Topological dynamics of one-dimensional cellular automata

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Contents

1 Glossary

Almost equicontinuous CA has an equicontinuous configuration.
Attractor: omega-limit of a clopen invariant set.
Blocking word interrupts information flow.
Closing CA: distinct asymptotic configurations have distinct images.
Column subshift: columns in space-time diagrams.
Cross section: one-sided inverse map.

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Directional dynamics: dynamics along a direction in the space-time diagram.
Equicontinuous configuration: nearby configurations remain close.
Equicontinuous CA: all configurations are equicontinuous.
Expansive CA: distinct configurations get away.
Finite time attractor is attained in finite time from its neighbourhood.
Jointly periodic configuration is periodic both for the shift and the CA.
Lyapunov exponents: asymptotic speed of information propagation.
Maximal attractor: omega-limit of the full space.
Nilpotent CA: maximal attractor is a sigleton.
Open CA: image of an open set is open.
Permutive CA: local rule permutes an extremal coordinate.
Quasi-attractor: a countable intersection of attractors.
Signal subshift: weakly periodic configurations of a given period.
Spreading set: clopen invariant set which propagates in both directions.
Subshift attractor: limit set of a spreading set.

2 Definition

A **topological dynamical system** is a continuous selfmap $F : X \to X$ of a topological space X. Topological dynamics studies iterations $F^n : X \to X$, or trajectories $(F^n(x))_{n\geq 0}$. Basic questions are how trajectories depend on initial conditions, whether they are dense in the state space X, whether they have limits, or what are their accumulation points. While cellular automata have been introduced in late forties by von Neumann [43] as regular infinite networks of finite automata, topological dynamics of cellular automata begins in 1969 with Hedlund [20] who viewed one-dimensional cellular automata in the context of symbolic dynamics as endomorphisms of the shift dynamical systems. In fact, the term "cellular automaton" never appears in his paper. Hedlund's main results are the characterizations of surjective and open cellular automata. In the early eighties Wolfram [44] produced space-time representations of one-dimensional cellular automata and classified them informally into four classes using dynamical concepts like periodicity, stability and chaos. Wolfram's classification stimulated mathematical research involving all the concepts of topological and measure-theoretical dynamics, and several formal classifications were introduced using dynamical concepts.

There are two well understood classes of cellular automata with remarkably different stability properties. Equicontinuous cellular automata settle into a fixed or periodic configuration depending on the initial condition and cannot be perturbed by fluctuations. This is a paradigm of stability. Positively expansive cellular automata, on the other hand, are conjugated (isomorphic) to one-sided full shifts. They have dense orbits, dense periodic configurations, positive topological entropy, and sensitive dependence on the initial conditions. This is a paradigm of chaotic behaviour. Between these two extreme classes there are many distinct types of dynamical behaviour which are understood much less. Only some specific classes or particular examples have been elucidated and a general theory is still lacking.

3 Introduction

Dynamical properties of CA are usually studied in the context of symbolic dynamics. Other possibilities are measurable dynamics (see Pivato [38]) or noncompact dynamics in Besicovitch or Weyl spaces (see Formenti and Kůrka [18]). In symbolic dynamics, the state space is the Cantor space of symbolic sequences. The Cantor space has distinguished topological properties which simplify some concepts of dynamics. This is the case of attractor and topological entropy. Cellular automata can be defined in context of symbolic dynamics as continuous mappings which commute with the shift map.

Equicontinuous and almost equicontinuous CA can be characterized using the concept of blocking words. While equicontinuous CA are eventually periodic, closely related almost equicontinuous automata, are periodic on a large (residual) subset of the state space. Outside of this subset, however, their behaviour can be arbitrarily complex.

A property which strongly constrains the dynamics of cellular automata is surjectivity. Surjective automata preserve the uniform Bernoulli measure, they are bounded-to-one, and their unique subshift attractor is the full space. An important subclass of surjective CA are (left- or right-) closing automata. They have dense sets of periodic configurations. Cellular automata which are both left- and right-closing are open: They map open sets to open sets, are n-to-one and have cross-sections. A fairly well understood class is that of positively expansive automata. A positively expansive cellular automaton is conjugated (isomorphic) to a one-sided full shift. Closely related are bijective expansive automata which are conjugated to two-sided subshifts, usually of finite type.

Another important concept elucidating the dynamics of CA is that of an attractor. With respect to attractors, CA clasiffy into two basic classes. In one class there are CA which have disjoint attractors. They have then countably infinite numbers of attractors and uncountable numbers of quasi-attractors i.e., countable intersections of attractors. In the other class there are CA which have either a minimal attractor or a minimal quasi-attractor which is then contained in any attractor. An important class of attractors are subshift attractors - subsets which are both attractors and subshifts. They have always non-empty intersection, so they form a lattice with maximal element.

Factors of CA which are subshifts are usefull because factor maps preserve many dynamical properties while they simplify the dynamics. In a special case of column subshifts, they are formed by sequences of words occurring in a column of a space-time diagram. Factor subshifts are instrumental in evaluating the entropy of CA and in characterizing CA with the shadowing property.

A finer classification of CA is provided by quantitative characteristics. Topological entropy measures the quantity of information available to an observer who can see a finite part of a configuration. Lyapunov exponents measure the speed of information propagation. The minimum preimage number provides a finer classification of surjective cellular automata. Sets of left- and right-expansivity directions provide finer classification for left- and right-closing cellular automata.

4 Topological dynamics

We review basic concepts of topological dynamics as exposed in Kůrka [25]. A **Cantor space** is any metric space which is **compact** (any sequence has a convergent subsequence), **totally disconnected** (distinct points are separated by disjoint **clopen**, i.e., closed and open sets), and **perfect** (no point is isolated). Any two Cantor spaces are homeomorphic. A **symbolic space** is any compact, totally disconnected metric space, i.e., any closed subspace of a Cantor space. A **symbolic dynamical system** (SDS) is a pair (X, F) where X is a symbolic space and $F : X \to X$ is a continuous map. The n-th **iteration** of F is denoted by F^n. If F is bijective (invertible), the negative iterations are defined by $F^{-n} = (F^{-1})^n$. A set $Y \subseteq X$ is **invariant**, if $F(Y) \subseteq Y$ and **strongly invariant** if $F(Y) = Y$.

A **homomorphism** $\varphi : (X, F) \to (Y, G)$ of SDS is a continuous map $\varphi : X \to Y$ such that $\varphi \circ F = G \circ \varphi$. A **conjugacy** is a bijective homomorphism. The systems (X, F) and (Y, G) are **conjugated**, if there exists a conjugacy between them. If φ is surjective, we say that (Y, G) is a **factor** of (X, F). If φ is injective, (X, F) is a

subsystem of (Y,G). In this case $\varphi(X) \subseteq Y$ is a closed invariant set. Conversely, if $Y \subseteq X$ is a closed invariant set, then (Y,F) is a subsystem of (X,F).

We denote by $d : X\times X \to [0,\infty)$ the metric and by $B_\delta(x) = \{y \in X : \ d(y,x) < \delta\}$ the ball with center x and radius δ. A finite sequence $(x_i \in X)_{0\le i\le n}$ is a δ**-chain** from x_0 to x_n, if $d(F(x_i), x_{i+1}) < \delta$ for all $i < n$. A point $x \in X$ ε**-shadows** a sequence $(x_i)_{0\le i\le n}$, if $d(F^i(x), x_i) < \varepsilon$ for all $0 \le i \le n$. A SDS (X,F) has the **shadowing property**, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that every δ-chain is ε-shadowed by some point.

Definition 1 *Let (X,F) be a SDS. The* **orbit relation** $\mathcal{O}_F$*, the* **recurrence relation** $\mathcal{R}_F$*, the* **nonwandering relation** $\mathcal{N}_F$*, and the* **chain relation** $\mathcal{C}_F$ *are defined by*

$$
\begin{aligned}
(x,y) \in \mathcal{O}_F &\iff \exists n > 0, y = F^n(x)\\
(x,y) \in \mathcal{R}_F &\iff \forall \varepsilon > 0, \exists n > 0, d(y, F^n(x)) < \varepsilon\\
(x,y) \in \mathcal{N}_F &\iff \forall \varepsilon, \delta > 0, \exists n > 0, \exists z \in B_\delta(x), d(F^n(z), y) < \varepsilon\\
(x,y) \in \mathcal{C}_F &\iff \forall \varepsilon > 0, \exists \varepsilon-\textit{chain from } x \textit{ to } y
\end{aligned}
$$

We have $\mathcal{O}_F \subseteq \mathcal{R}_F \subseteq \mathcal{N}_F \subseteq \mathcal{C}_F$. The diagonal of a relation $S \subseteq X \times X$ is $|S| := \{x \in X : \ (x,x) \in S\}$. We denote by $S(x) := \{y \in X : \ (x,y) \in S\}$ the S-image of a point $x \in X$. The orbit of a point $x \in X$ is $\mathcal{O}_F(x) := \{F^n(x) : \ n > 0\}$. It is an invariant set, so its closure $(\overline{\mathcal{O}_F(x)}, F)$ is a subsystem of (X,F). A point $x \in X$ is **periodic** with period $n > 0$, if $F^n(x) = x$, i.e., if $x \in |\mathcal{O}_F|$. A point $x \in X$ is **eventually periodic**, if $F^m(x)$ is periodic for some **preperiod** $m \ge 0$. The points in $|\mathcal{R}_F|$ are called **recurrent**, the points in $|\mathcal{N}_F|$ are called **nonwandering** and the points in $|\mathcal{C}_F|$ are called **chain-recurrent**. The sets $|\mathcal{N}_F|$ and $|\mathcal{C}_F|$ are closed and invariant, so $(|\mathcal{N}_F|, F)$ and $(|\mathcal{C}_F|, F)$ are subsystems of (X,F). The set of **transitive** points is $\mathcal{T}_F := \{x \in X : \ \overline{\mathcal{O}(x)} = X\}$. A system (X,F) is **minimal**, if $\mathcal{R}_F = X \times X$. This happens iff each point has a dense orbit, i.e., if $\mathcal{T}_F = X$. A system is **transitive**, if $\mathcal{N}_F = X \times X$, i.e., if for any nonempty open sets $U, V \subseteq X$ there exists $n > 0$ such that $F^n(U) \cap V \ne \emptyset$. A system is transitive iff it has a transitive point, i.e., if $\mathcal{T}_F \ne \emptyset$. In this case the set of transitive points $\mathcal{T}_F$ is **residual**, i.e., it contains a countable intersection of dense open sets. An infinite system is **chaotic**, if it is transitive and has a dense set of periodic points. A system (X,F) is **weakly mixing**, if $(X\times X, F\times F)$ is transitive. It is **strongly transitive**, if (X,F^n) is transitive for any $n > 0$. A system (X,F) is **mixing**, if for every nonempty open sets $U,V \subseteq X$, $F^n(U) \cap V \ne \emptyset$ for all sufficiently large n. A system is **chain-transitive**, if $\mathcal{C}_F = X \times X$, and **chain-mixing**, if for any $x, y \in X$ and any $\varepsilon > 0$ there exist chains from x to y of arbitrary, large enough length. If a system (X,F) has the shadowing property, then $\mathcal{N}_F = \mathcal{C}_F$. It follows that a chain-transitive system with the shadowing property is transitive, and a chain-mixing system with the shadowing property is mixing.

