PHASE TRANSITIONS OF LATTICE MODELS (RENNES LECTURES)

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I. INTRODUCTION

My goal in these lectures is to try to explain in a consistent way a piece of machinery used to study phase transitions of lattice models. Techniques I have in mind here can be characterized as those that are stable with respect to small perturbations. In particular, no symmetries or particular restrictions on monotonicity or sign of interactions should play a role. The main tool is a geometric characterization of phases and a study of the probability distributions of underlying geometric objects. I will try to restrict myself to simplest possible cases bearing some general features.

When physicists talk about phase transitions they have in mind some discontinuity (or at least nonanalyticity) of thermodynamic functions describing the state of the system (say, the density for the liquid-gas transition) as function of external parameters (temperature, pressure ...). The goal is to describe these effects starting from the microscopic level.

A simplest model featuring phase transitions is the Ising model:
we have random variables σ_i (called "spins") labeled by sites of a ν-dimensional lattice, $i \in \mathbb{Z}^\nu$, attaining only the values $+1$ and -1, $\sigma_i = \pm 1$. Their probability distribution $\mu(\sigma) = \mu(\{\sigma_i\})$ is given by the *Boltzmann-Gibbs weights*

$$\frac{e^{-\beta H(\sigma)}}{Z}, \tag{I.1}$$

where the *Hamiltonian* (energy) is

$$H(\sigma) = -\sum_{\langle i,j \rangle} (\sigma_i \sigma_j - 1) - h \sum_i \sigma_i \tag{I.2}$$

with the first sum taken over (unordered) pairs of nearest neighbours, $||i-j|| = 1$, and the normalizing *partition function* is

$$Z = \sum_\sigma e^{-\beta H(\sigma)}. \tag{I.3}$$

(To give a sense to these formulas, we consider here a finite lattice $\Lambda \subset \mathbb{Z}^\nu$.) The parameter β is, in physicists language, inverse temperature, the parameter h is an external magnetic field. Of course, there is no singularity whatever if we only consider finite sets Λ. To really "see phase transitions", one has to go to ∞ volume (thermodynamical limit)[1].

[1]Serious doubts that the phase transitions, say a liquid-gas transition, can be described by a theory based only on a universal underlying Hamiltonian that does not *a priori* distinguish the phases existed not so long time ago. This was witnessed by an annecdotical ballot at the van der Waals Centenary Conference in 1937. The question was "can infinite volume limit yield, starting from a single Hamiltonian, two different phases?" The outcome was a slight majority for "YES".

The most straightforward statement about the existence of phase transitions is based on considering the *free energy*

$$f(\beta, h) = -\frac{1}{\beta} \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log Z_\Lambda. \tag{I.4}$$

Here Z_Λ is the partition function [I.3] with the dependence on Λ explicitly shown. (Both, the existence of this limit and its independence on the eventual boundary conditions is easy to establish.)

The function f turns out to be analytic in all the plane (β, h) except (if $\nu \geq 2$) a halfline $\{(\beta, h) \mid \beta \geq \beta_c, h = 0)\}$. The derivative $\frac{\partial f}{\partial h}$ has a discontinuity when crossing this line at $\beta > \beta_c$. Since it is a first derivative that is discontinuous, we speak about first-order phase transitions. At the point $(\beta = \beta_c, h = 0)$ the first derivatives are continuous and the singularity is revealed by higher derivatives — we speak about a continuous transition.

To bring you in touch with high school physics, let me summarize what was said above about the phase transitions of the Ising model in a drawing that might recall what you have seen in your textbooks when discussing the liquid-gas phase transitions:

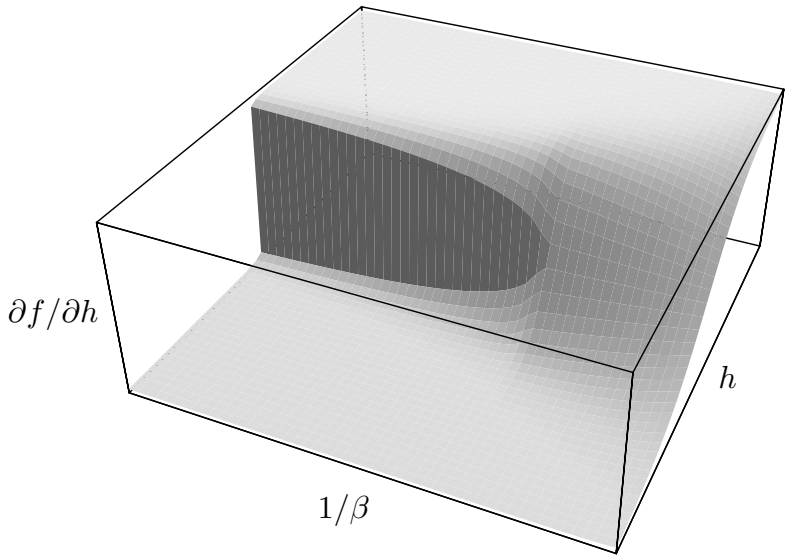


FIG. 1

II. GIBBS STATES

A convenient mathematically rigorous way of formulating the existence of phase transitions is in terms of Gibbs states. Sticking still to the Ising model we introduce the probability of a configuration $\sigma_\Lambda \equiv \{\sigma_i\}_{i\in\Lambda}$, $\sigma_i \in \{-1, +1\}$, on a finite lattice $\Lambda \subset \mathbb{Z}^\nu$, under a fixed boundary conditions $\overline{\sigma}_{\Lambda^c} = \{\overline{\sigma}_i\}_{i\in\mathbb{Z}^\nu\setminus\Lambda}$, by

$$\mu_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c}) = \frac{e^{-\beta H_\Lambda(\sigma_\Lambda|\overline{\sigma}_{\Lambda^c})}}{Z_\Lambda(\overline{\sigma}_{\Lambda^c})}, \tag{II.1}$$

with the energy

$$H_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c}) = -\sum_{\substack{\langle i,j\rangle \\ i,j\in\Lambda}} (\sigma_i\sigma_j - 1) - \sum_{\substack{\langle i,j\rangle \\ i\in\Lambda, j\in\Lambda^c}} (\sigma_i\overline{\sigma}_j - 1) - h\sum_{i\in\Lambda}\sigma_i \tag{II.2}$$

and the partition function

$$Z_\Lambda(\overline{\sigma}_{\Lambda^c}) = \sum_{\sigma_\Lambda} e^{-\beta H_\Lambda(\sigma_\Lambda|\overline{\sigma}_{\Lambda^c})}. \tag{II.3}$$

Gibbs states are defined as all those measures on $\{-1,1\}^{\mathbb{Z}^\nu}$ that, conditioned on the configuration $\overline{\sigma}_{\Lambda^c}$ outside Λ, yield in Λ the probability [II.1]. In other words, μ is a Gibbs state if for each cylindrical function φ on $\{-1,+1\}^{\mathbb{Z}^\nu}$, one has[2]

$$\int \mu_\Lambda(\varphi \mid \overline{\sigma}) d\mu(\overline{\sigma}) = \mu(\varphi) \tag{II.4}$$

once Λ is sufficiently large.

It can be shown[3] that for fixed parameters (β, h), the set $\mathcal{G}(\beta,h)$ of all Gibbs states is a convex closed set forming a Choquet simplex. Moreover, the set of all limits of the measures (I.1) with all possible sequences of boundary conditions yields the set of measures whose convex envelope is dense in $\mathcal{G}(\beta,h)$. Actually, if μ is an extremal point of $\mathcal{G}(\beta,h)$, then for μ-a.e. $\overline{\sigma}$, in the weak topology,

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu} \mu_\Lambda(\cdot \mid \overline{\sigma}_{\Lambda^c}) = \mu. \tag{II.5}$$

We say that a system undergoes a first order phase transition for a particular value (β,h) if $|\mathcal{G}(\beta,h)| > 1$. In other words, the fact that the Gibbs state is not unique, $|\mathcal{G}(\beta,h)| > 1$, means that there is a certain instability with respect to boundary conditions — a small change on the boundary may leads to a dramatical change in the limiting measure on $\{-1,1\}^{\mathbb{Z}^\nu}$.

At high temperatures, β small, the random variables σ_i are 'almost independent'[4] and as a result there is a unique weak limit μ of (II.1), for $\Lambda \nearrow \mathbb{Z}^d$, independent of boundary conditions (or sequence of boundary conditions $\{\overline{\sigma}_{\Lambda^c}\}_\Lambda$).

On the other hand, at low temperatures, β large, the variables σ_i are strongly dependent — the favoured configurations are those that do not differ too much from *ground configurations* minimizing the energy[5]: the configurations $\sigma = +\underline{1}$, ($\sigma_i = +1$ for all i) and $\sigma = -\underline{1}$. As a result a first-order phase transition occurs that reveals itself in the fact that, for for vanishing external field, $h = 0$, the particular boundary conditions corresponding to the ground configurations lead to two different limiting measures μ_+ and μ_- (see Section III.2).

[2]This definition of infinite volume Gibbs states was proposed by Dobrushin [Dob3] and Lanford and Ruelle [LR]. The equations (II.4) are DLR equations.

[3]See e. g. [Geo, Rue, Sim] and also [EFS] for these generalities.

[4]A precise meaning can be given to this statement [DS] and, in particular, it can be shown that different standard results for independent random variables (central limit theorem, large deviations, ...) can be extended to the random variables $\{\sigma_i\}$.

[5]More precisely, considering configurations σ that differ from, say $+\underline{1}$, on a finite number of sites, one has $H_\Lambda(\sigma_\Lambda \mid +\underline{1}) - H_\Lambda(+\underline{1}_\Lambda \mid +\underline{1}) \geq 0$ (supposing Λ is large enough).

III. A CLOSER LOOK ON ISING MODEL

III.1. High temperatures.

How to evaluate the behaviour of weakly dependent random variables $\{\sigma_i\}$ for small inverse temperatures β?

Observing that the characteristic function of the variable $\sum \sigma_i$ is just the ratio $\frac{Z_\Lambda(\beta,h+i\frac{t}{\beta})}{Z_\Lambda(\beta,h)}$, it is clear that to get a good control over the collective behaviour of $\{\sigma_i\}$, one has to be able to evaluate well the partition function (or rather its logarithm) for complex h. Considering here the partition function Z_Λ with free boundary conditions (the second term on the right hand side of (II.2) omitted) and using the obvious identity

$$e^{\beta\epsilon} = \cosh\beta + \epsilon \sinh\beta,\ \epsilon = \pm 1, \tag{III.1}$$

we can rewrite

$$\begin{aligned} Z_\Lambda &= \sum_{\sigma_\Lambda} \prod_{\langle i,j\rangle} \cosh\beta\big(1+\sigma_i\sigma_j\tanh\beta\big) \prod_i \cosh(\beta h)\big(1+\sigma_i\tanh(\beta h)\big) = \\ &= 2^{|\Lambda|}(\cosh\beta)^{|B(\Lambda)|}(\cosh(\beta h))^{|\Lambda|} \sum_G \big(\tanh\beta\big)^{|B(G)|}\big(\tanh(\beta h)\big)^{|V(G)|}. \end{aligned} \tag{III.2}$$

Here the sum goes over all pairs $G = (B, V)$, where $B \equiv B(G)$ is a set of bonds $B \subset B(\Lambda) = \{\langle i,j\rangle, i,j \in \Lambda\}$ and $V \equiv V(G)$ a set of vertices, $V \subset \Lambda$, fulfilling the following condition[6]:

each site $i \in V$ is contained in an odd number of bonds from B and every $i \notin V$ is contained in an even (possibly 0) number of bonds from B.

Splitting G into its connected components, $G = \{\Gamma_1, \Gamma_2, \dots\}$, to be called *polymers*, we get

$$Z_\Lambda = 2^{|\Lambda|}(\cosh\beta)^{|B(\Lambda)|}(\cosh(\beta h))^{|\Lambda|} \sum_G \prod_{\Gamma\in G} z(\Gamma) \tag{III.3}$$

with

$$z(\Gamma) = \big(\tanh\beta\big)^{|B(\Gamma)|}\big(\tanh(\beta h)\big)^{|V(\Gamma)|}. \tag{III.4}$$

What we are getting here is a representation of the partition function in terms of polymers that are *essentially independent* — up to the fact that different polymers must be disjoint, they contribute to the overall weight in a multiplicative manner. In the same time they are sufficiently *dumped* — their weights (III.4) decay exponentially with their size (supposing that β is small enough).

This polymer representation allows one to use the powerful techniques of cluster expansions to show, among other things, analyticity (in β and h) of the free energy defined by (I.4).

[6]The contribution from all other G's vanishes after the sum over σ_Λ is executed.

Intermezzo: cluster expansions.

