Subshift attractors of cellular automata

Enrico Formenti[1] **and Petr Kůrka**[1] [2]

[1] Université de Nice Sophia Antipolis, Département d'Informatique,
Parc Valrose, F-06108 Nice Cedex 2, France.
[2] Center for Theoretical Study, Charles University in Prague,
Jilská 1, CZ-11000 Praha 1, Czechia.

Abstract. A subshift attractor is a two-sided subshift which is an attractor of a cellular automaton. We prove that each subshift attractor is chain-mixing, contains a configuration which is both F-periodic and σ-periodic and the complement of its language is recursively enumerable. We prove that a subshift of finite type is an attractor of a cellular automaton iff it is mixing. We identify a class of chain-mixing sofic subshifts which are not subshift attractors. We construct a cellular automaton whose maximal attractor is a non-sofic mixing subshift, answering a question raised in Maass [8]. We show that a cellular automaton is surjective on its small quasi-attractor which is the nonempty intersection of all subshift attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$.

AMS classification scheme numbers: 37B15, 37B10, 54H20.

1. Introduction

Cellular automata are dynamical systems with very rich spectrum of dynamical properties. Many concepts of topological dynamics have been used to describe and classify them. When a cellular automaton is not surjective, the concept of attractor is essential for its understanding. Some attractors of cellular automata are subshifts and some are not. These two kinds of attractors have quite different properties (see e.g., Hurley [3] or Kůrka [4]). It seems that the existence of attractors which are not subshifts is rather an artefact of a definition (of inward set) which was originally coined for classical (e.g., differentiable) dynamical systems. In the totally disconnected Cantor space, the condition of inwardness reduces to simple invariance. This has a consequence that, for example, every clopen set is an attractor for the identity map - a fact which does not correspond to intuition.

In contrast, subshift attractors of cellular automata are generated by clopen sets which are not only invariant but also spreading (see Kůrka [6] or Section 3). In the present paper, we address the question how to characterize those subshifts, which are attractors of cellular automata. The special case of this question, which subshifts are maximal attractors (limit sets) has received great attention. There are several constraints on maximal attractors. For example, a maximal attractor necessarily

contains a σ-fixed configuration (of the form a^∞) and the complement of its language is recursively enumerable (Culik, Hurd and Yu [1]). When the maximal attractor is attained in a finite number of steps (this happens in the so called stable cellular automata), it is a mixing sofic subshift. In the opposite case of an unstable cellular automata, the maximal attractor cannot be a subshift of finite type (see Hurd [2]). Maass [8] proved that any mixing almost of finite type subshift (a subclass of sofic subshifts) possessing a receptive fixed point is the maximal attractor of some stable cellular automaton.

For a more general concept of a subshift attractor, things seem to be simpler. There are three general properties of subshift attractors: they are chain-mixing, contain configurations which are both F-periodic and σ-periodic, and the complements of their languages are recursively enumerable. It follows that every subshift attractor of finite type is mixing. Conversely, we prove that every mixing subshift of finite type is an attractor of some cellular automaton. Thus a mixing subshift of finite type without σ-fixed configuration is a subshift attractor but never a maximal one.

In a more general case of sofic subshifts, there are well known examples of non-mixing maximal attractors, so the characterization of subshift attractors of finite type cannot be generalized. We identify a class of chain-mixing sofic subshifts which cannot be subshift attractors. For example, all skew-Sturmian subshifts (see e.g., Kůrka [5]) are in this class. Outside of the class of sofic subshifts there are many both mixing and non-mixing subshifts which are not subshift attractors. Culik, Hurd and Yu [1] have shown that there exist maximal attractors with arbitrarily complex languages. However, none of these examples is transitive. We answer in positive a question raised in Maass [8] and construct a cellular automaton with mixing non-sofic maximal attractor.

Finally, we treat the small quasi-attractor $\mathcal{Q}_F$ which has been defined in Kůrka [6] as the (nonempty) intersection of all subshift attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$. We show that $\mathcal{Q}_F$ is a chain-mixing F-invariant subshift and that F is surjective on $\mathcal{Q}_F$.

2. Cellular automata and subshifts

For a finite alphabet A, denote by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A, and by $A^+ := \bigcup_{n>0} A^n = A^* \setminus \{\lambda\}$ the set of nonempty words. The length of a word $u \in A^n$ is denoted by $|u| := n$. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{Z}}$ the space of **A-configurations**, or doubly-infinite sequences of letters of A equipped with the metric

$$d(x, y) := 2^{-n}, \text{ where } n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}.$$

For any $u \in A^+$ we have a periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. The **cylinder** $[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l,l+|u|)} = u\}$ of a word $u \in A^*$

located at $l \in \mathbb{Z}$ is a clopen set (closed and open). Any clopen set is a finite union of cylinders.

The **shift map** $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i := x_{i+1}$. A **cellular automaton** is a continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which commutes with the shift map, i.e., $F\sigma = \sigma F$. For any cellular automaton F there exists a **radius** $r \geq 0$ and a **local rule** $F_0 : A^{2r+1} \to A$ such that $F(x)_i = F_0(x_{[i-r,i+r]})$.

A set $Y \subseteq A^{\mathbb{Z}}$ is **F-invariant** if $F(Y) \subseteq Y$, and **strongly F-invariant** if $F(Y) = Y$. The **omega-limit** of a closed invariant set Y is $\Omega_F(Y) := \bigcap_{n \geq 0} F^n(Y)$. A set $Y \subseteq A^{\mathbb{Z}}$ is an **attractor** of F if there exists a clopen F-invariant set V such that $\Omega_F(V) = Y$. The maximal attractor $\Omega_F := \Omega_F(A^{\mathbb{Z}})$ is also called the **limit set** of F.

A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{Z}}$ which is strongly σ-invariant and closed. For a subshift Σ there exists a set $D \subseteq A^+$ of **forbidden words** such that $\Sigma = \Sigma_D := \{x \in A^{\mathbb{Z}} : \forall u \sqsubseteq x, u \notin D\}$. A subshift Σ_D is of **finite type** (SFT), if D is finite. A subshift is uniquely determined by its **language**

$$\mathcal{L}(\Sigma) := \bigcup_{n \geq 0} \mathcal{L}^n(\Sigma), \text{ where } \mathcal{L}^n(\Sigma) := \{u \in A^n : \exists x \in \Sigma, u \sqsubseteq x\}.$$

A subshift is called **sofic** if its language is regular. Sofic subshifts are usually described by labelled graphs. A **labelled graph** over an alphabet A is a structure $G = (V, E, s, t, l)$, where V is a finite set of vertices, E is a finite set of edges, $s, t : E \to V$ are the **source** and **target** maps, and $l : E \to A$ is a **labelling function**. A finite or infinite word $w \in E^* \cup E^{\mathbb{Z}}$ is a **path** in G if $t(w_i) = s(w_{i+1})$ for all i. The source and target of a finite path $w \in E^n$ are $s(w) := s(w_0)$, $t(w) := t(w_{n-1})$. The label $l(w) \in A^* \cup A^{\mathbb{Z}}$ of a path w is defined by $l(w)_i := l(w_i)$. A subshift Σ is sofic iff there exists a labelled graph G and $\Sigma = \Sigma_G$ is the set of labels of all doubly infinite paths in G. The underlying (unlabelled) graph $|G| = (V, E, s, t)$ determines a subshift of finite type $\Sigma_{|G|} \subseteq E^{\mathbb{Z}}$ consisting of all paths of $|G|$. The labelling function extends to a factor map $l : \Sigma_{|G|} \to \Sigma_G$ (see e.g., Lind and Marcus [7]).