A **clopen partition** of a symbolic space X is a finite system of disjoint clopen sets whose union is X. The **join** of clopen partitions $\mathcal{U}$ and $\mathcal{V}$ is $\mathcal{U}\vee\mathcal{V} = \{U\cap V : \ U \in \mathcal{U}, V \in \mathcal{V}\}$. The inverse image of a clopen partition $\mathcal{U}$ by F is $F^{-1}(\mathcal{U}) = \{F^{-1}(U) : U \in \mathcal{U}\}$. The entropy $H(X,F,\mathcal{U})$ of a partition and the **entropy** $\mathfrak{h}(X,F)$ of a system are defined by

$$
\begin{aligned}
H(X,F,\mathcal{U}) &= \lim_{n\to\infty} \frac{\ln |\mathcal{U} \vee F^{-1}(\mathcal{U}) \vee \cdots \vee F^{-(n-1)}(\mathcal{U})|}{n},\\
\mathfrak{h}(X,F) &= \sup\{H(X,F,\mathcal{U}) : \ \mathcal{U} \text{ is a clopen partition of } X\}.
\end{aligned}
$$

5 Symbolic dynamics

An **alphabet** is any finite set with at least two elements. The cardinality of a finite set A is denoted by $|A|$. We frequently use alphabet $\mathbf{2} = \{0,1\}$ and more generally

$\mathbf{n} = \{0, \ldots, n-1\}$. A word over A is any finite sequence $u = u_0 \ldots u_{n-1}$ of elements of A. The length of u is denoted by $|u| := n$ and the word of zero length is denoted by λ. The set of all words of length n is denoted by A^n. The set of all nonzero words and the set of all words are

$$A^+ = \bigcup_{n>0} A^n, \quad A^* = \bigcup_{n\geq 0} A^n.$$

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of nonnegative integers, by $\mathbb{N}^+$ the set of positive integers, by $\mathbb{Q}$ the set of rational numbers, and by $\mathbb{R}$ the set of real numbers. The set of one-sided configurations (infinite words) is $A^{\mathbb{N}}$ and the set of two-sided configurations (biinfinite words) is $A^{\mathbb{Z}}$. If u is a finite or infinite word and $I = [i,j]$ is an interval of integers on which u is defined, put $u_{[i,j]} = u_i \ldots u_j$. Similarly for open or half-open intervals $u_{[i,j)} = u_i \ldots u_{j-1}$. We say that v is a **subword** of u and write $v \sqsubseteq u$, if $v = u_I$ for some interval $I \subseteq \mathbb{Z}$. If $u \in A^n$, then $u^\infty \in A^{\mathbb{Z}}$ is the infinite repetition of u defined by $(u^\infty)_{kn+i} = u_i$. Similarly $x = u^\infty.v^\infty$ is the configuration satisfying $x_{i+k|u|} = u_i$ for $k < 0$, $0 \leq i < |u|$ and $x_{i+k|v|} = v_i$ for $k \geq 0$, $0 \leq i < |v|$. On $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ we have metrics

$$\begin{aligned} d(x,y) &= 2^{-n} \text{ where } n = \min\{i \geq 0 : x_i \neq y_i\}, \quad x,y \in A^{\mathbb{N}} \\ d(x,y) &= 2^{-n} \text{ where } n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}, \quad x,y \in A^{\mathbb{Z}} \end{aligned}$$

Both $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ are Cantor spaces. In $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ the cylinder sets of a word $u \in A^n$ are $[u] := \{x \in A^{\mathbb{N}} : x_{[0,n)} = u\}$, and $[u]_k := \{x \in A^{\mathbb{Z}} : x_{[k,k+n)} = u\}$, where $k \in \mathbb{Z}$. Cylinder sets are clopen and every clopen set is a finite union of cylinders. The shift maps $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ and $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ are continuous. While the two-sided shift is bijective, the one-sided shift is not: every configuration has $|A|$ preimages. A **one-sided subshift** is any non-empty closed set $\Sigma \subseteq A^{\mathbb{N}}$ which is shift-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. A **two-sided subshift** is any non-empty closed set $\Sigma \subseteq A^{\mathbb{Z}}$ which is strongly shift-invariant, i.e., $\sigma(\Sigma) = \Sigma$. Thus a subshift Σ represents a SDS (Σ, σ). Systems $(A^{\mathbb{Z}}, \sigma)$ and $(A^{\mathbb{N}}, \sigma)$ are called **full shifts**. Given a set $D \subseteq A^*$ of **forbidden words**, the set $\mathcal{S}_D := \{x \in A^{\mathbb{N}} : \forall u \in D, u \not\sqsubseteq x\}$ is a one-sided subshift, provided it is nonempty. Any one-sided subshift has this form. Similarly, $\mathcal{S}_D := \{x \in A^{\mathbb{Z}} : \forall u \in D, u \not\sqsubseteq x\}$ is a two-sided subshift, and any two-sided subshift has this form. A (one- or two-sided) subshift is of **finite type** (SFT), if the set D of forbidden words is finite. The **language of a subshift** Σ is the set of all subwords of configurations of Σ,

$$\begin{aligned} \mathcal{L}^n(\Sigma) &= \{u \in A^n : \exists x \in \Sigma, u \sqsubseteq x\}, \\ \mathcal{L}(\Sigma) &= \bigcup_{n\geq 0} \mathcal{L}^n(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\} \end{aligned}$$

The entropy of a subshift Σ is $\mathbf{h}(\Sigma, \sigma) = \lim_{n\to\infty} \ln |\mathcal{L}^n(\Sigma)|/n$. A word $w \in \mathcal{L}(\Sigma)$ is **intrinsically synchronizing**, if for any $u, v \in A^*$ such that $uw, wv \in \mathcal{L}(\Sigma)$ we have $uwv \in \mathcal{L}(\Sigma)$. A subshift is of finite type iff all sufficiently long words are intrinsically synchronizing (see Lind and Marcus [30]).

A subshift Σ is **sofic**, if $\mathcal{L}(\Sigma)$ is a regular language, i.e., if $\Sigma = \Sigma_{\mathcal{G}}$ is the set of labels of paths of a **labelled graph** $\mathcal{G} = (V, E, s, t, l)$, where V is a finite set of vertices, E is a finite set of edges, $s, t : E \to V$ are the source and target map, and $l : E \to A$ is a labelling function. The labelling function extends to a function $\ell : E^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\ell(x)_i = l(x_i)$. A graph $\mathcal{G} = (V, E, s, t, l)$ determines a SFT $\Sigma_{|\mathcal{G}|} = \{u \in E^{\mathbb{Z}}, \forall i \in \mathbb{Z}, t(u_i) = s(u_{i+1})\}$ and $\Sigma_{\mathcal{G}} = \{\ell(u) : u \in \Sigma_{|\mathcal{G}|}\}$ so that $\ell : (\Sigma_{|\mathcal{G}|}, \sigma) \to (\Sigma_{\mathcal{G}}, \sigma)$ is a factor map. If $\Sigma = \Sigma_{\mathcal{G}}$, we say that $\mathcal{G}$ is a **presentation** of Σ. A graph $\mathcal{G}$ is **right-resolving**, if different outgoing edges of a vertex are labelled differently, i.e., if $l(e) \neq l(e')$ whenever $e \neq e'$ and $s(e) = s(e')$. A word w

is **synchronizing** in $\mathcal{G}$, if all paths with label w have the same target, i.e., if $t(u) = t(u')$ whenever $\ell(u) = \ell(u') = w$. If w is synchronizing in $\mathcal{G}$, then w is intrinsically synchronizing in $\Sigma_{\mathcal{G}}$. Any transitive sofic subshift Σ has a unique **minimal right-resolving presentation** $\mathcal{G}$ which has the smallest number of vertices. Any word can be extended to a word which is synchronizing in $\mathcal{G}$ (see Lind and Marcus [30]).

A **deterministic finite automaton (DFA)** over an alphabet A is a system $\mathcal{A} = (Q, \delta, q_0, q_1)$, where Q is a finite set of states, $\delta : Q \times A \to Q$ is a transition function and q_0, q_1 are the initial and rejecting states. The transition function extends to $\delta : Q \times A^* \to Q$ by $\delta(q, \lambda) = q$, $\delta(q, ua) = \delta(\delta(q, u), a)$. The language accepted by $\mathcal{A}$ is $\mathcal{L}(\mathcal{A}) := \{u \in A^* : \delta(q_0, u) \neq q_1\}$ (see e.g., Hopcroft and Ullmann [21]). The DFA of a labelled graph $\mathcal{G} = (V, E, s, t, l)$ is $\mathcal{A}(\mathcal{G}) = (\mathcal{P}(V), \delta, V, \emptyset)$, where $\mathcal{P}(V)$ is the set of all subsets of V and $\delta(q, a) = \{v \in V : \exists u \in q, u \xrightarrow{a} v\}$. Then $\mathcal{L}(\mathcal{A}(\mathcal{G})) = \mathcal{L}(\Sigma_G)$. We can reduce the size of $\mathcal{A}(\mathcal{G})$ by taking only those states which are accessible from the initial state V.

A **periodic structure** $\mathfrak{n} = (n_i)_{i \geq 0}$ is a sequence of integers greater than 1. For a given periodic structure $\mathfrak{n}$, let $X_{\mathfrak{n}} := \prod_{i \geq 0} \{0, \ldots, n_i - 1\}$ be the product space with metric $d(x, y) = 2^{-n}$ where $n = \min\{i \geq 0 : x_i \neq y_i\}$. Then $X_{\mathfrak{n}}$ is a Cantor space. The **adding machine** (odometer) of $\mathfrak{n}$ is a SDS $(X_{\mathfrak{n}}, F)$ given by the formula

$$F(x)_i = \begin{cases} (x_i + 1) \bmod n_i & \text{if} \quad \forall j < i, x_j = n_j - 1 \\ x_i & \text{if} \quad \exists j < i, x_j < n_j - 1 \end{cases}$$

Each adding machine is minimal and has zero topological entropy.

Definition 2 *A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a* **cellular automaton (CA)** *if there exist integers $m \leq a$ (***memory** *and* **anticipation***) and a* **local rule** *$f : A^{a-m+1} \to A$ such that for any $x \in A^{\mathbb{Z}}$ and any $i \in \mathbb{Z}$, $F(x)_i = f(x_{[i+m, i+a]})$. Call $r = \max\{|m|, |a|\} \geq 0$ the* **radius** *of F and $d = a - m \geq 0$ its* **diameter***.*

By a theorem of Hedlund [20], a map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton iff it is continuous and commutes with the shift, i.e., $\sigma \circ F = F \circ \sigma$. This means that $F : (A^{\mathbb{Z}}, \sigma) \to (A^{\mathbb{Z}}, \sigma)$ is a homomorphism of the full shift and $(A^{\mathbb{Z}}, F)$ is a SDS. We can assume that the local rule acts on a symmetric neighbourhood of 0, so $F(x)_i = f(x_{[i-r, i+r]})$, where $f : A^{2r+1} \to A$. There is a trade-off between the radius and the size of the alphabet. Any CA is conjugated to a CA with radius 1. Any σ-periodic configuration of a CA $(A^{\mathbb{Z}}, F)$ is F-eventually periodic. Hence the set of F-eventually periodic configurations is dense. Thus a cellular automaton is never minimal, because it has always an F-periodic configuration. A configuration $x \in A^{\mathbb{Z}}$ is **weakly periodic**, if there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}^+$ such that $F^q \sigma^p(x) = x$. A configuration $x \in A^{\mathbb{Z}}$ is **jointly periodic**, if it is both F-periodic and σ-periodic. A CA $(A^{\mathbb{Z}}, F)$ is **nilpotent**, if $F^n(A^{\mathbb{Z}})$ is a singleton for some $n > 0$.