There exist different ways to state the main assertion about the *cluster expansion* as well as different strategies for the proof. For example those in [GK, Sei, KP2], to name a few.

Very recently a remarkably elegant and simple proof was presented by R. Dobrushin [Dob5]. Even though I am not going to reproduce all details of his proof here, I would like to indicate the main ideas. The statement itself is for our purposes slightly simplified. In particular, we do not consider the most general abstract setting and the dumping assumption (III.7) is not stated in its weakest form.

We may have the representation (III.3) as a typical example of a polymer model in mind. However, the concrete form of polymers and their weight factors is not very important. Sometimes it is useful to consider more complicated structures — standard polymers 'decorated' by some additional sets etc. Polymer models can be easily introduced in a more abstract way [KP2] covering these situations. In particular, the condition of simple disjointness can be replaced by some more complicated condition of mutual compatibility of polymers. However, an important feature that has to be valid is that compatibility is defined pairwise — a collection G of polymers is compatible if all pairs of polymers from G are compatible.

Here we will restrict ourselves only to a limited class models. Namely, we suppose that polymer Γ is specified by an area $A(\Gamma)$ it "covers" on the lattice and, possibly, by an additional "configuration" on this set. For the particular case of the polymers above, we have $A(\Gamma) = \{i \mid i \in b \text{ for some bond } b \in B(\Gamma)\}$ (notice that, necessarily, $V(\Gamma) \subset A(\Gamma)$). A "configuration" on $A(\Gamma)$ is given by specifying the sets $V(\Gamma)$ and $B(\Gamma)$. In addition, we always suppose (as is true in the special case above) that $A(\Gamma)$ is connected (pathwise by sequences of bonds) and consider two polymers Γ_1 and Γ_2 to be *compatible* iff their areas are disjoint, $A(\Gamma_1) \cap A(\Gamma_2) = \emptyset$. A collection G of polymers in Λ is called *compatible* if the polymers from G are mutually compatible. Moreover, we assume that the number of contours with a fixed area A is bounded by $C^{|A|}$ with a suitable constant C,

$$\big|\{\Gamma\colon A(\Gamma) = A\}\big| \le C^{|A|}. \tag{III.5}$$

Let us suppose now that a weight factor z assigning a complex number $z(\Gamma)$ to every polymer Γ is given. The partition function of the *polymer model* defined by the weight factor z, is defined by

$$\mathcal{Z}(\Lambda; z) = \sum_{G \in \Lambda} \prod_{\Gamma \in G} z(\Gamma) \tag{III.6}$$

with the sum over all compatible collections G of polymers in Λ. The contribution of the empty configuration $G = \emptyset$ is taken to be 1 by definition. Notice that the definition (III.6) is meaningful also when taking for Λ any finite set of polymers (and not just a set of all polymers contained in a given volume). If the weights z are real non-negative numbers, we can speak about the the probability of any compatible collection G of

polymers in Λ by taking

$$\mu_\Lambda(G;z) = \frac{1}{\mathcal{Z}(\Lambda;z)} \prod_{\Gamma\in G} z(\Gamma) \tag{III.7}$$

The terms of the cluster expansion are usually labeled by *multiindices* X on the set K of all polymers, $X\colon K \to \{0,1,2,\dots\}$. Denoting $\operatorname{supp} X = \{\Gamma \in K, X(\Gamma) \neq 0\}$ we use $\mathfrak{X}$ for the set of all multiindices X with finite support and, for any finite $\Lambda \subset K$, we use $\mathfrak{X}(\Lambda)$ for the set of all $X \in \mathfrak{X}$ with $\operatorname{supp} X \subset \Lambda$. For $X \in \mathfrak{X}$, let $A(X)$ be the joint *area* covered by polymers from $\operatorname{supp} X$, $A(X) = \cup_{\Gamma\in\operatorname{supp} X} A(\Gamma)$. Finally, a multiindex X is called a *cluster* if $A(X)$ is connected.

Proposition III.1. *There exist a positive number τ_0, depending only on the dimension ν and the constant C from (III.5), and a function $\phi\colon \mathfrak{X} \to \mathbb{R}$ such that for any polymer model with a complex weight factor z satisfying the bound*

$$|z(\Gamma)| \leq e^{-\tau|A(\Gamma)|} \tag{III.8}$$

with $\tau \geq \tau_0$ and for every polymer $\Gamma \in K$, the partition function $Z(\Lambda;z)$ does not vanish for any finite $\Lambda \subset K$ and[7]

$$\log \mathcal{Z}(\Lambda;z) = \sum_{X\in\mathfrak{X}(\Lambda)} \phi(X) \prod_{\Gamma\in\operatorname{supp} X} z(\Gamma)^{X(\Gamma)}. \tag{III.9}$$

The function $\phi(X)$ vanishes whenever X is not a cluster and it satisfies the bound

$$|\phi(X)| \leq |\operatorname{supp} X| \prod_{\Gamma\in\operatorname{supp} X} e^{\tau_0|A(\Gamma)|X(\Gamma)}. \tag{III.10}$$

If the polymer model is translation invariant[8] *and $\tau > \tau_0$ is sufficiently large, the limit defining the 'free energy', $g(z) = -\beta^{-1}\lim\{\Lambda^{-1}\log\mathcal{Z}(\Lambda;z)\}$, exists and is explicitely given by the absolutely convergent series*

$$-\beta g(z) = \sum_{X: i\in A(X)} \frac{\phi(X)}{|A(X)|} \prod_{\Gamma\in\operatorname{supp} X} z(\Gamma)^{X(\Gamma)} \tag{III.11}$$

with the sum running over all multiindices $X \in \mathfrak{X}$ whose support contains a given fixed site. Further, there exists a constant K depending only on the constant C and the dimension ν, such that

$$\big|g(z)\big| \leq Ke^{-(\tau-\tau_0)}. \tag{III.12}$$

[7] With a branch of the logarithm chosen so that $\log\mathcal{Z}(\Lambda; z\equiv 0) = 0$.

[8] I.e., $z(\Gamma) = z(\Gamma + i)$ for any polymer Γ and any shift i. A correspondingly modified statement is true for polymer model satisfying some condition of periodicity.

Though I am presenting here the main idea of Dobrushin's proof in a form that should suffice to reconstruct it, I suggest that the reader consults his original paper [Dob 5] for a fully general statement and the proof under weaker assumptions.

Proof. The main idea of Dobrushin's proof is first to show, by induction in the number of elements in Λ, that $\mathcal{Z}(\Lambda; z) \neq 0$ and that the bound

$$\left| \log \left| \frac{\mathcal{Z}(\Lambda; z)}{\mathcal{Z}(\Lambda'; z)} \right| \right| \leq \sum_{\Gamma \in \Lambda \setminus \Lambda'} e^{-(\tau_0 - 1)|A(\Gamma)|} \tag{III.13}$$

is satisfied for all Λ finite and $\Lambda' \subset \Lambda$. The crucial point is that if $\Lambda' = \Lambda \setminus \{\Gamma_0\}$, we get directly from the definition (III.6) the equality

$$\mathcal{Z}(\Lambda; z) = \mathcal{Z}(\Lambda'; z) + z(\Gamma_0)\mathcal{Z}(\Lambda \setminus [\Gamma_0]; z). \tag{III.14}$$

Here $[\Gamma_0]$ is the set of all polymers that are not not compatible with Γ_0. Dividing (III.14) by $\mathcal{Z}(\Lambda'; z)$ and using the induction assumption and the bound

$$\sum_{\Gamma \in [\Gamma_0]} e^{-(\tau_0 - 1)|A(\Gamma)|} \leq A(\Gamma_0) \tag{III.15}$$

(this follows from (III.5) and the assumption that τ_0 is sufficiently large), we get the needed bound with the help of the inequality $|\log(1 + z)| \leq -\log(1 - |z|)$ valid for any complex z such that $|z| < 1$.

Notice that the bound (III.13) means in particular ($\Lambda' = \emptyset$) that

$$\big|\log |\mathcal{Z}(\Lambda; z)|\big| \leq |\Lambda|. \tag{III.16}$$

Viewing now $\log \mathcal{Z}(\Lambda; z)$ as a complex function of $|\Lambda|$ variables $z(\Gamma)$ holomorphic in the interior of the polydisc $W(\Lambda) = \{z = (z(\Gamma), \Gamma \in \Lambda) \colon |z(\Gamma)| \leq e^{-\tau_0 |A(\Gamma)|}\}$ and considering its Taylor expansion around $z \equiv 0$, we get

$$\log \mathcal{Z}(\Lambda; z) = \sum_{X \in \mathfrak{X}(\Lambda)} \phi_\Lambda(X) \prod_{\Gamma \in \operatorname{supp} X} z(\Gamma)^{X(\Gamma)}, \tag{III.17}$$

with the Taylor coefficients $\phi_\Lambda(X)$ that *a priori* depend on Λ. Inspecting, however, their definition in terms of the corresponding derivatives of $\log \mathcal{Z}(\Lambda; z)$ at $z \equiv 0$, it is easy to realize, taking into account that $\mathcal{Z}(\Lambda; z)$ given by (III.6) is a polynomial in z, that we get exactly the same coefficients from $\log \mathcal{Z}(\operatorname{supp} X; z)$ (the variables not appearing in the derivatives may be skipped already before derivatives are taken). Hence

$$\phi_\Lambda(X) = \phi_{\operatorname{supp} X}(X). \tag{III.18}$$

It means that the notation $\phi(X)$ indicating coefficients not depending on Λ is substantiated. The fact that $\phi(X)$ vanishes if X is not a cluster follows from the observation that

$\mathcal{Z}(\operatorname{supp} X; z) = \mathcal{Z}(\operatorname{supp} X_1; z)\mathcal{Z}(\operatorname{supp} X_2; z)$ once $X = X_1 + X_2$ and $A(X_1) \cap A(X_2) = \emptyset$. Finally, the bound (III.10) follows from the Cauchy formula (with the integration contour $\{|z(\Gamma)| = e^{-\tau_0|A(\Gamma)|}, \gamma \in \operatorname{supp} X\}$) for the Taylor coefficient $\phi_{\operatorname{supp} X}(X)$ with the help of the bound (III.16) on $\big|\log \mathcal{Z}(\operatorname{supp} X; z)\big|$.

The explicit expression (III.11) for $g(z)$ follows immediately from (III.9) once we realize that the series on the right hand side of (III.11) is absolutely convergent. This can be shown[9] "by hand", using the bound (III.10). Namely, bounding $|\operatorname{supp} X| \leq 2^{|\operatorname{supp} X|}$ and $|A(X)| \geq 1$, we dominate the series of absolute values by

$$\sum_{X: i \in A(X)} \prod_{\Gamma \in \operatorname{supp} X} \big(2e^{-(\tau-\tau_0)|A(\Gamma)|X(\Gamma)}\big) \tag{III.19}$$

that can be bounded by

$$\sum_{A \ni i} \sum_{\{A_\alpha\}: \cup A_\alpha = A} \prod_\alpha \Big(2 \sum_{\Gamma: A(\Gamma) = A_\alpha} \sum_{X(\Gamma)=1}^{\infty} e^{-(\tau-\tau_0)|A_\alpha|X(\Gamma)}\Big) \leq \sum_{A \ni i} \sum_{\{A_\alpha\}: \cup A_\alpha = A} \prod_\alpha \epsilon^{|A_\alpha|} \tag{III.20}$$

with $\epsilon = 4Ce^{-(\tau-\tau_0)}$ and τ taken to be large enough to assure that $1 - e^{-(\tau-\tau_0)} \geq \frac{1}{2}$. This can be further, for any $\delta > 0$, bounded by

$$\sum_{A \ni i} \big(\frac{\epsilon}{\delta}\big)^{|A|} \sum_n \sum_{\{A_1, \ldots, A_n\}} \prod_{\alpha=1}^n \delta^{|A_\alpha|} \leq \sum_{A \ni i} \big(\frac{\epsilon}{\delta}\big)^{|A|} \sum_n \frac{1}{n!} \big(\sum_{\bar{A} \subset A} \delta^{|\bar{A}|}\big)^n. \tag{III.21}$$

Observe now that

$$\sum_{\bar{A} \subset A} \delta^{|\bar{A}|} \leq \sum_{j \in A} \sum_{\bar{A} \ni j} \delta^{|\bar{A}|} \leq 2C(\nu)|A|\delta, \tag{III.22}$$

where $C(\nu)^n$ is a bound on the number of connected sets in $\mathbb{Z}^\nu$ containing n sites including a fixed site i and we suppose that δ is small enough to assure the lower bound $1 - C(\nu)\delta \geq \frac{1}{2}$. Taking thus $\delta = \frac{1}{2C(\nu)}$ we get the bound

$$\sum_{A \ni i} \big(2C(\nu)\epsilon\big)^{|A|} e^{|A|} \leq \big(2C(\nu)\big)^2 e\epsilon \tag{III.23}$$

(supposing that $1 - 2eC(\nu)\epsilon \geq \frac{1}{2}$) and (III.12) follows. □

Coming now back to the high temperature representation (III.3) and observing the analyticity of $z(\Gamma)$'s (as functions of β or h), we can conclude that the free energy f represented by the uniformly convergent sum (III.11) is also analytic, in the region of β and h small.

