CLUSTER EXPANSION

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The method of cluster expansions in statistical physics provides a systematic way of computing power series for thermodynamic potentials (logarithms of partition functions) as well as correlations. It originated from the works of Mayer and others devoted to expansions for dilute gas.

1. Mayer expansion

Consider a system of interacting particles with Hamiltonian

$$H_N(\vec{p}_1,\dots,\vec{p}_N,\vec{r}_1,\dots,\vec{r}_N)=\sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{2m}+\sum_{i,j=1}^{N}\Phi(\vec{r}_i-\vec{r}_j), \tag{1}$$

where Φ is a *stable* and *regular* pair potential. Namely, we assume that there exists $B\geq 0$ such that

$$\sum_{i,j=1}^{N}\Phi(\vec{r}_i-\vec{r}_j)\geq -BN \tag{2}$$

for all $N=2,3,\dots$ and all $(\vec{r}_1,\dots,\vec{r}_N)\in\mathbb{R}^{3N}$, and that

$$C(\beta)=\int\left|e^{-\beta\Phi(\vec{r})}-1\right|d^3\vec{r}<\infty \tag{3}$$

for some $\beta>0$ (and hence all $\beta>0$). Basic thermodynamic quantities are given in terms of the *grand-canonical partition function*

$$\begin{aligned}Z(\beta,\lambda,V)&=\sum_{N=0}^{\infty}\frac{z^N}{N!}\int_{\mathbb{R}^{3N}\times V^N}e^{-\beta H_N}\frac{\prod d^3\vec{p}_i\prod d^3\vec{r}_i}{h^{3N}}=\\&=\sum_{N=0}^{\infty}\frac{\lambda^N}{N!}\int_{V^N}e^{-\beta\sum_{i,j}\Phi(\vec{r}_i-\vec{r}_j)}\prod d^3\vec{r}_i.\end{aligned} \tag{4}$$

In the second expression we absorbed the factor resulting from the integration over impulses into (configurational) activity $\lambda=\left(\frac{2\pi m}{\beta h^2}\right)^{\frac{3}{2}}z$. In particular, the *pressure* p and the *density* ρ are defined by the thermodynamic limits (with $V\to\infty$ in the sense of Van Hove)

$$p(\beta,\lambda)=\frac{1}{\beta}\lim_{V\to\infty}\frac{1}{|V|}\log Z(\beta,\lambda,V) \tag{5}$$

and

$$\rho(\beta,\lambda)=\lim_{V\to\infty}\frac{1}{|V|}\lambda\frac{\partial}{\partial\lambda}\log Z(\beta,\lambda,V). \tag{6}$$

Mayer series are the expansions of p and ρ in powers of λ,

$$\beta p(\beta,\lambda) = \sum_{n=1}^{\infty} b_n \lambda^n \tag{7}$$

and

$$\rho(\beta,\lambda) = \sum_{n=1}^{\infty} n b_n \lambda^n. \tag{8}$$

Mayer's idea for a systematic computation of coefficients b_n was based on a reformulation of partition function $Z(\beta,\lambda,V)$ in terms of *cluster integrals*. Introducing the function

$$f(\vec{r}) = e^{-\beta\Phi(\vec{r})} - 1 \tag{9}$$

and using $\mathcal{G}[N]$ to denote the set of all graphs on N vertices $\{1,\dots,N\}$, we get

$$\begin{aligned} Z(\beta,\lambda,V) &= \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int_{V^N} \prod_{i,j=1}^{N} \big(1 + f(\vec{r}_i - \vec{r}_j)\big) \prod d^3\vec{r}_i = \\ &= \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{g\in\mathcal{G}[N]} w(g), \end{aligned} \tag{10}$$

where

$$w(g) = \int_{V^N} \prod_{\{i,j\}\in g} f(\vec{r}_i - \vec{r}_j) \prod d^3\vec{r}_i. \tag{11}$$