A subshift Σ is **transitive**, if for any $x, y \in \Sigma$ and any $\varepsilon > 0$ there exists $k > 0$ and $z \in \Sigma$ such that $d(x, z) < \varepsilon$ and $d(y, \sigma^k(z)) < \varepsilon$. A subshift Σ is **mixing**, if for any $x, y \in \Sigma$ and any $\varepsilon > 0$ there exists $m > 0$ such that for every $k \geq m$ there exists $z \in \Sigma$ such that $d(x, z) < \varepsilon$ and $d(y, \sigma^k(z)) < \varepsilon$. In sofic subshifts, these properties depend on the structure of their labelled graph.

A labelled graph $G = (V, E, s, t, l)$ is **connected** if for any two vertices $x, y \in V$ there exists a path from x to y. A **subgraph** of a graph G is a graph $G' = (V', E', s', t', l')$, such that $V' \subseteq V$, $E' = \{e \in E : s(e) \in V' \ \& \ t(e) \in V'\}$, and s', t', l' coincide respectively with s, t, l on E'. A **connected component** of G is a subgraph of G which is connected and maximal with respect to this property. The subshift of a connected graph is transitive. Conversely, for every transitive sofic subshift $\Sigma \subseteq A^{\mathbb{Z}}$ there exists a connected graph G such that $\Sigma = \Sigma_G$. If the subshift of a connected graph G is non-mixing, then there exists a period $p > 1$ and a partition $V = V_0 \cup \cdots \cup V_{p-1}$ of the set of vertices such that if $x \in V_i$ and there exists an arrow

from x to y, then $y \in V_{(i+1) \bmod p}$. A configuration $x \in A^{\mathbb{Z}}$ is **periodic** if $\sigma^p(x) = x$ for some $p > 0$. This happens iff $x = u^\infty$ for some $u \in A^+$. A finite subshift consists only of periodic configurations. A finite subshift is transitive iff it consists of a unique periodic orbit $\{\sigma^i(u^\infty) : \ 0 \leq i < |u|\}$. It is mixing iff $|u| = 1$.

There are weaker versions of transitivity and mixing based on ε-chains. An ε-chain (of length k from z_0 to z_k) is a sequence $z_0, \ldots, z_k$ of configurations such that $d(\sigma(z_i), z_{i+1}) < \varepsilon$ for $0 \leq i < k$. A subshift Σ is **chain-transitive** if for any $x, y \in \Sigma$ and any $\varepsilon > 0$ there exists an ε-chain from x to y. A subshift Σ is **chain-mixing**, if for any $x, y \in \Sigma$ and any $\varepsilon > 0$ there exists $m > 0$ such that for any $k \geq m$ there exists an ε-chain of length k from x to y.

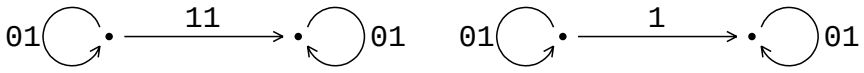

Figure 1. A chain-transitive and a chain-mixing subshifts.

Example 1 *The sofic subshifts in Figure 1 are not transitive. The left subshift is chain-transitive but not chain-mixing. The right subshift is chain-mixing.*

Transitivity and mixing can be expressed in terms of languages. A subshift Σ is transitive iff for any $u, v \in \mathcal{L}(\Sigma)$ there exists $w \in A^*$ such that $uwv \in \mathcal{L}(\Sigma)$. If $|w|$ can be any number larger than a (mixing) constant, then Σ is mixing. A subshift Σ is chain-transitive, iff for any $u, v \in \mathcal{L}(\Sigma)$ and any $q > 0$ there exists $w \in A^*$ such that $(uwv)_{[i,i+q)} \in \mathcal{L}(\Sigma)$ for all $i \leq |uwv| - q$. If $|w|$ can be any number larger than a constant, then Σ is chain-mixing. In the sequel we shall need the following Lemma.

Lemma 2 *Let $(\Sigma_n)_{n \geq 0}$ be a sequence of subshifts such that $\Sigma_{n+1} \subseteq \Sigma_n \subseteq A^{\mathbb{Z}}$. If all Σ_n are chain-mixing, then their intersection $\Sigma := \bigcap_{n \geq 0} \Sigma_n$ is also chain-mixing.*

Proof: For a given $\varepsilon > 0$ there exists a positive $\delta < \varepsilon/3$ such that if $d(x, y) < \delta$ then $d(\sigma(x), \sigma(y)) < \varepsilon/3$. By compactness there exists n such that $\Sigma_n \subseteq B_\delta(\Sigma) := \{y \in A^{\mathbb{Z}} : \exists x \in \Sigma, d(x, y) < \delta\}$. Let $x, x' \in \Sigma$ and $k > 0$. Since $x, x' \in \Sigma_n$ and Σ_n is chain-mixing, for each $m \geq k$ there exists an $\frac{\varepsilon}{3}$-chain $x = y_0, \ldots, y_m = x'$ in Σ_n. Since $\Sigma_n \subseteq B_\delta(\Sigma)$, there exist $x_i \in \Sigma$ such that $d(x_i, y_i) < \delta$. For $i = 0, m$ we set $x_0 := x$, $x_m := x'$. Then $x = x_0, \ldots, x_m = x'$ is an ε-chain in Σ since

$$d(\sigma(x_i), x_{i+1}) \leq d(\sigma(x_i), \sigma(y_i)) + d(\sigma(y_i), y_{i+1}) + d(y_{i+1}, x_{i+1}) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \delta < \varepsilon. \qquad \square$$

3. Spreading sets

A **subshift attractor** of a cellular automaton is an attractor which is a subshift. For example, the maximal attractor Ω_F is a subshift attractor. It has been proved in Kůrka [6] that the only subshift attractor of a surjective cellular automaton is the full space. We characterize now clopen invariant sets whose omega limits are subshifts.

Definition 3 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. We say that a clopen F-invariant set $U \subseteq A^{\mathbb{Z}}$ is* **spreading** *if $F^k(U) \subseteq \sigma^{-1}(U) \cap U \cap \sigma(U)$ for some $k > 0$. We denote by $\widetilde{\Omega}_\sigma(U) := \bigcap_{i\in\mathbb{Z}} \sigma^i(U)$ the σ-limit set of U.*

If U is a spreading set then $\widetilde{\Omega}_\sigma(U)$ is an F-invariant SFT, $\Omega_F(U) \subseteq \widetilde{\Omega}_\sigma(U)$ and $\Omega_F(U) = \Omega_F(\widetilde{\Omega}_\sigma(U))$.