6 Equicontinuity

A point $x \in X$ of a SDS (X, F) is **equicontinuous**, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall n \geq 0, d(F^n(y), F^n(x)) < \varepsilon.$$

The set of equicontinuous points is denoted by $\mathcal{E}_F$. A system is **equicontinuous**, if $\mathcal{E}_F = X$. In this case it is uniformly equicontinuous, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X, (d(x, y) < \delta \Rightarrow \forall n \geq 0, d(F^n(x), F^n(y)) < \varepsilon).$$

A system (X, F) is **almost equicontinuous**, if $\mathcal{E}_F \neq \emptyset$. A system is **sensitive**, if

$$\exists \varepsilon > 0, \forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, d(F^n(y), F^n(x)) \geq \varepsilon.$$

Clearly, a sensitive system has no equicontinuous points. The converse is not true in general but holds for transitive systems (Akin et al., [2]).

Definition 3 *A word $u \in A^+$ with $|u| \geq s \geq 0$ is* s**-blocking** *for a CA $(A^{\mathbb{Z}}, F)$, if there exists an* **offset** *$k \in [0, |u| - s]$ such that*

$$\forall x, y \in [u]_0, \forall n \geq 0, F^n(x)_{[k,k+s)} = F^n(y)_{[k,k+s)}.$$

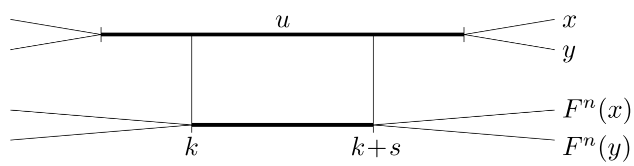


Figure 1: A blocking word

Theorem 4 (Kůrka [28]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius $r \geq 0$. The following conditions are equivalent.*

(1) $(A^{\mathbb{Z}}, F)$ is not sensitive.

(2) $(A^{\mathbb{Z}}, F)$ has an r-blocking word.

(3) $\mathcal{E}_F$ is **residual**, *i.e., a countable intersection of dense open sets.*

(4) $\mathcal{E}_F \neq \emptyset$.

For a nonempty set $B \subseteq A^*$ define

$$\begin{aligned}\mathcal{T}_\sigma^n(B) &:= \{x \in A^{\mathbb{Z}} : (\exists j > i > n, x_{[i,j)} \in B) \,\&\, (\exists j < i < -n, x_{[j,i)} \in B)\},\\ \mathcal{T}_\sigma(B) &:= \bigcap_{n \geq 0} \mathcal{T}_\sigma^n(B).\end{aligned}$$

Each $\mathcal{T}_\sigma^n(B)$ is open and dense, so the set $\mathcal{T}_\sigma(B)$ of B-**recurrent** configurations is residual. If B is the set of r-blocking words, then $\mathcal{E}_F = \mathcal{T}_\sigma(B)$.

Theorem 5 (Kůrka [28]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius $r \geq 0$. The following conditions are equivalent.*

(1) $(A^{\mathbb{Z}}, F)$ is equicontinuous, i.e., $\mathcal{E}_F = A^{\mathbb{Z}}$.

(2) There exists $k > 0$ such that any $u \in A^k$ is r-blocking.

(3) There exists a preperiod $q \geq 0$ and a period $p > 0$, such that $F^{q+p} = F^q$.

In particular every CA with radius $r = 0$ is equicontinuous. A configuration is equicontinuous for F iff it is equicontinuous for F^n, i.e., $\mathcal{E}_F = \mathcal{E}_{F^n}$. This fact enables to consider equicontinuity along rational directions $\alpha = \frac{p}{q}$.

Definition 6 *The sets of* **equicontinuous directions** *and* **almost equicontinuous directions** *of a CA $(A^{\mathbb{Z}}, F)$ are defined by*

$$\begin{aligned}\mathfrak{E}(F) &= \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}^+, \mathcal{E}_{F^q\sigma^p} = A^{\mathbb{Z}}\right\},\\ \mathfrak{A}(F) &= \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}^+, \mathcal{E}_{F^q\sigma^p} \neq \emptyset\right\}.\end{aligned}$$

Clearly, $\mathfrak{E}(F) \subseteq \mathfrak{A}(F)$, and both sets $\mathfrak{E}(F)$ and $\mathfrak{A}(F)$ are convex (Sablik [39]): If $\alpha_0 < \alpha_1 < \alpha_2$ and $\alpha_0, \alpha_2 \in \mathfrak{A}(F)$, then $\alpha_1 \in \mathfrak{A}(F)$. Sablik [39] considers also equicontinuity along irrational directions.

Proposition 7 *Let $(A^{\mathbb{Z}}, F)$ be an equicontinuous CA such that there exists $0 \neq \alpha \in \mathfrak{A}(F)$. Then $(A^{\mathbb{Z}}, F)$ is nilpotent.*

Proof: We can assume $\alpha < 0$. There exist $0 \le k < m$ and $w \in A^m$, such that for all $x, y \in A^{\mathbb{Z}}$ and for all $i \in \mathbb{Z}$ we have

$$\begin{aligned} x_{[i,i+m)} = y_{[i,i+m)} \quad &\Longrightarrow \quad \forall n \ge 0, F^n(x)_{i+k} = F^n(y)_{i+k}, \\ w = x_{[i,i+m)} = y_{[i,i+m)} \quad &\Longrightarrow \quad \forall n \ge 0, F^n\sigma^{\lfloor n\alpha \rfloor}(x)_{i+k} = F^n\sigma^{\lfloor n\alpha \rfloor}(y)_{i+k}. \end{aligned}$$

Take n such that $l := \lfloor n\alpha \rfloor + m \le 0$. There exists $a \in A$ such that $F^n\sigma^{\lfloor n\alpha \rfloor}(z)_k = a$ for every $z \in [w]_0$. Let $x \in A^{\mathbb{Z}}$ be arbitrary. For a given $i \in \mathbb{Z}$, take a configuration $y \in [x_{[i,i+m)}]_i \cap [w]_{i-l+m}$. Then $z := \sigma^{i-\lfloor n\alpha \rfloor}(y) = \sigma^{i-l+m}(y) \in [w]_0$ and $F^n(x)_{i+k} = F^n(y)_{i+k} = F^n\sigma^{\lfloor n\alpha \rfloor}(z)_k = a$. Thus $F^n(x) = a^\infty$ for every $x \in A^{\mathbb{Z}}$, and $F^{n+t}(x) = F^n(F^t(x)) = a^\infty$ for every $t \ge 0$, so $(A^{\mathbb{Z}}, F)$ is nilpotent (see also Sablik [40]).

Theorem 8 *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory m and anticipation a, i.e., $F(x)_i = f(x_{[i+m,i+a]})$. Then exactly one of the following conditions is satisfied.*

(1) $\mathfrak{E}(F) = \mathfrak{A}(F) = \mathbb{Q}$. This happens iff $(A^{\mathbb{Z}}, F)$ is nilpotent.

(2) $\mathfrak{E}(F) = \emptyset$ and there exist real numbers $\alpha_0 < \alpha_1$ such that

$$(\alpha_0, \alpha_1) \subseteq \mathfrak{A}(F) \subseteq [\alpha_0, \alpha_1] \subseteq [-a, -m].$$

(3) There exists $-a \le \alpha \le -m$ such that $\mathfrak{A}(F) = \mathfrak{E}(F) = \{\alpha\}$.

(4) There exists $-a \le \alpha \le -m$ such that $\mathfrak{A}(F) = \{\alpha\}$ and $\mathfrak{E}(F) = \emptyset$.

(5) $\mathfrak{A}(F) = \mathfrak{E}(F) = \emptyset$.

This follows from Theorems II. 2 and II.5 in Sablik [39] and from Proposition 7. The zero CA of Example 1 belongs to class (1). The product CA of Example 4 belongs to class (2). The identity CA of Example 2 belongs to class (3). The Coven CA of Example 18 belongs to class (4). The sum CA of Example 11 belongs to class (5).

Sensitivity can be expressed quantitatively by **Lyapunov exponents** which measure the speed of information propagation. Let $(A^{\mathbb{Z}}, F)$ be a CA. The left and right **perturbation sets** of $x \in A^{\mathbb{Z}}$ are

$$\begin{aligned} W_s^-(x) &= \{y \in A^{\mathbb{Z}} : \forall i \le s, y_i = x_i\}, \\ W_s^+(x) &= \{y \in A^{\mathbb{Z}} : \forall i \ge s, y_i = x_i\}. \end{aligned}$$

The left and right **perturbation speeds** of $x \in A^{\mathbb{Z}}$ are

$$\begin{aligned} I_n^-(x) &= \min\{s \ge 0 : \forall i \le n, F^i(W_s^-(x)) \subseteq W_0^-(F^i(x))\}, \\ I_n^+(x) &= \min\{s \ge 0 : \forall i \le n, F^i(W_{-s}^+(x)) \subseteq W_0^+(F^i(x))\}. \end{aligned}$$

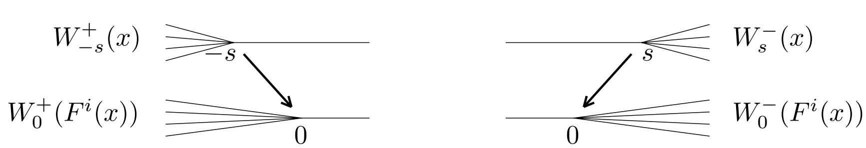


Figure 2: Perturbation speeds

Thus $I_n^-(x)$ is the minimum distance of a perturbation of the left part of x which cannot reach the zero site by time n. Both $I_n^-(x)$ and $I_n^+(x)$ are nondecreasing. If $0 < s < t$, and if $x_{[s,t]}$ is an r-blocking word (where r is the radius), then $\lim_{n\to\infty} I_n^-(x) \le t$. Similarly, if $s < t < 0$ and if $x_{[s,t]}$ is an r-blocking word, then $\lim_{n\to\infty} I_n^+(x) \le |s|$. In particular, if $x \in \mathcal{E}_F$, then both $I_n^-(x)$ and $I_n^+(x)$ have finite limit. If $(A^{\mathbb{Z}}, F)$ is sensitive, then $\lim_{n\to\infty}(I_n^-(x) + I_n^+(x)) = \infty$ for every $x \in A^{\mathbb{Z}}$.

Definition 9 *The left and right* **Lyapunov exponents** *of a CA* $(A^{\mathbb{Z}}, F)$ *and* $x \in A^{\mathbb{Z}}$ *are*

$$\lambda_F^-(x) = \liminf_{n\to\infty} \frac{I_n^-(x)}{n}, \quad \lambda_F^+(x) = \liminf_{n\to\infty} \frac{I_n^+(x)}{n}.$$

If F has memory m and anticipation a, then $\lambda_F^-(x) \leq \max\{a, 0\}$ and $\lambda_F^+(x) \leq \max\{-m, 0\}$ for all $x \in A^{\mathbb{Z}}$. If $x \in \mathcal{E}_F$, then $\lambda_F^+(x) = \lambda_F^-(x) = 0$. If F is right-permutive (see Section 8) with $a > 0$, then $\lambda_F^-(x) = a$ for every $x \in A^{\mathbb{Z}}$. If F is left-permutive with $m < 0$, then $\lambda_F^+(x) = -m$ for every $x \in A^{\mathbb{Z}}$.

Theorem 10 (Bressaud and Tisseur [9]) *For a positively expansive CA (see Section 9) there exists a constant* $c > 0$*, such that for all* $x \in A^{\mathbb{Z}}$*,* $\lambda^-(x) \geq c$ *and* $\lambda^+(x) \geq c$*.*

Conjecture 11 (Bressaud and Tisseur [9]) *Any sensitive CA has a configuration* x *such that* $\lambda^-(x) > 0$ *or* $\lambda^+(x) > 0$*.*

7 Surjectivity

Let $(A^{\mathbb{Z}}, F)$ be a CA with diameter $d \geq 0$ and a local rule $f : A^{d+1} \to A$. We extend the local rule to a function $f : A^* \to A^*$ by $f(u)_i = f(u_{[i,i+d]})$ for $i < |u| - d$, so $|f(u)| = \max\{|u| - d, 0\}$. A **diamond** for f (Figure 3 left) consists of words $u, v \in A^d$ and distinct $w, z \in A^+$ of the same length, such that $f(uwv) = f(uzw)$.