[9] Unfortunately, the needed bound that arises as a byproduct of the proof in [KP] has to be explicitely evaluated if we use the Dobrushin's approach.

III.2. Low temperatures.

Now we turn to the low temperature case, β large, where (once $\nu \geq 2$) we have genuinely strongly correlated random variables $\{\sigma_i\}$ and a phase transition at $h = 0$.

The proof of the existence of (at least) two different Gibbs states for $h = 0$ and β large is by the famous *Peierls argument* based on a reformulation of the model in terms of probabilities of particular spatial patterns in the configurations. Namely, one considers configurations $\partial = \{\Gamma\}$ of *contours* Γ introduced for a spin configuration σ as connected components of the boundary between areas of pluses and minuses[10]. For a fixed boundary condition (say $+\underline{1}$) the correspondence between spin configurations and collections ∂ of mutually disjoint contours is one to one and the probability of a contour configuration ∂ under the measure $\mu_\Lambda(\cdot \mid +\underline{1})$ (recall that here we consider $h = 0$) is given just by a polymer model[11] (in the present context often called a *contour model*)

$$\mu_\Lambda(\partial \mid +\underline{1}) = \frac{1}{\mathcal{Z}(\Lambda)} \prod_{\Gamma \in \partial} e^{-2\beta|\Gamma|}. \tag{III.24}$$

Here

$$\mathcal{Z}(\Lambda) = \sum_{\partial \subset \Lambda} \prod_{\Gamma \in \partial} e^{-2\beta|\Gamma|} \tag{III.25}$$

with the sum over collections of mutually disjoint contours in Λ. It differs from $Z_\Lambda(+\underline{1})$ ($\equiv Z_\Lambda(-\underline{1})$) by a factor that equals the contribution of the configuration $+\underline{1}$ to $Z_\Lambda(+\underline{1})$. Namely, one has

$$Z_\Lambda(+\underline{1}) = e^{-\beta H_\Lambda(+\underline{1}_\Lambda \mid +\underline{1})} \mathcal{Z}(\Lambda), \tag{III.26}$$

where $H_\Lambda(+\underline{1}_\Lambda \mid +\underline{1})$ equals to the number of bonds in the set B_Λ of all bonds that contain sites from Λ.

The typical configurations σ of the measure μ_+ obtained as the infinite volume limit of $\mu_\Lambda(\,\cdot \mid +\underline{1})$ can be characterized by proving that, in the limiting probability obtained from (III.24), the typical contour configurations ∂ are such that for every $\Gamma \in \partial$ there exists the most external contour surrounding it. (No infinite 'cascades' of contours immersed one into another exist.) This fact is proven, with the help of the Borel–Cantelli lemma, by evaluating the probability of every contour surrounding a fixed site in such a way that the sum of these probabilities can be shown to converge (see below). As a result, one characterizes the typical configurations σ of the measure μ_+ as

[10] In the two-dimensional case, contours are connected sets consisting of edges of the dual lattice $(\mathbb{Z}^2)^* \equiv \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ such that every vertex of $(\mathbb{Z}^2)^*$ is contained in even number (0, 2, or 4) of its edges. Similarly, in the three-dimensional case, contours are connected sets consisting of plaquettes — elementary squares with vertices on the dual lattice — that separate pairs of plus and minus spins. .... We use $|\Gamma|$ to denote the "lengths" of the contour: the number of edges (plaquettes, ...) for $\nu = 2$ ($\nu = 3$, ...).

[11] It is a polymer model in the sense of the previous section if we take for $A(\Gamma)$ the set of all sites of the dual lattice that are contained in Γ. Mutually disjoint contours Γ_1 and Γ_2 cover disjoint areas, $A(\Gamma_1) \cap A(\Gamma_2) = \emptyset$. Notice that $\frac{|\Gamma|}{\nu} \leq |A(\Gamma)| \leq 2^{\nu-1}|\Gamma|$ and thus the bound (III.8) is for $z(\Gamma) = e^{-2\beta|\Gamma|}$ satisfied once β is sufficiently large.

consisting of a connected sea of pluses containing finite islands of minuses. Or, in other words, in a typical configuration of μ_+ the pluses percolate (and minuses do not)[12]. This situation can be described as the *stability* of plus phase. By the same reasoning we can show that also the minus phase is stable and characterize the measure μ_- as supported by configurations consisting of a sea of minuses with islands of pluses. The measures μ_+ and μ_- thus differ — we say that two different phases coexist for $h = 0$ and β large or that phase transition of the first order occurs for $h = 0$.

Thus, the trick that allows one to describe the typical configurations in spite of the fact that the variables σ_i are actually strongly dependent, is based on replacing them by 'contour variables' and viewing their probability distribution (III.24) as a perturbation of a contour-free (empty) configuration that corresponds to the ground spin state $+\underline{1}$ in the case of μ_+. The crucial fact for the Ising model is its plus-minus symmetry. It follows not only that the phase transition should be expected to occur for vanishing external field, $h = 0$, but also that the contours distributed by (III.24) are *essentially independent* and we have a polymer model in the sense of the preceding section.

More concretely, in Peierls argument we evaluate the probability that the lattice site $0 \in \Lambda$ is occupied by -1 under the finite volume distribution $\mu_\Lambda(\,\cdot \mid \overline{\sigma}_{\Lambda^c} = +\underline{1})$ with the $+$ boundary conditions (cf. (II.1)). To have $\sigma_0 = -1$, there necessarily must exist at least one contour Γ that is encircling the site 0,

$$\mu_\Lambda(\sigma_0 = -1 \mid \overline{\sigma}_{\Lambda^c} = +\underline{1}) \le \sum_{\Gamma \text{ enc. } 0} \mu_\Lambda(\Gamma \in \partial \mid +\underline{1}) =$$

$$= \sum_{\Gamma \text{ enc. } 0} \frac{e^{-2\beta|\Gamma|} \sum_{\partial \subset \Lambda, \partial \text{ compatible with } \Gamma} \prod_{\overline{\Gamma} \in \partial} e^{-2\beta|\overline{\Gamma}|}}{\mathcal{Z}(\Lambda)} \le \sum_{\Gamma \text{ enc. } 0} e^{-2\beta|\Gamma|}. \quad \text{(III.27)}$$

In the last inequality we are using the fact that a configuration with a particular contour skipped is again a possible configuration (under fixed boundary conditions $+\underline{1}$ there exists a uniquely defined corresponding spin configuration in Λ) and the weights of remaining contours do not change. The second main ingredient is the fact that the long contours are sufficiently *dumped* — the weight factor of a given contour Γ (in our case $e^{-2\beta|\Gamma|}$) satisfies the bound (III.8) with τ sufficiently large. This is a direct consequence of the fact that the difference of the energy of a configuration and the ground state configuration (say $+\underline{1}$) is proportional to the length of its contours (Peierls condition).

It is now easy to conclude the bound by observing that the number of contours of length k encircling a given site is not larger than $\overline{C}(\nu)^k$, with a fixed constant $\overline{C}(\nu)$ depending only on the dimension ν, yielding

$$\mu_\Lambda(\sigma_0 = -1 \mid \overline{\sigma}_{\Lambda^c} = +\underline{1}) \le \sum_{k=4}^{\infty} \overline{C}(\nu)^k e^{-2\beta k} \quad \text{(III.28)}$$

[12]This is without any reservation true only for low temperatures. It is known [Hig] that at intermediate temperatures for the three-dimensional Ising model both pluses and minuses $*$-percolate (i.e. create infinite $*$-connected clusters, where $*$-connected means pathwise connected with pathes composed of sequences of pairs of next nearest neighbours).

uniformly in Λ.

Notice finally, that from the explicit expression (III.26) of the partition function $Z_\Lambda(+\underline{1})$ in terms of the contour model partition function (III.25) we can get the analyticity properties of the free energy f applying (III.11) to the contour model.

Namely, we get the free energy explicitely as a uniformly convergent series

$$f(\beta, h=0) = -2\nu + \frac{1}{\beta} \sum_{X: i \in A(X)} \frac{\phi(X)}{|A(X)|} \prod_{\Gamma \in \operatorname{supp} X} e^{-2\beta X(\Gamma)|\Gamma|} \tag{III.29}$$

under the assumption that that $\operatorname{Re}\beta$ is large enough. Hence, f is a holomorphic function of β in this region.

IV. Asymmetric models

IV.1. Perturbed Ising model.

In the case of the Ising ferromagnet with vanishing external field we were fortunate to get immediately the representation (III.24) in terms of a polymer model. This, in its consequence, allowed to construct the *phase diagram* — to understand, at least for some domains of β and h, what is the number $|\mathcal{G}_0(\beta, h)|$ of extremal periodic Gibbs states.

However, even a small perturbation to the Hamiltonian (I.2) may introduce a "long-distance interaction" among contours that will spoil the representation (III.24). To see what I mean by that, consider a simple plus-minus symmetry breaking term, say,

$$-\kappa \sum_{(i,j,k)} \sigma_i \sigma_j \sigma_k, \tag{IV.1}$$

added to the Hamiltonian (I.2). Here the sum is over all triangles consisting of a site j and two its nearest neighbours i and k such that the edges (ij) and (jk) are orthogonal. We consider all triplets with at least one of the sites i, j, k in Λ; σ for those sites that are outside Λ is to be interpreted as the corresponding boundary condition $\overline{\sigma}$ (say $+\underline{1}$). Rewriting the model in terms of contours we obtain

$$\mu_\Lambda(\partial \mid +\underline{1}) = \frac{1}{Z_\Lambda(+\underline{1})} \prod_{\gamma \in \partial} \rho(\gamma) e^{-\beta e_+ |V_\Lambda^+(\partial)| - \beta e_- |V_\Lambda^-(\partial)| - \beta E_{\partial\Lambda}(+\underline{1})} \ . \tag{IV.2}$$

Here $V_\Lambda^+(\partial)$ (resp. $V_\Lambda^-(\partial)$) is the number of sites in Λ occupied, for the configuration corresponding to ∂, by pluses (resp. minuses), $e_+ = -\nu - h - \kappa 2\nu(\nu-1)$ (resp. $e_- = -\nu + h + \kappa 2\nu(\nu-1)$) is the average energy per site of the configuration $+\underline{1}$ (resp. $-\underline{1}$), and $E_{\partial\Lambda}(+\underline{1})$ is the boundary energy in Λ of the configuration $+\underline{1}$,

$$E_{\partial\Lambda}(+\underline{1}) = H_\Lambda(+\underline{1}_\Lambda \mid +\underline{1}) - e_+ |\Lambda| \tag{IV.3}$$

(and similarly for $E_{\partial\Lambda}(-\underline{1})$). Notice that the weights $\rho(\gamma)$ actually depend not only on the geometrical form of the contour, but also on whether γ is surrounded from outside by pluses or minuses. For example for the contour surrounding a single plus spin

immersed in minuses we obtain $\rho(\gamma) = e^{-\beta(8+8\kappa\nu)}$, while for the contour surrounding a single minus spin immersed in pluses we obtain $\rho(\gamma) = e^{-\beta(8-8\kappa\nu)}$. As a result we have to label the contours by the signature of the spins surrounding it from outside (in (IV.2) we anticipated this and introduced *labeled contours* $\gamma = (\Gamma, \varepsilon)$ consisting of a geometrical shape Γ labeled by the sign $\varepsilon = \pm 1$ of the outer spins). In (IV.2) we thus again obtained a representation in terms of contours — labeled contours with the weights ρ that are dumped (once κ and h are small). However, the condition of essential independence has been lost. The order of labeled contours matters — if a plus-contour is surrounded by another plus-contour, there must be a minus-contour immersed between them. Unlike in unperturbed case, this *matching condition* introduces certain 'long-range hard core' — a minus-contour γ surrounded by a disjoint plus-contour $\overline{\gamma}$ 'knows' about its presence. Erasing $\overline{\gamma}$ (by flipping all spins inside $\overline{\gamma}$) would turn γ into plus-contour and thereby change its weight $\rho(\gamma)$.

IV.2. Potts model.

The representation (IV.2) of a lattice model in terms of a probability distribution of matching collections of labeled contours is not restricted to our simple perturbed Ising model. There exists a large class of models that naturally yield such a representation which is actually the starting point of the Pirogov–Sinai theory. Before discussing how to recover essential independence and to transform this representation into a polymer model, let us consider an example of slightly different type — the Potts model — that leads, however, to a representation of the same type as (IV.2).