Observing that the weight w is multiplicative in connected components (clusters) $g_1,\dots,g_k$ of the graph g,

$$w(g) = \prod_{\ell=1}^{k} w(g_\ell), \tag{12}$$

we can rewrite

$$Z(\beta,\lambda,V) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{\{g_\ell\}} \prod_{\ell} w(g_\ell) \tag{13}$$

with the sum running over all disjoint collections $\{g_\ell\}$ of connected graphs with vertices in $\{1,\dots,N\}$. A straightforward exponential expansion can be used to show that, at least in the sense of formal power series,

$$\log Z(\beta,\lambda,V) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \sum_{g\in\mathcal{C}[n]} w(g), \tag{14}$$

where $\mathcal{C}[n]$ is the set of all connected graphs on n vertices. Using $b_n^{(V)}$ to denote the coefficients

$$b_n^{(V)} = \frac{1}{|V|}\frac{1}{n!} \sum_{g\in\mathcal{C}[n]} w(g) \tag{15}$$

and observing that the limits $\lim_{V\to\infty} \frac{1}{|V|} w(g)$ of cluster integrals exist, we get $b_n = \lim_{V\to\infty} b_n^{(V)}$. The convergence of Mayer series can be be controlled directly by combinatorial estimates on the coefficients $b_n^{(V)}$. As a result, the diameter of convergence of the series (7) and (8) can be proven to be at least $\big(C(\beta)e^{2\beta B+1}\big)^{-1}$.

A less direct proof is based on an employment of linear integral Kirkwood-Salsburg equations in a suitable Banach space of correlation functions.

Similar combinatorial methods are available also for evaluation of coefficients of the *virial expansion* of pressure in powers of gas density,

$$\beta p(\beta,\rho) = \sum_{n=1}^{\infty} \beta_n \rho^n \tag{16}$$

obtained by inverting (8) (notice that $b_1 = 1$) and inserting it into (7). One is getting $\beta_n = \lim_{V\to\infty} \beta_n^{(V)}$ with

$$\beta_n^{(V)} = \frac{1}{|V|}\frac{1}{n!} \sum_{g\in\mathcal{B}[n]} w(g), \tag{17}$$

where $\mathcal{B}[n] \subset \mathcal{C}[n]$ is the set of all 2-connected graphs on $\{1,\dots,n\}$; namely, those graphs that cannot be split into disjoint subgraphs by erasing one vertex (and all adjacent edges). The diameter of convergence of the virial expansion turns out to be no less than $\big(C(\beta)e(e^{2\beta B}+1)\big)^{-1}$.

2. Abstract polymer models

An application of the ideas of Mayer expansions to lattice models is based on a reformulation of the partition function in terms of a *polymer model*, a formulation akin to (13) above. Namely, the partition function is rewritten as a sum over collections of pairwise compatible geometric objects—polymers. Most often the compatibility means simply their disjointness.

While the reformulation of "physical partition function" in terms of a polymer model (including the definition of compatibility) depends on particularities of a given lattice model and on the considered region of parameters—high-temperature, low-temperature, large external fields, etc—the essence and results of cluster expansion may be conveniently formulated in terms of an *abstract polymer model*.

Let $G = (V, E)$ be any (possibly infinite) countable graph and suppose that a map $w : V \to \mathbb{C}$ is given. Vertices $v \in V$ are called *abstract polymers*, with two abstract polymers connected by an edge in the graph G called *incompatible*. We shall refer to $w(v)$ as to the *weight* of the abstract polymer v. For any finite $W \subset V$, we consider the induced subgraph $G[W]$ of G spanned by W and define

$$Z_W(w) = \sum_{I\subset W} \prod_{v\in I} w(v). \tag{18}$$

Here the sum runs over all collections I of compatible abstract polymers—or, in other words, the sum is over all *independent sets* I of vertices in W (no two vertices in I are connected by an edge).

The partition function $Z_W(w)$ is an entire function in $w = \{w(v)\}_{v\in W} \in \mathbb{C}^{|W|}$ and $Z_W(0) = 1$. Hence, it is nonvanishing in some neighbourhood of the origin $w = 0$ and its logarithm is, on this neighbourhood, an analytic function yielding a convergent Taylor series

$$\log Z_W(w) = \sum_{X\in\mathcal{X}(W)} a_W(X) w^X. \tag{19}$$

Here, $\mathcal{X}(W)$ is the set of all multi-indices $X : W \to \{0, 1, \dots\}$ and $w^X = \prod_v w(v)^{X(v)}$. Inspecting the formula for $a_W(X)$ in terms of corresponding derivatives of $\log Z_W(w)$, it is easy to show that the Taylor coefficients $a_W(X)$ actually do not depend on W: $a_W(X) = a_{\operatorname{supp} X}(X)$, where $\operatorname{supp} X = \{v \in V : X(v) \neq 0\}$. As a result, one is getting the existence of coefficients $a(X)$ such that

$$\log Z_W(w) = \sum_{X \in \mathcal{X}(W)} a(X) w^X \tag{20}$$

for every finite $W \subset V$.