Proposition 4 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and U a clopen F-invariant set. Then $\Omega_F(U)$ is a subshift attractor iff U is spreading.*

Proof: If $\Omega_F(U)$ is a subshift, then $\Omega_F(U) = \Omega_F(\sigma(U)) \subseteq \sigma(U)$, and by compactness there exists $k > 0$ such that $F^k(U) \subseteq \sigma(U)$. A similar argument applies to $\sigma^{-1}(U)$. Conversely, assume that $F^k(U) \subseteq \sigma^{-1}(U) \cap U \cap \sigma(U)$. If $y \in \Omega_F(U)$ and $n \geq 0$ then $y \in F^{n+k}(U)$. So there exists $x \in U$ with $y = F^{n+k}(x) = F^n(F^k(x)) \in F^n(\sigma^{-1}(U))$, and $y \in \Omega_F(\sigma^{-1}(U)) = \bigcap_{n\geq 0} \sigma^{-1}F^n(U)$. Thus $\sigma(y) \in \Omega_F(U)$, and $\sigma(\Omega_F(U)) \subseteq \Omega_F(U)$. Similarly we prove $\sigma^{-1}(\Omega_F(U)) \subseteq \Omega_F(U)$, so $\Omega_F(U)$ is a subshift. □

The zero cellular automaton $F(x)_i = 0$ has the unique subshift attractor $\Omega_F = F(A^{\mathbb{Z}}) = \{0^\infty\}$. However, any clopen set which contains a cylinder of the form $[0^n]_l$ is spreading. Thus, a spreading set may contain many superfluous cylinders which can be omitted without changing the attractor. When we do this, the resulting σ-limit set $\widetilde{\Omega}_\sigma(U)$ is mixing. This is proved in Proposition 5.

Proposition 5 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and U a spreading set. There exists a spreading set $W \subseteq U$ such that $\Omega_F(U) = \Omega_F(W)$ and $\widetilde{\Omega}_\sigma(W)$ is a mixing subshift of finite type.*

Proof: We can assume that U is a union of cylinders of length $q+1$ located at 0. Construct a labelled graph $G = (V, E, s, t, l)$, where

$$E = \{u \in A^{q+1} : [u]_0 \subseteq U\},\ V = \{u \in A^q : \exists a \in A, ([ua]_0 \subseteq U \text{ or } [au]_0 \subseteq U)\},$$

$s, t : E \to V$ are the prefix and the suffix maps, and the label is $l(u) = u_q$. Then, $\widetilde{\Omega}_\sigma(U) = \Sigma_G$ is F-invariant. If G_0 is a connected component of G, then $F(\Sigma_{G_0})$ is a transitive subshift and there exists a connected component G_1 of G such that $F(\Sigma_{G_0}) \subseteq \Sigma_{G_1}$. In this way we get a dynamical system $\mathcal{F}$ on the finite set $\mathcal{G}$ of connected components of G. This system has some preperiod $m \geq 0$ and period $n > 0$, so $\mathcal{F}^{m+n} = \mathcal{F}^m$. For every $G_0 \in \mathcal{F}^m(\mathcal{G})$ we have $F^n(\Sigma_{G_0}) \subseteq \Sigma_{G_0}$. We show that $\mathcal{F}^m(\mathcal{G})$ consists of a unique connected component. Assume by contradiction that there exist different connected components G_0, G_1 such that $F^n(\Sigma_{G_i}) \subseteq \Sigma_{G_i}$ for $i = 0, 1$. Take some $x \in \Sigma_{G_0}$, $y \in \Sigma_{G_1}$ and define a configuration $z \in A^{\mathbb{Z}}$ by $z_{(-\infty,0)} := x_{(-\infty,0)}$, $z_{[0,\infty)} := y_{[0,\infty)}$. Since U is spreading, there exists $k > 0$ such that $F^k(z) \in \Sigma_G$. If r is the radius of F, then $F^{kn}(z)_{[-q-knr,-knr]} \in \Sigma_{G_0}$, and $F^{kn}(z)_{[knr,knr+q]} \in \Sigma_{G_1}$. This means that there is a path from G_0 to G_1 in G. Similarly, we prove that there exists a path from G_1 to G_0. Thus $G_0 = \mathcal{F}^n(\mathcal{G})$ is the unique connected component

of G. We show that Σ_{G_0} is mixing. If not, then there exists a period $p > 1$ and a disjoint union of closed sets $\Sigma_{G_0} = Y_0 \cup \cdots \cup Y_{p-1}$ such that $\sigma(Y_i) \subseteq Y_{(i+1) \bmod p}$. Take some $x \in Y_0$, $y \in Y_1$ and define a configuration $z \in A^{\mathbb{Z}}$ by $z_{(-\infty,0)} := x_{(-\infty,0)}$, $z_{[0,\infty)} := y_{[0,\infty)}$. Since U is spreading, there exists k such that $F^k(z) \in \Sigma_{G_0}$. However, $F^k(z)$ does not belong to any Y_i. This is a contradiction, so Σ_{G_0} is mixing. Set $W := \bigcup_{u \in E_{G_0}} [u]_0$. Then $W \subseteq U$ is spreading and $\widetilde{\Omega}_\sigma(U) = \widetilde{\Omega}_\sigma(W) = \Sigma_{G_0}$ is mixing. Clearly $\Omega_F(U) = \Omega_F(\widetilde{\Omega}_\sigma(U)) = \Omega_F(\widetilde{\Omega}_\sigma(W)) = \Omega_F(W)$. □

4. Subshift attractors

In this section we prove some general properties of subshift attractors.

Theorem 6 *Each subshift attractor is chain-mixing.*

Proof: Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. Let U be a spreading set and $\Sigma := \Omega_F(U)$. By Proposition 5, there exists a spreading set $W \subseteq U$ such that $\Sigma_0 := \widetilde{\Omega}_\sigma(W)$ is a mixing SFT and $\Sigma = \Omega_F(W) = \Omega_F(\Sigma_0)$. It is easy to see that each $F^n(\Sigma_0)$ is mixing, so their intersection Σ is chain-mixing by Lemma 2. □

Theorem 7 *Each subshift attractor contains a configuration which is both F-periodic and σ-periodic.*

Proof: Let U be a spreading set which is a union of cylinders of length p. Then the set $U \cap \{x \in A^{\mathbb{Z}} : \sigma^p(x) = x\}$ is nonempty, finite, and F-invariant. Therefore, it contains an F-periodic configuration. □

As an immediate consequence of Theorem 7, no minimal infinite subshift can be a subshift attractor of a cellular automaton. We generalize now a result of Culik, Hurd and Yu [1].

Theorem 8 *If $\Sigma \subseteq A^{\mathbb{Z}}$ is a subshift attractor, then the language $A^* \setminus \mathcal{L}(\Sigma)$ is recursively enumerable.*

Proof: Let F be a cellular automaton. Let U be a spreading set which is a union of cylinders of length p located at 0, and assume that $\Sigma = \Omega_F(U)$. A word $u \in A^*$ does not belong to $\mathcal{L}(\Sigma)$ iff there exists $n > 0$ such that each $v \in f^{-n}(u)$ contains a subword of length p whose cylinder located at 0 is disjoint with U. □

5. Subshifts of finite type

By a theorem of Walters (see e.g., Kůrka [5]), a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is a SFT iff it has the shadowing property. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for

any finite δ-chain $z_0, \ldots, z_k$ there exists $x \in \Sigma$ such that $d(\sigma^i(x), z_i) < \varepsilon$ for all $i \leq k$. It follows that an SFT is mixing iff it is chain-mixing. From Theorem 6 we obtain immediately Corollary 9. Theorem 10 shows that the converse of Corollary 9 is also true.