Theorem 12 (Hedlund [20], Moothathu[34]) *Let* $(A^{\mathbb{Z}}, F)$ *be a CA with local rule* $f : A^{d+1} \to A$*. The following conditions are equivalent.*

(1) $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is surjective.*

(2) For each $x \in A^{\mathbb{Z}}$*,* $F^{-1}(x)$ *is finite or countable.*

(3) For each $x \in A^{\mathbb{Z}}$*,* $F^{-1}(x)$ *is a finite set.*

(4) For each $x \in A^{\mathbb{Z}}$*,* $|F^{-1}(x)| \leq |A|^d$*.*

(5) $f : A^* \to A^*$ *is surjective.*

(6) For each $u \in A^+$*,* $|f^{-1}(u)| = |A|^d$*.*

(7) For each $u \in A^+$ *with* $|u| \leq d \cdot \log_2 |A| \cdot (2d + |A|^{2d})$*,* $|f^{-1}(u)| = |A|^d$*.*

(8) There exists no diamond for f*.*

It follows that any injective CA is surjective and hence bijective. Although (6) asserts equality, the inequality in (4) may be strict. Another equivalent condition states that the uniform Bernoulli measure is invariant for F. In this form, Theorem 12 has been generalized to CA on mixing SFT (see Theorem 2B.1 in Pivato [38]).

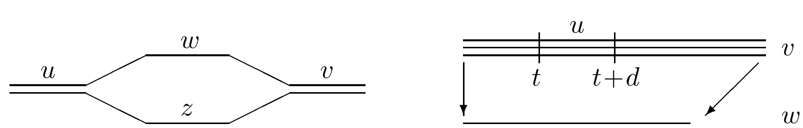


Figure 3: A diamond (left) and a magic word (right)

Theorem 13 (Blanchard and Tisseur [5])

(1) Any configuration of a surjective CA is nonwandering, i.e., $|\mathbf{N}_F| = A^{\mathbb{Z}}$*.*

(2) Any surjective almost equicontinuous CA has a dense set of F*-periodic configurations.*

(3) If $(A^{\mathbb{Z}}, F)$ is an equicontinuous and surjective CA, then there exists $p > 0$ such that $F^p = \mathrm{Id}$. In particular, F is bijective.

Theorem 14 (Moothathu [33]) *Let $(A^{\mathbb{Z}}, F)$ be a surjective CA.*

(1) $|\mathfrak{R}_F|$ is dense in $A^{\mathbb{Z}}$.

(2) F is **semiopen**, *i.e., $F(U)$ has nonemty interior for any open $U \neq \emptyset$.*

(3) If $(A^{\mathbb{Z}}, F)$ is transitive, then it is weakly mixing, and hence totally transitive and sensitive.

Conjecture 15 *Every surjective CA has a dense set of F-periodic configurations.*

Proposition 16 (Acerbi et al., [1]) *If every mixing CA has a dense set of F-periodic configurations, then every surjective CA has a dense set of jointly periodic configurations.*

Definition 17 *Let $(A^{\mathbb{Z}}, F)$ be a CA with local rule $f : A^{d+1} \to A$.*

(1) The **minimum preimage number** *(Figure 3 right) $\mathbf{p}(F)$ is defined by*

$$\begin{aligned} p(F, w) &= \min_{t \le |w|} |\{u \in A^d :\ \exists v \in f^{-1}(w), v_{[t,t+d)} = u\}|, \\ \mathbf{p}(F) &= \min\{p(F, w) :\ w \in A^+\}. \end{aligned}$$

(2) A word $w \in A^+$ is **magic**, *if $p(F, w) = \mathbf{p}(F)$.*

Recall that $\mathfrak{T}_\sigma(w)$ is the (residual) set of configurations which contain an infinite number of occurences of w both in $x_{(-\infty,0)}$ and in $x_{(0,\infty)}$. Configurations $x, y \in A^{\mathbb{Z}}$ are **d-separated**, if $x_{[i,i+d)} \neq y_{[i,i+d)}$ for all $i \in \mathbb{Z}$.

Theorem 18 (Hedlund [20], Kitchens [24]) *Let $(A^{\mathbb{Z}}, F)$ be a surjective CA with diameter d and minimum preimage number $\mathbf{p}(F)$.*

(1) If $w \in A^+$ is a magic word, then any $z \in \mathfrak{T}_\sigma(w)$ has exactly $\mathbf{p}(F)$ preimages. These preimages are pairwise d-separated.

(2) Any configuration $z \in A^{\mathbb{Z}}$ has at least $\mathbf{p}(F)$ pairwise d-separated preimages.

(3) If every $y \in A^{\mathbb{Z}}$ has exactly $\mathbf{p}(F)$ preimages, then all long enough words are magic.

Theorem 19 *Let $(A^{\mathbb{Z}}, F)$ be a CA and $\Sigma \subseteq A^{\mathbb{Z}}$ a sofic subshift. Then both $F(\Sigma)$ and $F^{-1}(\Sigma)$ are sofic subshifts. In particular, the first image subshift $F(A^{\mathbb{Z}})$ is sofic.*

See e.g., Formenti and Kůrka [16] for a proof. The **first image graph** of a local rule $f : A^{d+1} \to A$ is $\mathcal{G}(f) = (A^d, A^{d+1}, s, t, f)$, where $s(u) = u_{[0,d)}$ and $t(u) = u_{[1,d]}$. Then $F(A^{\mathbb{Z}}) = \Sigma_{\mathcal{G}(f)}$. It is algorithmically decidable whether a given CA is surjective. One decision procedure is based on the Moothathu result in Theorem 12(7). Another procedure is based on the construction of the DFA $\mathcal{A}(\mathcal{G}(f))$ (see Section 5). A CA with local rule $f : A^{d+1} \to A$ is surjective iff the rejecting state $\emptyset$ cannot be reached from the initial state A^d in $\mathcal{A}(\mathcal{G}(f))$. See Morita [35] for further information on bijective CA.

8 Permutive and closing cellular automata

Definition 20 *Let $(A^{\mathbb{Z}}, F)$ be a CA, and let $f : A^{d+1} \to A$ be the local rule for F with smallest diameter.*

(1) F is **left-permutive** *if $\forall u \in A^d, \forall b \in A, \exists! a \in A, f(au) = b$.*

(2) F is **right-permutive** *if* $\forall u \in A^d, \forall b \in A, \exists! a \in A, f(ua) = b$.

(3) F is **permutive** *if it is either left-permutive or right-permutive.*

(4) F is **bipermutive** *if it is both left- and right-permutive.*

Permutive CA can be seen in Examples 8, 10, 11, 18.

Definition 21 *Let* $(A^{\mathbb{Z}}, F)$ *be a CA.*

(1) Configurations $x, y \in A^{\mathbb{Z}}$ *are* **left-asymptotic***, if* $\exists n, x_{(-\infty,n)} = y_{(-\infty,n)}$.

(2) Configurations $x, y \in A^{\mathbb{Z}}$ *are* **right-asymptotic***, if* $\exists n, x_{(n,\infty)} = y_{(n,\infty)}$.

(3) $(A^{\mathbb{Z}}, F)$ *is* **right-closing** *if* $F(x) \neq F(y)$ *for any left-asymptotic* $x \neq y \in A^{\mathbb{Z}}$.

(4) $(A^{\mathbb{Z}}, F)$ *is* **left-closing** *if* $F(x) \neq F(y)$ *for any right-asymptotic* $x \neq y \in A^{\mathbb{Z}}$.

(5) A CA is **closing** *if it is either left- or right-closing.*

Proposition 22

(1) Any right-permutive CA is right-closing.

(2) Any right-closing CA is surjective.

(3) A CA $(A^{\mathbb{Z}}, F)$ *is right-closing iff there exists* $m > 0$ *such that for any* $x, y \in A^{\mathbb{Z}}$, $x_{[-m,0)} = y_{[-m,0)}$ & $F(x)_{[-m,m]} = F(y)_{[-m,m]} \implies x_0 = y_0$ *(see Figure 4 left).*

See e.g., Kůrka [25] for a proof. The proposition holds with obvious modification for left-permutive and left-closing CA. The multiplication CA from Example 14 is both left- and right-closing but neither left-permutive nor right-permutive. The CA from Example 15 is surjective but not closing.

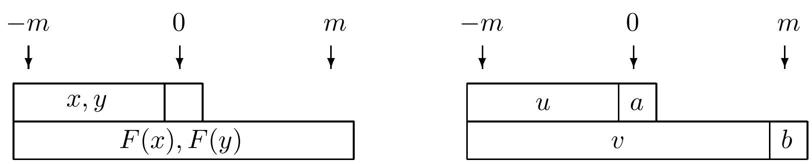


Figure 4: Closingness

Proposition 23 *Let* $(A^{\mathbb{Z}}, F)$ *be a right-closing CA. For all sufficiently large* $m > 0$*, if* $u \in A^m$, $v \in A^{2m}$*, and if* $F([u]_{-m}) \cap [v]_{-m} \neq \emptyset$*, then (Figure 4 right)*

$$\forall b \in A, \exists! a \in A, F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset.$$

See e.g., Kůrka [25] for a proof.

Theorem 24 (Boyle and Kitchens [6]) *Any closing CA* $(A^{\mathbb{Z}}, F)$ *has a dense set of jointly periodic configurations.*

Theorem 25 (Coven, Pivato and Yassawi [14]) *Let F be a left-permutive CA with memory 0.*

(1) If $\mathcal{O}(x)$ *is infinite and* $x_{[0,\infty)}$ *is fixed, i.e., if* $F(x)_{[0,\infty)} = x_{[0,\infty)}$*, then* $(\overline{\mathcal{O}(x)}, F)$ *is conjugated to an adding machine.*

(2) If F is not bijective, then the set of configurations such that $(\overline{\mathcal{O}(x)}, F)$ *is conjugated to an adding machine is dense.*

A SDS (X, F) is **open**, if $F(U)$ is open for any open $U \subseteq X$. A **cross section** of a SDS (X, F) is any continuous map $G : X \to X$ such that $F \circ G = \mathrm{Id}$. If F has a cross section, it is surjective. In particular, any bijective SDS has a cross section.

Theorem 26 (Hedlund [20]) *Let $(A^{\mathbb{Z}}, F)$ be a CA. The following conditions are equivalent.*

(1) $(A^{\mathbb{Z}}, F)$ is open.

(2) $(A^{\mathbb{Z}}, F)$ is both left- and right-closing.

(3) For any $x \in A^{\mathbb{Z}}$, $|F^{-1}(x)| = \mathbf{p}(F)$

(4) There exist cross sections $G_1, \ldots, G_{\mathbf{p}(F)} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, such that for any $x \in A^{\mathbb{Z}}$, $F^{-1}(x) = \{G_1(x), \ldots, G_{\mathbf{p}(F)}(x)\}$ and $G_i(x) \neq G_j(x)$ for $i \neq j$.

In general, the cross sections G_i are not CA as they need not commute with the shift. Only when $\mathbf{p}(F) = 1$, i.e., when F is bijective, the inverse map F^{-1} is a CA. Any CA which is open and almost equicontinuous is bijective (Kůrka [25]).