The model is a straightforward generalization of the Ising model by considering spins attaining q values, $\sigma_i = 1, \dots, q$, with Hamiltonian

$$H_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c}) = - \sum_{\substack{\langle i,j\rangle \\ i,j\in\Lambda}} (\delta_{\sigma_i,\sigma_j} - 1) - \sum_{\substack{\langle i,j\rangle \\ i\in\Lambda, j\in\Lambda^c}} (\delta_{\sigma_i,\overline{\sigma}_j} - 1). \tag{IV.4}$$

A remarkable fact is that, once q is large enough, the transition is first order also in temperature. Namely, there exists a temperature β_t at which ordered low temperature phases coexist with disordered high temperature phase. There exist $q+1$ different Gibbs states for $\beta = \beta_t$, q ordered and one disordered. Contours separating ordered and disordered regions were used already in the original proof of existence of this transition [KS]. However, to evaluate their probability one has to estimate the loss of entropy resulting when introducing a contour — such an estimate was accomplished there with the help of so called chessboard estimates. A treatment by the Pirogov–Sinai theory has been presented, among others, in [KLMR, BKL] and [Mar]. A simplification based on the Fortuin–Kasteleyn representation was suggested in [LMMRS] and here I will use the reformulation from [BKM].

The first step is to rewrite the model in terms of the Fortuin–Kasteleyn random cluster representation [FK]. Namely, using the identity

$$e^{\beta(\delta_{\sigma_i,\sigma_j}-1)} = e^{-\beta} + \delta_{\sigma_i,\sigma_j}(1 - e^{-\beta}) \tag{IV.5}$$

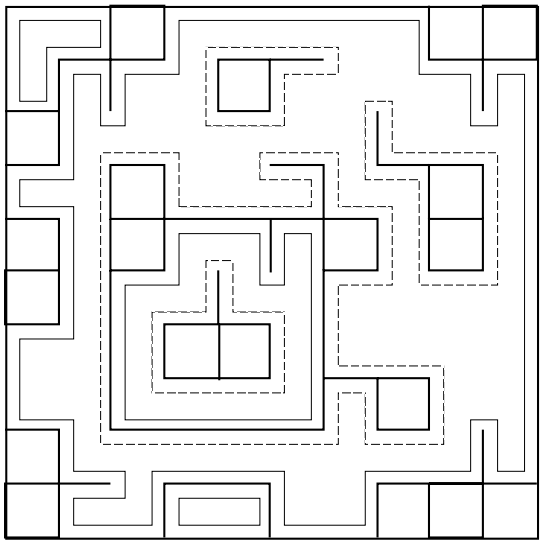


FIG. 2. Contours for a configuration of occupied bonds ω in the random cluster representation of the Potts model under the wired boundary conditions. Thick lines correspond to the bonds from ω. Plain thin lines denote the ordered contours (ω from outside), while dashed thin lines denote the disordered contours (ω from inside).

we get

$$Z_\Lambda(\overline{\sigma}) = \sum_{\omega \subset B_\Lambda} (e^{-\beta})^{|B_\Lambda \setminus \omega|} (1-e^{-\beta})^{|\omega|} \sum_{\overline{\sigma}_\Lambda} \prod_{\substack{\langle i,j\rangle \in \omega \\ i,j \in \Lambda}} \delta_{\sigma_i,\sigma_j} \prod_{\substack{\langle i,j\rangle \in \omega \\ i \in \Lambda, j \in \Lambda^c}} \delta_{\sigma_i,\overline{\sigma}_j}. \qquad \text{(IV.6)}$$

Here the sum runs over all subsets ω of the set B_Λ of all bonds that intersect Λ.

Considering, in particular, the ordered boundary conditions (say $\overline{\sigma} = \underline{1}$) or free boundary conditions (the second term in the right hand side of (IV.4) absent)[13], we get (up to the factor q in the ordered case) the partition functions

$$Z_\Lambda(b) = e^{-\beta |B_\Lambda(b)|} \sum_\omega (e^\beta - 1)^{|\omega|} q^{c(\omega,b)}. \qquad \text{(IV.7)}$$

Here $c(\omega, b)$ is the number of components (each site not touched by ω is counted as one additional component) of the set ω under the boundary conditions b (for $b = f$, the free boundary conditions, all sites outside Λ are considered to be disjoint; for $b = w$, the wired boundary conditions, all sites outside Λ connected) and $B_\Lambda(w) = B_\Lambda$, while $B_\Lambda(f) = B(\Lambda)$.

For every set of bonds ω we can introduce contours in the following way: consider first the closed set $\overline{\omega}$ consisting of the union of all bonds from ω with all unit squares

[13] In this case we actually have to skip the second product in (IV.6) as well as to replace B_Λ by $B(\Lambda)$, the set of all bonds with both endpoints in Λ.

whose all four sides belong to ω, all unit cubes whose all twelve edges belong to ω etc. Taking now the $\frac{1}{4}$-neighbourhood $U_{1/4}(\overline{\omega})$ of $\overline{\omega}$ we define the contours as connected components of the boundary of $U_{1/4}(\overline{\omega})$. This procedure is illustrated in Fig. 2. The contours are boundaries between regions occupied by ω (*ordered* regions) and empty (*disordered*) regions whose each site contributes by the factor q to the partition function (IV.7) (it represents the component attributed to a site unattached to ω). Denoting by $V^0_\Lambda(\partial) \equiv \omega$ the set of bonds in the former and $V^d_\Lambda(\partial) \equiv B_\Lambda(b) \setminus \omega$ in the latter, we get

$$Z_\Lambda(b) = e^{-\beta|B_\Lambda(b)|} \sum_\partial (e^\beta - 1)^{|V^0_\Lambda(\partial)|} q^{|V^d_\Lambda(\partial)|/\nu} \prod_{\gamma\in\partial} \rho(\gamma). \tag{IV.8}$$

Here the weights of contours $\rho(\gamma)$ depend on the surrounding regions — if γ is surrounded by the order (i.e., ω) from outside (plain thin lines in Fig. 2), we have

$$\rho(\gamma) = q^{-|\gamma|/(2\nu)}, \tag{IV.9}$$

while for γ surrounded by the disorder from outside (dashed lines in Fig. 2) we have

$$\rho(\gamma) = q^{-|\gamma|/(2\nu)+1}. \tag{IV.10}$$

Again, in (IV.8) we have a similar representation as in (IV.2). Taking q large enough allows to get sufficiently small weights ρ above. Notice also that the role of the ground state energies $e_\pm$ is played here by the free energies (per bond) $-\frac{1}{\beta}\log(e^\beta - 1)$ and $-\frac{1}{\beta\nu}\log q$ of the ordered and entirely disordered states, respectively.

The asymmetry here is a rather weak one. If it were not for the factor q that makes (IV.10) to be different from (IV.9), we would be able to argue just by Peierls argument that the transition occurs exactly when $e^\beta - 1 = q^{1/\nu}$!

V. Phase diagram — Pirogov-Sinai theory

V.1. Recovering essential independence.

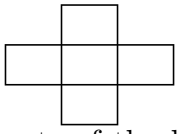
The general case (represented for us by the perturbed Ising model) is treated with the help of a trick introduced by Pirogov and Sinai. An important fact is that we cannot apply standard cluster expansions directly — we first have to get rid off the above described long-range dependence of labeled contours. However, this is rather easy to achieve either by certain fixed point arguments [PS] or relying on inductive definitions [Zah, KP1]. Namely, in the latter approach one introduces two polymer weight factors $z_+(\Gamma)$ and $z_-(\Gamma)$ (by using the notation Γ we want to stress that we really have a polymer weight, unlike for $\rho(\gamma)$, the dependence will be only on the shape of the contour[14] (the label being delegated to the subscript of z))

$$z_+(\Gamma) = \rho((\Gamma,+)) e^{-\beta(e_- - e_+)|\partial_{\rm I}\Gamma|} \frac{Z_{\mathrm{Int}\,\Gamma}(-\underline{1})}{Z_{\mathrm{Int}\,\Gamma}(+\underline{1})} e^{-\beta(E_{\partial\,\mathrm{Int}\,\Gamma}(+\underline{1}) - E_{\partial\,\mathrm{Int}\,\Gamma}(-\underline{1}))} \tag{V.1}$$

[14]Here we are obtaining polymer models with contours playing the role of polymers. Hence we are occassionally referring to contour models instead of polymer models.

and

$$z_-(\Gamma) = \rho((\Gamma,-))e^{-\beta(e_+ - e_-)|\partial_{\rm I}\Gamma|}\frac{Z_{\rm Int\,\Gamma}(+\underline{1})}{Z_{\rm Int\,\Gamma}(-\underline{1})}e^{-\beta(E_{\partial\,\rm Int\,\Gamma}(-\underline{1})-E_{\partial\,\rm Int\,\Gamma}(+\underline{1}))}. \tag{V.2}$$

To fix the notation in the above formulas, we first use $\partial_{1/2}\Gamma$ to denote the set of all sites attached from inside to the contour Γ (those sites from $\mathbb{Z}^\nu$ inside Γ whose distance from Γ in the maximum metric equals $\frac{1}{2}$). Splitting $\partial_{1/2}\Gamma$ into components (by considering the intersections of $\partial_{1/2}\Gamma$ with components of $\mathbb{R}^\nu \setminus \Gamma$), we notice that, whenever Γ is a contour of a configuration σ, the values of spins are constant through each component of $\partial_{\rm I}\Gamma$ — on some of them they agree with the spin outside the contour while on the remaining components the label has the opposite sign[15]. (Think of a contour of the form  with one component of $\partial_{1/2}\Gamma$ of the former type and four components of the latter type.) We use $\partial_{\rm I}\Gamma$ in (V.1, 2) to denote the union of the latter and $\mathrm{Int}\,\Gamma$ the union of corresponding components of $\mathbb{Z}^\nu \cap (\mathbb{R}^\nu \setminus \Gamma)$. All the components labeled by the same sign as the outside spin yield $\mathrm{Ext}_\Lambda\,\Gamma = \Lambda\setminus(\partial_{\rm I}\Gamma\cup\mathrm{Int}\,\Gamma)$ and $\mathrm{Ext}\,\Gamma = \mathbb{Z}^\nu \setminus (\partial_{\rm I}\Gamma \cup \mathrm{Int}\,\Gamma)$. The term $E_{\partial\,\rm Int\,\Gamma}(\pm\underline{1})$ is defined by $E_{\partial\,\rm Int\,\Gamma}(\pm\underline{1}) = H_{\rm Int\,\Gamma}(\pm\underline{1}_{\rm Int\,\Gamma} \mid \pm\underline{1}) - e_\pm|\,\mathrm{Int}\,\Gamma|$ in agreement with (IV.3).

With the help of the weights (V.1) and (V.2) we get the original partition functions in terms of polymer models (the formulas below replace (III.26) of the symmetric case).