Corollary 9 *Every subshift attractor of finite type is mixing.*

Theorem 10 *If $\Sigma \subseteq A^{\mathbb{Z}}$ is a mixing subshift of finite type, then there exists a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that Σ is an attractor of F, and $F(x) = x$ for every $x \in \Sigma$.*

Proof: Given a configuration $x \in A^{\mathbb{Z}} \setminus \Sigma$ and $u \sqsubseteq x$, write u as $u = vzw$, where v is the maximal prefix of u which belongs to $\mathcal{L}(\Sigma)$ and w is the maximal suffix of u which belongs to $\mathcal{L}(\Sigma)$. If z is sufficiently long, the automaton changes the first and last letters of z so that the interval outside of $\mathcal{L}(\Sigma)$ shortens. If z is short, the automaton replaces it by a word z' of the same length, such that $vz'w$ belongs to $\mathcal{L}(\Sigma)$.

Let k be the order of Σ, so $x \in \Sigma$ iff $x_{[i,i+k)} \in \mathcal{L}(\Sigma)$ for all $i \in \mathbb{Z}$. Let $m \geq 2$ be a mixing constant, so for every $u, v \in \mathcal{L}(\Sigma)$ and for every $n \geq m$ there exists $w \in A^n$ such that $uwv \in \mathcal{L}(\Sigma)$. For $l, r \in \mathbb{Z}$ set $\Sigma_l := \{x_{(-\infty,l)} : \ x \in \Sigma\}$, and $\Sigma^r := \{x_{[r,\infty)} : \ x \in \Sigma\}$. We say that $x \in A^{\mathbb{Z}}$ is Σ-asymptotic, if $x \notin \Sigma$ and there exist $l, r \in \mathbb{Z}$ such that $x_{(-\infty,l)} \in \Sigma_l$ and $x_{[r,\infty)} \in \Sigma^r$. If x is a Σ-asymptotic configuration, set

$$l_x := \max\{l \in \mathbb{Z} : \ x_{(-\infty,l)} \in \Sigma_l\}, \ r_x := \min\{r \in \mathbb{Z} : \ x_{[r,\infty)} \in \Sigma^r\}.$$

Since there exists $a \in \mathbb{Z}$ such that $x_{[a,a+k)} \notin \mathcal{L}(\Sigma)$, we get $l_x \leq a + k - 1$, $r_x \geq a + 1$ and therefore $r_x \geq l_x - k + 2$. Consider a (correcting) set

$$C := \{x_{[l_x-k,l_x+k+m)} : \ x \text{ is } \Sigma - \text{asymptotic, and } r_x - l_x \leq m\} \subseteq A^{2k+m}.$$

Clearly, the words of C do not belong to $\mathcal{L}(\Sigma)$. Since Σ is mixing of constant m, there exists a function $g_0 : C \to \mathcal{L}^{2k+m}(\Sigma)$ such that $g_0(u)_{[0,k)} = u_{[0,k)}$ and $g_0(u)_{[|u|-k,|u|)} = u_{[|u|-k,|u|)}$. There exist also right and left extending functions $g_r, g_l : \mathcal{L}(\Sigma) \to A$, such that for each $u \in \mathcal{L}(\Sigma)$, both $ug_r(u)$ and $g_l(u)u$ belong to $\mathcal{L}(\Sigma)$. We construct a local rule with radius $c = k + m - 1$. For a given $u \in A^{2c+1}$, set

$$l_u := \max\{l \geq 0 : \ u_{[0,l)} \in \mathcal{L}(\Sigma)\}, \ r_u := \min\{r \leq |u| : \ u_{[r,|u|)} \in \mathcal{L}(\Sigma)\}.$$

Define the local rule $f : A^{2c+1} \to A$ by

$$f(u) := \begin{cases} g_r(u_{[0,c)}) & \text{if } r_u - l_u > m \text{ and } l_u = c \\ g_l(u_{(c,2c]}) & \text{if } r_u - l_u > m \text{ and } r_u = c + 1, \\ g_0(u_{[l_u-k,l_u+k+m)})_{c-l_u+k} & \text{if } r_u - l_u \leq m \text{ and } l_u \leq c < l_u + m \\ u_c & \text{otherwise.} \end{cases}$$

We show that for $p = 2k + m - 2$, the clopen set $U = \bigcup_{u \in \mathcal{L}^p(\Sigma)} [u]_0$ is spreading and that Σ is its omega-limit set. First we show that if $x \in U$ then $F(x)_i = x_i$ for all $0 \leq i < p$.

Assume by contradiction that $x_{[0,p)} \in \mathcal{L}(\Sigma)$ and that for some $0 \leq i < p$, $F(x)_i \neq x_i$. Denote by $u = x_{[i-c,i+c]}$. We distinguish three cases:

1. $F(x)_i = g_r(x_{[i-c,i)})$. Then $l_u = c$, $u_{[0,c)} \in \mathcal{L}(\Sigma)$, and $u_{[0,c]} \notin \mathcal{L}(\Sigma)$, so $x_{[i-k+1,i]} = u_{[c-k+1,c]} \notin \mathcal{L}(\Sigma)$. Since $x_{[0,p)} \in \mathcal{L}(\Sigma)$, we get $i < k-1$. On the other hand $r_u > m + l_u = m + c$, so $x_{[i+m,i+c]} = u_{[m+c,2c]} \notin \mathcal{L}(\Sigma)$. Since $x_{[0,p)} \in \mathcal{L}(\Sigma)$, $i + c > p - 1$, so $i > p - c - 1 = k - 2$, and this is a contradiction.

2. $F(x)_i = g_l(x_{(i,i+c]})$. Then $r_u = c + 1$, so $u_{(c,2c]} \in \mathcal{L}(\Sigma)$ and $u_{[c,2c]} \notin \mathcal{L}(\Sigma)$. It follows that $x_{[i,i+k)} = u_{[c,c+k)} \notin \mathcal{L}(\Sigma)$ and therefore $i + k > p$, so $i > k + m - 2$. On the other hand, $l_u < r_u - m = k$, so $x_{[i-c,i-c+k)} = u_{[0,k)} \notin \mathcal{L}(\Sigma)$, and $i < c = k + m - 1$ and we get again a contradiction.

