Stern-Brocot graph in Möbius number systems

Petr Kůrka

Center for Theoretical Study, Academy of Sciences and Charles University in Prague, Jilská 1, CZ-11000 Praha 1, Czechia

Abstract. We characterize interval Möbius number systems with sofic expansion subshifts and show that they can be obtained as factors of interval Möbius number systems with expansion subshifts of finite types. The endpoints of interval cylinders of such systems can be computed by a simple formula which generalizes the computation of Farey fractions in the Stern-Brocot graph. We treat in detail the bimodular number system which has many nice properties and could be used for exact real computer arithmetic.

AMS classification scheme numbers: 54H20, 37F30

1. Introduction

Möbius number systems (MNS) have been introduced in Kůrka [4] and [5] as a generalization of both positional number systems and continued fractions. Real numbers are represented by infinite words from a one-sided subshift. The letters of the alphabet represent real Möbius transformations and the concatenation of letters corresponds to the composition of transformations. In Kůrka and Kazda [8] we have investigated interval MNS which are determined by an interval cover or almost-cover indexed by the alphabet. Given a number x, we find an interval to which x belongs, take the inverse image of x by the corresponding transformation and repeat the procedure. The expansion subshift consists of all infinite words obtained. Using the concept of expansion quotient, we have given conditions which ensure that the extended real line is a factor of the expansion subshift. In Kůrka [6] we have investigated rational MNS in which rational numbers have periodic or preperiodic expansions.

In the present paper we study interval MNS whose expansion subshifts are of finite type or sofic. It turns out that the expansion subshift cannot be of finite type when the interval almost-cover is actually a cover. The important class of redundant MNS which admit efficient arithmetic algorithms thus cannot be of finite type. Nevertheless, any system with sofic expansion subshift is a factor of a system with the expansion subshift of finite type. For expansion subshifts of finite type we have a simple formula which computes the endpoints of interval cylinders from the endpoints of the parent interval cylinders. This is a generalization of the computation of Farey fractions in the Stern-Brocot graph, which works for the parabolic modular system (see Niqui [10] and Kůrka [6]). However, modular systems whose transformations have unit determinant cannot be redundant and arithmetic algorithms do not work for them.

In order that a redundant interval MNS has a sofic expansion subshift and admits the generalized Stern-Brocot formula, it must have some symmetries and satisfy some constraints. We treat in detail the simplest system with these properties. It consists of the only eight transformations with integer entries, determinant 2, norm 6, and trace 3. It has several sofic expansion subshifts, some of them being redundant. The generalized Stern-Brocot formula works nicely in this system.

2. Möbius transformations

The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ can be regarded as a projective space, i.e., the space of one-dimensional subspaces of the two-dimensional vector space. On $\overline{\mathbb{R}}$ we have homogenous coordinates $x = (x_0, x_1) \in \widehat{\mathbb{R}} = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\overline{\mathbb{R}} = \widehat{\mathbb{R}}/\approx$, where $x \approx y$ iff $\det(x,y) = x_0y_1 - x_1y_0 = 0$. We regard $x \in \widehat{\mathbb{R}}$ as a column vector, and write it usually as $x = x_0/x_1$, for example $\infty = 1/0$. The stereograhic projection $\mathbf{h}(z) = (iz+1)/(z+i)$ maps $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$ conformally to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Define the *circle distance* on $\overline{\mathbb{R}}$ by

$$\varrho(x,y) = 2\arcsin\frac{|x_0y_1 - y_0x_1|}{\sqrt{(x_0^2+x_1^2)(y_0^2+y_1^2)}},$$

which is the length of the shortest arc joining $\mathbf{h}(x)$ and $\mathbf{h}(y)$ in $\partial\mathbb{D}$.

A real orientation-preserving *Möbius transformation* (MT) is a self-map of $\overline{\mathbb{R}}$ of the form $M_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d} = \frac{ax_0+bx_1}{cx_0+dx_1}$ where $a,b,c,d \in \mathbb{R}$ and $\det(M_{(a,b,c,d)}) = ad-bc > 0$. MT act also on the upper half-plane $\mathbb{U}$. On $\overline{\mathbb{D}} = \mathbb{D}\cup\partial\mathbb{D}$ we get *disc Möbius transformation* defined by $\widehat{M}_{(a,b,c,d)}(z) = \mathbf{h} \circ M_{(a,b,c,d)} \circ \mathbf{h}^{-1}(z) = (\alpha z + \beta)/(\overline{\beta}z + \overline{\alpha})$, where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. The *circle derivation* and the *expansion interval* of M are defined by

$$M^\bullet(x) := \lim_{y\to x}\frac{\varrho(M(y),M(x))}{\varrho(y,x)} = \frac{\det(M)\cdot||x||^2}{||M(x)||^2},$$
$$\mathbf{V}(M) := \{x \in \overline{\mathbb{R}} : (M^{-1})^\bullet(x) > 1\},$$

where $||x|| = \sqrt{x_0^2+x_1^2}$ is the norm of x. If $M = R_\alpha = M_{(\cos\frac{\alpha}{2},\sin\frac{\alpha}{2},-\sin\frac{\alpha}{2},\cos\frac{\alpha}{2})}$ is a rotation by α then $M^\bullet(x) = 1$ for all $x \in \overline{\mathbb{R}}$ and $\mathbf{V}(M)$ is empty. Otherwise $\mathbf{V}(M)$ is a proper set interval, i.e., a nonempty open connected set properly included in $\overline{\mathbb{R}}$. A Möbius transformation M is hyperbolic if it has the stable fixed point $s_M \in \overline{\mathbb{R}}$ with $M^\bullet(s_M) < 1$ and the unstable fixed point $u_M \in \overline{\mathbb{R}}$ with $M^\bullet(u_M) > 1$. A transformation is parabolic if it has a unique fixed point $s_M \in \overline{\mathbb{R}}$ with $M^\bullet(s_M) = 1$. It is elliptic if it has no fixed point in $\overline{\mathbb{R}}$.

3. Intervals

A *set interval* is an open connected subset of $\overline{\mathbb{R}}$. A proper interval is a nonempty set interval properly included in $\overline{\mathbb{R}}$. We represent proper intervals by (2×2)-matrices whose columns are their left and right endpoints. In the calculation of the Farey fractions by the Stern-Brocot formula, an interval $I = (x,y) = (\frac{x_0}{x_1}, \frac{y_0}{y_1})$ is cut into two intervals $I_0 = (x, \frac{x_0+y_0}{x_1+y_1})$ and $I_1 = (\frac{x_0+y_0}{x_1+y_1}, y)$ (see Figure 2). These two intervals can be obtained from I by matrix multiplication

$$I_0 = \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad I_1 = \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

To carry out such computations, we regard the endpoints of intervals as elements of $\widehat{\mathbb{R}}$ rather than elements of $\overline{\mathbb{R}} = \widehat{\mathbb{R}}/\approx$. Equivalent but different representations would give different results. The stereographic projection applied to $x = \frac{r\sin\alpha}{r\cos\alpha} \in \widehat{\mathbb{R}}$ gives $\mathbf{h}(x) = \sin 2\alpha - i\cos 2\alpha = e^{i(2\alpha - \frac{\pi}{2})}$, so it duplicates the angles. Intervals with endpoints $x = \frac{r\sin\alpha}{r\cos\alpha}$, $y = \frac{s\sin\beta}{s\cos\beta}$ where $0 \le \alpha < 2\pi$, $\alpha < \beta < \alpha + \pi$ can therefore represent any proper interval. Since $\det(x,y) = x_0y_1 - x_1y_0 = rs\sin(\alpha-\beta) < 0$, we define matrix intervals as (2×2)-matrices with negative determinant, which we write as pairs $I = (\frac{a}{c}, \frac{b}{d})$ of their left and right endpoints $\mathfrak{l}(I) = \frac{a}{c}$, $\mathfrak{r}(I) = \frac{c}{d}$. The set of *matrix intervals* is therefore

$$\mathcal{I}(\mathbb{R}) = \{(\tfrac{a}{c}, \tfrac{b}{d}) \in GL(\mathbb{R}, 2) :\ ad - bc < 0\}$$

The length of an interval is defined by $|(\frac{a}{c}, \frac{b}{d})| = \pi + 2\arctan\frac{ab+cd}{ad-bc}$. Then we get $|(\frac{r\sin\alpha}{r\cos\alpha}, \frac{s\sin\beta}{s\cos\beta})| = 2(\beta - \alpha)$, provided $0 < \beta - \alpha < \pi$. A matrix interval defines an open and closed set interval by

$$\begin{aligned} z \in I &\Leftrightarrow \det(\mathfrak{l}(I), z)\cdot\det(z, \mathfrak{r}(I)) > 0, \\ z \in \overline{I} &\Leftrightarrow \det(\mathfrak{l}(I), z)\cdot\det(z, \mathfrak{r}(I)) \ge 0. \end{aligned}$$

If $I = (\frac{r\sin\alpha}{r\cos\alpha}, \frac{s\sin\beta}{s\cos\beta})$, then $z = \frac{t\sin\gamma}{t\cos\gamma} \in I$ iff either $\alpha < \gamma < \beta$ or $\alpha + \pi < \gamma < \beta + \pi$. For two intervals $I, J \in \mathcal{I}(\mathbb{R})$ we have $J \subseteq I$, (i.e., $\forall x \in J, x \in I$) iff either

$$\begin{aligned} &\det(\mathfrak{l}(I), \mathfrak{l}(J)) \le 0,\ \det(\mathfrak{l}(J), \mathfrak{r}(I)) < 0,\ \det(\mathfrak{l}(I), \mathfrak{r}(J)) < 0,\ \det(\mathfrak{r}(J), \mathfrak{r}(I)) \le 0, \text{ or} \\ &\det(\mathfrak{l}(I), \mathfrak{l}(J)) \ge 0,\ \det(\mathfrak{l}(J), \mathfrak{r}(I)) > 0,\ \det(\mathfrak{l}(I), \mathfrak{r}(J)) > 0,\ \det(\mathfrak{r}(J), \mathfrak{r}(I)) \ge 0. \end{aligned}$$

We write $I \approx J$ if $I \subseteq J$ and $J \subseteq I$. The intersection of two intervals need not be an interval. However, if $|I| + |J| < 2\pi$ then $I \cap J$ is a (possibly empty) interval. When we transform intervals, we work with the matrix representations of MT rather than with the transformations themselves. Möbius transformations are represented by matrices

$$\mathcal{M}(\mathbb{R}) = \{M_{(a,b,c,d)} \in GL(\mathbb{R}, 2) :\ ad - bc > 0\}$$

which act on vectors $x \in \widehat{\mathbb{R}}$ by multiplication $x \mapsto Mx$. Two matrices represent the same MT if one is a nonzero multiple of the other. If $M \in \mathcal{M}(\mathbb{R})$ and $I \in \mathcal{I}(\mathbb{R})$, then both MI and IM are intervals. While $MI = M(I)$ represents the M-image of the set interval of I, IM is the interval cut from I by M.