9 Expansive cellular automata

Definition 27 *Let $(A^{\mathbb{Z}}, F)$ be a CA.*

(1) F is **left-expansive***, if there exists $\varepsilon > 0$ such that if $x_{(-\infty,0]} \neq y_{(-\infty,0]}$, then $d(F^n(x), F^n(y)) \geq \varepsilon$ for some $n \geq 0$.*

(2) F is **right-expansive***, if there exists $\varepsilon > 0$ such that if $x_{[0,\infty)} \neq y_{[0,\infty)}$, then $d(F^n(x), F^n(y)) \geq \varepsilon$ for some $n \geq 0$.*

(3) F is **positively expansive***, if it is both left- and right-expansive, i.e., if there exists $\varepsilon > 0$ such that for all $x \neq y \in A^{\mathbb{Z}}$, $d(F^n(x), F^n(y)) \geq \varepsilon$ for some $n > 0$.*

Any left-expansive or right-expansive CA is sensitive and (by Theorem 12) surjective, because it cannot contain a diamond. A bijective CA is **expansive**, if

$$\exists \varepsilon > 0, \forall x \neq y \in A^{\mathbb{Z}}, \exists n \in \mathbb{Z}, d(F^n(x), F^n(y)) \geq \varepsilon.$$

Proposition 28 *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory m and anticipation a.*

(1) If $m < 0$ and if F is left-permutive, then F is left-expansive.

(2) If $a > 0$ and if F is right-permutive, then F is right-expansive.

(3) If $m < 0 < a$ and if F is bipermutive, then F is positively expansive.

See e.g., Kůrka [25] for a proof.

Theorem 29 (Nasu[36],[37])

(1) Any positively expansive CA is conjugated to a one-sided full shift.

(2) A bijective expansive CA with memory 0 is conjugated to a two-sided SFT.

Conjecture 30 *Every bijective expansive CA is conjugated to a two-sided SFT.*

Definition 31 *Let $(A^{\mathbb{Z}}, F)$ be a CA. The left- and right-***expansivity direction sets** *are defined by*

$$\begin{aligned}
\mathfrak{X}^-(F) &= \left\{ \frac{p}{q} : \; p \in \mathbb{Z}, q \in \mathbb{N}^+, \; F^q\sigma^p \text{ is left-expansive} \right\}, \\
\mathfrak{X}^+(F) &= \left\{ \frac{p}{q} : \; p \in \mathbb{Z}, q \in \mathbb{N}^+, \; F^q\sigma^p \text{ is right-expansive} \right\}, \\
\mathfrak{X}(F) &= \mathfrak{X}^-(F) \cap \mathfrak{X}^+(F).
\end{aligned}$$

All these sets are convex and open. Moreover, $\mathfrak{X}^-(F) \cap \mathfrak{A}(F) = \mathfrak{X}^+(F) \cap \mathfrak{A}(F) = \emptyset$ (Sablik [39]).

Theorem 32 (Sablik [39]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory m and anticipation a.*

(1) If F is left-permutive, then $\mathfrak{X}^-(F) = (-\infty, -m)$.

(2) If F is right-permutive, then $\mathfrak{X}^+(F) = (-a, \infty)$.

(3) If $\mathfrak{X}^-(F) \neq \emptyset$ then there exists $\alpha \in \mathbb{R}$ such that $\mathfrak{X}^-(F) = (-\infty, \alpha) \subseteq (-\infty, -m)$.

(4) If $\mathfrak{X}^+(F) \neq \emptyset$ then there exists $\alpha \in \mathbb{R}$ such that $\mathfrak{X}^+(F) = (\alpha, \infty) \subseteq (-a, \infty)$.

(5) If $\mathfrak{X}(F) \neq \emptyset$ then there exists $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $\mathfrak{X}(F) = (\alpha_0, \alpha_1) \subseteq (-a, -m)$.

Theorem 33 *Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton.*

(1) F is left-closing iff $\mathfrak{X}^-(F) \neq \emptyset$.

(2) F is right-closing iff $\mathfrak{X}^+(F) \neq \emptyset$.

(3) If $\mathfrak{A}(F)$ is an interval, then F is not surjective and $\mathfrak{X}^-(F) = \mathfrak{X}^+(F) = \emptyset$.

Proof: (1) The proof is the same as the following proof of (2).
(2$\Leftarrow$) If F is not right-closing and $\varepsilon = 2^{-n}$, then there exist distinct left-asymptotic configurations such that $x_{(-\infty,n]} = y_{(-\infty,n]}$ and $F(x) = F(y)$. It follows that $d(F^i(x), F^i(y)) < \varepsilon$ for all $i \geq 0$, so F is not right-expansive. The same argument works for any $F^q\sigma^p$, so $\mathfrak{X}^+(F) = \emptyset$.
(2$\Rightarrow$) Let F be right-closing, and let $m > 0$ be the constant from Proposition 23. Assume that $F^n(x)_{[-m+(m+1)n, m+(m+1)n]} = F^n(y)_{[-m+(m+1)n, m+(m+1)n]}$ for all $n \geq 0$. By Proposition 23, $F^{n-1}(x)_{m+(m+1)(n-1)+1} = F^{n-1}(y)_{m+(m+1)(n-1)+1}$. By induction we get $x_{[-m,m+n]} = y_{[-m,m+n]}$. This holds for every $n > 0$, so $x_{[0,\infty)} = y_{[0,\infty)}$. Thus $F\sigma^{m+1}$ is right-expansive, and therefore $\mathfrak{X}^+(F) \neq \emptyset$.
(3) If there are blocking words for two different directions, then the CA has a diamond and therefore is not surjective by Theorem 12.

Corollary 34 *Let $(A^{\mathbb{Z}}, F)$ be an equicontinuous CA. There are three possibilities.*

(1) If F is surjective, then $\mathfrak{A}(F) = \mathfrak{E}(F) = \{0\}$, $\mathfrak{X}^-(F) = (-\infty, 0)$, $\mathfrak{X}^+(F) = (0, \infty)$.

(2) If F is neither surjective nor nilpotent, then $\mathfrak{A}(F) = \mathfrak{E}(F) = \{0\}$, $\mathfrak{X}^-(F) = \mathfrak{X}^+(F) = \emptyset$.

(3) If F is nilpotent, then $\mathfrak{A}(F) = \mathfrak{E}(F) = \mathfrak{R}$, $\mathfrak{X}^-(F) = \mathfrak{X}^+(F) = \emptyset$.

The proof follows from Proposition 7 and Theorem 13 (see also Sablik [40]). The identity CA is in class (1). The product CA of Example 3 is in class (2). The zero CA of Example 1 is in class (3).

10 Attractors

Let (X, F) be a SDS. The **limit set** of a clopen invariant set $V \subseteq X$ is $\Omega_F(V) := \bigcap_{n\geq 0} F^n(V)$. A set $Y \subseteq X$ is an **attractor**, if there exists a nonempty clopen invariant set V such that $Y = \Omega_F(V)$. We say that Y is a **finite time attractor**, if $Y = \Omega_F(V) = F^n(V)$ for some $n > 0$ (and a clopen invariant set V). There exists always the largest attractor $\Omega_F := \Omega_F(X)$. Finite time maximal attractors are also called **stable limit sets** in the literature. The number of attractors is at most countable. The union of two attractors is an attractor. If the intersection of two attractors is nonempty, it contains an attractor. The **basin** of an attractor $Y \subseteq X$ is the set $\mathcal{B}(Y) = \{x \in X;\ \lim_{n\to\infty} d(F^n(x), Y) = 0\}$. An attractor $Y \subseteq X$ is a **minimal attractor**, if no proper subset of Y is an attractor. An attractor is a minimal attractor iff it is chain-transitive. A periodic point $x \in X$ is **attracting** if its orbit $\mathfrak{O}(x)$ is an attractor. Any attracting periodic point is equicontinuous. A **quasi-attractor** is a nonempty set which is an intersection of a countable number of attractors.

Theorem 35 (Hurley [23])

(1) If a CA has two disjoint attractors, then any attractor contains two disjoint attractors and an uncountably infinite number of quasi-attractors.

(2) If a CA has a minimal attractor, then it is a subshift, it is contained in any other attractor, and its basin of attraction is a dense open set.

(3) If $x \in A^{\mathbb{Z}}$ is an attracting F-periodic configuration, then $\sigma(x) = x$ and $F(x) = x$.

Corollary 36 *For any CA, exactly one of the following statements holds.*

(1) There exist two disjoint attractors and a continuum of quasi-attractors.

(2) There exists a unique quasi-attractor. It is a subshift and it is contained in any attractor.

(3) There exists a unique minimal attractor contained in any other attractor.

Both equicontinuity and surjectivity yield strong constraints on attractors.

Theorem 37 (Kůrka [25])

(1) A surjective CA has either a unique attractor or a pair of disjoint attractors.

(2) An equicontinuous CA has either two disjoint attractors or a unique attractor which is an attracting fixed configuration.

(3) If a CA has an attracting fixed configuration which is a unique attractor, then it is equicontinuous.

We consider now subshift attractors of CA, i.e., those attractors which are subshifts. Let $(A^{\mathbb{Z}}, F)$ be a CA. A clopen F-invariant set $U \subseteq A^{\mathbb{Z}}$ is **spreading**, if there exists $k > 0$ such that $F^k(U) \subseteq \sigma^{-1}(U) \cap \sigma(U)$. If U is a clopen invariant set, then $\Omega_F(U)$ is a subshift iff U is spreading (Kůrka [26]). Recall that a language is **recursively enumerable**, if it is a domain (or a range) of a recursive function (see e.g., Hopcroft and Ullmann [21]).

Theorem 38 (Formenti and Kůrka [17]) *Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a subshift attractor of a CA $(A^{\mathbb{Z}}, F)$.*

(1) $A^* \setminus \mathcal{L}(\Sigma)$ is a recursively enumerable language.

(2) Σ contains a jointly periodic configuration.

(3) (Σ, σ) is chain-mixing.

Theorem 39 (Formenti and Kůrka [17])

(1) The only subshift attractor of a surjective CA is the full space.

(2) A subshift of finite type is an attractor of a CA iff it is mixing.

(3) Given a CA $(A^{\mathbb{Z}}, F)$, the intersection of all subshift attractors of all $F^q\sigma^p$, where $q \in \mathbb{N}^+$ and $p \in \mathbb{Z}$, is a nonempty F-invariant subshift called the **small quasi-attractor** *$\mathcal{Q}_F$. $(\mathcal{Q}_F, \sigma)$ is chain-mixing and $F : \mathcal{Q}_F \to \mathcal{Q}_F$ is surjective.*

The system of all subshift attractors of a given CA forms a lattice with join $\Sigma_0 \cup \Sigma_1$ and meet $\Sigma_0 \wedge \Sigma_1 := \Omega_F(\Sigma_0 \cap \Sigma_1)$. There exist CA with infinite number of subshift attractors (Kůrka [27]).

Proposition 40 (Di Lena [29]) *The basin of a subshift attractor is a dense open set.*

By a theorem of Hurd [22], if Ω_F is SFT, then it is stable, i.e., $\Omega_F = F^n(A^{\mathbb{Z}})$ for some $n > 0$. We generalize this theorem to subshift attractors.

Theorem 41 *Let U be a spreading set for a CA $(A^{\mathbb{Z}}, F)$.*

(1) There exists a spreading set $W \subseteq U$ such that $\Omega_F(W) = \Omega_F(U)$ and $\widetilde{\Omega}_\sigma(W) := \bigcap_{i\in\mathbb{Z}} \sigma^i(W)$ is a mixing subshift of finite type.

(2) If $\Omega_F(W)$ is a SFT, then $\Omega_F(W) = F^n(\widetilde{\Omega}_\sigma(W))$ for some $n \geq 0$.