The coefficients $a(X)$ can be obtained explicitly. One can pass from (18) to (20) in a similar way as passing from (10) to (13). The starting point is to replace the restriction to compatible collections of abstract polymers in the sum (18) by the factor $\prod_{v,v' \in W} \big(1 + F(v, v')\big)$ with

$$F(v, v') = \begin{cases} 0 \text{ if } v \text{ and } v' \text{ are compatible} \\ -1 \text{ otherwise } \ (v \text{ and } v' \text{ connected by an edge from } G), \end{cases} \tag{21}$$

and to expand the product afterwards. The resulting formula is

$$a(X) = (X!)^{-1} \sum_{H \subset G(X)} (-1)^{|E(H)|}. \tag{22}$$

Here, $G(X)$ is the graph with $|X| = \sum |X(v)|$ vertices induced from $G[\operatorname{supp} X]$ by replacing each its vertex v by the complete graph on $|X(v)|$ vertices and $X!$ is the multi-factorial $X! = \prod_{v \in \operatorname{supp} X} X(v)!$. The sum is over all connected subgraphs $H \subset G(X)$ spanned by the set of vertices of $G(X)$ and $|E(H)|$ is the number of edges of the graph H.

A useful property of the coefficients $a(X)$ is their *alternating sign*,

$$(-1)^{|X|+1} a(X) \geq 0. \tag{23}$$

More important than an explicit form of the coefficients $a(X)$ are the convergence criteria for the series (20). One way to proceed is to find direct combinatorial bounds on the coefficients as expressed by (22). While doing so, one has to take into account the cancellations arising in view of the presence of terms of opposite signs in (22). Indeed, disregarding them would lead to a failure since, as it is easy to verify, the number of connected graphs on $|X|$ vertices is bounded from below by $2^{\frac{(|X|-1)(|X|-2)}{2}}$. An alternative approach is to prove the convergence of (20) on polydiscs $\mathcal{D}_{W,\boldsymbol{R}} = \{w : |w(v)| \leq R(v) \text{ for } v \in W\}$ by induction in $|W|$, once a proper condition on the set of radii $\boldsymbol{R} = \{R(v); v \in V\}$ is formulated. The most natural for the inductive proof (leading in the same time to the strongest claim) turns out to be the Dobrushin condition:

There exists a function $r : V \to [0, 1)$ *such that, for each* $v \in V$,

$$R(v) \leq r(v) \prod_{v' \in \mathcal{N}(v)} \big(1 - r(v')\big). \tag{24}$$

Here $\mathcal{N}(v)$ is the set of vertices $v' \in V$ adjacent in graph G to the vertex v.

Using $\mathcal{X}$ to denote the set of all multi-indices $X : V \to \{0, 1, \dots\}$ with finite $|X| = \sum |X(v)|$ and saying that $X \in \mathcal{X}$ is a cluster if the graph $G(\operatorname{supp} X)$ is

connected, we can summarise the cluster expansion claim for an abstract polymer model in the following way:

Theorem (Cluster expansion). *There exists a function $a : \mathcal{X} \to \mathbb{R}$ that is nonvanishing only on* clusters*, so that for any sequence of diameters $\boldsymbol{R}$ satisfying the condition* (24) *with a sequence $\{r(v)\}$, the following holds true:*

(i) *For every finite $W \subset V$, and any contour weight $w \in \mathcal{D}_{W,\boldsymbol{R}}$, one has $Z_W(w) \neq 0$ and*

$$\log Z_W(w) = \sum_{X \in \mathcal{X}(W)} a(X) w^X;$$

(ii) $\sum_{X \in \mathcal{X} : \operatorname{supp} X \ni v} |a(X)||w|^X \leq -\log\bigl(1 - r(v)\bigr)$.

Notice that, we have got not only an absolute convergence of the Taylor series of $\log Z_W$ in the closed polydisc $\mathcal{D}_{W,\boldsymbol{R}}$, but also the bound (ii) (uniform in W) on the sum over all terms containing a fixed vertex v. Such a bound turns out to be very useful in applications of cluster expansions. It yields, eventually, bounds on various error terms, avoiding a need of an explicit evaluation of the number of clusters of "given size".

The restriction to compatible collections of polymers can be actually relaxed. Namely, replacing (25) by

$$Z_W(w) = \sum_{W' \subset W} \prod_{v \in W'} w(v) \prod_{v,v' \in W'} U(v,v'), \tag{25}$$

with $U(v,v') \in [0,1]$ (soft repulsive interaction), and the condition (24) by

$$R(v) \leq r(v) \prod_{v' \neq v} \frac{1 - r(v')}{1 - U(v,v') r(v')}, \tag{26}$$

one can prove that the partition function $Z_W(w)$ does not vanish on the polydisc $\mathcal{D}_{W,\boldsymbol{R}}$ implying thus that the power series of $\log Z_W(w)$ converges absolutely on $\mathcal{D}_{W,\boldsymbol{R}}$.