4. Subshifts

For a finite alphabet A denote by $A^* = \bigcup_{m\ge 0} A^m$ the set of finite words and by $A^+ = \bigcup_{m>0} A^m$ the set of finite non-empty words. The length of a word $u = u_0 \dots u_{m-1} \in A^m$ is $|u| = m$. We denote by $A^{\mathbb{N}}$ the Cantor space of infinite words with the metric $d(u,v) = 2^{-k}$, where $k = \min\{i \ge 0 : u_i \ne v_i\}$. We say that $v \in A^*$ is a subword of $u \in A^* \cup A^{\mathbb{N}}$ and write $v \sqsubseteq u$, if $v = u_{[i,j)} = u_i \dots u_{j-1}$ for some $0 \le i \le j \le |u|$. Given $u \in A^m$ and $v \in A^p$ with $p > 0$, denote by $u.v \in A^{\mathbb{N}}$ the *periodic word* with preperiod u and period v defined by $(u.v)_i = u_i$ for $i < m$ and $(u.v)_{m+kp+i} = v_i$ for $i < p$, $k \ge 0$. The set of periodic words is denoted by $\mathbb{P}_A = \{u \in A^{\mathbb{N}} :\ \exists m \ge 0, \exists p > 0, \sigma^{m+p}(u) = \sigma^m(u)\}$ The cylinder of $u \in A^n$ is the set $[u] = \{v \in A^{\mathbb{N}} :\ v_{[0,n)} = u\}$. The *shift map* $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A *subshift* is a nonempty set $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$.

If $D \subseteq A^+$ then $\Sigma_D = \{x \in A^{\mathbb{N}} : \forall u \sqsubseteq x, u \notin D\}$ is the subshift with *forbidden set* D. Any subshift can be obtained in this way. A subshift is uniquely determined by its *language* $\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$. Denote by $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n$; in particluar we assume $\mathcal{L}^1(\Sigma) = A$. A map $F : \Sigma_0 \to \Sigma_1$ between two subshifts is called a morphism, if there exists $r > 0$ and a local rule $f : \mathcal{L}^r(\Sigma_0) \to \mathcal{L}^1(\Sigma_1)$ such that $F(x)_i = f(x_{[i,i+r)})$ for each $x \in \Sigma_0$ and $i \in \mathbb{N}$. A surjective morphism is called a *factor*. For $u \in A^+$ define $\sigma_u : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by $\sigma_u(v) = uv$. Given a subshift $\Sigma \subseteq A^{\mathbb{N}}$ denote by $\mathcal{O}_u = \{v \in \Sigma : uv \in \Sigma\}$ the follower set of $u \in \mathcal{L}(\Sigma)$, so $[u] \cap \Sigma = \sigma_u(\mathcal{O}_u)$. If $uv \in \mathcal{L}(\Sigma)$ then $\sigma_v(\mathcal{O}_{uv}) \subseteq \mathcal{O}_u$.

A *labelled graph* over an alphabet A is a structure $G = (V, E, s, t, \ell)$, where $V = |G|$ is the set of vertices, E is the set of edges, $s, t : E \to V$ are the source and target maps, and $\ell : E \to A$ is a labelling function. The subshift $\Sigma_G \subseteq A^{\mathbb{N}}$ of G consists of labels of all infinite paths of G. A subshift is *sofic*, if it is the subshift of a finite labelled graph. This happens iff the set $\{\mathcal{O}_u : u \in A^*\}$ of its follower sets is finite. In this case the graph with labelled edges $\mathcal{O}_u \xrightarrow{a} \mathcal{O}_{ua}$ presents Σ (see Lind and Marcus [9]). A subshift is of *finite type* (SFT) of *order* p, if its forbidden words have length at most p, i.e., if there exists a forbidden set $D \subset A^p$ such that $\Sigma = \Sigma_D$. In this case $u \in A^{\mathbb{N}}$ belongs to Σ iff all subwords of u of length p belong to $\mathcal{L}(\Sigma)$. A subshift is sofic iff it is a factor of a subshift of finite type (see Lind and Marcus [9]).

5. Möbius number systems

Definition 1 *A Möbius iterative system is a map $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, or a family of orientation-preserving Möbius transformations $(F_u : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{u \in A^*}$ satisfying $F_{uv} = F_u \circ F_v$ and $F_\lambda = \mathrm{Id}$, where λ is the empty word. The convergence space $\mathbb{X}_F \subseteq A^{\mathbb{N}}$ and the symbolic representation $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ are defined by*

$$\begin{aligned} \mathbb{X}_F &= \{u \in A^{\mathbb{N}} : \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}, \\ \Phi(u) &= \lim_{n\to\infty} F_{u_{[0,n)}}(i), \end{aligned}$$

where $i \in \mathbb{U}$ is the imaginary unit. If $\Sigma \subseteq \mathbb{X}_F$ is a subshift such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective, then we say that (F, Σ) is a Möbius number system (MNS). We say that a Möbius number system is redundant, if for every continuous map $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $\Phi f = g\Phi$.

Redundancy is necessary for the existence of exact arithmetical algorithms (see Weihrauch [14], Vuillemin [13], Kornerup and Matula [3], Potts [11] or Potts et al. [12]). If $u \in \mathbb{X}_F$ then $\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(z)$ for every $z \in \mathbb{U}$ (see Kazda [2]). For $v \in A^+$, $w \in A^{\mathbb{N}}$ we have $vw \in \mathbb{X}_F$ iff $w \in \mathbb{X}_F$ and then $\Phi(vw) = F_v(\Phi(w))$. If $\Sigma \subseteq \mathbb{X}_F$ then $v\Sigma = \{vw : w \in \Sigma\} \subseteq \mathbb{X}_F$, $\Phi([v] \cap \Sigma) = F_v \Phi(\mathcal{O}_v)$, and the following diagrams commute

$$\begin{array}{ccc} X_F & \xrightarrow{\sigma_v} & X_F \\ \Phi\downarrow & & \downarrow\Phi \\ \overline{\mathbb{R}} & \xrightarrow{F_v} & \overline{\mathbb{R}} \end{array} \qquad \begin{array}{ccc} \Sigma & \xrightarrow{\sigma_v} & v\Sigma \\ \Phi\downarrow & & \downarrow\Phi \\ \overline{\mathbb{R}} & \xrightarrow{F_v} & \overline{\mathbb{R}} \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{uv} & \xrightarrow{\sigma_v} & \mathcal{O}_u \\ \Phi\downarrow & & \downarrow\Phi \\ \Phi(\mathcal{O}_{uv}) & \xrightarrow{F_v} & \Phi(\mathcal{O}_u) \end{array}$$

Definition 2 *An open almost-cover for a Möbius iterative system $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a family of matrix intervals $\mathcal{W} = \{W_a : a \in A\}$ such that $\bigcup_{a \in A} \overline{W_a} = \overline{\mathbb{R}}$. If for each*

$a, b \in A$ either $W_a \approx W_b$ or $W_a \cap W_b = \emptyset$ then we say that $\mathcal{W}$ is a multipartition. If moreover $W_a \not\approx W_b$ for $a \neq b$ then we say that $\mathcal{W}$ is a partition. If $\bigcup_{a\in A} W_a = \overline{\mathbb{R}}$ then we say that $\mathcal{W}$ is a cover. Denote by $\mathcal{E}(\mathcal{W}) = \{\mathfrak{l}(W_a), \mathbf{r}(W_a) : a \in A\}$ the set of endpoints of $\mathcal{W}$. The expansion subshift $\mathcal{S}_\mathcal{W}$ is defined by

$$\mathcal{S}_\mathcal{W} = \{u \in A^\mathbb{N} : \forall k > 0, W_{u_{[0,k)}} \neq \emptyset\}, \text{ where}$$

$$W_u = W_{u_0} \cap F_{u_0} W_{u_1} \cap F_{u_{[0,2)}} W_{u_2} \cap \cdots \cap F_{u_{[0,n)}} W_{u_n},\ u \in A^{n+1}.$$

We call W_u the interval cylinder of u.

It follows $W_{uv} = W_u \cap F_u W_v$ for each $uv \in \mathcal{L}(\mathcal{S}_\mathcal{W})$. Multipartitions are used in the construction of the extension MNS in Definition 9.

Theorem 3 (Kůrka and Kazda [8]) *Let $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and assume that $\mathcal{W} = \{W_a : a \in A\}$ is an almost-cover of $\overline{\mathbb{R}}$ such that $W_a \subseteq \mathbf{V}(F_a)$ for all $a \in A$. Then $(F, \mathcal{S}_\mathcal{W})$ is a Möbius number system. It is redundant provided $\mathcal{W}$ is a cover. For each $u \in \mathcal{S}_\mathcal{W}$ and $v \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ we have*

$$\{\Phi(u)\} = \bigcap_{n\geq 0} \overline{W_{u_{[0,n)}}},\ \Phi([v] \cap \mathcal{S}_\mathcal{W}) = \overline{W_v}.$$

A stronger theorem which uses the concept of expansion quotient has been proved in Kůrka and Kazda [8]. Nevertheless our examples satisfy the condition of Theorem 3, so we adopt it as a definition:

Definition 4 *An interval Möbius number system over alphabet A is a pair $(F, \mathcal{W})$, where $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Möbius iterative system and $\mathcal{W} = \{W_a : a \in A\}$ is an almost-cover of $\overline{\mathbb{R}}$ such that $W_a \subseteq \mathbf{V}(F_a)$ for each $a \in A$.*

If $(F, \mathcal{W})$ is an interval MNS then $\lim_{n\to\infty} \max\{|W_u|\ u \in \mathcal{L}^n(\mathcal{S}_\mathcal{W})\} = 0$. This is an immediate consequence of the uniform continuity of $\Phi : \mathcal{S}_\mathcal{W} \to \overline{\mathbb{R}}$.