3. $F(x)_i = g_0(u_{[l_u-k,l_u+k+m)})_{c-l_u+k}$. Then $x_{[i-c+l_u-k+1,i-c+l_u+1)} = u_{[l_u-k+1,l_u+1)} \notin \mathcal{L}(\Sigma)$, and $x_{[i-c+r_u-1,i-c+r_u-1+k)} = u_{[r_u-1,r_u-1+k)} \notin \mathcal{L}(\Sigma)$, so $i - c + l_u - k + 1 \leq -1$ and $i - c + r_u - 1 + k \geq p + 1$ which implies $p + 2 + c - r_u - k \leq i \leq c - l_u + k - 2$. We get $r_u - l_u \geq p + 4 - 2k = m + 2$ and this is in a contradiction with the assumption $r_u - l_u \leq m$. We have proved that U is an invariant set and that any $x \in \Sigma = \bigcap_{i\in\mathbb{Z}} \sigma^i(U)$ is a fixed point for F. We show now that $F(U) \subseteq \sigma^{-1}(U) \cap U \cap \sigma(U)$. Assume by contradiction that $x_{[0,p)} \in \mathcal{L}(\Sigma)$ and $F(x)_{[1,p]} \notin \mathcal{L}(\Sigma)$, so $x_{[1,p]} \notin \mathcal{L}(\Sigma)$. Set $u = x_{[p-c,p+c]}$, so $l_u = c$. If $r_u - l_u > m$, then $F(x)_p = g_r(x_{[p-c,p)})$ and $F(x)_{[0,p]} \in \mathcal{L}(\Sigma)$ and this is a contradiction. If $r_u - l_u \leq m$, then $F(x)_p = g_0(u_{[l_u-k,l_u+k+m)})_{p-l_u+k}$ and $F(x)_{[p-c,p+c]} \in \mathcal{L}(\Sigma)$ so we get again a contradiction. Thus U spreads to the right, and one can similarly prove that U also spreads to the left. Thus $\Omega_F \subseteq \bigcap_{i\in\mathbb{Z}} \sigma^i(U) = \Sigma$. Since every $x \in \Sigma$ is a fixed point, we get $\Sigma \subseteq \Omega_F$. □

Subshift attractors are more general than maximal attractors. Subshifts of finite type which are maximal attractors have been characterized by Maass [8]: A subshift of finite type $\Sigma \subseteq A^{\mathbb{Z}}$ is a maximal attractor iff it is mixing and contains a receptive fixed point. Recall that a word $u \in \mathcal{L}(\Sigma)$ is called **magic** (see e.g., Lind and Marcus [7]), if for all $vu \in \mathcal{L}(\Sigma)$ and $uw \in \mathcal{L}(\Sigma)$ we have $vuw \in \mathcal{L}(\Sigma)$. A σ-fixed point $a^\infty \in \Sigma$ is called **receptive**, if there exist magic words u, v such that for all large enough n, $ua^nv \in \mathcal{L}(\Sigma)$.

6. Sofic subshifts

Corollary 9 and Theorem 10 cannot be generalized to sofic subshifts. There are many non-mixing sofic subshifts which are maximal attractors. An example is the product rule $F(x)_i = x_{i-1}x_ix_{i+1}$ in the binary alphabet, whose maximal attractor is $\Omega_F = \{x \in \{0,1\}^{\mathbb{Z}} : \forall n > 0, 10^n1 \not\sqsubset x\}$. Since each sofic subshift contains a σ-periodic configuration and the complement of its language is recursively enumerable, the only constraint of Section 4 is that a sofic subshift attractor must be chain-mixing. This is, however, not a characterization. Proposition 11 shows that there exists a class of chain-mixing sofic subshifts which are not subshift attractors.

Proposition 11 *Let G be a labelled graph with two connected components G_0 and G_1 such that $\Sigma_G \neq \Sigma_{G_0} = \Sigma_{G_1} = \{\sigma^i(u^\infty) : \ i < |u|\}$ for some $u \in A^+$. Then Σ_G is not a subshift attractor.*

Proof: Assume that F is a cellular automaton, U is an F-spreading set and $\Sigma_G = \Omega_F(U)$. Since $F(\Sigma_{G_0})$ is transitive, we get $F(\Sigma_{G_0}) \subseteq \Sigma_{G_0}$. Every $x \in \Sigma_G \setminus \Sigma_{G_0}$ is clearly both left and right Σ_{G_0}-asymptotic (see the proof of Theorem 10 for definitions). Moreover, there exists a constant c such that for every $x \in \Sigma_G \setminus \Sigma_{G_0}$, $r_x - l_x < c$. Choose some $x_0 \in \Sigma_G \setminus \Sigma_{G_0}$. Since F is surjective on Σ_G, there exists a sequence of configurations $x_i \in \Sigma_G$ such that $F(x_{i+1}) = x_i$. Since Σ_{G_0} is F-invariant, all x_i belong to $\Sigma_G \setminus \Sigma_{G_0}$, and, for some nonnegative integers $i < j$, $(x_i)_{[l_{x_i}-|u|, r_{x_i}+|u|)} = (x_j)_{[l_{x_j}-|u|, r_{x_j}+|u|)}$. This implies in particular that $r_{x_i} - l_{x_i} = r_{x_j} - l_{x_j}$. It follows that for $q = j - i$ and some $p \in \mathbb{Z}$, $F^q\sigma^p(x_i) = x_i$. Take a subword $w \sqsubseteq x_i$ such that $(x_i)_{[l_{x_i}-|u|, r_{x_i}+|u|)} \sqsubseteq w$, and for some $k > 0$, u^k is both a prefix and a suffix of w. If k is large enough, then $z = w^\infty \in U$ satisfies $F^q\sigma^p(z) = z$ and $F^n(z)$ does not converge to Σ_G. This is a contradiction. □

The subshift in Figure 1 right is an example of a chain-mixing sofic subshift which is not subshift attractor. The hypothesis of Proposition 11 is satisfied by every skew-Sturmian subshift (see e.g., Kůrka [5]). Some characterizations of sofic subshifts which are maximal attractors have been given in Maass [8]. A cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is called **stable**, if $\Omega_F(A^{\mathbb{Z}}) = F^n(A^{\mathbb{Z}})$ for some n. Otherwise, F is called **unstable**.

Definition 12 *A sofic subshift Σ is* **almost of finite type**, *if there exists a connected labelled graph G such that $\Sigma = \Sigma_G$, and the factor map $l : \Sigma_{|G|} \to \Sigma_G$ is biclosing. This means that for every $u, v \in \Sigma_{|G|}$ such that $l(u) = l(v)$ and either $u_{(-\infty,0)} = v_{(-\infty,0)}$ or $u_{[0,\infty)} = v_{[0,\infty)}$ we have $u = v$. If moreover $\{x \in \Sigma : \ |l^{-1}(x)| > 1\}$ is a finite set of σ-periodic points, then Σ is called* **near Markov**.

In [8] Maass proves that every mixing sofic subshift which is almost of finite type and possesses a receptive fixed point, is a stable limit set of a cellular automaton. An example is the even subshift $\Sigma = \{x \in \{0,1\}^{\mathbb{Z}} : \ 01^{2n+1}0 \not\sqsubseteq x\}$. Another theorem of [8] says that no near Markov subshift is the unstable limit set of a cellular automaton. The proof of this fact, however, cannot be generalized to subshift attractors.

7. Non-sofic subshifts

Because of the constraints given by Theorems 6, 7, 8, there are many subshifts (both mixing and non-mixing) which are not subshift attractors. Culik, Hurd and Yu [1] construct cellular automata whose maximal attractors have arbitrarily high language complexity. However, none of these examples is transitive. We construct a cellular automaton whose maximal attractor is mixing but not sofic.

Example 13 *There exists a cellular automaton whose maximal attractor is a non-sofic mixing subshift.*

x10:0,	x11:0,	x20:3,	x21:3,	34x:3,	44x:3,	35x:0,	45x:0,
x14:2,	14x:5,	124:0,	154:3,	1zz:2,	yy4:5,	12x:1,	x54:4,
x1x:0,	x4x:3,	xyz:2,	yzx:5,	x02:2,	53x:5,	z2x:1,	x5y:4.