Lemma V.1. *For every finite Λ one has*

$$e^{\beta E_{\partial\Lambda}(+\underline{1})}Z_\Lambda(+\underline{1}) = e^{-\beta e_+|\Lambda|}\sum_{\partial\subset\Lambda}\prod_{\Gamma\in\partial} z_+(\Gamma) \tag{V.3}$$

and

$$e^{\beta E_{\partial\Lambda}(-\underline{1})}Z_\Lambda(-\underline{1}) = e^{-\beta e_-|\Lambda|}\sum_{\partial\subset\Lambda}\prod_{\Gamma\in\partial} z_-(\Gamma). \tag{V.4}$$

Proof. Indeed, resumming in the expression (cf. (IV.2))

$$Z_\Lambda(+\underline{1}) = \sum_{\partial\subset\Lambda}\prod_{\gamma\in\partial}\rho(\gamma)e^{-\beta e_+|V_\Lambda^+(\partial)|-\beta e_-|V_\Lambda^-(\partial)|-\beta E_{\partial\Lambda}(+\underline{1})} \tag{V.5}$$

over all ∂ with a fixed collection ϑ of the most external (plus-)contours, we get

$$Z_\Lambda(+\underline{1}) = \sum_{\vartheta\subset\Lambda} e^{-\beta(e_+|\,\mathrm{Ext}_\Lambda\,\vartheta|+E_{\partial\Lambda}(+\underline{1}))}\prod_{(\Gamma,+)\in\vartheta}\rho((\Gamma,+))e^{-\beta(e_-|\partial_{\rm I}\Gamma|-E_{\partial\,\rm Int\,\Gamma}(-\underline{1}))}Z_{\rm Int\,\Gamma}(-\underline{1}) \tag{V.6}$$

[15]For the two-dimensional perturbed Ising model, we could actually have considered contours with a single component of $\mathrm{Int}\,\Gamma$ labeled by opposite spin as that outside Γ. Namely, 'rounding the corners' in a prescribed way whenever 4 bonds meet, the contours would turn into closed selfavoiding paths. However, in general situation it is natural that different components of $\mathrm{Int}\,\Gamma$ have different labels. Here we have chosen (cf. [Zah]) to include the components labeled by the same spin as the outside to the exterior of the contour.

with $\mathrm{Ext}_\Lambda\,\vartheta = \cap_{\Gamma\in\vartheta}\mathrm{Ext}_\Lambda\,\Gamma$. Notice that the partition function $Z_{\mathrm{Int}\,\Gamma}(-\underline{1})$ has the fixed minus boundary condition on $\partial_{\mathrm{I}}\Gamma$ and every contour contributing to it is disjoint from Γ. Multiplying each term on the right-hand side of (V.6) by

$$e^{-\beta(e_+|\partial_{\mathrm{I}}\Gamma|-e_+|\partial_{\mathrm{I}}\Gamma|)}\,\frac{Z_{\mathrm{Int}\,\Gamma}(+\underline{1})}{Z_{\mathrm{Int}\,\Gamma}(+\underline{1})}e^{\beta(E_{\partial\,\mathrm{Int}\,\Gamma}(-\underline{1})-E_{\partial\,\mathrm{Int}\,\Gamma}(-\underline{1}))},$$

using the definition (V.1), and proceeding in proving (V.3) and (V.4) by induction in the number of sites in Λ, we use (V.3) for $Z_{\mathrm{Int}\,\Gamma}(+\underline{1})$ on the right-hand side valid by induction hypothesis ($\mathrm{Int}\,\Gamma\subsetneq\Lambda$) and obtain thus (V.3) for the full volume Λ. □

As a result, in (V.3) and (V.4) we succeeded in rewriting the partition functions $Z_\Lambda(+\underline{1})$ and $Z_\Lambda(-\underline{1})$ in terms of partition functions $\mathcal{Z}(\Lambda;z_+)$ and $\mathcal{Z}(\Lambda;z_-)$ of contour models z_+ and z_-. These are polymer models according to our definition from Section 2 with the condition of essential independence fulfilled.

Two questions may now arise. First, in the formulas (V.3) and (V.4) we rewrote only the partition functions. Moreover, we did so in terms of rather artificial polymer model (formally speaking, we suppose that inside a plus contour there are again only plus contours). Thus, even if we have the corresponding polymer models under control, will it suffice to say something, for example, about typical configurations of the measures μ_+ and μ_-? The answer is positive. Namely, it is clear that the polymer models z_+ and z_- introduced above, not only lead to the same (up to a factor) partition functions as the original model, but also yield exactly the same probabilities that a given set ϑ of external contours is present, namely,

$$\mu_\Lambda(\vartheta\mid\pm\underline{1}) = \frac{1}{\mathcal{Z}(\Lambda;z_\pm)}\prod_{\Gamma\in\vartheta}z_\pm(\Gamma)\mathcal{Z}(\mathrm{Int}\,\Gamma;z_\pm). \tag{V.7}$$

Once we know that the corresponding polymer model, say z_+, is dumped (satisfies the bound (III.8)), we can control the limit $\Lambda\nearrow\mathbb{Z}^d$ and with the help of the equality (V.7), show that there are no infinite cascades of contours in the limiting measure μ_+ and the plus phase is stable.

However, and this is the second question, it is not clear whether, even though the original weights ρ were dumped, the newly defined weights z_+ and z_- will be also dumped. The answer depends on the values of the parameters β and h. It turns out that for a fixed (sufficiently large) β there exists a value $h_t\equiv h_t(\beta)$ such that for $h=h_t$ both z_+ and z_- are dumped and thus both plus and minus phases are stable, while for $h>h_t$ only z_+ is dumped and for $h<h_t$ only z_- is dumped.

The description of this *transition point* $h_t(\beta)$ actually yields the *phase diagram* in the case of the perturbed Ising model[16].

[16]The 'tuning parameter' (driving field) here was the external field h. For the Potts model, one can closely follow our treatment of the perturbed Ising model. The role of 'tuning parameter' is played by the (inverse) temperature β and, to get dumped weights $\rho(\gamma)$, we have to suppose that q is large enough.

Our next task thus will be to find the transition point with the above formulated properties. Sometimes, in presence of a symmetry, the value of the transition point can be guessed. For example, for the unperturbed Ising model we expected $h_t = 0$. Indeed, for $h = 0$ we got $e_+ = e_- = 0$, $Z_\Lambda(+\underline{1}) = Z_\Lambda(-\underline{1})$, and thus directly

$$z_+(\Gamma) = z_-(\Gamma) = e^{-2\beta|\Gamma|}.$$

Before turning to the general situation, when h_t is *a priori* unknown, let us consider a case for which the value of the transition point can be guessed from some symmetry of the model.

V.2. Use of a symmetry.

One such example is the Ising antiferromagnet with the Hamiltonian

$$H_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c}) = \sum_{\substack{\langle i,j\rangle \\ i,j\in\Lambda}} (\sigma_i\sigma_j + 1) + \sum_{\substack{\langle i,j\rangle \\ i\in\Lambda, j\in\Lambda^c}} (\sigma_i\overline{\sigma}_j + 1) - h\sum_{i\in\Lambda}\sigma_i. \qquad \text{(V.8)}$$

It is known [Dob 2] that for sufficiently small temperatures and small external field h, there exist two antiferromagnetic phases corresponding to two ground configurations. Namely, the configuration with pluses on even lattice sites ($i = (i_1, i_2, \dots, i_\nu)$ such that $\|i\| = |i_1| + |i_2| + \cdots + |i_\nu|$ is even) and minuses on odd sites — let us call it the *even* ground configuration (and use the subscript 'e' to refer to it) — and the same configuration shifted by a unit vector — the *odd* ground configuration (the subscript 'o').

Let us prove that, indeed, both phases are stable once, for β large enough, the external field h is sufficiently small. In spite of its simplicity, there are two good reasons for including this model here. The proof that polymer weights for both coexisting phases are really dumped is not immediate and it actually involves an important ingredient of the general case. Moreover, a similar reasoning might be useful also in other more complex situations — actually, recently it was used when describing the phase transitions in diluted spin systems [CKS] and in a discussion of renormalization group transformations for large external field [EFK]. Finally, there are other cases with a less obvious symmetry (see for example [HDP]) for which this type of reasoning is also valid.

As a byproduct of the proof of coexistence of two phases one can use the resulting polymer representation to prove that the free energy of Ising antiferromagnet is analytic in h and β in the concerned region. Again, this statement is a precursor of similar claims in more complex situations. For example, it was used recently [BJK] to show that there is no phase transition between two seemingly disparate regions in the phase diagram of the extended Hubbard model in the atomic limit.

Let us take, say, the odd ground configuration as the boundary condition for (V.8). To introduce contours, we again consider the boundaries between regions with even and odd ground configurations. However, this time we take as belonging to the same contour all components whose distance, in maximum metric, is one. Thus, for the configuration

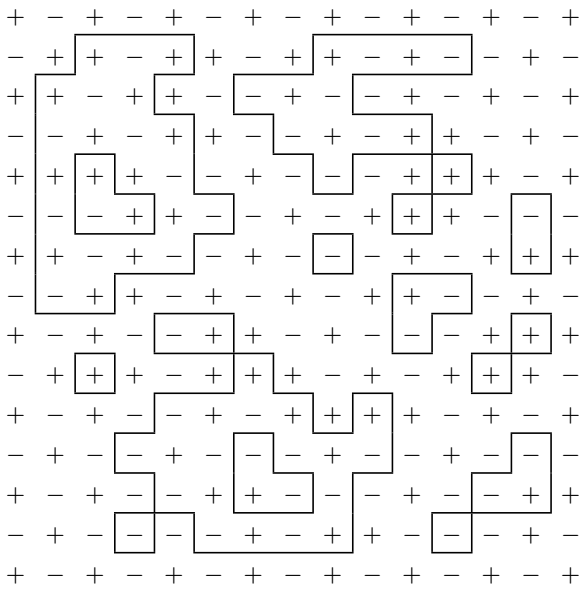


FIG. 3

on Fig. 2 we have just one contour. Again, under fixed boundary conditions we have a one-to-one correspondence between spin configurations and collections of contours. Notice that even though we consider, in general, a non-vanishing h, the energies of the ground configuration are (contrary to the case of Ising ferromagnet) equal, $e_o = e_e = 0$.

Thus we have a particularly simple form of labeled contour model with

$$Z_\Lambda(e) = \sum_{\partial \subset \Lambda} \prod_{\gamma \in \partial} \rho(\gamma). \tag{V.9}$$

Even though this formula is reminiscent of that for the ferromagnet with vanishing field (cf. (III.25)), we *do not have* a polymer model here — the weights of labeled contours γ depend on the label (which ground configuration surrounds them from outside)! To compute the weight $\rho(\gamma)$, one has to compute the energy of the configuration σ for which γ is the single contour. Consider, for the configuration σ, all pairs i, j of nearest neighbour sites such that $j = i + (1, 0, \dots, 0)$ ($j_1 = i_1 + 1$, $j_k = i_k$, $k = 2, \dots, \nu$) and $\sigma_i = 1$, $\sigma_j = -1$. The remaining unpaired sites are necessarily attached to the contour. Denoting $S(\gamma) = \sum \sigma_i$, with the sum over all these unpaired sites, we clearly have

$$\rho(\gamma) = e^{-2\beta|\gamma| - \beta h S(\gamma)}. \tag{V.10}$$

The reason for gluing together different components[17] of the boundary between ground configurations was that otherwise these unpaired sites might be shared by different

[17]The idea of gluing together different connected components is in the general Pirogov–Sinai approach automatically carried out by considering 'thick contours' that would consist, for the present case, of components of the union of all those $2 \times 2 \times \cdots \times 2$ cubes for which the configuration σ restricted to it differs from both ground configurations.

contours. Had we chosen the standard definition of contour, the weight $\rho(\gamma)$ would depend on whether the contour γ is isolated or there are other contours around whose distance from γ is 1 and they share some unpaired sites. Notice, for future use, that the complement of a contour, say an even contour $\gamma = (\Gamma, e)$, may have several components (cf. Fig. 3). We take for the interior of Γ, $\operatorname{Int}\Gamma$, only those sites that are in the configuration σ (the configuration whose single contour is γ) occupied by the odd ground configuration and whose distance, again in maximum metric, from Γ is larger than $\frac{3}{2}$. Notice also that the weight of a labeled contour γ equals the weight of the contour shifted by a unit vector but labeled by the other ground configuration,

$$\rho((\Gamma, e)) = \rho((\Gamma + (1, 0, \dots, 0), o)). \tag{V.11}$$

We can again use the strategy of the preceding section and introduce the weights

$$z_e(\Gamma) = \rho((\Gamma, e)) \frac{Z_{\operatorname{Int}\Gamma}(o)}{Z_{\operatorname{Int}\Gamma}(e)} e^{\beta(E_{\partial \operatorname{Int}\Gamma}(o) - E_{\partial \operatorname{Int}\Gamma}(e))} \tag{V.12}$$

and

$$z_o(\Gamma) = \rho((\Gamma, o)) \frac{Z_{\operatorname{Int}\Gamma}(e)}{Z_{\operatorname{Int}\Gamma}(o)} e^{\beta(E_{\partial \operatorname{Int}\Gamma}(e) - E_{\partial \operatorname{Int}\Gamma}(o))}, \tag{V.13}$$

for which

$$e^{\beta E_{\partial\Lambda}(e)} Z_\Lambda(e) = \mathcal{Z}(\Lambda; z_e), \quad e^{\beta E_{\partial\Lambda}(o)} Z_\Lambda(o) = \mathcal{Z}(\Lambda; z_o). \tag{V.14}$$

Showing now that both z_e and z_o are dumped, we will prove that both phases are stable[18].

