aggr}}(\mathfrak{A}(M,a), \mathfrak{X}_B^{\text{aggr}}(p,q)|s;H) > 1 - \frac{1}{2}\varepsilon,$$

with

$$\mathfrak{A}(M,a) = \{\mathcal{A} \in \mathfrak{X}_B^{\text{aggr}}(p,q); [\mathbf{w} \in \mathcal{W}(\mathcal{A}) \text{ and}$$
$$V(W) \cap \pi^{-1}(M) \neq \emptyset] \Rightarrow \operatorname{diamW} \leq \mathrm{a}\}.$$

Comparing μ^{aggr} and μ (Proposition 4.4 (i)), we get the bound

$$\mu\Big(X(M,a), \pi^{-1}(B)|y;H\Big) > 1 - \frac{1}{2}\varepsilon \tag{4.47}$$

for

$$X(M,a) = \Big\{x \in X; \big[\mathbf{w} \in \mathcal{W}(\boldsymbol{I}(x)) \text{ and } V(W) \cap \pi^{-1}(M) \neq \emptyset\big] \Rightarrow \operatorname{diamW} \leq \mathrm{a}\Big\}.$$

Given a finite $\Lambda \subset \mathbb{Z}^{\nu}$ with projection $\pi(\Lambda) = M$, x_Λ is fully determined if we know the contours γ with $V(\Gamma) \cap \Lambda \neq \emptyset$ and walls $\mathbf{w}$ with $V(W) \cap \Lambda \neq \emptyset$. Let $\boldsymbol{I}$ be a given interface compatible with the boundary condition y on Λ^c. If γ is a contour such that $V(\Gamma) \cap \Lambda \neq \emptyset$ and, in the same time, $V(\Gamma) \subset \operatorname{Int} W = V(W) \setminus W$

for some wall, then necessarily $V(W)\cap\Lambda \neq \emptyset$. Thus it suffices to consider contours that are either above or below I, $\Gamma \subset (\bigcup_j \mathrm{Int}_j\, I \cup I)^c$. Introducing

$$X(\boldsymbol{I},\Lambda,b) = \{x \in X; \boldsymbol{I}(x) = \boldsymbol{I} \text{ and}$$
$$[\gamma \text{ is a labeled contour of } x, \Gamma \subset (\bigcup_j \mathrm{Int}_j\, I \cup I)^c, V(\Gamma)\cap\Lambda \neq \emptyset] \Rightarrow \mathrm{diam}\Gamma \leq \mathrm{b}\},$$

we get

$$\mu\Big(X(\boldsymbol{I},\Lambda,b),\pi^{-1}(B)|y,I;H\Big) > 1 - \frac{1}{2}\varepsilon, \tag{4.48}$$

independently of I for $B \subset \mathbb{Z}^{\nu-1}$ with $\mathrm{dist}(M,B^c) > d$ for some $b = b(M,\varepsilon)$ and $d = d(M,\varepsilon) > d'$. Here we consider the conditional probability under the condition that the interface with support I is present. The bounds (4.47) and (4.48) together yield (4.45).

Remark. Notice that having (4.45) for Λ's that are singletons, we get that almost all configurations with respect to the limit Gibbs state are perturbations of y.

In fact, we may prove an explicit expression for the limit probability $\mu(\cdot|y;H)$ by means of the limit conditional probability "$\mu(\cdot,\pi^{-1}|I)$" and the limit probability on families of walls "$\lim_{B_n\nearrow\mathbb{Z}^{\nu-1}} \mu^I(\cdot,\pi^{-1}(B_n)|y;H)$" similarly as (2.1) was proved in [HKZ, Section 6.2.2]. However, it is a little bit more complicated in our more general situation. The probabilities μ^I can be expressed using their projections to families of external walls. Also we don't have the exponential estimates as (a) and (b) of (i) and (ii) in [HKZ, Theorem 2]. However, one may check that the above estimate (4.45) is sufficient to carry on the proof of the respective identity.

References

[BCF 1] C. Borgs, J. Chayes and J. Fröhlich, Dobrushin states for classical spin systems with complex interactions, *Jour. Stat. Phys.* **89**, 895 (1997).

[BCF 2] C. Borgs, J. Chayes and J. Fröhlich, Dobrushin states in quantum lattice systems *Jour. Stat. Phys.* **89**, 591 (1997).

[BI] C. Borgs and J. Imbrie, A Unified Approach to Phase Diagrams in Field Theory and Statistical Mechanics, *Commun. Math. Phys.* **123**, 305–328 (1989).

[BK] C. Borgs and R. Kotecký, A Rigorous Theory of Finite-Size Scaling at First-Order Phase Transitions, *Journ. Stat. Phys.* **61**, 79–119 (1990).

[BL] J. Bricmont and J. L. Lebowitz, Wetting in Potts and Blume-Capel Models, *Journ. Stat. Phys.* **46**, 1015–1029 (1987).

[BS] J. Bricmont and J. Slawny, Phase transitions in systems with a finite number of dominant ground states, *Journ. Stat. Phys.* **54**, 89–161 (1989).

[Br] D. C. Brydges, A short course on cluster expansions, Critical phenomena, Random Systems, Gauge Theories, Les Houches 1984, North-Holland, Amsterdam, 129–183 (1986).

[D 96] R. L. Dobrushin, Estimates of Semi-invariants for the Ising Model at Low Temperatures, *Amer. Math. Soc. Transl.* (2) **177**, 59–81 (1996).

[D 72] R. L. Dobrushin, Gibbs State Describing Coexistence of Phases for a Three-Dimensional Ising Model, *Teor. Ver. Pril.* **17**, 619–639 (1972), *English transl. in Theor. Probability Appl.* **17**, 582–600 (1972).

[DKS] R. L. Dobrushin, R. Kotecký and S. Shlosman, The Wulff construction: a global shape from local interactions AMS, Translations of Mathematical Monographs **104**, Providence, Rhode Island (1992).

[EFS] A.C.D. van Enter, R. Fernández and A. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, *J. Stat. Phys.* **72**, 879–1167 (1993).

[HKZ] P. Holický, R. Kotecký and M. Zahradník, Rigid interfaces for lattice models at low temperatures, *Journ. Stat. Phys.* **50**, 755–812 (1988).

[HZ] P. Holický and M. Zahradník, Stratified low-temperature phases of stratified spin models. A general Pirogov-Sinai approach, preprint.

[KP] R. Kotecký and D. Preiss, Cluster Expansion for Abstract Polymer Models, *Commun. Math. Phys.* **103**, 491–498 (1986).

[PS] S. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems, *Theor. Mat. Phys.*, **25**, 1185–1192 (1975), **26**, 39–49 (1976).

[Se] E. Seiler, Gauge theories as a problem of constructive quantum field theory and statistical mechanics *Lecture notes in physics*, Vol. 159, Springer Verlag Berlin, Heidelberg, New York (1982).

[S] Ya. G. Sinai, Theory of phase transitions: Rigorous results, Pergamon Press, Oxford-New York-etc (1982).

[Z] M. Zahradník, An Alternate Version of Pirogov-Sinai Theory, *Commun. Math. Phys.* **93**, 559–581 (1984).

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