6. Expansion subshifts of finite type

Theorem 5 *Assume that $(F, \mathcal{W})$ is an interval MNS. Then $\mathcal{S}_\mathcal{W}$ is a SFT of order at most $m+1$ iff $\forall a \in A, \forall u \in \mathcal{L}^m(\mathcal{S}_\mathcal{W}), (F_a W_u \cap W_a \neq \emptyset \Rightarrow F_a W_u \subseteq W_a)$. In this case $W_u = F_{u_{[0,n-m]}} W_{u_{(n-m,n]}}$ for each $u \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ of length at least $m+1$. In particular, if $m = 1$ then $W_u = F_{u_{[0,n)}} W_{u_n}$ for each $u \in \mathcal{L}^{n+1}(\mathcal{S}_\mathcal{W})$.*

Proof: Assume that $(F, \mathcal{W})$ satisfies the condition. Let $u \in A^{n+1}$, and suppose that for all $v \sqsubseteq u$ with $|v| = m+1$ we have $W_v \neq \emptyset$. Then

$$\begin{aligned} W_u &= W_{u_0} \cap F_{u_0} W_{u_{[1,m]}} \cap F_{u_{[0,m]}} W_{u_{[m+1,n]}} \\ &= F_{u_0} W_{u_{[1,m]}} \cap F_{u_{[0,m]}} W_{u_{[m+1,n]}} \\ &= F_{u_0} W_{u_{[1,n]}} = \cdots = F_{u_{[0,n-m]}} W_{u_{(n-m,n]}}, \end{aligned}$$

and $W_{u_{(n-m,n]}} \neq \emptyset$, so $W_u \neq \emptyset$. Conversely, assume by contradiction that $\mathcal{S}_\mathcal{W}$ is a SFT of order at most $m+1$ and that there exist $a \in A$ and $u \in \mathcal{L}^m(\mathcal{S}_\mathcal{W})$ with $F_a W_u \cap W_a \neq \emptyset$ and $F_a W_u \not\subseteq W_a$. Thus $F_a W_u \setminus W_a$ is nonempty and therefore $F_u^{-1} W_u \setminus F_{au}^{-1} W_a$ is nonempty. Since $\lim_{|v|\to\infty} |W_v| = 0$, there exists $v \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ such that $W_v \subset F_u^{-1} W_u \setminus F_{au}^{-1} W_a$. It follows $W_{uv} = W_u \cap F_u W_v \neq \emptyset$ and $W_{auv} = W_a \cap F_a W_u \cap F_{au} W_v = \emptyset$, so $au \in \mathcal{L}(\mathcal{S}_\mathcal{W})$, $uv \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ but $auv \notin \mathcal{L}(\mathcal{S}_\mathcal{W})$. This is a contradiction. □

Corollary 6 *Let $(F, \mathcal{W})$ be an interval MNS with a multipartition $\mathcal{W}$ whose endpoint set $\mathcal{E}(\mathcal{W})$ is invariant in the sense that $\forall a \in A, \forall x \in \overline{W_a} \cap \mathcal{E}(\mathcal{W}), F_a^{-1}x \in \mathcal{E}(\mathcal{W})$. Then $\mathcal{S}_\mathcal{W}$ is a SFT of order 2.*

Proof: For $m = 1$, the condition of Theorem 5 can be equivalently stated as

$$\forall a, b \in A, (W_b \cap F_a^{-1}W_a \neq \emptyset \;\Rightarrow\; W_b \subseteq F_a^{-1}W_a)$$

If $W_b \cap F_a^{-1}W_a$ and $W_b \not\subseteq F_a^{-1}W_a$, then W_b contains an endpoint of $F_a^{-1}W_a$ which, by the assumption, belongs to $\mathcal{E}(\mathcal{W})$. Thus $\mathcal{W}$ cannot be a multipartition which is a contradiction.

Proposition 7 *Assume that $(F, \mathcal{W})$ is an interval MNS with the expansion subshift $\mathcal{S}_\mathcal{W}$ of finite type of order 2. For $ab \in \mathcal{L}^2(\mathcal{S}_\mathcal{W})$ define the cut matrices by $\Psi_{ab} = W_a^{-1}F_aW_b$. Then for each $uab \in \mathcal{L}_\mathcal{W}$ we have $W_{uab} = W_{ua}\Psi_{ab}$.*

Indeed, $W_{uab} = F_uF_aW_b = F_uW_aW_a^{-1}F_aW_b = W_{ua}\Psi_{ab}$. Thus for every $u \in \mathcal{L}^{n+1}(\mathcal{S}_\mathcal{W})$ we have two ways of computing W_u:

$$W_u = F_{u_0}F_{u_1}\cdots F_{u_{n-1}}W_{u_n} = W_{u_0}\Psi_{u_0u_1}\Psi_{u_1u_2}\cdots\Psi_{u_{n-1}u_n}.$$

In arithmetical algorithms (see Kůrka and Kazda [8]), the latter way is more efficient, since we search for the first n such that the interval cylinder of $u_{[0,n)}$ is sufficiently small. Unfortunately, redundant MNS in which these arithmetical algorithms work cannot have expansion subshifts of finite type:

Theorem 8 *If $(F, \mathcal{W})$ is an interval MNS and $\mathcal{W}$ is a cover of $\overline{\mathbb{R}}$ then $\mathcal{S}_\mathcal{W}$ is not a SFT.*

Proof: By the assumption, $\{W_u : u \in \mathcal{L}^m(\mathcal{S}_\mathcal{W})\}$ is a cover of $\overline{\mathbb{R}}$ for each m. If x is an endpoint of some $F_a^{-1}W_a$ and $m > 0$, then there exists $u \in \mathcal{L}^m(\mathcal{S}_\mathcal{W})$ with $x \in W_u$, so $W_u \cap F_a^{-1}W_a \neq \emptyset$ but $W_u \not\subseteq F_a^{-1}W_a$. Thus $\mathcal{S}_\mathcal{W}$ is not a SFT of order $m + 1$. □

7. Sofic expansion subshifts

One of the characterizations of sofic subshifts is that they are factors of subshifts of finite type. We extend this characterization to MNS. Each interval MNS with sofic expansion subshift is a factor of an interval MNS with expansion subshift of finite type and order 2. The extension is obtained by cutting the interval almost-cover into a sufficiently fine multipartition. There are many redundant MNS with sofic expansion subshifts and their arithmetical algorithms can be simplified by using their extension MNS with expansion subshifts of order 2.

Definition 9 *Let $(F, \mathcal{W})$ be an interval MNS over an alphabet A. A partition $\mathcal{P} = \{P_c : c \in C\}$ is a refinement of $\mathcal{W} = \{W_a : a \in A\}$, if for each $a \in A$, $c \in C$ we have $P_c \subseteq W_a$ whenever $P_c \cap W_a \neq \emptyset$. The extension of $(F, \mathcal{W})$ by $\mathcal{P}$ is the pair $(G, \mathcal{V})$ over the alphabet $B = \{(a, c) \in A \times C : P_c \subseteq W_a\}$ defined by $V_{(a,c)} = P_c$ and $G_{(a,c)} = F_a$. The projection map $\ell : B \to A$ is given by $\ell(a, c) = a$ and extends to the map $\ell : \mathcal{L}(\mathcal{S}_\mathcal{V}) \to \mathcal{L}(\mathcal{S}_\mathcal{W})$ and to the morphism $\ell : \mathcal{S}_\mathcal{V} \to \mathcal{S}_\mathcal{W}$ by $\ell(v)_i = \ell(v_i)$.*

Theorem 10 *Assume that $(F,\mathcal{W})$ is an interval MNS over an alphabet A, $\mathcal{P}$ is a partition refinement of $\mathcal{W}$ and $(G,\mathcal{V})$ is the extension of $(F,\mathcal{W})$ by $\mathcal{P}$. Then $(G,\mathcal{V})$ is an interval MNS, $\mathcal{V}$ is a multipartition, the projection $\ell : \mathcal{S}_\mathcal{V} \to \mathcal{S}_\mathcal{W}$ is a surjective factor map which commutes with Φ, i.e., $\Phi_G = \Phi_F \circ \ell$, and $\overline{W_u} = \bigcup\{\overline{V_v} : \ell(v) = w\}$ for each $u \in \mathcal{L}(\mathcal{S}_\mathcal{W})$. Each $w \in \mathcal{S}_\mathcal{W}$ has at most 2 ℓ-preimages. There exists $p > 0$ such that for each $w \in \mathcal{L}_\mathcal{W}$, the set $\{v_{[0,|w|-p]} : \ell(v) = w\}$ has at most two elements.*