Proposition V.2. *Let $h < 2$ and β be sufficiently large (depending on h). Then both z_e and z_o are dumped and both phases are stable. The free energy f is holomorphic in h and β in this region.*

Proof. We will prove the bound (III.8) for z_e and z_o simultaneously by induction on $\operatorname{diam}\Gamma$. Let us suppose that both z_e and z_o satisfy (III.8) for all $\overline{\Gamma}$ such that $\operatorname{diam}\overline{\Gamma} < n$. Considering now Γ with $\operatorname{diam}\Gamma = n$, we apply (V.14) for $Z_{\operatorname{Int}\Gamma}(e)$ and $Z_{\operatorname{Int}\Gamma}(o)$. By the induction hypothesis we can use the cluster expansion (III.9) for $\mathcal{Z}(\operatorname{Int}\Gamma; z_e)$ and $\mathcal{Z}(\operatorname{Int}\Gamma; z_o)$ yielding

$$\frac{\mathcal{Z}(\operatorname{Int}\Gamma; z_e)}{\mathcal{Z}(\operatorname{Int}\Gamma; z_o)} = \exp\Big\{ \sum_{X \in \mathfrak{X}(\operatorname{Int}\Gamma)} \phi(X) \Big(\prod_{\overline{\Gamma} \in \operatorname{supp} X} z_e(\overline{\Gamma})^{X(\overline{\Gamma})} - \prod_{\overline{\Gamma} \in \operatorname{supp} X} z_o(\overline{\Gamma})^{X(\overline{\Gamma})} \Big) \Big\}. \tag{V.15}$$

Observing first that for every $\overline{\Gamma}$ one has $z_e(\overline{\Gamma}) = z_o(\overline{\Gamma} + (1, 0, \dots, 0))$ as a direct consequence of the equality $Z_V(e) = Z_{V+(1,0,\dots,0)}(o)$ implied by (V.11), the terms in the

[18] This is true for a range of values of the field h — the field h does not break the symmetry between the phases. For a 'tuning parameter' that is able to discriminate between these two phases one has to introduce an additional field, for example a *staggered field* in the form of the term $g \sum (-1)^{\|i\|} \sigma_i$ added to the Hamiltonian. Here we are actually taking the transition value $g_t = 0$.

exponent on the right hand side of (V.15) with $A(X)$ not too near to the boundary of $\operatorname{Int}\Gamma$ will be canceled. To bound the remaining terms we notice that, since only contours $\overline{\Gamma}$ with $\operatorname{diam}\overline{\Gamma} < n$ contribute in (V.15), the bound (III.10) for $\phi(X)$ as well as (III.8) for z_e and z_o are satisfied[19] by the induction hypothesis. As a consequence we obtain

$$\exp\{-\varepsilon|\Gamma|\} \le \frac{\mathcal{Z}(\operatorname{Int}\Gamma; z_e)}{\mathcal{Z}(\operatorname{Int}\Gamma; z_o)} \le \exp\{\varepsilon|\Gamma|\} \tag{V.16}$$

with ε of the order $Ke^{-(\tau-\tau_0)}$. Taking into account that, clearly, $|S(\gamma)| \le |\gamma|$ and $|E_{\partial\Lambda}(e) - E_{\partial\Lambda}(o)| = O(h)|\partial\Lambda|$, we get the bound (III.7) once $h < 2$. □

V.3. General case.

Finally, we consider again the perturbed Ising model as a representative of the general case for which the value of the transition point h_t is not known.

The task is to decide, for given values of parameters h, κ, and β, which of the phases is stable, or in other words, which of the polymer weights z_+ or z_- is dumped. Following the reformulation of Pirogov–Sinai theory by Zahradník [Zah] (or, rather, the version by Borgs and Imbrie [BI1]) we introduce "metastable free energies" by suppressing all contours whose weights are not dumped. Putting thus

$$\overline{z}_\pm(\Gamma) = \begin{cases} z_\pm(\Gamma) & \text{if } |z_\pm(\Gamma)| \le e^{-\tau(\Gamma)}, \\ 0 & \text{otherwise,} \end{cases} \tag{V.17}$$

we define

$$\overline{Z}_\Lambda(\pm\underline{1}) = e^{-\beta H_\Lambda(\pm\underline{1}_\Lambda|\pm\underline{1})}\mathcal{Z}(\Lambda; \overline{z}_\pm). \tag{V.18}$$

Notice that both weights $\overline{z}_+$ and $\overline{z}_-$ are automatically dumped, and it follows that the cluster expansion can be employed to control the limit $g(\overline{z}_\pm) = -\beta^{-1}\lim\{|\Lambda|^{-1}\log\mathcal{Z}(\Lambda;\overline{z}_\pm)\}$ (see (III.11)). Comparing the explicit expressions (III.9) and (III.11), we get $\log\mathcal{Z}(\Lambda;\overline{z}_\pm) = -\beta|\Lambda|g(\overline{z}_\pm) + \varepsilon|\partial\Lambda|$ with ε (as well as βg) of the order $Ke^{-(\tau-\tau_0)}$ and thus

$$e^{\beta E_{\partial\Lambda}(\pm\underline{1})}\overline{Z}_\Lambda(\pm\underline{1}) = \exp\{-\beta f_\pm|\Lambda| + \varepsilon|\partial\Lambda|\} \tag{V.19}$$

with

$$f_\pm = e_\pm + g(\overline{z}_\pm). \tag{V.20}$$

The *metastable free energies* defined by the equality (V.20) play an important role in determining which phase is stable — it turns out that the stable phase is characterized by having the minimal metastable free energy. Namely, defining

$$a_\pm = \beta\big(f_\pm - \min(f_+, f_-)\big), \tag{V.21}$$

[19] Formally, we may consider $z^{(n)}_{e(o)}$ defined by

$$z^{(n)}_{e(o)}(\overline{\Gamma}) = \begin{cases} z_{e(o)}(\overline{\Gamma}) & \text{if } \operatorname{diam}\overline{\Gamma} < n \\ 0 & \text{otherwise,} \end{cases}$$

notice that $\mathcal{Z}\big(V; z^{(n)}_{e(o)}\big) = \mathcal{Z}(V; z_{e(o)})$ for every V such that $\operatorname{diam} V < n$ by the induction hypothesis, and that only those X contribute in (V.15) for which $\operatorname{diam} A(X) < n$.

we claim that z_+ is dumped once $a_+ = 0$ (and similarly for the minus phase). To prove this assertion we prove by induction on n the following crucial lemma (stated in our particular case of the perturbed Ising model).

Lemma V.3 [Zah, BI]**.** *Let κ and h be sufficiently small and let $a_+ = 0$. Then, for sufficiently large β, for every n:*

(i) *if* $\operatorname{diam}\Lambda \le n$ *and* $a_- \operatorname{diam}\Lambda \le 1$, *then* $z_-(\Gamma) \le e^{-\tau|\Gamma|}$ *for every* Γ *in* Λ,

(ii) $z_+(\Gamma) \le e^{-\tau|\Gamma|}$ *for every* Γ *with* $\operatorname{diam}\Gamma \le n$.

Remarks. (1) Notice that, by definition, $\min(a_+, a_-) = 0$. Thus, by this lemma, always at least one of the phases is stable (the plus phase above). Moreover, by (ii) one actually has $\overline{z}_+ \equiv z_+$ and $\overline{Z}_\Lambda(+\underline{1}) = Z_\Lambda(+\underline{1})$. Thus

$$f_+ = -\beta^{-1} \lim\bigl\{|\Lambda|^{-1} \log Z_\Lambda(+\underline{1})\bigr\} \equiv f; \tag{V.22}$$

the metastable free energy of the stable phase equals the standard free energy of the original model (which, actually, does not depend on the boundary conditions).
(2) The transition point h_t is characterized by the equation $a_+ = a_- = 0$. The parameter $\max(a_+, a_-)$ can be viewed as a measure of distance from the transition point. For sufficiently large volumes is the unstable phase suppressed — the system with unstable (minus) boundary conditions prefers to flip to the plus phase over a long contour encircling large part of the volume Λ. Even though the energy cost of such a large contour is of the order $|\partial\Lambda|$, there is a volume gain $a_-|\Lambda|$. The statement (i) of Lemma V.3 then says that for 'small volumes' ($a_- \operatorname{diam}\Lambda \le 1$) the system prefers to stay in the minus phase. The closer to the transition point (i.e., the smaller is the parameter a_-) the larger volumes are able to support the unstable phase. Very close to the transition point, both phases seem to be stable from the point of view of small volumes (we are saying that the unstable phase (minus) is metastable in small volumes) and, only coming to large volumes, the system is able to distinguish which phase is really stable.

Proof. (i) By the induction hypothesis we can replace $Z_{\operatorname{Int}\Gamma}(\pm\underline{1})$ by $\overline{Z}_{\operatorname{Int}\Gamma}(\pm\underline{1})$. Applying then the equality (V.19) and the bound $|E_{\partial\Lambda}(+\underline{1}) - E_{\partial\Lambda}(-\underline{1})| \le O(\kappa)|\partial\Lambda|$ in the definition (V.2), we get

$$\begin{aligned} \frac{Z_{\operatorname{Int}\Gamma}(+\underline{1})}{Z_{\operatorname{Int}\Gamma}(-\underline{1})} = \frac{\overline{Z}_{\operatorname{Int}\Gamma}(+\underline{1})}{\overline{Z}_{\operatorname{Int}\Gamma}(-\underline{1})} &\le \exp\{-\beta(f_+ - f_-)|\operatorname{Int}\Gamma| + 2\varepsilon|\Gamma|\} \\ &= \exp\{a_-|\operatorname{Int}\Gamma| + 2\varepsilon|\Gamma|\} \le \exp\{(1+2\varepsilon)|\Gamma|\} \end{aligned} \tag{V.23}$$

with ε of the order $O(\beta\kappa + e^{-(\tau-\tau_0)}$. In the last bound we used the inequality $|\operatorname{Int}\Gamma| \le |\Gamma|\operatorname{diam}\Gamma$ and the assumption $a_- \operatorname{diam}\Lambda \le 1$. Taking into account that

$$\rho(\gamma) \le \exp\{-2\beta(1 - O(\kappa))|\gamma|\}, \tag{V.24}$$

we get (III.7) for $z_+(\Gamma)$ once $|\kappa|$ and $|h|$ are small and β is large enough.

(ii) Let us call *small* those contours that satisfy the condition $a_- \operatorname{diam}\Gamma \le 1$. The remaining contours will be called *large*. Resumming in (V.5) over all collections ∂ of contours with a fixed set ϑ of large external contours and using the induction hypothesis, we get

$$\frac{Z_{\operatorname{Int}\Gamma}(-\underline{1})}{Z_{\operatorname{Int}\Gamma}(+\underline{1})} = \sum_{\substack{\vartheta \text{ large} \\ \vartheta\subset\operatorname{Int}\Gamma}} \frac{Z^{\mathrm{small}}_{\operatorname{Ext}_{\operatorname{Int}\Gamma}(\vartheta)}(-\underline{1}) \prod_{\overline{\gamma}\in\vartheta} \rho(\overline{\gamma}) e^{-\beta e_+ |\partial_{\mathrm{I}}\overline{\gamma}|} \overline{Z}_{\operatorname{Int}\overline{\gamma}}(+\underline{1})}{\overline{Z}_{\operatorname{Int}\Gamma}(+\underline{1})} \le$$

$$\le e^{2\varepsilon|\Gamma|} \sum_{\substack{\vartheta\text{large} \\ \vartheta\subset\operatorname{Int}\Gamma}} \exp\Big\{-|\operatorname{Ext}_{\operatorname{Int}\Gamma}(\vartheta)|\beta(f^{\mathrm{small}}_- - f_+)\Big\} \prod_{\overline{\gamma}\in\vartheta} \rho(\overline{\gamma}) e^{3\varepsilon|\overline{\gamma}|}. \tag{V.25}$$

(One $\varepsilon|\overline{\gamma}|$ in the last term comes from the bound on $\beta|e_+ - f_+||\partial_{\mathrm{I}}\overline{\gamma}|$.) In this equation, $Z^{\mathrm{small}}_{\operatorname{Ext}_\Lambda(\vartheta)}(-\underline{1})$ is the partition function with sum taken only over small contours and f^{small}_- is the corresponding metastable free energy. To evaluate the right hand side of (V.25) we consider an auxiliary polymer model with the weight

$$\tilde{z}(\overline{\Gamma}) = \begin{cases} \big\{\rho((\overline{\Gamma},+)) + \rho((\overline{\Gamma},-))\big\} e^{|\Gamma|} & \text{if } \overline{\Gamma} \text{ is large} \\ 0 & \text{otherwise.} \end{cases} \tag{V.26}$$

We will bound (V.25) by $e^{2\varepsilon|\Gamma|}$ once we show that

$$\sum_{\vartheta\subset\operatorname{Int}\Gamma} \exp\big\{-|\operatorname{Ext}_{\operatorname{Int}\Gamma}(\vartheta)|a\big\} \prod_{\overline{\Gamma}\in\vartheta} \tilde{z}(\overline{\Gamma}) \le 1, \tag{V.27}$$

where $a = \beta(f^{\mathrm{small}}_- - f_+)$. Notice first that

$$\frac{\mathcal{Z}(\operatorname{Int}\Gamma;\tilde{z}) \exp\big\{-|\operatorname{Ext}_{\operatorname{Int}\Gamma}(\vartheta)|a\big\}}{\mathcal{Z}(\operatorname{Int}\vartheta;\tilde{z})} \le 1 \tag{V.28}$$

for any ϑ. Indeed,

$$\mathcal{Z}(\operatorname{Int}\Gamma;\tilde{z}) = \sum_{\partial\subset\operatorname{Int}\Gamma} \prod_{\overline{\Gamma}\in\partial} \tilde{z}(\overline{\Gamma}) \le \Big(\sum_{\partial'\subset\operatorname{Int}\vartheta} \prod_{\overline{\Gamma}\in\partial'} \tilde{z}(\overline{\Gamma})\Big)\Big(1 + \sum_{\partial''\cap\operatorname{Ext}\vartheta\neq\emptyset} \prod_{\overline{\Gamma}\in\partial''} \tilde{z}(\overline{\Gamma})\Big). \tag{V.29}$$