Proof: (1) See Formenti and Kůrka [17].
(2) Let D be a finite set of forbidden words for $\Omega_F(W)$. For each $u \in D$ there exists $n_u > 0$ such that $u \notin \mathcal{L}(F^{n_u}(\widetilde{\Omega}_\sigma))$. Take $n := \max\{n_u : u \in D\}$.

By Theorem 5, every equicontinuous CA has a finite time maximal attractor.

Definition 42 *Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a mixing sofic subshift, let $\mathcal{G} = (V, E, s, t, l)$ be its minimal right-resolving presentation with factor map $\ell : (\Sigma_{|\mathcal{G}|}, \sigma) \to (\Sigma, \sigma)$.*

(1) A homogenous configuration $a^\infty \in \Sigma$ is **receptive***, if there exist intrinsically synchronizing words $u, v \in \mathcal{L}(\Sigma)$ and $n \in \mathbb{N}$ such that $ua^m v \in \mathcal{L}(\Sigma)$ for all $m > n$.*

(2) Σ is **almost of finite type (AFT)***, if $\ell : \Sigma_{|\mathcal{G}|} \to \Sigma$ is one-to-one on a dense open set of $\Sigma_{|G|}$.*

(3) Σ is **near-Markov***, if $\{x \in \Sigma : \ |\ell^{-1}(x)| > 1\}$ is a finite set of σ-periodic configurations.*

(3) is equivalent to the condition that ℓ is left-closing, i.e., that $\ell(u) \neq \ell(v)$ for distinct right-asymptotic paths $u, v \in E^{\mathbb{Z}}$. Each near-Markov subshift is AFT.

Theorem 43 (Maass [31]) *Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a mixing sofic subshift with a receptive configuration $a^\infty \in \Sigma$.*

(1) If Σ is either SFT or AFT, then there exists a CA $(A^{\mathbb{Z}}, F)$ such that $\Sigma = \Omega_F = F(A^{\mathbb{Z}})$.

(2) A near-Markov subshift cannot be an infinite time maximal attractor of a CA.

On the other hand, a near-Markov subshift can be a finite time maximal attractor (see Example 21). The language $\mathcal{L}(\Omega_F)$ can have arbitrary complexity (see Culik et al., [15]). A CA with nonsofic mixing maximal attractor has been constructed in Formenti and Kůrka [17].

Definition 44 *Let $f : A^{d+1} \to A$ be a local rule of a cellular automaton. We say that a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ has* **decreasing preimages***, if there exists $m > 0$ such that for each $u \in A^* \setminus \mathcal{L}(\Sigma)$, each $v \in f^{-m}(u)$ contains as a subword a word $w \in A^* \setminus \mathcal{L}(\Sigma)$ such that $|w| < |u|$.*

Proposition 45 (Formenti and Kůrka [16]) *If $(A^{\mathbb{Z}}, F)$ is a CA and $\Sigma \subseteq A^{\mathbb{Z}}$ has decreasing preimages, then $\Omega_F \subseteq \Sigma$.*

For more information about attractor-like objects in CA see Section 3B of Pivato [38].

11 Subshifts and entropy

Definition 46 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton.*

(1) Denote by $\mathbf{S}_{(p,q)}(F) := \{x \in A^{\mathbb{Z}} : \ F^q\sigma^p(x) = x\}$ the set of all weakly periodic configurations of F with period $(p, q) \in \mathbb{Z} \times \mathbb{N}^+$.

(2) A **signal subshift** *is any non-empty $\mathbf{S}_{(p,q)}(F)$.*

(3) *The* **speed subshift** *of* F *with speed* $\alpha = \frac{p}{q} \in \mathbb{Q}$ *is* $\mathcal{S}_\alpha(F) = \overline{\bigcup_{n>0} \mathcal{S}_{(np,nq)}(F)}$.

Note that both $\mathcal{S}_{(p,q)}(F)$ and $\mathcal{S}_\alpha(F)$ are closed and σ-invariant. However, $\mathcal{S}_{(p,q)}(F)$ can be empty, so it need not be a subshift.

Theorem 47 *Let* $(A^{\mathbb{Z}}, F)$ *be a cellular automaton with memory* m *and anticipation* a, *so* $F(x)_i = f(x_{[i+m,i+a]})$.

(1) *If* $\mathcal{S}_{(p,q)}(F)$ *is nonempty, then it is a subshift of finite type.*

(2) *If* $\mathcal{S}_{(p,q)}(F)$ *is infinite, then* $-a \le p/q \le -m$.

(3) *If* $p_0/q_0 < p_1/q_1$, *then* $\mathcal{S}_{(p_0,q_0)}(F) \cap \mathcal{S}_{(p_1,q_1)}(F) \subseteq \{x \in A^{\mathbb{Z}} : \sigma^p(x) = x\}$, *where* $p = q(\frac{p_1}{q_1} - \frac{p_0}{q_0})$ *and* $q = \mathbf{lcm}(q_0, q_1)$ *(the least common multiple).*

(4) $\mathcal{S}_{(p,q)}(F) \subseteq \mathcal{S}_{\frac{p}{q}}(F) \subseteq \Omega_F$ *and* $\mathcal{S}_{\frac{p}{q}}(F) \neq \emptyset$.

(5) *If* $\mathfrak{X}(F) \neq \emptyset$ *or if* $(A^{\mathbb{Z}}, F)$ *is nilpotent, then* F *has no infinite signal subshifts.*

Proof: (1), (2) and (3) have been proved in Formenti and Kůrka [16].
(4) Since F is bijective on each signal subshift, we get $\mathcal{S}_{(p,q)}(F) \subseteq \Omega_F$, and therefore $\mathcal{S}_{\frac{p}{q}}(F) \subseteq \Omega_F$. Since every $F^q\sigma^p$ has a periodic point, we get $\mathcal{S}_{\frac{p}{q}}(F) \neq \emptyset$.
(5) It has been proved in Kůrka [26], that a positively expansive CA has no signal subshifts. This property is preserved when we compose F with a power of the shift map. If $(A^{\mathbb{Z}}, F)$ is nilpotent, then each $\mathcal{S}_{(p,q)}(F)$ contains at most one element.

The Identity CA has a unique infinite signal subshift $\mathcal{S}_{(0,1)}(\mathrm{Id}) = A^{\mathbb{Z}}$. The CA of Example 17 has an infinite number of infinite signal subshifts of the same speed. A CA with infinitely many in finite signal subshifts with infinitely many speeds has been constructed in Kůrka [26]. In some cases, the maximal attractor can be constructed from signal subshifts (see Theorem 49).

Definition 48 *Given an integer* $c \ge 0$, *the* c**-join** $\Sigma_0 \overset{c}{\vee} \Sigma_1$ *of subshifts* $\Sigma_0, \Sigma_1 \subseteq A^{\mathbb{Z}}$ *consists of all configurations* $x \in A^{\mathbb{Z}}$ *such that either* $x \in \Sigma_0 \cup \Sigma_1$, *or there exist integers* b, a *such that* $b - a \ge c$, $x_{(-\infty,b)} \in \mathcal{L}(\Sigma_0)$, *and* $x_{[a,\infty)} \in \mathcal{L}(\Sigma_1)$.

The operation of join is associative, and the c-join of sofic subshifts is sofic.

Theorem 49 (Formenti, Kůrka [16]) *Let* $(A^{\mathbb{Z}}, F)$ *be a CA and let* $\mathcal{S}_{(p_1,q_1)}(F)$, $\ldots$, $\mathcal{S}_{(p_n,q_n)}(F)$ *be signal subshifts with decreasing speeds, i.e.,* $p_i/q_i > p_j/q_j$ *for* $i < j$. *Set* $q := \mathbf{lcm}\{q_1, \ldots q_n\}$ *(the least common multiple). There exists* $c \ge 0$ *such that for* $\Sigma := \mathcal{S}_{(p_1,q_1)}(F) \overset{c}{\vee} \cdots \overset{c}{\vee} \mathcal{S}_{(p_n,q_n)}(F)$ *we have* $\Sigma \subseteq F^q(\Sigma)$ *and therefore* $\Sigma \subseteq \Omega_F$. *If moreover* $F^{nq}(\Sigma)$ *has decreasing preimages for some* $n \ge 0$, *then* $F^{nq}(\Sigma) = \Omega_F$.

Definition 50 *Let* $(A^{\mathbb{Z}}, F)$ *be a CA.*

(1) *The* k**-column homomorphism** $\varphi_k : (A^{\mathbb{Z}}, F) \to ((A^k)^{\mathbb{N}}, \sigma)$ *is defined by* $\varphi_k(x)_i = F^i(x)_{[0,k)}$.

(2) *The* k**-th column subshift** *is* $\Sigma_k(F) = \varphi_k(A^{\mathbb{Z}}) \subseteq (A^k)^{\mathbb{N}}$.

(3) *If* $\psi : (A^{\mathbb{Z}}, F) \to (\Sigma, \sigma)$ *is a factor map, where* Σ *is a one-sided subshift, we say that* Σ *is a* **factor subshift** *of* $(A^{\mathbb{Z}}, F)$.

Thus each $(\Sigma_k(F), \sigma)$ is a factor of $(A^{\mathbb{Z}}, F)$ and each factor subshift is a factor of some $\Sigma_k(F)$. Any positively expansive CA $(A^{\mathbb{Z}}, F)$ with radius $r > 0$ is conjugated to $(\Sigma_{2r+1}(F), \sigma)$. This is an SFT which by Theorem 29 is conjugated to a full shift.

Proposition 51 (Shereshevski and Afraimovich [41]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with negative memory and positive anticipation $m < 0 < a$. Then $(A^{\mathbb{Z}}, F)$ is bipermutive iff it is positively expansive and $\Sigma_{a-m+1}(F) = (A^{a-m+1})^{\mathbb{N}}$ is the full shift.*

Theorem 52 (Blanchard and Maass [4], Di Lena, [29]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and memory m.*

(1) If $m \geq 0$ and $\Sigma_r(F)$ is sofic, then any factor subshift of $(A^{\mathbb{Z}}, F)$ is sofic.

(2) If $\Sigma_{2r+1}(F)$ is sofic, then any factor subshift of $(A^{\mathbb{Z}}, F)$ is sofic.

If $(x_i)_{i \geq 0}$ is a 2^{-m}-chain in a CA $(A^{\mathbb{Z}}, F)$, then for all i, $F(x_i)_{[-m,m]} = (x_{i+1})_{[-m,m]}$, so $u_i = (x_i)_{[-m,m]}$ satisfy $F([u_i]_{-m}) \cap [u_{i+1}]_{-m} \neq \emptyset$. Conversely, if a sequence $(u_i \in A^{2m+1})_{i \geq 0}$ satisfies this property and $x_i \in [u_i]_{-m}$, then $(x_i)_{i \geq 0}$ is a 2^{-m}-chain.

Theorem 53 (Kůrka [28]) *Let $(A^{\mathbb{Z}}, F)$ be a CA.*

(1) If $\Sigma_k(F)$ is an SFT for any $k > 0$, then $(A^{\mathbb{Z}}, F)$ has the shadowing property.

(2) If $(A^{\mathbb{Z}}, F)$ has the shadowing property, then any factor subshift is sofic.

Any factor subshift of the Coven CA from Example 18 is sofic, but the CA has not the shadowing property (Blanchard and Maass [4]). A CA with shadowing property whose factor subshift is not SFT has been constructed in Kůrka [25].

Proposition 54 *Let $(A^{\mathbb{Z}}, F)$ be a CA.*

(1) $\mathbf{h}(A^{\mathbb{Z}}, F) = \lim_{k \to \infty} \mathbf{h}(\Sigma_k(F), \sigma)$.