Proof: Since $V_b \subseteq W_{\ell(b)} \subseteq \mathbf{V}(F_{\ell(b)}) = \mathbf{V}(G_b)$, the pair $(G,\mathcal{V})$ is an interval MNS and $\mathcal{V}$ is clearly a multipartition. We show by induction that $V_v \subseteq W_{\ell(v)}$ for each $v \in \mathcal{L}(\mathcal{S}_\mathcal{V})$. By the assumption the statement holds for $|v| = 1$. If the statement holds for v and $av \in \mathcal{L}(\mathcal{S}_\mathcal{V})$ then $V_{av} = V_a \cap G_a V_v \subseteq W_{l(a)} \cap F_{\ell(a)} W_{\ell(v)} = W_{l(av)}$. If $v \in \mathcal{S}_\mathcal{V}$ then $G_{v_{[0,n)}}(i) = F_{\ell(v)_{[0,n)}}(i)$ for each n (here i is the imaginary unit), so $v \in \mathbb{X}_G$ and $\Phi_G(v) = \Phi_F(\ell(v))$. We show that $\ell : \mathcal{S}_\mathcal{V} \to \mathcal{S}_\mathcal{W}$ is surjective and each $w \in \mathcal{S}_\mathcal{W}$ has at most two ℓ-preimages. For a given $w \in \mathcal{S}_\mathcal{W}$ denote by $x_0 = \Phi_F(w)$. If $x_0 \notin \mathcal{E}(\mathcal{V})$, then there exists a unique v_0 with $x_0 \in V_{v_0}$ and $\ell(v_0) = w_0$, so $G_{v_0} = F_{w_0}$. If $x_1 = F_{w_0}^{-1}(x_0) \notin \mathcal{E}(\mathcal{V})$, then there exists a unique v_1 with $x_1 \in V_{v_1}$ and $\ell(v_1) = w_0$ and we continue the construction of v_i by induction. If n is the first index for which $x_n \in \mathcal{E}(\mathcal{V})$, then there exist unique v_n, v'_n with $x_n = \mathbf{r}(V_{v_n}) = \mathfrak{l}(V_{v'_n})$. If $x_m \in \mathcal{E}(\mathcal{V})$ for some $m > n$, then we cannot choose v_m with $x_m = \mathfrak{l}(V_{v_m})$, since in this case V_v would be empty. Thus we have unique choice for v_m, v'_m with $x_m = \mathbf{r}(V_{v_m}) = \mathfrak{l}(V_{v'_m})$. It follows that there exist only two $v, v' \in \mathcal{S}_\mathcal{V}$ with $\ell(v) = \ell(v') = w$. Consider now the map $\ell : \mathcal{L}(\mathcal{S}_\mathcal{V}) \to \mathcal{L}(\mathcal{S}_\mathcal{W})$ which preserves the length of words. Let p be the smallest integer such that $\forall w \in \mathcal{L}^p(\mathcal{S}_\mathcal{W}), \forall b \in B, |W_v| < |V_b|$. We show that for each $w \in \mathcal{L}_\mathcal{W}$ with $|w| = n > p$, the set $\{v_{[0,n-p]} : \ell(v) = w\}$ has at most two elements. If $\ell(v) = w$ then $\emptyset \neq W_w \cap V_v = W_{w_{[0,n-p)}} \cap F_{w_{[0,n-p)}} W_{w_{[n-p,n)}} \cap G_{v_{[0,n-p)}} V_{v_{[n-p,n)}}$. Since $G_{v_{[0,n-p)}} = F_{w_{[0,n-p)}}$, we get $W_{w_{[n-p,n)}} \cap V_{v_{n-p}} \supseteq W_{w_{[n-p,n)}} \cap V_{v_{[n-p,n)}} \neq \emptyset$ and there exist at most two v_{n-p} with this property. Since $G_{v_{n-p-1}} V_{n-p} \subseteq V_{v_{n-p-1}}$, v_{n-p-1} is uniquely determined by v_{n-p} and by induction, every v_m with $m < n-p$ is uniquely determined by v_{n-p}. □

An immediate consequence of Theorem 10 is that there exists $m > 0$, such that each $w \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ has at most m preimages and that all these preimages can be obtained by a finite transducer whose algorithm is based on the formula

$$\ell^{-1}(wc) = \{vab : \ va \in \ell^{-1}(w) \ \& \ b \in \ell^{-1}(c) \ \& \ ab \in \mathcal{L}(\mathcal{S}_\mathcal{W})\}.$$

Theorem 11 *Assume that $(F,\mathcal{W})$ is an interval MNS with a sofic subshift $\mathcal{S}_\mathcal{W}$. Then there exists a refinement partition $\mathcal{P}$ of $\mathcal{W}$ such that in the extension $(G,\mathcal{V})$ of $(F,\mathcal{W})$ by $\mathcal{P}$, $\mathcal{S}_\mathcal{V}$ is a SFT of order 2. The sofic subshift $\mathcal{S}_\mathcal{W}$ is presented by a graph whose vertices are sets $P \in \mathcal{P}$ and labelled edges are $P \xrightarrow{a} Q$ where $P \subseteq W_a$ and $F_a(Q) \subseteq P$.*

Proof: Assume that $u, v \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ have the same follower sets $\mathcal{O}_u = \mathcal{O}_v$. For $w \in A^+$ we have $W_u \cap F_u W_w = W_{uw} \neq \emptyset$ iff $W_v \cap F_v W_w = W_{vw} \neq \emptyset$, so $F_u^{-1} W_u \cap W_w \neq \emptyset$ iff $F_v^{-1} W_v \cap W_w \neq \emptyset$. Since $\{W_w : \ w \in \mathcal{L}^n(\mathcal{S}_\mathcal{W})\}$ is an almost-cover for each n and since $\lim_{|w|\to\infty} |W_w| = 0$, we get $F_u^{-1} W_u \approx F_v^{-1} W_v$. Since $\mathcal{S}_\mathcal{W}$ is sofic, the set of its follower sets is finite, so the set $\{F_u^{-1} W_u : \ u \in \mathcal{L}(\mathcal{S}_\mathcal{W})\}$ is finite as well. Denote by $\mathcal{P}$ the partition whose endpoints $\mathcal{E}(\mathcal{P})$ are all endpoints of all W_a and of all $F_u^{-1} W_u$, so $\mathcal{P}$ is a refinement of $\mathcal{W}$. We show that if $P \in \mathcal{P}$ and $P \subseteq W_a$, then both endpoints of $F_a^{-1} P$ belong to $\mathcal{E}(\mathcal{P})$. An endpoint of $F_a^{-1} P$ is either an endpoint of some $F_a^{-1} W_a$ or an endpoint of some $F_a^{-1}(F_u^{-1}W_u \cap W_a) = F_{ua}^{-1}(W_u \cap F_u W_a) = F_{ua}^{-1}W_{ua}$. In both cases such an endpoint belongs to $\mathcal{E}(\mathcal{P})$. By Corollary 6, $\mathcal{S}_\mathcal{V}$ is a SFT of order 2. Assume that $P_0 \xrightarrow{u_0} P_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} P_n$ is a labelled path, so $P_i \subseteq W_{u_i}$ and $F_{u_i}(P_{i+1}) \subseteq P_i$. Then

$$F_{u_{[0,n)}} P_n \subseteq F_{u_{[0,n-1)}} P_{n-1} \subseteq \cdots\cdots \subseteq F_{u_0} P_1 \subseteq P_0,$$
$$F_{u_{[0,n)}} P_n \subseteq F_{u_{[0,n-1)}} W_{u_{n-1}} \cap \cdots\cdots \cap F_{u_0} W_{u_1} \cap W_{u_0} = W_{u_{[0,n)}},$$

so $W_{u_{[0,n)}} \neq \emptyset$ and $u_{[0,n)} \in \mathcal{L}(\mathcal{S}_\mathcal{W})$. Conversely assume that $W_{u_{[0,n)}} \neq \emptyset$ and $x_0 \in W_{u_{[0,n)}}$. There exist $P_i \in \mathcal{P}$ such that $x_0 \in P_0 \subseteq W_{u_0}$, $x_1 = F_{u_0}^{-1}(x_0) \in P_1 \subseteq W_{u_1}$ and similarly $x_i = F_{u_{[0,i)}}^{-1}(x_0) \in P_i \subseteq W_{u_i}$ for all $i \leq n$. Since $x_i \in F_{u_i} P_{i+1} \cap P_i \neq \emptyset$, $P_0 \xrightarrow{u_0} P_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} P_n$ is a path in the expansion graph. □

Some examples of labelled graphs constructed by Theorem 11 are in Figure 4.

8. Integer Möbius number systems

Denote by $\mathbb{Z}$ the set of integers and by $\widehat{\mathbb{Q}} = \mathbb{Z}^2 \setminus \{\frac{0}{0}\}$. For $x = \frac{x_0}{x_1} \in \widehat{\mathbb{Q}}$ we denote by $\gcd(x) > 0$ the greatest common divisor of x_0, x_1. Denote by $\overline{\mathbb{Q}} = \{x \in \widehat{\mathbb{Q}} :\ \gcd(x) = 1\}$. Each rational number has two representations in $\overline{\mathbb{Q}}$. We have the map $\mathbf{d} : \widehat{\mathbb{Q}} \to \overline{\mathbb{Q}}$ defined by $\mathbf{d}(x) = \frac{x_0/\gcd(x)}{x_1/\gcd(x)}$. Set

$$\mathcal{M}(\mathbb{Z}) = \{M \in GL(\mathbb{Z}, 2) :\ \det(M) > 0,\ \gcd(M) = 1\},$$
$$\mathcal{I}(\mathbb{Z}) \ \ = \{I \in GL(\mathbb{Z}, 2) :\ \det(I) < 0,\ \gcd(I) = 1\},$$

where $\gcd(M) > 0$ is the greatest common divisor of the entries of M. The norm of $x \in \widehat{\mathbb{Q}}$ is $||x|| = x_0^2 + x_1^2$ and the norm of $M_{(a,b,c,d)}$ is $||M|| = a^2 + b^2 + c^2 + d^2$. For $M = M_{(a,b,c,d)} \in GL(\mathbb{Z}, 2)$ denote by $\mathbf{d}(M) = M_{(a/g, b/g, c/g, d/g)} \in \mathcal{M}(\mathbb{Z})$, where $g = \gcd(M)$. The pseudoinverse of M is $M^{-1} = M_{(d,-b,-c,a)}$. With the multiplication operation $M, N \mapsto \mathbf{d}(MN)$ and pseudoinvers M^{-1}, $\mathcal{M}(\mathbb{Z})$ is a group. Each MT with integer entries has two representations in $\mathcal{M}(\mathbb{Z})$ and each interval with integer entries has two representations in $\mathcal{I}(\mathbb{Z})$. We say that $(F, \mathcal{W})$ is an integer MNS, if $F_a \in \mathcal{M}(\mathbb{Z})$ and $W_a \in \mathcal{I}(\mathbb{Z})$ for each $a \in A$. In integer MNS we have an algorithm for expansion of rational numbers.

Definition 12 *The rational expansion graph of an integer MNS $(F, \mathcal{W})$ is a labelled graph whose vertices are $(x, s) \in \widehat{\mathbb{Q}} \times \{-, 0, +\}$ and labelled edges are*

$$(x, s) \xrightarrow{a} (F_a^{-1}x, s),\ \textit{if}\, x \in W_a \ \&\ s \in \{-, 0, +\},$$
$$(x, s) \xrightarrow{a} (F_a^{-1}x, -),\ \textit{if}\, x = \mathbf{r}(W_a) \ \&\ s \in \{-, 0\},$$
$$(x, s) \xrightarrow{a} (F_a^{-1}x, +),\ \textit{if}\, x = \mathbf{l}(W_a) \ \&\ s \in \{0, +\}.$$

Proposition 13 *Let $(F, \mathcal{W})$ be an interval MNS and $x \in \widehat{\mathbb{Q}}$. Then a word $u \in A^\mathbb{N}$ is the label of a path with source $(x, 0)$ iff $u \in \mathcal{S}_\mathcal{W}$ and $\Phi(u) = x$.*

See Kůrka [6] for a proof. The expansion algorithm works in $\overline{\mathbb{Q}}$ if we replace the edges of the expansion graph by $(x, s) \xrightarrow{a} (\mathbf{d}(F_a^{-1}x), t)$. Figure 5 shows the expansion graph of rational numbers of the bimodular system $(F, \mathcal{R})$ in $\overline{\mathbb{Q}}$ (left) and in $\widehat{\mathbb{Q}}$ (right).