Table 1. Local rule for Example 13.

Proof: The automaton uses the alphabet $A := A_0 \cup A_1$, where $A_0 := \{0, 1, 2\}$ and $A_1 := \{3, 4, 5\}$. The radius is $r = 1$, so $F(x)_i = F_0(x_{[i-1,i+1]})$. The local rule $F_0 : A^3 \to A$ is given in Table 1, where $x \in A$, $y \in A_0$, $z \in A_1$, and the first applicable rule is used, otherwise the letter is left unchanged. A simulation can be seen in Figure 2.

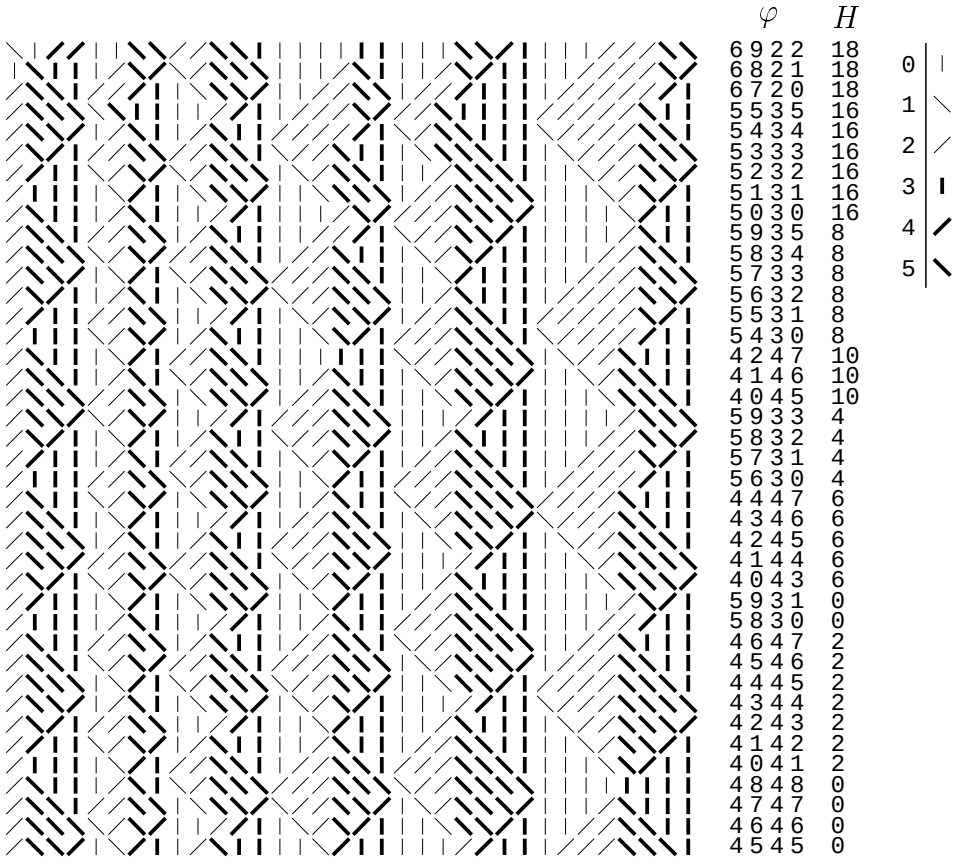

Figure 2. A cellular automaton with mixing non-sofic maximal attractor.

A structure of a configuration of Ω_F is pictured in Figure 3 left. The ordering of the letters in the maximal attractor is given by a mixing SFT Σ_D (Figure 3 right) with forbidden words $D = \{10, 11, 20, 21, 34, 44, 35, 45\}$. For each $u \in D$ there is a production which eliminates u from Ω_F. It can be verified by a computer program that $F^3(A^{\mathbb{Z}}) \subseteq \Sigma_D$. The program and its output can be found at http:\\www.cts.cuni.cz\~kurka\prog.html. Given a configuration $x \in A^{\mathbb{Z}}$, we say that an interval $I \subseteq \mathbb{Z}$ is an A_j-interval, if $\forall i \in I, x_i \in A_j$, and I is maximal with this property. There is at most one left-unbounded A_j-interval of the form $(-\infty, i)$ and at most one right-unbounded A_j-interval

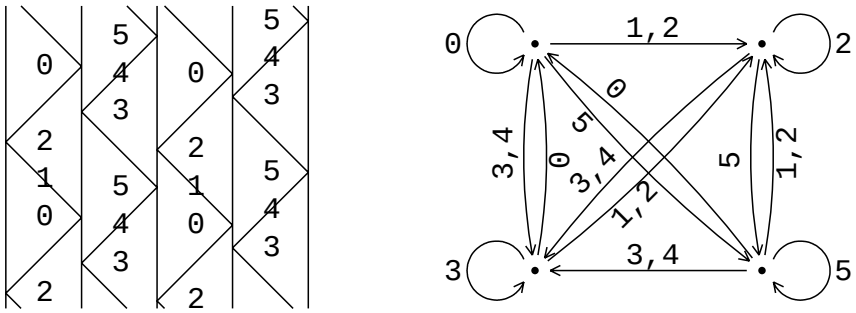

Figure 3. The structure of the maximal attractor and its SFT approximation.

of the form $[i, \infty)$. Each border between an A_0-interval and A_1-interval (in this order) generates a particle 2 via xyz:2, which propagates to the left via x02:2. On encounter with another A_1-interval, 2 changes to 1 via z2x:1 which goes to the right while erasing all 2 particles via 12x:1 and x1x:0. A border between an A_0-interval and A_1-interval generates also a particle 5 via yzx:5, which propagates to the right via 53x:5. On encounter with another A_0-interval, 5 changes to 4 via x5y:4 which goes to the left while erasing all 5 particles via x54:4 and x4x:3. When particles 1 and 4 meet, they transform to the left-going particles 2 and 5 via x14:2, 14x:5. If 1 and 4 meet one field apart, they disappear via productions 124:0 and 154:3. When 1 encounters a border with an A_1-interval without the presence of 4, and when this A_1-interval has width at least 2, the border is shifted to the right by 1zz:2. Similarly when 4 encounters an A_0-interval without the presence of 1, and when this A_0-interval has width at least 2, the border is shifted to the left by yy4:5. From the time 3 onwards, the borders between A_1-intervals and A_0-intervals (in this order) do not move. The borders between A_0-intervals and A_1-intervals are shifted (provided both these intervals have width at least 2) alternatively to the right and to the left until both regions have nearly equal width. The claims of the Example are proved in following Lemmas. □

Lemma 14 *Any word of the form $auvb \in \mathcal{L}(\Sigma_D)$, where $a \in A_1$, $u \in A_0^+$, $v \in A_1^+$, $b \in A_0$ is blocking and determines the whole column below uv. This means that there exists a function $f : A_0^+A_1^+ \cap \Sigma_D \to A_0^+A_1^+ \cap \Sigma_D$ such that if $x \in \Sigma_D$, $x_{[i,j)} = uv$, $x_{i-1} \in A_1$, $x_j \in A_0$, then $F^n(x)_{[i,j)} = f^n(uv)$.*