(2) If F has radius r, then

$$\mathbf{h}(A^{\mathbb{Z}}, F) \leq 2 \cdot \mathbf{h}(\Sigma_r(F), \sigma) \leq 2r \cdot \mathbf{h}(\Sigma_1(F), \sigma) \leq 2r \cdot \ln |A|.$$

(3) If $0 \leq m \leq a$, then $\mathbf{h}(A^{\mathbb{Z}}, F) = \mathbf{h}(\Sigma_a(F), \sigma)$

See e.g., Kůrka [25] for a proof.

Conjecture 55 *If $(A^{\mathbb{Z}}, F)$ is a CA with radius r, then $\mathbf{h}(A^{\mathbb{Z}}, F) = \mathbf{h}(\Sigma_{2r+1}(F), \sigma)$.*

Conjecture 56 (Moothathu [33]) *Any transitive CA has positive topological entropy.*

Definition 57 *The* **directional entropy** *of a CA $(A^{\mathbb{Z}}, F)$ along a rational direction $\alpha = p/q$ is $\mathbf{h}_\alpha(A^{\mathbb{Z}}, F) := \mathbf{h}(A^{\mathbb{Z}}, F^q \sigma^p)/q$.*

The definition is based on the equality $\mathbf{h}(X, F^n) = n \cdot \mathbf{h}(X, F)$ which holds for every SDS. Directional entropies along irrational directions have been introduced in Milnor [32].

Proposition 58 (Courbage and Kamiński [11], Sablik [39]) *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory m and anticipation a.*

(1) If $\alpha \in \mathcal{E}(F)$, then $\mathbf{h}_\alpha(F) = 0$.

(2) If $\alpha \in \mathfrak{X}^-(F) \cup \mathfrak{X}^+(F)$, then $\mathbf{h}_\alpha(F) > 0$.

(3) $\mathbf{h}_\alpha(F) \leq (\max(a + \alpha, 0) - \min(m + \alpha, 0)) \cdot \ln |A|$.

(4) If F is bipermutive, then $\mathbf{h}_\alpha(F) = (\max\{a + \alpha, 0\} - \min\{m + \alpha, 0\}) \cdot \ln |A|$.

(5) If F is left-permutive, and $\alpha < -a$, then $\mathbf{h}_\alpha(F) = |m + \alpha| \cdot \ln |A|$.

(6) If F is right-permutive, and $\alpha > -m$, then $\mathbf{h}_\alpha(F) = (a + \alpha) \cdot \ln |A|$.

The directional entropy is not necessarily continuous (see Smillie [42]).

Theorem 59 (Boyle and Lind [8]) *The function $\alpha \mapsto \mathbf{h}_\alpha(A^{\mathbb{Z}}, F)$ is convex and continuous on $\mathfrak{X}^-(F) \cup \mathfrak{X}^+(F)$.*

12 Examples

Cellular automata with binary alphabet $\mathbf{2} = \{0,1\}$ and radius $r = 1$ are called **elementary** (Wolfram [45]). Their local rules are coded by numbers between 0 and 255 by

$$f(000) + 2 \cdot f(001) + 4 \cdot f(010) + \cdots + 32 \cdot f(101) + 64 \cdot f(110) + 128 \cdot f(111)$$

Example 1 (The zero rule ECA0) $F(x) = 0^\infty$.

The zero CA is an equicontinuous nilpotent CA. Its equicontinuity directions are $\mathfrak{E}(F) = \mathfrak{A}(F) = (-\infty, \infty)$.

Example 2 (The identity rule ECA204) $\mathrm{Id}(x) = x$.

The identity is an equicontinuous surjective CA which is not transitive. Every clopen set is an attractor and every configuration is a quasi-attractor. The equicontinuity and expansivity directions are $\mathfrak{A}(\mathrm{Id}) = \mathfrak{E}(\mathrm{Id}) = \{0\}$, $\mathfrak{X}^-(\mathrm{Id}) = (-\infty, 0)$, $\mathfrak{X}^+(\mathrm{Id}) = (0, \infty)$. The directional entropy is $h_\alpha(\mathrm{Id}) = |\alpha|$.

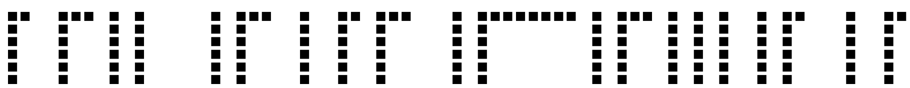


Figure 5: ECA12

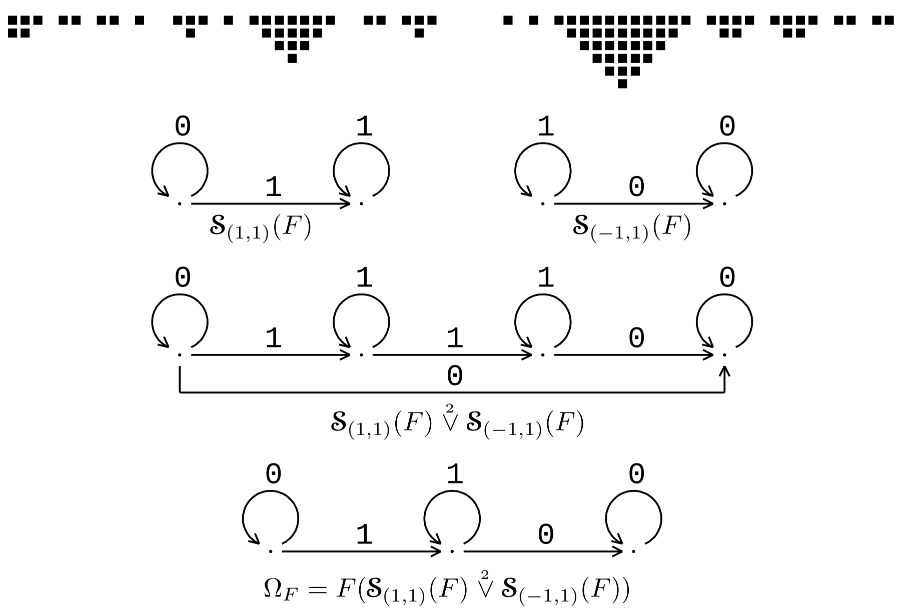


Figure 6: Signal subshifts of ECA128.

Example 3 (An equicontinuous rule ECA12) $F(x)_i = (1 - x_{i-1})x_i$.

000:0, 001:0, 010:1, 011:1, 100:0, 101:0, 110:0, 111:0.

The ECA12 is equicontinuous: the preperiod and period are $m = p = 1$. The automaton has finite time maximal attractor $\Omega_F = F(A^{\mathbb{Z}}) = \Sigma_{\{11\}} = \mathcal{S}_{(0,1)}(F)$ which is called the **golden mean subshift**.

Example 4 (A product rule ECA128) $F(x)_i = x_{i-1}x_ix_{i+1}$.

The ECA128 is almost equicontinuous and 0 is a 1-blocking word. The first column subshift is $\Sigma_1(F) = \mathcal{S}_{\{01\}}$. Each column subshift $\Sigma_k(F)$ is an SFT with zero entropy, so F has the shadowing property and zero entropy. The n-th image

$$F^n(\mathbf{2}^{\mathbb{Z}}) = \{x \in \mathbf{2}^{\mathbb{Z}} : \forall m \in [1, 2n], 10^m1 \not\sqsubseteq x\}$$

is a SFT. The first image graph can be seen in Figure 10 left. The maximal attractor $\Omega_F = \mathcal{S}_{\{10^n1:\, n>0\}}$ is a sofic subshift and has decreasing preimages. The only other attractor is the minimal attractor $\{0^\infty\} = \Omega_F([0]_0)$, which is also the minimal quasi-attractor. The equicontuinity directions are $\mathfrak{E}(F) = \emptyset$ and $\mathfrak{A}(F) = [-1, 1]$. For Lyapunov exponents we have $\lambda^-_F(0^\infty) = \lambda^+_F(0^\infty) = 0$ and $\lambda^-_F(1^\infty) = \lambda^+_F(1^\infty) = 1$. The only infinite signal subshifts are nontransitive subshifts $\mathbf{S}_{(1,1)}(F) = \mathcal{S}_{\{10\}}$ and $\mathbf{S}_{(-1,1)}(F) = \mathcal{S}_{\{01\}}$. The maximal attractor can be constructed using the join construction $\Omega_F = F(\mathbf{S}_{(1,1)}(F) \overset{2}{\vee} \mathbf{S}_{(-1,1)}(F))$ (Figure 6).

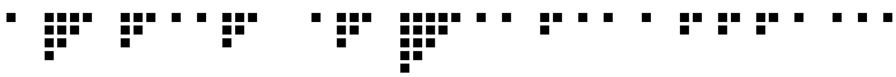


Figure 7: ECA136

Example 5 (A product rule ECA136) $F(x)_i = x_ix_{i+1}$.

The ECA136 is almost equicontinuous since 0 is a 1-blocking word. As in Example 4, we have $\Omega_F = \mathcal{S}_{\{10^k1:\, k>0\}}$. For any $m \in \mathbb{Z}$, $[0]_m$ is a clopen invariant set, which is spreading to the left but not to the right. Thus $Y_m = \Omega_F([0]_m) = \{x \in \Omega_F : \forall i \leq m, x_i = 0\}$ is an attractor but not a subshift. We have $Y_{m+1} \subset Y_m$ and $\bigcap_{m\geq 0} Y_m = \{0^\infty\}$ is the unique minimal quasi-attractor. Since $F^2\sigma^{-1}(x)_i = x_{i-1}x_ix_{i+1}$ is the ECA128 which has a minimal subshift attractor $\{0^\infty\}$, F has the small quasi-attractor $\mathcal{Q}_F = \{0^\infty\}$. The almost equicontinuity directions are $\mathfrak{A}(F) = [-1, 0]$.

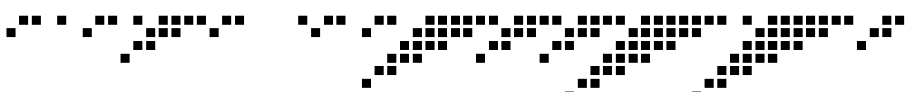


Figure 8: A unique attractor $F(x)_i = x_{i+1}x_{i+2}$

Example 6 (A unique attractor) $(\mathbf{2}^{\mathbb{Z}}, F)$ *where* $F(x)_i = x_{i+1}x_{i+2}$

The system is sensitive and has a unique attractor $\Omega_F = \mathcal{S}_{\{10^k1:\, k>0\}}$ which is not F-transitive. If $x \in [10]_0 \cap \Omega_F(\mathbf{2}^{\mathbb{Z}})$, then $x_{[0,\infty)} = 10^\infty$, so for any $n > 0$, $F^n(x) \notin [11]_0$. However, $(A^{\mathbb{Z}}, F)$ is chain-transitive, so it does not have the shadowing property. The small quasi-attractor is $\mathcal{Q}_F = \{0^\infty\}$. The topological entropy is zero. The factor subshift $\Sigma_1(F) = \{x \in \mathbf{2}^{\mathbb{N}} : \forall n \geq 0, (x_{[n,n+1]} = 10 \implies x_{[n,2n+1]} = 10^{n+1})\}$ is not sofic (Gilman [19]).

Example 7 (The majority rule ECA232) $F(x)_i = \left\lfloor \frac{x_{i-1}+x_i+x_{i+1}}{2} \right\rfloor$.

000:0, 001:0, 010:0, 011:1, 100:0, 101:1, 110:1, 111:1.