Definition 14 *The rational expansion interval of $M \in \mathcal{M}(\mathbb{Z})$ is defined by*

$$\mathbf{R}(M) = \{x \in \widehat{\mathbb{R}} : \ (M^{-1})^{\bullet}(x) > \det(M)\}.$$

We say that an integer MNS $(F, \mathcal{W})$ is rational, if $W_a \subseteq \mathbf{R}(F_a)$ for each $a \in A$.

Note that if $x \in \mathbf{R}(M)$ and $y = M^{-1}x$ then $||y|| \leq ||x||$. For each $M \in \mathcal{M}$ we have $\mathbf{R}(M) \subseteq \mathbf{V}(M)$ and either $\mathbf{R}(M) \subseteq (\frac{0}{1}, \frac{1}{0})$ or $\mathbf{R}(M) \subseteq (\frac{-1}{0}, \frac{0}{1})$ (see Kůrka [6]). Thus if $\mathcal{W}$ is a cover, then $(F, \mathcal{W})$ cannot be a rational MNS.

Theorem 15 (Kůrka [6]) *In a rational Möbius number system, every expansion of every rational number is periodic. i.e., $\Phi^{-1}(\widehat{\mathbb{Q}}) \subseteq \mathbb{P}_A \cap \mathcal{S}_{\mathcal{W}}$.*

This follows from the fact that if $(x_0, 0) \stackrel{u_0}{\to} (x_1, s_1) \stackrel{u_1}{\to} \cdots$ is a path in the expansion graph then $||x_{i+1}|| \leq ||x_i||$. Theorem 15 holds in the space $\overline{\mathbb{Q}}$ as well. For $d \in \mathbb{Z} \setminus \{0\}$ define $\psi_d : \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}}$ by $\psi_d(x) = dx_0/dx_1$.

Theorem 16 *Every rational MNS has a sofic expansion subshift. In this case the partition $\mathcal{P}$ of the extension SFT has a $\widehat{\mathbb{Q}}$-invariant endpoint set $\mathcal{E}(\mathcal{P}) \subset \widehat{\mathbb{Q}}$ such that $\forall x \in \mathcal{E}(\mathcal{P}) \cap \overline{W_a}, F_a^{-1}x \in \mathcal{E}(\mathcal{P})$, and if $x, y \in \mathcal{E}(\mathcal{P})$, $x \approx y$, then either $x = y$ or $x = \psi_{-1}(y) = -y_0/-y_1$.*

Proof: Start with the set

$$\mathcal{E}_0 = \{x \in \widehat{\mathbb{Q}} : \ \exists a \in A, (x \approx \mathfrak{l}(W_a) \text{ or } x \approx \mathbf{r}(W_a)), \ \gcd(x) = 1\}.$$

Thus $\mathcal{E}_0$ contains every endpoint of every W_a in two versions x and $\psi_{-1}(x)$ such that x_0 is coprime with x_1. Let $\mathcal{E}_1 \subset \widehat{\mathbb{Q}}$ be the smallest subset of $\widehat{\mathbb{Q}}$ which contains $\mathcal{E}_0$ and has the property that $y = F_a^{-1}x \in \mathcal{E}_1$ whenever $x \in \mathcal{E}_1 \cap \overline{W_a}$. Since $||y|| \leq ||x||$ in this case, the set $\mathcal{E}_1$ is finite. The set $\mathcal{E}_2 = \{x \in \mathcal{E}_1 : \ \forall p > 1, \psi_p(x) \notin \mathcal{E}_1\}$. is still invariant. Let $\mathcal{P}$ be the partition with endpoints $\mathcal{E}(\mathcal{P}) = \mathcal{E}_2$. By Corollary 6, the extension of the system by $\mathcal{P}$ is a SFT of order 2. □

The $\widehat{\mathbb{Q}}$-invariant endpoint set of the bimodular system $(F, \mathcal{R})$ is show in thick in Figure 5 right.

Proposition 17 *Let $(F, \mathcal{W})$ be an integer MNS whose expansion subshift is a SFT of order 2 and assume that all F_a have the same determinant $\det(F_a) = d$. Then if W_u and W_{ua} have the common left endpoint, then $\mathfrak{l}(W_{ua}) = \psi_d(\mathfrak{l}(W_u))$. Similarly for the right endpoints.*

Proof: If $ab \in \mathcal{L}^2(\mathcal{S}_{\mathcal{W}})$ and $\mathfrak{l}(W_{ab}) = \mathfrak{l}(W_a)$, then $F_a^{-1}(\mathfrak{l}(W_a)) = \mathfrak{l}(W_b)$ and $\mathfrak{l}(W_{ab}) = \mathfrak{l}(F_a W_b) = F_a \mathfrak{l}(W_b) = F_a F_a^{-1} \mathfrak{l}(W_a) = \psi_d(\mathfrak{l}(W_a))$. If $\mathfrak{l}(W_{uab}) \approx \mathfrak{l}(W_{ua})$ then $\mathfrak{l}(W_{uab}) = F_u \mathfrak{l}(W_{ab}) = F_u \psi_d(\mathfrak{l}(W_a)) = \psi_d(F_u \mathfrak{l}(W_a)) = \psi_d(\mathfrak{l}(W_{ua}))$. □

If the condition of Proposition 17 is satisfied, the neighboring interval cylinders have the same $\widehat{\mathbb{Q}}$-endpoints. Then the system of endpoints of cylinders can be regarded as a generalized Stern-Brocot graph. Its edges are $\mathfrak{l}(u) \to \mathfrak{l}(ua)$ and $\mathbf{r}(u) \to \mathbf{r}(ua)$. See Figure 2 for the Stern-Brocot graph of the parabolic modular system, and Figure 6 for the Stern-Brocot graph of the bimodular system $(F, \mathcal{R})$.

In integer MNS with expansion subshifts of order 2, the arithmetical algorithms can be simplified. A general algorithm for the computation of a MT $M \in \mathcal{M}(\mathbb{Z})$ in a redundant integer MNS has been given in Kůrka and Kazda [8]. Its simplified version is in Proposition 19.

Definition 18 *Let $(F, \mathcal{W})$ be a redundant integer MNS. The linear graph of $(F, \mathcal{W})$ is defined as follows: Its vertices are (M, a), where $M \in \mathcal{M}$ and $a \in A \cup \{\lambda\}$. The labelled edges are*

$$\begin{array}{rlcll} \textit{emission:} & (M,a) & \xrightarrow{(c,\lambda)} & (F_c^{-1}M, a) & \textit{if} \quad MW_a \subseteq W_c, \\ \textit{absorption:} & (M,a) & \xrightarrow{(\lambda,b)} & (MF_a, b) & \end{array}$$

For the empty word λ we set $W_\lambda = \overline{\mathbb{R}}$ and $F_\lambda = \mathrm{Id}$. The labels of paths are concatenation of the labels of their edges. They are pairs (w, u), where $u \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ is the input word and $w \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ is the output word. Given $M \in \mathcal{M}$ and $u \in \mathcal{S}_\mathcal{W}$, the lazy algorithm which computes $w \in \mathcal{S}_\mathcal{W}$ with $\Phi(w) = M\Phi(u)$ starts at the vertex (M, λ), applies the emission action whenever possible and the absorption action otherwise.

Proposition 19 *If (w, u) is the label of a finite path with source (M, λ) and $u \in \mathcal{L}(\mathcal{S}_\mathcal{W})$, then $w \in \mathcal{L}(\mathcal{S}_\mathcal{W})$ and $M(\Phi([u])) \subseteq \Phi([w])$. If (w, u) is the label of an infinite path with source (M, λ), $u \in \mathcal{S}_\mathcal{W}$, and $w \in A^\mathbb{N}$, then $w \in \mathcal{S}_\mathcal{W}$ and $\Phi(w) = M(\Phi(u))$.*

Proof: We show by induction that when there is a path with source (M, λ) and label $(w, ua) \in A^* \times \mathcal{L}(\mathcal{S}_\mathcal{W})$, then $M(W_{ua}) \subseteq W_w$ and its target is $(F_w^{-1}MF_u, a)$. Since $W_\lambda = \overline{\mathbb{R}}$, the first edge $(M, \lambda) \to (M, a)$ has label (λ, a), so $M(W_\lambda) \subseteq W_\lambda$ is satisfied. Suppose that the assumption holds for (w, ua), and consider an edge $(F_w^{-1}MF_u, a) \to (F_w^{-1}MF_{ua}, b)$ with label (λ, b). Then $MW_{uab} \subseteq MW_{ua} \subseteq W_w$, so the statement holds for the path label (w, uab). Consider an edge $(F_w^{-1}MF_u, a) \to (F_{wc}^{-1}MF_u, a)$, with label (c, λ), so $F_w^{-1}MF_uW_a \subseteq W_c$. Then $MW_{ua} \subseteq MF_uW_a \subseteq F_wW_c$. Since $MW_{ua} \subseteq MW_u \subseteq W_w$, we get $MW_{ua} \subseteq W_w \cap F_wW_c = W_{wc}$, so the statement holds for the path label (wc, ua). By Theorem 3 we get $M(\Phi([u])) \subseteq \Phi([w])$. If u, w are infinite words, then for each n there exists k_n such that $MW_{u_{[0,k_n)}} \subseteq W_{w_{[0,n]}}$ and $M\Phi(u) \in M(\overline{W_{u_{[0,k_n)}}}) \subseteq \overline{W_{w_{[0,n]}}}$ and therefore $M\Phi(u) = \Phi(w)$.