Proof: The first eight productions (in the first row of Table 1) cannot be applied to words of $\mathcal{L}(\Sigma_D)$, so the only productions which change the alphabet of a site are 1zz:2 and yy4:5. Both these productions change the border from A_0-interval to A_1-interval, so the borders between A_1-intervals and A_0-intervals remain fixed. The only productions applicable to the first position of an A_0-interval are x14:2, x1x:0, xyz:2, x02:2, z2x:1, x5y:4. Each of these productions begins either with x or z, so it gives the same value for all letters of A_1. A similar argument applies for the last letter of uv. This shows that $auvb$ is a blocking word. □

Lemma 15 *For $u \in A_0^+ \cap \Sigma_D$, $v \in A_1^+ \cap \Sigma_D$ set $\varphi(uv) := (|u|, x(u), |v|, y(v))$, where*

$$x(0^i 12^j) := j,\ x(0^i 2^j) := 2i + j,\ y(5^j 43^i) := j,\ y(5^j 3^i) := 2i + j.$$

Then φ is injective and

$$W := \varphi(A_0^+ A_1^+ \cap \Sigma_D) = \{(n, x, m, y) \in \mathbb{N}^4 :\ n, m > 0, 0 \le x \le 2n, 0 \le y \le 2m\}.$$

The proof is straightforward. The phases $x(u)$ and $y(v)$ correspond to the time before the length of u or v is extended. The value of φ (of the last A_0 and A_1-intervals) together with the value of function H defined in the proof of Lemma 16 can be seen in Figure 2.

Lemma 16 *Define a map $g : W \to W$ by $g := \varphi^{-1} \circ f \circ \varphi$ (By Lemma 15, $\varphi : A_0^+ A_1^+ \cap \Sigma_D \to W$ is bijective). Denote by*

$$W^a := \{(n, x, m, y) \in W :\ m + n = a\}$$
$$W_k := \{(n, x, m, y) \in W :\ m - n = k\},\ W_{[i,j]} := \bigcup_{k=i}^{j} W_k.$$

Then each W^a is a finite g-invariant set. For $a \ge 6$, each cyclic trajectory of g is contained either in $W^a \cap W_{a-2}$ or in $W^a \cap W_{2-a}$ or in $W_{[-8,8]}$.

case	w	$g(w)$	condition	$H(g(w)) - H(w)$
1	(n, x, m, y)	$(n, x-1, m, y-1)$	$x, y > 0$	0
2	$(n, 0, m, y)$	$(n+1, 2n+1, m-1, y-2)$	$y > 1$	$2(n-m)+6$
3	$(n, 0, m, 1)$	$(n, 2n, m, 2m)$		$4(n-m)+2$
4	$(n, x, m, 0)$	$(n-1, x-2, m+1, 2m+1)$	$x > 1$	$2(n-m)-6$
5	$(n, 1, m, 0)$	$(n, 2n, m, 2m)$		$4(n-m)-2$
6	$(n, 0, m, 0)$	$(n, 2n-1, m, 2m-1)$		$4(n-m)$

Table 2. Dynamics of $g : W \to W$ and its effect on $H : W \to \mathbb{Z}$.

Proof: Define a function $H : W \to \mathbb{Z}$ by $H(n, x, m, y) := m^2 - 2y - n^2 + 2x$. Then $m^2 - 4m - n^2 \le H(n, x, m, y) \le m^2 - n^2 + 4n$ and

$$w \in W^a \cap W_k \Rightarrow (a-2)k - 2a \le H(w) \le (a-2)k + 2a$$

Formulas for g and H which hold for $n, m \ge 2$ are given in Table 2. If $w \in W_k$, then $g(w) \in W_{k-2} \cup W_k \cup W_{k+2}$. Clearly each W^a is a finite invariant set so the trajectory of g finally reaches a cycle. If for some $w = (n, x, m, y)$ we have $n = 1$, then the whole cycle is contained in $W^{m+1} \cap W_{m-1} = W^a \cap W_{a-2}$. Similarly if $m = 1$, then the whole cycle is contained in $W^{m+1} \cap W_{1-m} = W^a \cap W_{2-a}$. Assume therefore that $n, m \ge 2$ and $a \ge 6$. If $m - n \le -3$ then $H(g(w)) \ge H(w)$ with equality only in cases 1 and 2 of Table 2. However, cases 1 and 2 cannot be used alone in a whole cycle, so for some w

in such a cycle we get $H(g(w)) > H(w)$. Thus no cycle is contained in $W_{(-\infty,-3]}$. If a cycle contains an element of W_{-9}, then it must also contain an element of W_{-1} and there exists $w \in W_{-3}$, and $k > 0$ such that $g^i(w) \in W_{-3} \cup W_{-5} \cup W_{-7}$ for $0 \leq i < k$ and $g^k(w) \in W_{-9}$. The function H must increase somewhere along this trajectory so $H(g^k(w)) > H(w)$. On the other hand $H(g^k(w)) \leq -7a + 18 \leq -5a + 6 \leq H(w)$ and this is a contradiction. Thus no cycle can contain an element of $W_{(-\infty,-9]}$. Similarly it can be proved that no cycle can contain an element of $W_{[9,\infty)}$. □

Lemma 17 *Ω_F is not sofic.*

Proof: If $auvb \in \mathcal{L}(\Omega_F)$ where $a \in A_1$, $u \in A_0^+$, $v \in A_1^+$, $b \in A_0$, and $|uv| \geq 6$ then, by Lemma 16, either $|u| \leq 1$ or $|v| \leq 1$ or $-8 \leq |u| - |v| \leq 8$. A straightforward application of the Pumping lemma gives that $\mathcal{L}(\Omega_F)$ is not regular. □

Lemma 18 *Ω_F is mixing.*

Proof: Set $L := \{uv : \ u \in A_0^+, v \in A_1^+, \exists a \in A_1, \exists b \in A_0, auvb \in \mathcal{L}(\Omega_F)\}$, and let $L^* = \bigcup_{n>0} L^n$ be the set of all concatenations of words of L. By Lemma 14, $L^* \subseteq \mathcal{L}(\Omega_F)$. Let $u \in \mathcal{L}(\Omega_F)$, so there exists $x \in \Omega_F$ such that $u \sqsubseteq x$. It is easy to see that x can contain neither a left-unbounded A_0-interval nor a right-unbounded A_1-interval. We show that there exists $u' \in L^*$ such that $u \sqsubseteq u'$. This is clear if x does not contain any unbounded A_i-interval. Assume therefore that $(-\infty, j)$ is a left-unbounded A_1-interval. Then for any $i < j$ there exists $u \in A_0^{j-i}$ such that $ux_{[i,j)} \in L$. If $x_{[i,j)} = 5^k 3^l$ then $u = 0^l 2^k$. If $x_{[i,j)} = 5^k 43^l$ then $u = 0^l 12^k$. Similarly, if $x_{[i,\infty)}$ is a right-unbounded A_0-interval, then for any $j > i$ there exists $v \in A_1^{j-i}$ such that $x_{[i,j)}v \in L$. Thus for every $u, v \in \mathcal{L}(\Omega_F)$ there exist $u', v' \in L^*$ such that $u \sqsubseteq u'$ and $v \sqsubseteq v'$. For every $n > 1$ there exists $w \in L \cap A^n$ such that $u'wv' \in \mathcal{L}(\Omega_F)$, so Ω_F is mixing. □

8. The small quasi-attractor

The small quasi-attractor $\mathcal{Q}_F$ of a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ has been defined in [6] as the nonempty intersection of all subshift attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$. We show that $\mathcal{Q}_F$ is chain-mixing, F-invariant and F is surjective on $\mathcal{Q}_F$.