The first term is $\mathcal{Z}(\operatorname{Int}\vartheta;\tilde{z})$, while the second can be bounded by

$$\prod_{i\in\operatorname{Ext}\vartheta} \Big(1 + \sum_{\overline{\Gamma}\ni i} \tilde{z}(\overline{\Gamma})\Big) \le e^{a\operatorname{Ext}\vartheta} \tag{V.30}$$

since

$$\sum_{\substack{\overline{\Gamma}\ni i \\ \overline{\Gamma}\text{ large}}} \tilde{z}(\overline{\Gamma}) \le \sum_{\substack{\overline{\Gamma}\ni i \\ |\overline{\Gamma}|\ge\frac{1}{a_-}}} e^{-\tau|\overline{\Gamma}|} \le e^{-\frac{\tau}{2a_-}} \le a \tag{V.31}$$

(only large contours for which $|\overline{\Gamma}| \geq \operatorname{diam}\overline{\Gamma} \geq (a_-)^{-1}$ contribute). To see the last inequality, notice that

$$\beta|f_- - f_-^{\text{small}}| \leq \sum_{\substack{X: i\in A(X)\\ \operatorname{diam} A(X)\geq (a_-)^{-1}}} \frac{\phi(X)}{|A(X)|} \prod_{\overline{\Gamma}\in\operatorname{supp} X} z(\overline{\Gamma})^{X(\overline{\Gamma})} \leq K\exp\left\{-\frac{\tau-\tau_0}{a_-}\right\} \tag{V.32}$$

and thus

$$\beta(f_-^{\text{small}} - f_+) \geq a_- - K\exp\left\{-\frac{\tau-\tau_0}{a_-}\right\}. \tag{V.33}$$

Since, for τ large enough,

$$\exp\left\{-\frac{\tau}{2a_-}\right\} + K\exp\left\{-\frac{\tau-\tau_0}{a_-}\right\} \leq (K+1)\exp\left\{-\frac{\tau}{2a_-}\right\} \leq (K+1)\frac{2a_-}{\tau} \leq a_-, \tag{V.34}$$

we have

$$\exp\left\{-\frac{\tau}{2a_-}\right\} \leq a_- - K\exp\left\{-\frac{\tau-\tau_0}{a_-}\right\} \leq \beta\big(f_-^{\text{small}} - f_+\big). \tag{V.35}$$

Rewriting now the left hand side of (V.27) as

$$\sum_{\vartheta\subset\operatorname{Int}\Gamma} \frac{\frac{\mathcal{Z}(\operatorname{Int}\Gamma;\tilde{z})\exp\{-|\operatorname{Ext}_{\operatorname{Int}\Gamma}(\vartheta)|a\}}{\mathcal{Z}(\operatorname{Int}\vartheta;\tilde{z})}\mathcal{Z}(\operatorname{Int}\vartheta;\tilde{z})\prod_{\overline{\Gamma}\in\vartheta}\tilde{z}(\overline{\Gamma})}{\mathcal{Z}(\operatorname{Int}\Gamma;\tilde{z})} \tag{V.36}$$

and using (V.28) we get (V.27) since

$$\sum_{\vartheta\subset\operatorname{Int}\Gamma} \mathcal{Z}(\operatorname{Int}\vartheta;\tilde{z}) \prod_{\overline{\Gamma}\in\vartheta} \tilde{z}(\overline{\Gamma}) = \mathcal{Z}(\operatorname{Int}\Gamma;\tilde{z}). \tag{V.37}$$

Thus, referring again to the bound (V.24) and the definition (V.1), we conclude that $z_+(\Gamma)$ satisfies the bound (III.7). □

The free energies $f_\pm$ are, in view of the equality (V.20), close to the ground configuration energies $e_\pm$; the difference $\beta g(\bar{z}_\pm)$ is of the order $e^{-(\tau-\tau_0)}$ (cf. (III.12)). Moreover, while the ground state energies $e_\pm$ are linear in h, the functions $\beta g(\overline{z}_\pm)$ can be shown to be Lipschitz with the Lipschitz constant of the order $e^{-(\tau-\tau_0)}$. Indeed, using the definition of $g(\overline{z}_\pm)$, the (one sided) derivative $\frac{d}{dh}g(\overline{z}_\pm)$ can be expressed as the sum, over all contours Γ passing through a given site, of the product of the probability of the appearance of Γ (bounded by $e^{-\tau|\Gamma|}$) and the term $\frac{d}{dh}\log\overline{z}_\pm(\Gamma)$ (whenever $\overline{z}_\pm(\Gamma)\neq 0$). The latter can be bounded by $3|\Gamma| + 2|\operatorname{Int}\Gamma|$ as follows directly from the definition (V.1, V.2). To get the bound

$$\left|\frac{d}{dh}\log\frac{Z_{\operatorname{Int}\Gamma}(\pm\underline{1})}{Z_{\operatorname{Int}\Gamma}(\mp\underline{1})}\right| \leq 2|\operatorname{Int}\Gamma|, \tag{V.38}$$

one takes into account the explicit expressions (II.3) and (II.2) (see [Zah] for details).

Since the energies $e_\pm$ are linear in h, $e_\pm = \mp h \mp \kappa 2\nu(\nu-1)$, we infer that the free energies $f_\pm$ are 'almost linear' in h. As a result, there exists a unique solution h_t of the equation $f_+ = f_-$ and the difference of the value h_t from the value determined by equality of the ground configuration energies, $e_+ = e_-$, is of the order $e^{-(\tau-\tau_0)}$ (remember that $e^{-(\tau-\tau_0)}$ can be taken to be of the order, say $e^{-\beta}$)[20]. This fact can be stated in a more general form:

The phase diagram for large β is a deformation, of the order $e^{-\beta}$, of the phase diagram at vanishing temperature ($\beta = \infty$).

In this form the statement remains true also in a more general situation when there are r different ground configurations and one needs $(r-1)$ external fields to discriminate between them. One only has to assume that in a corresponding formula of the form (IV.2), the weights $\rho(\gamma)$ are dumped. This amounts to the *Peierls condition* — the assumptions that a contour between two ground configuratin regions leads to an increase of energy proprtional to the length of the concerned contour. The general statement of Pirogov–Sinai theory [PS] actually claims the above assertion for such case.

V.4. Completeness of the phase diagram.

Again, to formulate the problem and indicate its solution, we restrict ourselves, for simplicity, to the perturbed Ising model. In our proof we only have shown that for $h \geq h_t$ there exists the phase μ^+ and for $h \leq h_t$ there exists the phase μ^-. But nowhere we claimed that the phase μ^- does not exist for $h > h_t$ (and, similarly, that μ^+ does not exist for $h < h_t$) — actually a true statement.

Proposition V.4 [Zah, M]**.** *Let $|\kappa|$ and $|h|$ be sufficiently small, β sufficiently large. Then all translation invariant Gibbs states are convex combinations of μ_q with stable q (i. e. if $a_- \neq 0$ (resp. $a_+ \neq 0$), the state μ_+ (resp. μ_-) is the only translation invariant Gibbs state, while for $a_+ = a_- = 0$ there exist two translation invariant Gibbs states μ_+ and μ_-).*

[20]For the Potts model, the transition point β_t can be claimed, for q large, to be close (in the order $q^{-1/\nu}$) to the value yielded by the equation $(e^\beta - 1) = q^{1/\nu}$.

VI. Topics for which there is no place here

The aim of this last section is merely to indicate, in a very brief manner and without any claim for completeness, few topics for which the geometric point of view on configurations used in preceding sections played an important role. Passing by, we will mention several open problems.

Degeneracy.

Pirogov–Sinai theory works well whenever we have a finite number of ground configurations and the Peierls condition is satisfied.

Extensions to the situations with degenerated ground states [BS, Sla] (infinite number of ground configurations):

Successful: ANNNI model [DinS], microemulsions [DinM, KLMM].

ANNNI (anisotropic next nearest neighbour Ising)

$$H(\sigma) = -J_0 \sum_{\langle i,j\rangle \text{ horiz.}} \sigma_i\sigma_j - J_1 \sum_{\langle i,j\rangle \text{ vert.}} \sigma_i\sigma_j - J_2 \sum_{\langle\langle i,j\rangle\rangle \text{ vert.}} \sigma_i\sigma_j, \qquad \text{(VI.1)}$$

the first sum runs over horizontal pairs of nearest neighbours, the second one over vertical pairs, and the last sum is over vertically placed pairs i, j such that $|i - j| = 2$. At $\beta = \infty$:

$\delta = J_1 - 2J_2 > 0 \quad \to \quad$ 2 ground configurations $(+\underline{1}, -\underline{1})$,

$\delta < 0 \quad \to \quad$ 4 ground configurations,

$\delta = 0 \quad \to \quad$ infinite number of ground configurations — a degeneracy (layered configurations with planes + and −).

It is possible to use excitations to distinguish between them (determine dominant states). The trick is to introduce "restricted ensembles" — a particular ground configuration with excitations (with excess of energy of any 'local part' not exceeding a fixed level) — with their free energies playing the role of energies of ground states. Even when a boundary between two ground states might be introduced without a loss in energy, it might lead to a loss of free energy on the boundary between two restricted ensembles. This mechanism is responsible for a generalized Peierls condition to be valid.

Unsuccessful: Potts antiferromagnet — open problems even for $\beta = \infty$.

For example, one can consider a 3-dimensional Potts antiferromagnet with $q = 3$. (Hamiltonian like (IV.3) but with the opposite sign.) The conjecture (supported by heuristic arguments as well as computer simulations) is that while for $\nu = 2$ there exists a unique Gibbs state for β large, for $\nu = 3$ one expects 6 different periodic states — so called "broken sublattice symmetry" states. One of them is supported by the set of configurations for which, up to small islands, all spins on the even sublattice are aligned to be 1, while 2 and 3 are randomly distributed on the odd sublattice. The problem is open even for $\beta = \infty$: Taking the limiting finite volume states $\mu^\infty_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c}) = \lim_{\Lambda\to\mathbb{Z}^\nu} \mu^\beta_\Lambda(\sigma_\Lambda \mid \overline{\sigma}_{\Lambda^c})$ with μ^β_Λ defined by (II.1), the question is to determine the

number of Gibbs states defined by the DLR equations from μ_Λ^∞. In particular, taking μ_Λ^∞ with the boundary configuration $\overline{\sigma}$ with 1 at even sublattice (one can consider Λ whose boundary consists only from even sublattice sites) one can ask what is the probability[21] that the site 0 is occupied by spin $\sigma_0 = 1$. If $\mu_\Lambda^\infty(\sigma_0 = 1 \mid \bar{\sigma}_{\Lambda^c}) \geq \frac{1}{3} + \epsilon$ uniformly in Λ, the broken sublattice symmetry states exist.

Disorder.

There exists an alternative approach to Pirogov–Sinai theory based on an idea of renormalization group transformations applied to labeled polymer models (cf. (IV.2)–(IV.8)) [GKK]. Applying this type of the renormalization group reasoning, Bricmont and Kupiainen were able to justify the Imry–Ma argument (see below) and to prove the existence of phase transition for the three-dimensional random field Ising model [BKup].

Random Field Ising Model:

$$H(\sigma) = -\sum_{\langle i,j\rangle} \sigma_i\sigma_j - \sum_i h_i\sigma_i \tag{VI.2}$$

with i.i.d. random variables h_i.

The problem (for $\nu = 3$) was to show that for almost all $\{h_i\}$ there exist two different Gibbs states μ_+ and μ_-. In other words, the random external field does not suffice to "destroy" the spin order inherited from the case with $h_i \equiv 0$.

Imry-Ma argument goes roughly as follows: To erase a contour Γ of diameter $\sim L$ in the Peierls argument one pays by energy of the order

$$\Delta E = O(L^{\nu-1}) \pm O(L^{\nu/2}). \tag{VI.3}$$

The first term is deterministic;
the second term stems from typical fluctuations of $\sum_{i\in \mathrm{Int}\,\Gamma} h_i$.

For $\nu > 2$ the first term wins.