Definition 20 *Let $(F, \mathcal{W})$ be an integer MNS whose expansion subshift is a SFT of order 2. The linear cut graph of $(F, \mathcal{W})$ is defined as follows: Its vertices are (M, λ), where $M \in \mathcal{M}$ and (I, a) where $I \in \mathcal{I}(\mathbb{Z})$ and $a \in A$. The labelled edges are*

$$\begin{array}{rcll} (M,\lambda) & \xrightarrow{(\lambda,a)} & (MW_a, a), & \\ (I,a) & \xrightarrow{(\lambda,b)} & (I\Psi_{ab}, b) & \textit{if} \quad ab \in \mathcal{L}^2(\mathcal{S}_\mathcal{W}), \\ (I,a) & \xrightarrow{(c,\lambda)} & (F_c^{-1}I, a) & \textit{if} \quad I \subseteq W_c. \end{array}$$

Proposition 19 holds for the linear cut graph as well. The proof is based on the fact that if (w, ua) is the label of a path with source (M, λ), then its target is $(F_w^{-1}MF_uW_a, a)$ and $MF_uW_a \subseteq W_w$.

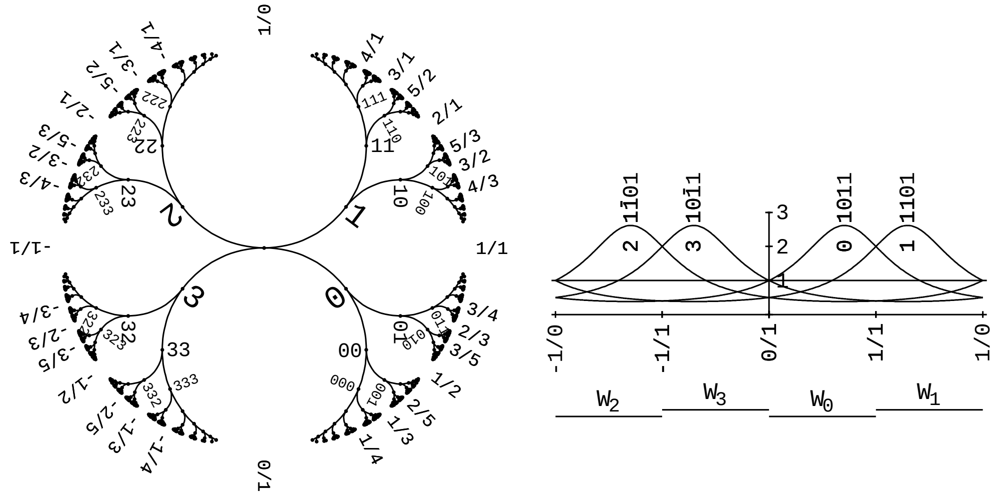

Figure 1. Parabolic modular system: means $\widehat{F}_u(0)$ (left) and circle derivations of F_a^{-1} with intervals W_a (right).

a	F_a	W_a	$F_a^{-1}W_a$	$ab : \Psi_{ab}$
0	$[1,0,1,1]$	$(\frac{0}{1}, \frac{1}{1})$	$(\frac{0}{1}, \frac{1}{0})$	$00 : (\frac{1}{0}, \frac{1}{1}), 01 : (\frac{1}{1}, \frac{0}{1})$
1	$[1,1,0,1]$	$(\frac{1}{1}, \frac{1}{0})$	$(\frac{0}{1}, \frac{1}{0})$	$10 : (\frac{1}{0}, \frac{1}{1}), 11 : (\frac{1}{1}, \frac{0}{1})$
2	$[1,-1,0,1]$	$(\frac{-1}{0}, \frac{-1}{1})$	$(\frac{-1}{0}, \frac{0}{1})$	$22 : (\frac{1}{0}, \frac{1}{1}), 23 : (\frac{1}{1}, \frac{0}{1})$
3	$[1,0,-1,1]$	$(\frac{-1}{1}, \frac{0}{1})$	$(\frac{-1}{0}, \frac{0}{1})$	$32 : (\frac{1}{0}, \frac{1}{1}), 33 : (\frac{1}{1}, \frac{0}{1})$

Table 1. Parabolic modular system: transformations, intervals, their inverse images, and cut matrices.

9. Parabolic modular system

Example 1 *The parabolic modular system with alphabet $A = \{0,1,2,3\}$ has transformations F_a and intervals W_a given in Table 1.*

All F_a are parabolic with fixed points ∞ or 0. The values $\widehat{F}_u(0)$ in the unit complex disc are given in Figure 1 left. The curves between $\widehat{F}_u(0)$ are constructed as follows. For each MT M there exists a family of MT $(M^t)_{t\in\mathbb{R}}$ such that $M^0 = \mathrm{Id}$, $M^1 = M$, and $M^{t+s} = M^t M^s$. Each value $\widehat{F}_u(0)$ in the diagram is joined to $\widehat{F}_{ua}(0)$ by the curve $(\widehat{F}_u \widehat{F}_a^t(0))_{0\le t\le 1}$. The labels $u \in A^+$ at $\widehat{F}_u(0)$ are written in the direction of the tangent vectors $\widehat{F}'_u(0)$. In Figure 1 right there are circle derivations of the inverse transformations F_a^{-1}. The system is rational and its expansion subshift $\mathcal{S}_{\mathcal{W}} = \{0,1\}^{\mathbb{N}} \cup \{2,3\}^{\mathbb{N}}$ is a SFT of order 2. The cut matrices are in Table 1. If $u \in \{0,1\}^+$ and $W_u = (x,y)$, then $W_{u0} = (x, \frac{x_0+y_0}{x_1+y1})$, $W_{u1} = (\frac{x_0+y_0}{x_1+y1}, y)$ (see Figure 2).

10. The bimodular group

For each integer $p \ge 1$, we have the group of p-modular Möbius transformations

$$\mathcal{M}_p = \{M \in \mathcal{M}(\mathbb{Z}) :\ \exists n \ge 0, \det(M) = p^n\}.$$

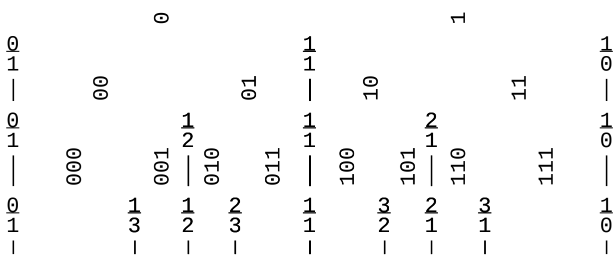

Figure 2. The Stern-Brocot graph of the parabolic modular system. Words u are written (in vertical) at the midpoints of intervals W_u. New endpoints are obtained from the old ones by $(x, y) \mapsto (x_0 + y_0)/(x_1 + y_1)$.

In particular the *modular group* consists of transformations with unit determinant and the *bimodular group* consists of transformations whose determinant is a power of 2. It is well-known that the transformations $S(x) = -1/x$, $T(x) = x + 1$ generate the modular group (see e.g., Coppel [1]). The proof can be generalized to the case of any p-modular group, where p is a prime.

Proposition 21 *If p is a prime, then the transformations $S(x) = -1/x$, $T(x) = x+1$ and $Q(x) = px$ generate the p-modular group and satisfy the identities $S^2 = \mathrm{Id}$, $(ST)^3 = \mathrm{Id}$, $(QS)^2 = \mathrm{Id}$, $QT = T^pQ$.*

Proof: Let $M = M_{(a,b,c,d)}$ be a p-modular transformation. We multiply it from the left by a sequence of generators until we get a generator. If $0 < |c| \leq |a|$ then $T^nM(x) = (a'x+b')/(c'x+d') = ((a+nc)x+b+nd)/(cx+d)$ and there exists n such that $|a'| + |c'| < |a| + |c|$. If $0 < |a| \leq |c|$ then $ST^nSM(x) = (a'x + b')/(c'x + d') = (ax+b)/((c-an)x+d-bn)$ and there exists n such that $|a'|+|c'| < |a|+|c|$. Thus after finitely many steps we obtain a matrix with $ac = 0$. If $c = 0$ then $M(x) = (p^nx+b)/p^m$ for some n, m and $M = Q^{-m}T^bQ^n$. If $a = 0$ then $M(x) = -p^n/(p^mx + d)$ and $M = SQ^{-n}T^dQ^m$. □

If we classify bimodular matrices $M_{(a,b,c,d)}$ according to their norm $n = a^2 + b^2 + c^2 + d^2$ and trace $t = |a+d|$, we get two rotations $M_{(1,-1,1,1)}$, $M_{(1,1,-1,1)}$ with $n = 4, t = 2$, two elliptic transformations $M_{(0,1,-2,0)}$, $M_{(0,2,-1,0)}$ with $n = 5, t = 0$, two hyperbolic transformations $M_{(1,0,0,2)}$, $M_{(2,0,0,1)}$ with $n = 5, t = 3$, eight elliptic transformations with $n = 6, t = 1$ and eight hyperbolic transformations with $n = 6, t = 3$. These eight transformation form a rational MNS with high symmetry and nice properties. Its transformations satisfy many identities and have several almost-covers with sofic subshifts. Moreover, arithmetical algorithms work faster in the bimodular system than in the classical positional binary system (see Kůrka [7]).

11. The bimodular system

Example 2 *The $(6, 3)$-bimodular Möbius iterative system consists of the transformations*

$$F_0 = M_{(1,0,1,2)}, \quad F_1 = M_{(1,1,0,2)}, \quad F_2 = M_{(2,0,1,1)}, \quad F_3 = M_{(2,1,0,1)},$$
$$F_4 = M_{(2,-1,0,1)}, \quad F_5 = M_{(2,0,-1,1)}, \quad F_6 = M_{(1,-1,0,2)}, \quad F_7 = M_{(1,0,-1,2)}.$$