Lemma 19 *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton and $U_1, \ldots, U_k$ spreading sets, then $U_1 \cap \cdots \cap U_k$ is a nonempty spreading set and $\Omega_F(U_1) \cap \cdots \cap \Omega_F(U_k)$ is a nonempty F-invariant subshift.*

Proof: Since a clopen set is a finite union of cylinders, there exists a configuration $x \in A^{\mathbb{Z}}$ such that the σ-orbit of x visits each U_i with bounded gaps, *i.e.*,

$$\exists n > 0, \forall i \leq k, \forall p \in \mathbb{Z}, \exists j < n, \sigma^{p+j}(x) \in U_i.$$

By Proposition 4, $F^m(x) \in \widetilde{U} := \widetilde{\Omega}_\sigma(U_1) \cap \cdots \cap \widetilde{\Omega}_\sigma(U_k)$ for some $m \geq 0$. Thus $\widetilde{U}$ is nonempty and F-invariant. The nonempty set $\Omega_F(\widetilde{U})$ is included in each $\Omega_F(U_i)$. □

Lemma 20 *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, $q > 0$, $p \in \mathbb{Z}$ and U a $F^q\sigma^p$-spreading set, then $\widetilde{\Omega}^q_\sigma(U) := \widetilde{\Omega}_\sigma(U \cap \cdots \cap F^{-(q-1)}(U))$ is nonempty and F-invariant, and*

$$\Omega_{F^q\sigma^p}(U \cap F^{-1}(U) \cap \cdots \cap F^{-(q-1)}(U)) = \Omega_F(\widetilde{\Omega}^q_\sigma(U)).$$

Proof: It can be easily verified that $F^{-m}(U)$ is a $F^q\sigma^p$-spreading set for any $m \geq 0$, so $\widetilde{\Omega}^q_\sigma(U)$ is nonempty by Lemma 19. It is easy to see that it is also F-invariant. It follows

$$\Omega_{F^q\sigma^p}(U \cap \cdots \cap F^{-(q-1)}(U)) = \Omega_{F^q\sigma^p}(\widetilde{\Omega}^q_\sigma(U)) = \Omega_F(\widetilde{\Omega}^q_\sigma(U)).$$ □

Proposition 21 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. If $\Sigma_1, \ldots, \Sigma_k \subseteq A^{\mathbb{Z}}$ are subshifts such that each Σ_i is an attractor of some $F^{q_i}\sigma^{p_i}$, where $q_i > 0$ and $p_i \in \mathbb{Z}$, then the intersection $\Sigma_1 \cap \cdots \cap \Sigma_k$ is nonempty.*

Proof: Let U_i be $F^{q_i}\sigma^{p_i}$-spreading sets, such that $\Sigma_i = \Omega_{F^{q_i}\sigma^{p_i}}(U_i)$. Since a clopen set is a finite union of cylinders, there exists a configuration $x \in A^{\mathbb{Z}}$ such that the σ-orbit of x visits each $U_i \cap \cdots \cap F^{-(q_i-1)}(U_i)$ with bounded gaps. By the assumption there exists $m > 0$, such that $F^m(x) \in U := \bigcap_{i \leq k} \widetilde{\Omega}^{q_i}_\sigma(U_i)$. Thus U is nonempty and by Lemma 20 it is F-invariant. The nonempty set $\Omega_F(U)$ is contained in all Σ_i. □

When a cellular automaton F is initialized on a random configuration x and if each word has positive probability, then this configuration contains with probability one each spreading set, not only of F, but also of all $F^q\sigma^p$. Therefore, the iterations of such a configuration converge to the intersection of all the attractors of all $F^q\sigma^p$.

Definition 22 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. The intersection of all subshift attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$, is called the* **small quasi-attractor** *of F and denoted by $\mathcal{Q}_F$.*

Theorem 23 *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, then $\mathcal{Q}_F$ is a nonempty F-invariant subshift and $F : \mathcal{Q}_F \to \mathcal{Q}_F$ is surjective.*

Proof: Let U_i be the sequence of all clopen sets which are $F^{q_i}\sigma^{p_i}$-spreading for some p_i, q_i. Set $W_i := \bigcap_{j \leq i} \widetilde{\Omega}^{q_i}_\sigma(U_i)$. By Lemma 20, the sets W_i are nonempty and F-invariant, so $\mathcal{Q}_F = \bigcap_{i>0} \Omega_F(W_i)$ is nonempty too. If $y \in \mathcal{Q}_F$, then there exists a sequence of configurations $x_i \in \Omega_F(W_i)$ such that $F(x_i) = y$. If x is a limit of a subsequence of x_i, then $x \in \mathcal{Q}_F$ and $y = F(x)$, so $F : \mathcal{Q}_F \to \mathcal{Q}_F$ is surjective. □

Theorem 24 *The small quasi-attractor $\mathcal{Q}_F$ is chain-mixing.*

Proof: This is a consequence of Theorem 6 and Lemma 2. □

Example 25 *The small quasi-attractor of F can be different from the intersection of all subshift attractors of F.*

Proof: The elementary cellular automaton $F(x)_i = x_i x_{i+1}$ has a unique subshift attractor $\Omega_F(A^{\mathbb{Z}}) = \{x \in \{0,1\}^{\mathbb{Z}} : \ \forall n, 10^n 1 \not\sqsubseteq x\}$. However, $\{0^\infty\}$ is an attractor of $F^2\sigma^{-1}(x)_i = x_{i-1}x_i x_{i+1}$, so $\mathcal{Q}_F = \{0^\infty\}$. □

9. Concluding remarks

Subshift attractors and their intersection, the small quasi-attractor $\mathcal{Q}_F$, give the basic information on the dynamics of non-surjective cellular automata. The small quasi-attractor manifests itself in simulations with random initial configurations. Indeed such an initial configuration contains with high probability all spreading sets of all $F^q\sigma^p$, so its iterations converge to $\mathcal{Q}_F$. Therefore it would be of great interest to obtain some more information on the structure and properties of $\mathcal{Q}_F$. The authors are currently investigating this subject.

Acknowledgments

The research was partially supported by the Research Program CTS MSM 0021620845.

References

[1] K. Culik, L. P. Hurd, and S. Yu. Computation theoretic aspects of cellular automata. *Physica D*, 45:357–378, 1990.

[2] L. P. Hurd. Recursive cellular automata invariant sets. *Complex Systems*, 4:119–129, 1990.

[3] M. Hurley. Attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 10:131–140, 1990.

[4] P. Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 17:417–433, 1997.

[5] P. Kůrka. *Topological and symbolic dynamics*, volume 11 of *Cours spécialisés*. Société Mathématique de France, Paris, 2003.

[6] P. Kůrka. On the measure attractor of a cellular automaton. *Discrete and Continuous Dynamical Systems*, supplement volume 2005:524–535, 2005.

[7] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[8] A. Maass. On the sofic limit set of cellular automata. *Ergodic Theory and Dynamical Systems*, 15:663–684, 1995.