There are many other types of disorder modeling impurities or nonhomogenities in the system, notably the spin glass, with Hamiltonian, say,

$$H(\sigma) = -\sum_{\langle i,j\rangle} J_{i,j}\sigma_i\sigma_j. \tag{VI.4}$$

Here, again, $J_{i,j}$ are i.i.d. random variables. The problem about the number of Gibbs states (for almost all $\{J_{i,j}\}$) is entirely open and even the intuition of different physicists is very disparate.

[21] In μ_Λ^∞ which amounts to the uniform distribution over all configurations σ_Λ such that $\sigma_\Lambda \cup \bar{\sigma}_{\Lambda^c}$ have nonaligned all pairs of nearest neighbours.

Interfaces.

Up to now we were discussing only translation invariant or periodic Gibbs states. Can a Gibbs state, say for the Ising model, describing an interface between two phases be constructed? For example, can one take the boundary conditions $\overline{\sigma}^{\pm}$ with $\overline{\sigma}^{\pm}_i = +1$ if $i_\nu > 0$ and $\overline{\sigma}^{\pm}_i = -1$ if $i_\nu \le 0$, and ask whether the limiting Gibbs state $\mu_\pm$ "feels the presence of an interface" at the 'horizontal' level $i_\nu \sim 0$.

To answer this question:

• rewrite $\mu_\Lambda(\,\cdot \mid \overline{\sigma}^{\pm})$ in terms of contours with one interface contour λ attached to the plane $i_\nu = 0$ outside Λ.

• using (III.9) for the partition functions "above" and "below" λ, we get

$$\begin{aligned} P_\Lambda(\lambda) &\sim \exp\Big\{-2\beta|\lambda| + \sum_{X: A(X)\cap\lambda\neq\emptyset} \phi(X) \prod_{\Gamma\in\operatorname{supp} X} e^{-2\beta|\Gamma|X(\Gamma)}\Big\} = \\ &= e^{-2\beta|\lambda|} \prod_{X: A(X)\cap\lambda\neq\emptyset} \big(e^{\phi(X)\prod_{\Gamma\in\operatorname{supp} X} e^{-2\beta|\Gamma|X(\Gamma)}} - 1 + 1\big) = \\ &= e^{-2\beta|\lambda|} \sum_{\{X_1,\dots,X_n\}} \prod_{\alpha=1}^{n} \big(e^{\phi(X_\alpha)\prod_{\Gamma\in\operatorname{supp} X_\alpha} e^{-2\beta|\Gamma|X_\alpha(\Gamma)}} - 1\big) \end{aligned} \tag{VI.5}$$

• spliting the interface λ with the decoration $\{X_1, \dots, X_n\}$ into purely horizontal non-decorated pieces and excitations — either "walls" where λ changes its level or "decorations" or both, one can rewrite this probability in terms of a "gas" of essentially independent excitations. (One actually takes for excitations the classes of objects equivalent by vertical shifts and notices that a family of excitations in this sense determines uniquely the interface.)

We are getting a polymer model in $\nu - 1$ dimensions.

The final claim now depends on the dimension ν:

For $\nu = 2$, the resulting gas of excitations is effectively a one-dimensional model — Gallavotti [Gal] — the interface does not survive since its height can be expressed as a sum of independent random variables[22]. As a result,

$$\lim_{\Lambda\to\infty} \mu_\Lambda(\,\cdot \mid \overline{\sigma}^{\pm}) = \frac{1}{2}(\mu_+ + \mu_-). \tag{VI.6}$$

For $\nu \ge 3$ the model of excitations is at least two-dimensional, the typical configurations are just small islands of nonconnected excitations in an otherwise plain interface — Dobrushin [Dob 4]. One can thus conclude that

$$\lim_{\Lambda\to\infty} \mu_\Lambda(\,\cdot \mid \overline{\sigma}^{\pm}) = \mu^{\pm} \tag{VI.7}$$

is a translation non-invariant Gibbs state yielding a positive mean, $\langle\sigma_i\rangle^{\pm} > 0$, for sites i with $i_\nu > 0$ and a negative one, $\langle\sigma_i\rangle^{\pm} < 0$, for sites i with $i_\nu \le 0$.

[22] Actually, if we rescale by $\operatorname{diam}\Lambda$ in the horizontal direction and by its root in the vertical dimension, the interface can be shown to approach asymptotically the Brownian bridge [Hig].

Notice also that one can use the above formulation in terms of lattice gas to yield an explicit expression for the surface free energy.

There are two open problems that can be mentioned here — one probably 'solvable' and one well known and deferring all attempts for a solution.

1. Prove (or disprove) for $\nu = 3$ Ising model that the set of all Gibbs states at β large consists of μ_+, μ_- and an ∞ number of 'interface' Gibbs states corresponding to different interfaces orthogonal to coordinate axes.

In particular, show that the boundary conditions $\overline{\sigma}^{\mathrm{diag}\pm}$ defined by

$$\overline{\sigma}_i^{\mathrm{diag}\pm}\begin{cases} +1 \text{ if } i_1+i_2+i_3 \geq 0 \\ -1 \text{ if } i_1+i_2+i_3 < 0, \end{cases} \tag{VI.7}$$

does not lead to a new Gibbs state.

2. Problem of the existence of a roughening transition.

Define $\beta_c(\nu) = \inf\{\beta; \mu_+ \neq \mu_-\}$ and $\beta_r(\nu) = \inf\{\beta; \mu_\pm(\sigma_{i_\nu=0}) \neq \mu_\pm(\sigma_{i_\nu=1})\}$. Prove that $\beta_r(3) > \beta_c(3)$. (Gallavotti's result says that $\beta_r(2) = \infty$ and it is known that $\beta_r(3) \leq \beta_c(2)$ (with $\beta_c(2) > \beta_c(3)$).)

Another topic where a geometrical description of contours plays a prominent role is the study of equilibrium crystal shapes. The problem can be formulated as a study of a state μ_Λ conditioned by a fixed magnetization: $\sum_{i\in\Lambda} = m|\Lambda|$. It turns out that a typical configuration features a droplet of a particular form (Wulff shape) of one phase immersed in the other. This statement was satisfactorily proved only for the 2-dimensional Ising model [DKS] and remains an open problem for $\nu = 3$.

Finite size effects.

Say, for perturbed Ising model $\to$ what is the asymptotic behaviour of the magnetization $m_L^{\mathrm{per}}(\beta,h) = \langle\sum_{i\in\Lambda}\sigma_i\rangle_L^{\mathrm{per}}$ in a finite cube, $|\Lambda| = L^d$, under periodic boundary conditions?

In the limit $L\to\infty$, the magnetization $m_\infty^{\mathrm{per}}(\beta,h)$ displays, as a function of h, a discontinuity at $h = h_t(\beta)$ (cf. Fig. 1). For finite L, the jump is smoothed into a steep increase in a neighbourhood of $h_t \to$ *rounding*. The clue to a systematic description of this rounding lies in a fact that the "metastable free energies" $f_+(\beta,h)$ and $f_-(\beta,h)$ from (V.20) can be replaced by $\overline{f}_+(\beta,h)$ and $\overline{f}_-(\beta,h)$ introduced in a smooth manner [BKot]. In the same time, applying cluster expansions, the partition function with the periodic boundary conditions $Z_L^{\mathrm{per}}(\beta,h)$ can be explicitely very accurately evaluated,

$$\left|Z_L^{\mathrm{per}}(\beta,h) - \exp\{-\beta\overline{f}_+L^\nu\} - \exp\{-\beta\overline{f}_-L^\nu\}\right| \leq \exp\{-\beta f L^\nu - b\beta L\}, \tag{VI.8}$$

with a fixed constant b.

First peculiar consequence of this claim: the limit

$$\lim_{L\to\infty}\frac{Z_L^{\mathrm{per}}(\beta,h)}{\exp\{-\beta f L^d\}} = N(\beta,h) \tag{VI.9}$$

exists and yields an integer that equals the number of phases[23].

The magnetization $m_\infty^{\rm per}(\beta,h)$ as well as the susceptibility $\chi_\infty^{\rm per}(\beta,h) = \lim_{L\to\infty} \frac{\partial m_L^{\rm per}(\beta,h)}{\partial h}$ (recall that the perturbed Ising model does not have the plus-minus symmetry) may have a discontinuity at $h = h_t$. Spontaneous magnetizations and susceptibilities:

$$m_\pm = \lim_{h\to h_t\pm} m_\infty^{\rm per}(\beta,h), m_0 = \tfrac12(m_+ + m_-), m = \tfrac12(m_+ - m_-), \tag{VI.10}$$

$$\chi_\pm = \frac{\partial m_\infty^{\rm per}(\beta,h)}{\partial h\pm}, \chi_0 = \tfrac12(\chi_+ + \chi_-), \chi = \tfrac12(\chi_+ - \chi_-). \tag{VI.11}$$

It turns out that, in spite of the asymmetry of the model, the finite volume magnetization $m_L^{\rm per}(\beta,h)$ has, as a consequence of (VI.8), a universal behaviour in the neighbourhood of the transition point h_t:

$$\begin{aligned} m_L^{\rm per}(\beta,h) &= m_0 + \chi_0(h-h_t) + \\ &+ (m + \chi(h-h_t)) \tanh\Big\{ L^\nu \beta \big[m(h-h_t) + \tfrac12 \chi (h-h_t)^2 \big] \Big\} + R(h,L). \end{aligned} \tag{VI.12}$$

The error can be bounded by $|R(h,L)| \le e^{-b\beta L} + K(h-h_t)^2$ whenever β is large enough (and $|\kappa|$ and $|h|$ are small).

Another consequence of (VI.8): Asymptotic behaviour of different variants of the finite volume approximations of the transition point (important for the interpretation of computer simulations). A natural choice for the transition point is the inflection point $h_{\max}(L)$ of $m_L^{\rm per}(\beta,h)$). Other possible definitions: the point $h_0(L)$ for which $m_L^{\rm per}(\beta,h) = m_0$ or the point $h_t(L)$ for which an approximation to (IX.4), say

$$N_L(\beta,h) = \left[\frac{Z_L^{\rm per}(\beta,h)^{2^\nu}}{Z_{2L}^{\rm per}(\beta,h)} \right]^{\frac{1}{2^\nu - 1}}, \tag{VI.13}$$

attains its maximum ($\equiv$ the point for which $m_L^{\rm per}(\beta,h) = m_{2L}^{\rm per}(\beta,h)$).

Proposition VI.1 [BKot]**.** *For a fixed constant δ, $|\kappa|$ and $|h|$ small, and β large enough, one has*

(i) $h_{\max}(L) = h_t + \dfrac{3\chi}{2\beta^2 m^3} L^{-2\nu} + O(L^{-3\nu})$,

[23] In particular, for perturbed Ising model we get $N(\beta,h) = 1$ for $h \ne h_t(\beta)$ and $N(\beta,h) = 2$ for $h = h_t(\beta)$. For the Potts model the limit $N(\beta)$ equals

$$N(\beta) = \begin{cases} q & \text{for } \beta > \beta_t, \\ q+1 & \text{for } \beta = \beta_t, \\ 1 & \text{for } \beta < \beta_t. \end{cases}$$

(ii) *in the interval* $[h_t - \delta, h_t + \delta]$, *there exists a unique* $h_0(L)$ *for which* $m_L^{\rm per}(\beta, h) = m_0$; *for this* $h_0(L)$ *one has* $h_0(L) = h_t + O(e^{-b_0 \beta L})$, *and*
(iii) $h_t(L) = h_t + O(e^{-b_0 \beta L})$.

Potts model with $d \geq 2$ and q large enough [BKM]:
the mean energy

$$E_L^{\rm per}(\beta) \approx E_0 + E \tanh\Big\{ E(\beta - \beta_t) L^\nu + \tfrac{1}{2} \log q \Big\}. \tag{VI.14}$$

The inverse temperature $\beta_{\max}(L)$ where the slope of $E_L^{\rm per}(\beta)$ is maximal is shifted by

$$\beta_{\max}(L) - \beta_t = -\frac{\log q}{2E} L^{-\nu} + O(L^{-2\nu}), \tag{VI.15}$$

while the inverse temperature $\beta_t(L)$ for which $N_L(\beta)$ is maximal differs from β_t only by an exponentially small error $O(q^{-bL})$.

The difference between the asymptotic behaviour of the shift of the inflection point for the perturbed Ising model and the Potts model settles a controversy [BL, CLB] about the order of the shift. The fact that the shift for the Potts model is of the order $L^{-\nu}$ can be traced down to the term $\log q$ in the argument of tanh, i.e., to the fact that at β_t we have coexistence of q low temperature phases with one high temperature phase. Perturbed Ising model corresponds in this sense to $q = 1$ (coexistence of one phase for $h \leq h_t$ with one phase for $h \geq h_t$) and the term of the order $L^{-\nu}$ multiplied by the factor $\log q$ vanishes.

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