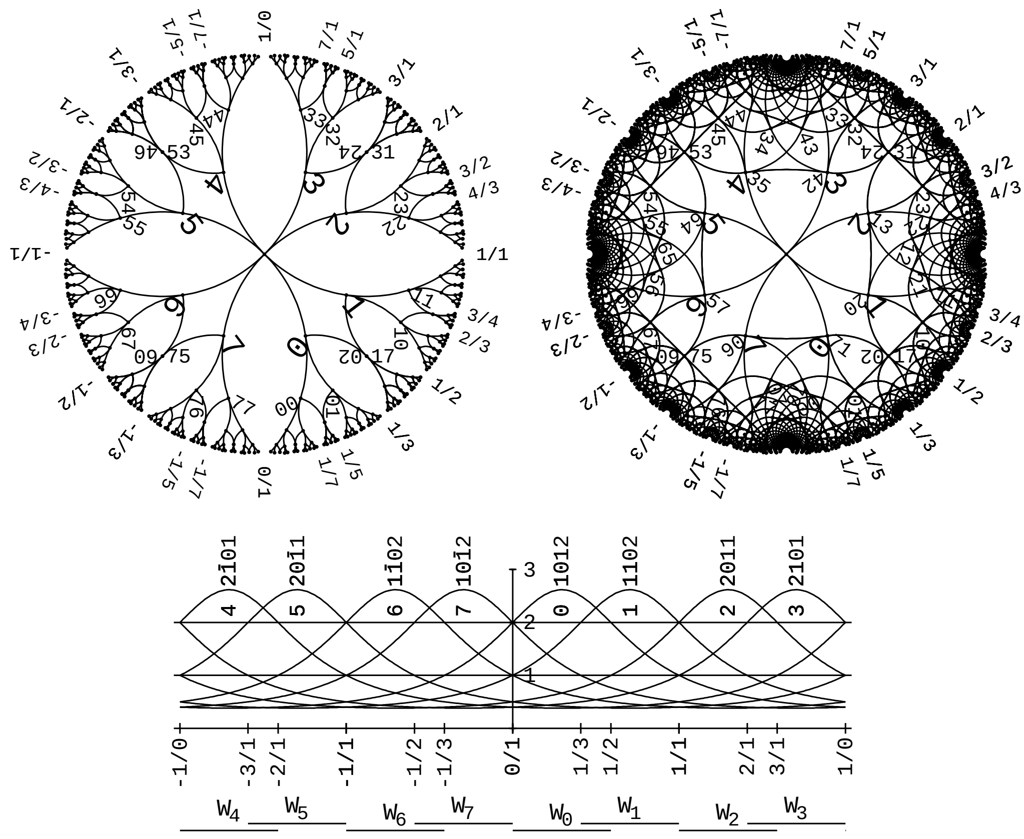

Figure 3. The bimodular systems $(F,\mathcal{R})$ (top left), $(F,\mathcal{W}_{23})$ (top right) and the circle derivations of F_a^{-1} with intervals of $\mathcal{R}$ (bottom).

over the alphabet $A = \{0,1,2,3,4,5,6,7\}$.

All transformations F_a are hyperbolic with stable fixed points 0, 1, 1, ∞, ∞, -1, -1, 0. The circle derivations of all these transformations have the same shape (Figure 3 bottom). The two pictures in Figure 3 top give values of the disc Möbius transformations $\widehat{F}_u(0)$ with the interval almost-cover $\mathcal{R} = \{\mathbf{R}(F_a) :\ a \in A\}$ (left) and with a cover $\mathcal{W}_{23}$ from Table 11 (right). The transformations of the bimodular system generate the bimodular group. Indeed for $S(x) = -1/x$, $T(x) = x+1$ and $Q(x) = 2x$ we have $S = F_{0260}$, $T = F_{13}$, $Q = F_{134}$.

There are several partitions, almost-covers and covers whose expansion subshifts are of finite type or sofic (see Table 2 bottom and Figure 4). If we start with midpoints of the intervals $(0,1)$, $(1,\infty)$, $(\infty,-1)$, $(-1,0)$, we get a partition $\mathcal{W}_1$ with endpoints $\mathcal{E}(\mathcal{W}_1) = \{0, \pm(\sqrt{2}-1), \pm 1, \pm(\sqrt{2}+1), \infty\}$ whose expansion subshift is of finite type of order 3. Its SFT extension has endpoints

$$\{0, \pm(3-2\sqrt{2}), \pm(\sqrt{2}-1), \pm\sqrt{2}/2, \pm 1, \pm\sqrt{2}, \pm(\sqrt{2}+1), \pm(3+2\sqrt{2}), \infty\}.$$

Several other almost-covers are obtained by choosing endpoints from the set $\mathcal{C} = \{0, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3, \infty\}$ whose many preimages remain in $\mathcal{C}$ (see Table 2 top). The subshifts of the partitions $\mathcal{W}_2$ and $\mathcal{W}_3$ are of finite type of order 2. The subshift

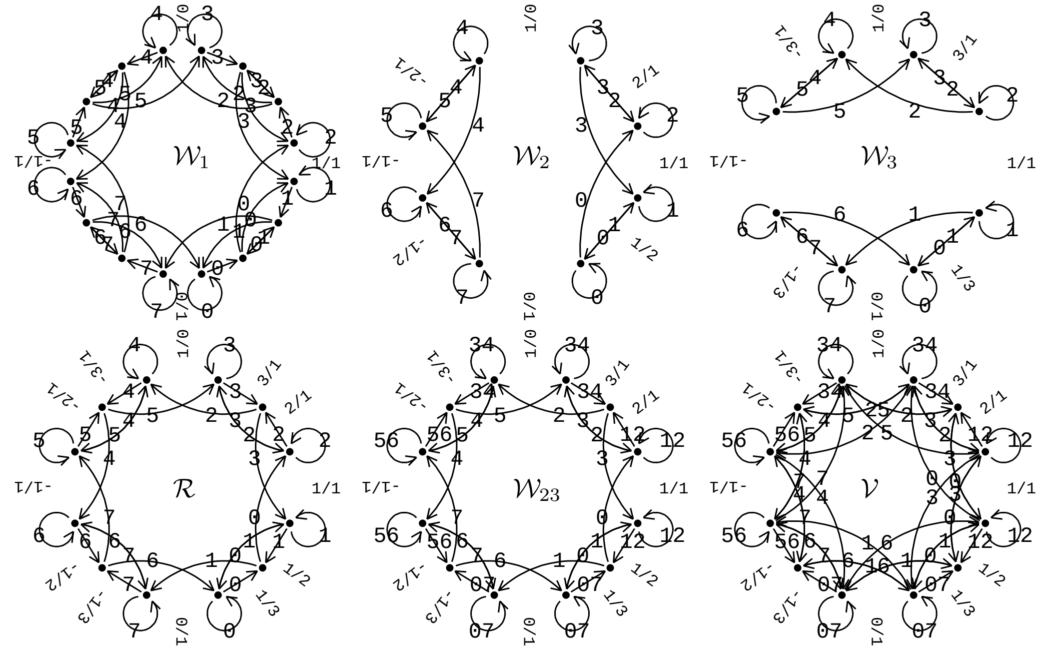

Figure 4. The labelled graphs for the sofic subshifts of the bimodular system constructed by Theorem 11.

x	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	∞	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$
$F_0^{-1}(x)$	0	1	2	∞	-4	-3	-2	$-\frac{3}{2}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{2}$
$F_1^{-1}(x)$	-1	$-\frac{1}{3}$	0	1	3	5	∞	-7	-5	-3	-2	$-\frac{5}{3}$
$F_2^{-1}(x)$	0	$\frac{1}{5}$	$\frac{1}{3}$	1	∞	-3	-1	$-\frac{3}{5}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{5}$	$-\frac{1}{7}$
$F_3^{-1}(x)$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	1	∞	-2	$-\frac{3}{2}$	-1	$-\frac{3}{4}$	$-\frac{2}{3}$
$F_4^{-1}(x)$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	∞	-1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{3}$
$F_5^{-1}(x)$	0	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	1	3	∞	-1	$-\frac{1}{3}$	$-\frac{1}{5}$
$F_6^{-1}(x)$	1	$\frac{5}{3}$	2	3	5	7	∞	-5	-3	-1	0	$\frac{1}{3}$
$F_7^{-1}(x)$	0	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{3}{2}$	2	3	4	∞	-2	-1

	W_0	W_1	W_2	W_3
$\mathcal{W}_1$	$(0, \sqrt{2}-1)$	$(\sqrt{2}-1, 1)$	$(1, \sqrt{2}+1)$	$(\sqrt{2}+1, \infty)$
$\mathcal{W}_2$	$(\frac{0}{1}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{1})$	$(\frac{1}{1}, \frac{2}{1})$	$(\frac{2}{1}, \frac{1}{0})$
$\mathcal{W}_3$	$(\frac{0}{1}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{1})$	$(\frac{1}{1}, \frac{3}{1})$	$(\frac{3}{1}, \frac{1}{0})$
$\mathcal{R}$	$(\frac{0}{1}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{1})$	$(\frac{1}{1}, \frac{3}{1})$	$(\frac{2}{1}, \frac{1}{0})$
$\mathcal{W}_{23}$	$(\frac{-1}{3}, \frac{1}{2})$	$(\frac{1}{3}, \frac{2}{1})$	$(\frac{1}{2}, \frac{3}{1})$	$(\frac{2}{1}, \frac{3}{-1})$
$\mathcal{V}$	$(\frac{-1}{3}, \frac{1}{1})$	$(\frac{0}{1}, \frac{2}{1})$	$(\frac{1}{2}, \frac{1}{0})$	$(\frac{1}{1}, \frac{3}{-1})$

Table 2. Preimages of the cutpoint set $\mathcal{C}$ (top) and interval almost-covers for the bimodular system (bottom). Intervals W_4, W_5, W_6, W_7 can be obtained from the symmetry. If $W_a = (x, y)$ then $W_{7-a} = (-y, -x)$.