Polymers that arise in typical applications are geometric objects endowed with a "support" in the considered lattice, say $\mathbb{Z}^d$, $d \geq 1$, and their weights satisfy the condition of translation invariance. Cluster expansions then yield an explicit power series for the *pressure* (resp. free energy) in the thermodynamic limit as well as its finite volume approximation.

To formulate it for an abstract polymer model, we assume that for each $x \in \mathbb{Z}^d$, an isomorphism $\tau_x : G \to G$ is given and that with each abstract polymer $v \in V$ a finite set $\Lambda(v) \subset \mathbb{Z}^d$ is associated so that $\Lambda((\tau_x(v)) = \Lambda(v) + x$ for every $v \in V$ and every $x \in \mathbb{Z}^d$. For any finite $W \subset V$ and any multi-index X, let $\Lambda(W) = \cup_{v \in W} \Lambda(v)$ and $\Lambda(X) = \Lambda(\operatorname{supp}(X))$. On the other hand, for any finite $\Lambda \subset \mathbb{Z}^d$, let $W(\Lambda) = \{v \in V \colon \Lambda(v) \subset \Lambda\}$. Assuming also that the weight $w : V \to \mathbb{C}$ is translation invariant—i.e. $w(v) = w(\tau_x(v))$ for every $v \in V$ and every $x \in \mathbb{Z}^d$—we get an explicit expression for the "pressure" of abstract polymer model in the thermodynamic limit

$$p = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z_{W(\Lambda)}(w) = \sum_{X : \Lambda(X) \ni 0} \frac{a(X) w^X}{|\Lambda(X)|}. \tag{27}$$

In addition, the finite volume approximation can be explicitly evaluated, yielding

$$\log Z_{W(\Lambda)}(w) = p|\Lambda| + \sum_{X:\Lambda(X)\cap\Lambda^c\neq\emptyset} a(X) w^X \frac{|\Lambda(X)\cap\Lambda|}{|\Lambda(X)|}. \tag{28}$$

Using the claim (ii), the second term can be bounded by const $|\partial\Lambda|$.

3. Cluster expansions for lattice models

There is a variety of applications of cluster expansions to lattice models. As noticed above, the first step is always to rewrite the model in terms of a polymer representation.

3.1. **High-temperature expansions.** Let us illustrate this point in the simplest case of the Ising model. Its partition function in volume $\Lambda \subset \mathbb{Z}^d$, with free boundary conditions and vanishing external field, is

$$Z_\Lambda(\beta) = \sum_{\sigma_\Lambda} \exp\Big\{ \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} \sigma_x\sigma_y \Big\}. \tag{29}$$

Using the identity

$$e^{\beta\sigma_x\sigma_y} = \cosh\beta + \sigma_x\sigma_y \sinh\beta, \tag{30}$$

it can be rewritten in the form

$$Z_\Lambda(\beta) = 2^{|\Lambda|}(\cosh\beta)^{|B(\Lambda)|} \sum_B (\tanh\beta)^{|B|}. \tag{31}$$

Here, the sum runs over all subsets B of the set $B(\Lambda)$ of all bonds in Λ (pairs of nearest neighbour sites from Λ) such that each site is contained in an even number of bonds from B. Using $\Lambda(B)$ to denote the set of sites contained in bonds from B, we say that $B_1, B_2 \subset B(\Lambda)$ are disjoint if $\Lambda(B_1)\cap\Lambda(B_2) = \emptyset$. Splitting now B into a collection $\mathcal{B} = \{B_1, \dots, B_k\}$ of its connected components called (high-temperature) *polymers* and using $\mathcal{B}(\Lambda)$ to denote the set of all polymers in Λ, we are getting

$$Z_\Lambda(\beta) = 2^{|\Lambda|}(\cosh\beta)^{|B(\Lambda)|} \sum_{\mathcal{B}\subset\mathcal{B}(\Lambda)} \prod_{B\in\mathcal{B}} (\tanh\beta)^{|B|} \tag{32}$$

with the sum running over all collections $\mathcal{B}$ of mutually disjoint polymers. This expression is exactly of the form (18), once we define compatibility of polymers by their disjointness. Introducing the weights

$$w(B) = \big(\tanh\beta\big)^{|B|}, \tag{33}$$

and taking the set $\mathcal{B}(\Lambda)$ of all polymers in Λ for W, we get the *polymer representation* $Z_\Lambda(\beta) = 2^{|\Lambda|}(\cosh\beta)^{|B(\Lambda)|} Z_{\mathcal{B}(\Lambda)}(w)$.

To apply the cluster expansion theorem, we have to find a function r such that the right hand side in (24) is positive and yields thus the radius of a polydisc of convergence. Taking $r(B) = \epsilon^{|B|}$ with a suitable ϵ, we get

$$\prod_{B'\in N(B)} (1 - r(B')) \geq e^{-2|B|} \tag{34}$$

allowing to choose $R(B) = r(B)e^{-2|B|} = \big(\epsilon e^{-2}\big)^{|B|}$. Indeed, to verify (34) we just notice that the number of polymers of size n containing a fixed site is bounded by κ^n with a suitable constant κ. Thus

$$\sum_{B':\Lambda(B')\ni x} \epsilon^{|B'|} \le \sum_{n=1}^{\infty} \kappa^n \epsilon^n \le \frac{1}{2} \tag{35}$$

once ϵ is sufficiently small, and thus

$$\sum_{B'\in N(B)} \epsilon^{|B'|} \le \frac{1}{2}|\Lambda(B)| \le |B| \tag{36}$$

yielding (34) ($1 - t > e^{-2t}$ for $0 < t < \frac{1}{2}$). To have $w \in \mathcal{D}_{W,\boldsymbol{R}}$ (for any W) is, for $R(B) = \big(\epsilon e^{-2}\big)^{|B|}$, sufficient to take $\beta \le \beta_0$ with $\tanh\beta_0 = \epsilon e^{-2}$.

As a consequence, for $\beta \le \beta_0$ we can use the cluster expansion theorem to obtain a convergent power series in powers of $\tanh\beta$. In particular, using $\Lambda(X) = \cup_{B\in \operatorname{supp} X}\Lambda(B)$, we get the pressure by the explicit formula

$$\beta p(\beta) = \log 2 + d\log(\cosh\beta) + \sum_{X:\Lambda(X)\ni x} \frac{a(X)}{|\Lambda(X)|} w^X \tag{37}$$

for any fixed $x \in \mathbb{Z}^d$ (by translation invariance of the contributing terms, the choice of x is irrelevant). The function $\beta p(\beta)$ is analytic on the region $\beta \le \beta_0$ since it is obtained as a uniformly absolutely convergent series of analytic terms $(\tanh\beta)^{|X|}$.

This type of *high-temperature cluster expansion* can be extended to a large class of models with Boltzmann factor in the form $\exp\{-\beta\sum_A U_A(\phi)\}$, where $\phi = (\phi_x; x \in \mathbb{Z}^d)$ is the configuration with *a priori* on site probability distribution $\nu(d\phi_x)$ and U_A, for any finite $A \subset \mathbb{Z}^d$, are the multi-site interactions (depending only on $(\phi_x; x \in A)$). Using the Mayer trick we can rewrite

$$\exp\{-\beta \sum_{A\subset\Lambda} U_A(\phi)\} = \prod_A \big(1 + f_A(\phi)\big) \tag{38}$$

with $f_A(\phi) = \exp\{-\beta U_A(\phi)\} - 1$. Expanding the product we will get a polymer representation with polymers $\mathcal{A}$ consisting of connected collections $\mathcal{A} = (A_1, \dots, A_k)$ with weights

$$w(\mathcal{A}) = \int \prod_{A\in\mathcal{A}} f_A(\phi) \prod_{x\in\cup_{A\in\mathcal{A}}A} \nu(d\phi_x). \tag{39}$$

Under appropriate bounds on the interactions U_A and for β small enough, using $\Lambda(\mathcal{A})$ to denote the set $\cup_{A\in\mathcal{A}}A$, we get

$$\sum_{\mathcal{A}:\Lambda(\mathcal{A})\ni x} |w(\mathcal{A})| \le 1. \tag{40}$$

This assumption allows, as before in the case of the high-temperature Ising model, to apply the cluster expansion theorem yielding an explicit series expansion for the pressure.