of the almost-cover $\mathcal{R} = \{\mathbf{R}(F_a) : a \in A\}$ is of finite type of order 3. Its $\widehat{\mathbb{Q}}$-invariant

b	ca	P_c	F_a	$bd : \Psi_{bd}$
0	00	$(\frac{0}{2}, \frac{1}{3})$	$[1, 0, 1, 2]$	$00 : (\frac{2}{0}, \frac{2}{1}), 01, 02 : (\frac{2}{1}, \frac{1}{1}), 03 : (\frac{1}{1}, \frac{0}{2})$
1	10	$(\frac{1}{3}, \frac{1}{2})$	$[1, 0, 1, 2]$	$14 : (\frac{2}{0}, \frac{0}{2})$
2	11	$(\frac{1}{3}, \frac{1}{2})$	$[1, 1, 0, 2]$	$2\mathrm{F} : (\frac{2}{0}, \frac{0}{2})$
3	21	$(\frac{1}{2}, \frac{2}{2})$	$[1, 1, 0, 2]$	$30 : (\frac{2}{0}, \frac{2}{1}), 31, 32 : (\frac{2}{1}, \frac{1}{1}), 33 : (\frac{1}{1}, \frac{0}{2})$
4	32	$(\frac{2}{2}, \frac{2}{1})$	$[2, 0, 1, 1]$	$44 : (\frac{2}{0}, \frac{1}{1}), 45, 46 : (\frac{1}{1}, \frac{1}{2}), 47 : (\frac{1}{2}, \frac{0}{2})$
5	42	$(\frac{2}{1}, \frac{3}{1})$	$[2, 0, 1, 1]$	$58 : (\frac{2}{0}, \frac{0}{2})$
6	43	$(\frac{2}{1}, \frac{3}{1})$	$[2, 1, 0, 1]$	$63 : (\frac{2}{0}, \frac{0}{2})$
7	53	$(\frac{3}{1}, \frac{2}{0})$	$[2, 1, 0, 1]$	$74 : (\frac{2}{0}, \frac{1}{1}), 75, 76 : (\frac{1}{1}, \frac{1}{2}), 77 : (\frac{1}{2}, \frac{0}{2})$
8	64	$(\frac{2}{0}, \frac{3}{-1})$	$[2, -1, 0, 1]$	$88 : (\frac{2}{0}, \frac{2}{1}), 89, 8\mathrm{A} : (\frac{2}{1}, \frac{1}{1}), 8\mathrm{B} : (\frac{1}{1}, \frac{0}{2})$
9	74	$(\frac{3}{-1}, \frac{2}{-1})$	$[2, -1, 0, 1]$	$9\mathrm{C} : (\frac{2}{0}, \frac{0}{2})$
A	75	$(\frac{3}{-1}, \frac{2}{-1})$	$[2, 0, -1, 1]$	$\mathrm{A}7 : (\frac{2}{0}, \frac{0}{2})$
B	85	$(\frac{2}{-1}, \frac{2}{-2})$	$[2, 0, -1, 1]$	$\mathrm{B}8 : (\frac{2}{0}, \frac{2}{1}), \mathrm{B}9, \mathrm{BA} : (\frac{2}{1}, \frac{1}{1}), \mathrm{BB} : (\frac{1}{1}, \frac{0}{2})$
C	96	$(\frac{2}{-2}, \frac{1}{-2})$	$[1, -1, 0, 2]$	$\mathrm{CC} : (\frac{2}{0}, \frac{1}{1}), \mathrm{CD}, \mathrm{CE} : (\frac{1}{1}, \frac{1}{2}), \mathrm{CF} : (\frac{1}{2}, \frac{0}{2})$
D	A6	$(\frac{1}{-2}, \frac{1}{-3})$	$[1, -1, 0, 2]$	$\mathrm{D}0 : (\frac{2}{0}, \frac{0}{2})$
E	A7	$(\frac{1}{-2}, \frac{1}{-3})$	$[1, 0, -1, 2]$	$\mathrm{EB} : (\frac{2}{0}, \frac{0}{2})$
F	B7	$(\frac{1}{-3}, \frac{0}{-2})$	$[1, 0, -1, 2]$	$\mathrm{FC} : (\frac{2}{0}, \frac{1}{1}), \mathrm{FD}, \mathrm{FE} : (\frac{1}{1}, \frac{1}{2}), \mathrm{FF} : (\frac{1}{2}, \frac{0}{2})$

Table 3. The extended system of the bimodular system $(F, \mathcal{R})$ constructed by Definition 9.

endpoint set

$$\{\tfrac{-2}{0}, \tfrac{-3}{1}, \tfrac{-2}{1}, \tfrac{-2}{2}, \tfrac{-1}{2}, \tfrac{-1}{3}, \tfrac{0}{2}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{2}{1}, \tfrac{3}{1}, \tfrac{2}{0}, \tfrac{3}{-1}, \tfrac{2}{-1}, \tfrac{2}{-2}, \tfrac{1}{-2}, \tfrac{1}{-3}, \tfrac{0}{-2}, \tfrac{-1}{-3}, \tfrac{-1}{-2}, \tfrac{-2}{-2}, \tfrac{-2}{-1}, \tfrac{-3}{-1}\}$$

is shown in thick in Figure 5 right. The extended system of $(F, \mathcal{R})$ is given in Table 3 and its Stern-Brocot graph is in Figure 6. The three systems $(F, \mathcal{W}_2)$, $(F, \mathcal{W}_3)$ and $(F, \mathcal{R})$ are rational. There exist also two covers $\mathcal{W}_{23}$ and $\mathcal{V} = \{\mathbf{V}(F_a) : a \in A\}$ with sofic expansion subshifts (see Table 2).

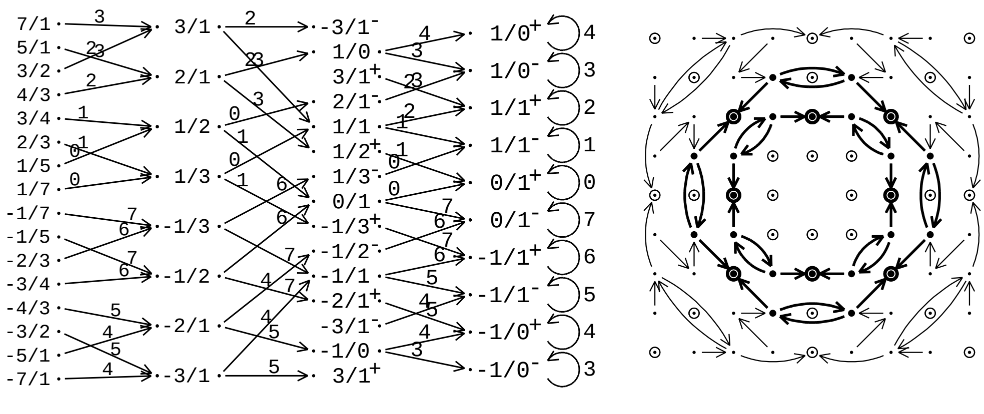

Figure 5. The expansion graphs of the bimodular system $(F, \mathcal{R})$ in $\overline{\mathbb{Q}}$ (left) and in $\widehat{\mathbb{Q}}$ (right). Fixed points are surrounded by circles. The $\widehat{\mathbb{Q}}$-invariant endpoint set is displayed in thick.

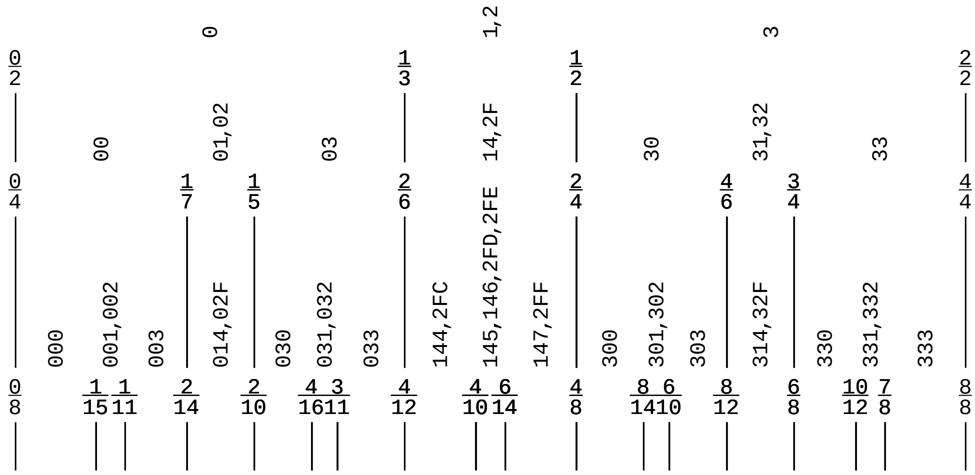

Figure 6. The Stern-Brocot graph of the extended system of the rational bimodular systems $(F, \mathcal{R})$.

12. Conclusions

In Kůrka [6] two more bimodular systems (each with eight transformations) have been considered, one with norm $n = 9$ and trace $t = 3$, the other with norm $n = 14$ and trace $t = 3$. The $(14, 3)$-system has the same rational expansion intervals as our $(6, 3)$-system and has also several interval covers with sofic subshifts. For a trimodular system with $\det(F_a) = 3$, we need at least 12 transformations to obtain a rational system. Since modular systems do not give redundant MNS, it seems that the simplest bimodular hyperbolic system with the smallest norm $n = 6$ is a good alternative for the implementation of computer arithmetic. In fact, arithmetical algorithms work faster in it than in the standard positional binary system (see Kůrka [7]).

Acknowledgments

The research was supported by the Research Program CTS MSM 0021620845 and by the Czech Science Foundation research project GAČR 201/09/0854. I thank Tomáš Hejda for valuable comments.

References

[1] W. A. Coppel. *Number Theory An Introduction to Mathematics.* Springer-Verlag, Berlin, 2009.

[2] A. Kazda. Convergence in Möbius number systems. *Integers*, 2:261–279, 2009.

[3] P. Kornerup and D. W. Matula. An algorithm for redundant binary bit-pipelined rational arithmetic. *IEEE Transactions on Computers*, 39(8):1106–1115, August 1990.

[4] P. Kůrka. A symbolic representation of the real Möbius group. *Nonlinearity*, 21:613–623, 2008.

[5] P. Kůrka. Möbius number systems with sofic subshifts. *Nonlinearity*, 22:437–456, 2009.

[6] P. Kůrka. Expansion of rational numbers in Möbius number systems. In S. Kolyada, Y. Manin, and M. Moller, editors, *Dynamical Numbers: Interplay between Dynamical Systems and Number Theory*, volume 532 of *Contemporary Mathematics*, pages 67–82. American Mathematical Society, 2010.

[7] P. Kůrka. Fast arithmetical algorithms in Möbius number systems. 2011. submitted.

[8] P. Kůrka and A.Kazda. Möbius number systems based on interval covers. *Nonlinearity*, 23:1031–1046, 2010.
[9] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, Cambridge, 1995.
[10] M. Niqui. Exact real arithmetic on the Stern-Brocot tree. *J. Discrete Algorithms*, 5(2):356–379, 2007.
[11] P. J. Potts. *Exact real arithmetic using Möbius transformations.* PhD thesis, University of London, Imperial College, London, 1998.
[12] P. J. Potts, A. Edalat, and M. H. Escardó. Semantics of exact real computation. In *Proceedings of the twelfth annual IEEE symposium in computer science*, pages 248–257, Warsaw, 1997.
[13] J. E. Vuillemin. Exact real computer arithmetic with continued fractions. *IEEE Transactions on Computers*, 39(8):1087–1105, August 1990.
[14] K. Weihrauch. *Computable analysis. An introduction.* EATCS Monographs on Theoretical Computer Science. Springer-Verlag, Berlin, 2000.