3.2. **Correlations.** Cluster expansions can be applied for evaluation of decay of correlations. Let us consider, for the class of models discussed above, the expectation

$$\langle\, \Psi \,\rangle_\Lambda = \frac{1}{Z_\Lambda} \int \Psi(\phi)\; e^{-\beta H_\Lambda(\phi)} \prod_{x\in\Lambda} \nu(d\phi_x) \tag{41}$$

with $H_\Lambda(\phi) = \sum_{A\subset\Lambda} U_A(\phi)$ and a function Ψ depending only on variables ϕ_x on sites x from a finite set $S \subset \Lambda \subset \mathbb{Z}^d$.

A convenient way of evaluating the expectation starts with introduction of the modified partition function

$$Z_{\Lambda,\Psi}(\alpha) = Z_\Lambda + \alpha Z_{\Lambda,\Psi} = Z_\Lambda\big(1 + \alpha\langle\, \Psi \,\rangle_\Lambda\big). \tag{42}$$

Clearly,

$$\langle\, \Psi \,\rangle_\Lambda = \frac{d\log Z_{\Lambda,\Psi}(\alpha)}{d\alpha}\Big|_{\alpha=0}. \tag{43}$$

Thus, one may get an expression for the expectation $\langle\, \Psi \,\rangle_\Lambda$, by forming a polymer representation of $Z_{\Lambda,\Psi}(\alpha)$ and isolating terms linear in α in the corresponding cluster expansion. For the first step, in the just cited hight-temperature case with general multi-site interactions, we first enlarge the original set $\mathfrak{A}(\Lambda)$ of all polymers in Λ (consisting of connected collections $\mathcal{A} = (A_1, \dots, A_k)$) to $\mathcal{W}_S(\Lambda) = \mathfrak{A}(\Lambda) \cup \mathfrak{A}_S(\Lambda)$, where $\mathfrak{A}_S(\Lambda)$ is the set of all collections $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ of polymers such that each of them intersects the set S (polymers $\mathcal{A}_1, \dots, \mathcal{A}_k$ are "glued" by S into a single entity). Compatibility is defined as before by disjointness; in addition, any two collections from $\mathfrak{A}_S(\Lambda)$ are declared to be incompatible as well as any polymer $\mathcal{A}$ from $\mathfrak{A}(\Lambda)$ intersecting S is considered to be incompatible with any collection from $\mathfrak{A}_S(\Lambda)$. Defining now $w_\alpha(\mathcal{A}) = w(\mathcal{A})$ for $\mathcal{A} \in \mathfrak{A}(\Lambda)$ and

$$w_\alpha(\mathcal{A}) = \alpha \int \Psi(\phi)\; e^{-\beta H_\Lambda(\phi)} \prod_{x\in\cup_{A\in\mathcal{A}_1\cup\dots\cup\mathcal{A}_k} A\cup S} \nu(d\phi_x) \tag{44}$$

for $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k) \in \mathfrak{A}_S(\Lambda)$, we get $Z_{\Lambda,\Psi}(\alpha)$ exactly in the form (18),

$$Z_{\Lambda,\Psi}(\alpha) = \sum_{\mathcal{I}\subset\mathcal{W}_S(\Lambda)} \prod_{\mathcal{A}\in\mathcal{I}} w_\alpha(\mathcal{A}). \tag{45}$$

As a result, we have

$$\log \mathcal{Z}_{\Lambda,\Psi}(\alpha) = \sum_{X\in\mathcal{X}(\mathcal{W}_S(\Lambda))} a(X) w_\alpha^X, \tag{46}$$

allowing easily to isolate terms linear in α: namely, the terms with multi-indices X with $\operatorname{supp} X \cap \mathfrak{A}_S(\Lambda)$ consisting of a single collection, say $\mathcal{A}_0$, that occurs with multiplicity one, $X(\mathcal{A}_0) = 1$. Explicitly, using

$$\mathcal{X}_{S,\mathcal{A}_0}(\Lambda) = \big\{X \in \mathcal{X}(\mathcal{W}_S(\Lambda))\colon\ \operatorname{supp} X \cap \mathfrak{A}_S(\Lambda) = \{\mathcal{A}_0\}, X(\mathcal{A}_0) = 1\big\}, \tag{47}$$

we get

$$\langle\, \Psi \,\rangle_\Lambda = \sum_{\mathcal{A}_0\in\mathfrak{A}_S(\Lambda)} \sum_{X\in\mathcal{X}_{S,\mathcal{A}_0}(\Lambda)} a(X) w^X. \tag{48}$$

It is easy to show that, for sufficiently small β, the series on the right hand side is absolutely convergent even if we extend $\mathfrak{A}_S(\Lambda)$ to $\mathfrak{A}_S = \cup_\Lambda \mathfrak{A}_S(\Lambda)$ and $\mathcal{X}_{S,\mathcal{A}_0}(\Lambda)$ to $\mathcal{X}_{S,\mathcal{A}_0} = \cup_\Lambda \mathcal{X}_{S,\mathcal{A}_0}(\Lambda)$. As a result, we have an explicit expression for the limiting

expectation $\langle\,\Psi\,\rangle$ in terms of an absolutely convergent power series. This can be immediately applied to show that $|\langle\,\Psi\,\rangle - \langle\,\Psi\,\rangle_\Lambda|$ decay exponentially in distance between S and the complement of Λ. Indeed, it suffices to find a suitable bound on $\sum_X |a(X)||w|^X$ with the sum running over all clusters X reaching from the set S to Λ^c. To this end one does not need to evaluate explicitly the number of clusters of given "diameter" $\mathrm{diam}(X) = \sum_{\mathcal{A}} X(\mathcal{A})\,\mathrm{diam}(\Lambda(\mathcal{A})) = m$ with $m \geq \mathrm{dist}(S, \Lambda^c)$. The needed estimate is actually already contained in the condition (ii) from the cluster expansion theorem. It just suffices to choose a suitable K and assume that β is small enough to assure validity of (40) in a stronger form, $\sum_{\mathcal{A}:\Lambda(\mathcal{A})\ni x} |w(\mathcal{A})|K^{|\Lambda(\mathcal{A})|} \leq 1$, yielding eventually

$$\sum_{X:\mathrm{diam}(X)\geq \mathrm{dist}(S,\Lambda^c)} |a(X)||w|^X \leq$$
$$\leq K^{-\mathrm{dist}(S,\Lambda^c)}|S| \sum_{X:\cup_{\mathcal{A}\in\mathrm{supp}\,X}\Lambda(\mathcal{A})\ni x} |a(X)||w|^X K^{\sum X(\mathcal{A})|\Lambda(\mathcal{A})|} \leq |S|K^{-\mathrm{dist}(S,\Lambda^c)}. \tag{49}$$

Exponential decay of correlations $\langle\,\Psi_1;\Psi_2\,\rangle_\Lambda = \langle\,\Psi_1\Psi_2\,\rangle_\Lambda - \langle\,\Psi_1\,\rangle_\Lambda\langle\,\Psi_2\,\rangle_\Lambda$ (and the limiting $\langle\,\Psi_1;\Psi_2\,\rangle$) in distance between the supports of Ψ_1 and Ψ_2 can be established in a similar way by isolating terms proportional to $\alpha_1\alpha_2$ in the cluster expansion of $\log Z_{\Lambda,\Psi_1,\Psi_2}(\alpha_1,\alpha_2)$ with

$$Z_{\Lambda,\Psi_1,\Psi_2}(\alpha_1,\alpha_2) = Z_\Lambda\big(1 + \alpha_1\langle\,\Psi_1\,\rangle_\Lambda + \alpha_2\langle\,\Psi_2\,\rangle_\Lambda + \alpha_1\alpha_2\langle\,\Psi_1\,\Psi_2\,\rangle_\Lambda\big). \tag{50}$$

Resulting claim can be readily generalised to a claim about the decay of the correlation $\langle\,\Psi_1;\ldots;\Psi_k\,\rangle$ in terms of the shortest tree connecting supports $S_1,\ldots,S_k$ of the functions $\Psi_1,\ldots,\Psi_k$.

3.3. **Low temperature expansions.** Finally, in some models with symmetries, we can apply cluster expansion also at low temperatures. Let us illustrate it again in the case of Ising model. This time, we take the partition function $Z^+_\Lambda(\beta)$ with plus boundary conditions. First, let us define for each nearest neighbour bond $\langle x,y\rangle$ its dual as the $(d-1)$-dimensional closed unit hypercube orthogonal to the segment from x to y and bisecting it at its centre. For a given configuration σ_Λ, we consider the boundary of the regions of constant spins consisting of the union $\partial(\sigma_\Lambda)$ of all hypercubes that are dual to nearest neighbour bonds $\langle x,y\rangle$ for which $\sigma_x \neq \sigma_y$. The *contours corresponding to* σ_Λ are now defined as the connected components of $\partial(\sigma_\Lambda)$. Notice that, under the fixed boundary condition, there is a one to one correspondence between configurations σ_Λ and sets Γ of mutually compatible (disconnected) contours in Λ.

Observing that the number of faces in $\partial(\sigma_\Lambda)$ is just the sum of the areas $|\gamma|$ of the contours $\gamma \in \Gamma$, we get the polymer representation

$$Z^+_\Lambda(\beta) = e^{\beta|E(\Lambda)|}\sum_\Gamma \exp\big(-\beta\sum_{\gamma\in\Gamma}|\gamma|\big) \tag{51}$$

with the sum is over all collections of disjoint contours in Λ. Here $E(\Lambda)$ is the set of all bonds $\langle x,y\rangle$ with at least one endpoint x,y in Λ.

The condition (24) with $r(\gamma) = \epsilon^\gamma$ yields a similar bound on the weights $w(\gamma) = e^{-\beta|\gamma|}$ as in the high-temperature expansion. To verify it, for β sufficiently large,

boils down to the evaluation of number of contours of size n that contain a fixed site.

As a result, we can employ the cluster expansion theorem to get

$$\log Z^+_\Lambda(\beta) = \beta|E(\Lambda)| + \sum_{X: X\in\mathcal{X}(\mathcal{C}(\Lambda))} a(X)w^X, \tag{52}$$

with an explicit formula for the limit

$$\beta p(\beta) = \beta d + \sum_{X: A(X)\ni 0} \frac{a(X)}{|A(X)|} w^X. \tag{53}$$

Here, $A(X)$ is the set of sites attached to contours from $\operatorname{supp} X$,

$$A(X) = \cup_{\gamma\in\operatorname{supp} X} A(\gamma) \tag{54}$$

with

$$A(\gamma) = \{x \in \mathbb{Z}^d \mid \text{such that } \operatorname{dist}(x,\gamma) \le 1/2\}. \tag{55}$$

As a consequence of the fact that (53) is, for large β, an absolutely convergent sum of analytic terms $a(X)w^X = a(X)e^{-\beta\sum_\gamma X(\gamma)|\gamma|}$ (considered as functions of β), the function $\beta p(\beta)$ is, for large β, analytic in β.

The fact that one can explicitly express the difference $\log Z^+_\Lambda(\beta) - |\Lambda|\beta p(\beta)$ (cf. (28)) found numerous applications in situations where one needs an accurate evaluation of the influence of the boundary of the region Λ on the partition function. One such example is a study of microscopic behaviour of interfaces. The main idea is to use the explicit expression in the form

$$Z^+_\Lambda(\beta) = \exp\bigl\{\beta p(\beta)|\Lambda|\bigr\} \exp\biggl\{ \sum_{X: A(X)\cap\Lambda^c\neq\emptyset} a(X)w^X \frac{|A(X)\cap\Lambda|}{|A(X)|} \biggr\} =$$

$$= \exp\bigl\{\beta p(\beta)|\Lambda|\bigr\} \prod_{X: A(X)\cap\Lambda^c\neq\emptyset} (1+f_X). \tag{56}$$

Noticing that $f_X = \exp\bigl\{a(X)w^X\frac{|A(X)\cap\Lambda|}{|A(X)|}\bigr\} - 1$ does not vanish only if $A(X)\cap\Lambda\neq\emptyset$, we can expand the product to obtain "decorations" of the boundary $\partial\Lambda$ by clusters f_X. In the case of interface these clusters can be incorporated into the weight of interface, while on a fixed boundary they yield a "wall free energy".

The possibility of the (low-temperature) polymer representation of the partition function in terms of contours is based on the $+ \leftrightarrow -$ symmetry of the Ising model. In absence of such a symmetry, cluster expansions can still be used, but in the framework of Pirogov-Sinai theory.

Bibliographical notes

Cluster expansions originated from the works of Ursell, Yvon, Mayer and others and were first studied in terms of formal power series. The combinatorial and enumeration problems considered in this framework were summarised in [UF62]. For related topics in modern language see [BLL98]. The convergence results for Mayer and virial expansions for dilute gas were first proven in works of Penrose, Lebowitz, Groenveld, and Ruelle; see [Rue69] for a detailed survey. General polymer models on lattice were discussed by Gruber and Kunz [GK71]. See also [Sim93] for discussion of high-temperature and low-temperature cluster expansions of lattice models.

Abstract polymer models were introduced by [KP86]. An elegant proof of a general claim presented by Dobrushin [Dob96] was further extended and summarised by Scott and Sokal [SS05]. We follow their reformulation of the Dobrushin condition. Cluster expansions with a view on applications in quantum field theory are reviewed in [Bry86].

See also: Equilibrium statistical mechanics, Pirogov-Sinai theory, Phase transitions in continuum systems, Wulff droplets

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