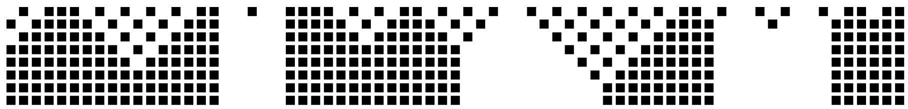


Figure 9: The majority rule ECA232

The majority rule has 2-blocking words 00 and 11, so it is almost equicontinuous. More generally, let $E = \{u \in \mathbf{2}^* : \ |u| \geq 2, u_0 = u_1, u_{|u|-2} = u_{|u|-1}, 010 \not\sqsubseteq u, 101 \not\sqsubseteq u\}$. Then for any $u \in E$ and for any $i \in \mathbb{Z}$, $[u]_i$ is a clopen invariant set, so its limit set $\Omega_F([u]_i)$ is an attractor. These attractors are not subshifts. There exists a subshift attractor given by the spreading set $U := \mathbf{2}^{\mathbb{Z}} \setminus ([010]_0 \cup [101]_0)$. We have $\Omega_F(U) = \mathbf{S}_{(0,1)}(F) = \mathcal{S}_{\{010,101\}}$. There are two more infinite signal subshifts $\mathbf{S}_{(-1,1)}(F) = \mathcal{S}_{\{001,110\}}$ and $\mathbf{S}_{(1,1)}(F) = \mathcal{S}_{\{011,100\}}$. The maximal attractor is $\Omega_F = \mathbf{S}_{(1,0)}(F) \cup (\mathbf{S}_{(1,1)}(F) \overset{3}{\vee} \mathbf{S}_{(-1,1)}(F)) = \mathcal{S}_{\{010^k1,10^k10,01^k01,101^k0,:\, k>1\}}$. All column subshifts are SFT, for example $\Sigma_1(F) = \mathcal{S}_{\{001,110\}}$ and the entropy is zero. The equicontinuity directions are $\mathfrak{E}(F) = \emptyset$, $\mathfrak{A}(F) = \{0\}$.

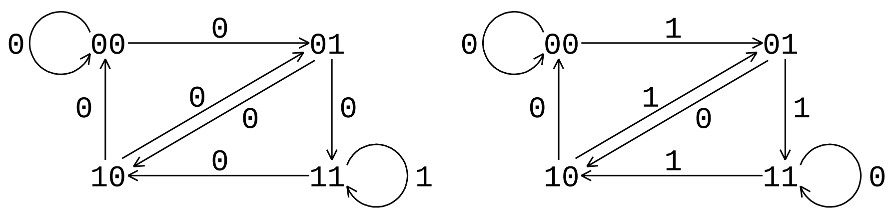


Figure 10: First image subshift of ECA128(left) and ECA106(right)

Example 8 (A right-permutive rule ECA106) $F(x)_i = (x_{i-1}x_i + x_{i+1}) \bmod 2$.

000:0, 001:1, 010:0, 011:1, 100:0, 101:1, 110:1, 111:0.

The ECA106 is transitive (see Kůrka [25]). The first image graph is in Figure 10 right. The minimum preimage number is $\mathbf{p}_F = 1$ and the word $u = 0101$ is magic. Its preimages are $f^{-1}(0101) = \{010101, 100101, 000101, 111001\}$ and for every $v \in f^{-1}(u)$ we have $v_{[4,5]} = 01$. This can be seen in Figure 11 bottom left, where all paths in the first image graph with label 0101 are displayed. Accordingly, $(01)^\infty$ has a unique preimage $F^{-1}((01)^\infty) = \{(10)^\infty\}$. On the other hand 0^∞ has two preimages $F^{-1}(0^\infty) = \{0^\infty, 1^\infty\}$ (Figure 11 bottom right) and 1^∞ has three preimages $F^{-1}(1^\infty) = \{(011)^\infty, (110)^\infty, (101)^\infty\}$. We have $\mathfrak{X}^-(F) = \emptyset$ and $\mathfrak{X}^+(F) = (-1, \infty)$ and there are no equicontinuity directions. For every x we have $\lambda_F^-(x) = 1$. On the other hand the right Lyapunov exponents are not constant. For example $\lambda_F^+(0^\infty) = 0$ while $\lambda_F^+((01)^\infty) = 1$. The only infinite signal subshift is the golden mean subshift $\mathbf{S}_{(-1,1)}(F) = \mathcal{S}_{\{11\}}$.

Example 9 (The shift rule ECA170) $\sigma(x)_i = x_{i+1}$.

The shift rule is bijective, expansive and transitive. It has a dense set of periodic configurations, so it is chaotic. Its only signal subshift is $\mathbf{S}_{(-1,1)}(\sigma) = \mathbf{2}^{\mathbb{Z}}$. The equicontinuity and expansivity directions are $\mathfrak{E}(\sigma) = \mathfrak{A}(\sigma) = \{-1\}$, $\mathfrak{X}^-(\sigma) = (-\infty, -1)$, $\mathfrak{X}^+(\sigma) = (-1, \infty)$. For any $x \in \mathbf{2}^{\mathbb{Z}}$ we have $\lambda_\sigma^-(x) = 0$, $\lambda_\sigma^+(x) = 1$.

Example 10 (A bipermutive rule ECA102) $F(x)_i = (x_i + x_{i+1}) \bmod 2$.

000:0, 001:1, 010:1, 011:0, 100:0, 101:1, 110:1, 111:0.

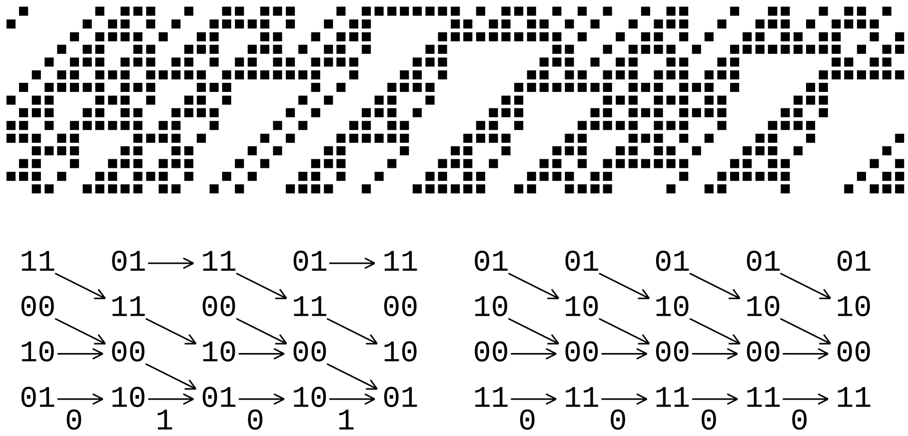


Figure 11: Preimages in ECA106

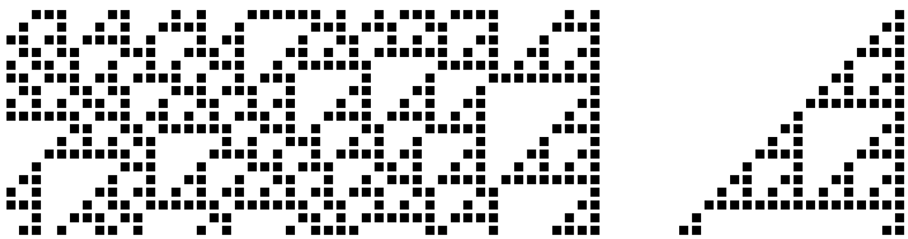


Figure 12: ECA102

The ECA102 is bipermutive with memory 0, so it is open but not positively expansive. The expansivity directions are $\mathfrak{X}^-(F) = (-\infty, 0)$, $\mathfrak{X}^+(F) = (-1, \infty)$, $\mathfrak{X}(F) = (-1, 0)$. If $x \in Y := W_0^+(0^\infty.10^\infty)$, i.e., if $x_0 = 1$ and $x_i = 0$ for all $i > 0$, then $(\overline{\mathcal{O}(x)}, F) = (Y, F)$ is conjugated to the adding machine with periodic structure $\mathfrak{n} = (2, 2, 2, \ldots)$. If $-2^n < i \le -2^{n-1}$, then $(F^m(x)_i)_{m \ge 0}$ is periodic with period 2^n. There are no signal subshifts. The minimum preimage number is $\mathbf{p}_F = 2$ and the two cross sections G_0, G_1 are uniquely determined by the conditions $G_0(x)_0 = 0$, $G_1(x)_0 = 1$. The entropy is $\mathbf{h}(A^{\mathbb{Z}}, F) = \ln 2$.

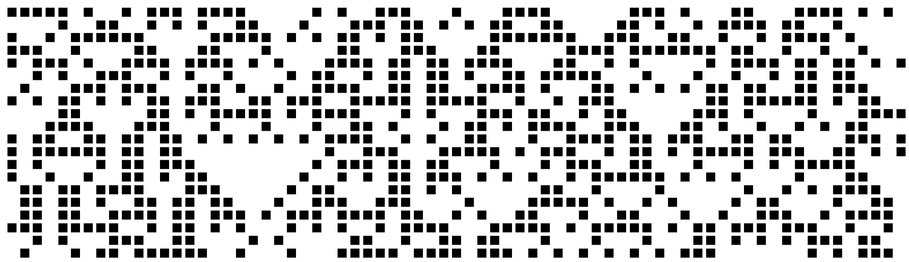
Example 11 (The sum rule ECA90) $F(x)_i = (x_{i-1} + x_{i+1}) \bmod 2$.

000:0, 001:1, 010:0, 011:1, 100:1, 101:0, 110:1, 111:0.

The sum rule is bipermutive with negative memory and positive anticipation. Thus it is open, positively expansive and mixing. It is conjugated to the full shift on four symbols $\Sigma_2(F) = \{00, 01, 10, 11\}^{\mathbb{N}}$. It has four cross-sections G_0, G_1, G_2, G_3 which are uniquely determined by the conditions $G_0(x)_{[0,1]} = 00$, $G_1(x)_{[0,1]} = 01$, $G_2(x)_{[0,1]} = 10$, and $G_3(x)_{[0,1]} = 11$. For every $x \in \mathbf{2}^{\mathbb{Z}}$ we have $\lambda_F^-(x) = \lambda_F^+(x) = 1$. The system has no almost equicontinuous directions and $\mathfrak{X}^-(F) = (-\infty, 1)$, $\mathfrak{X}^+(F) = (-1, \infty)$. The directional entropy is continuous and piecewise linear (Figure 13).

Example 12 (The traffic rule ECA184) $F(x)_i = 1$ *if* $x_{[i-1,i]} = 10$ *or* $x_{[i,i+1]} = 11$.

000:0 001:0 010:0 011:1 100:1 101:1 110:0 111:0



$$\mathfrak{h}_\alpha(F) = \begin{cases} (1-\alpha)\ln 2 & \text{for} \quad \alpha \le -1 \\ 2\ln 2 & \text{for} \quad -1 \le \alpha \le 1 \\ (1+\alpha)\ln 2 & \text{for} \quad \alpha \ge 1 \end{cases}$$

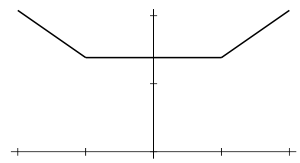


Figure 13: Directional entropy of ECA90

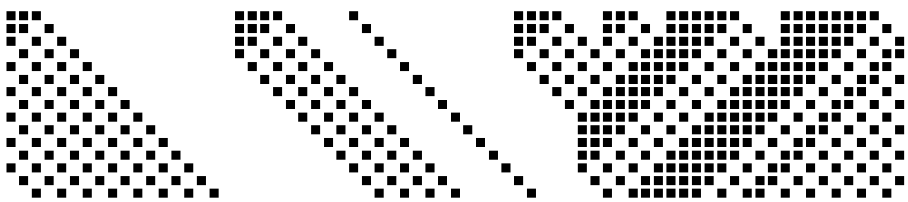


Figure 14: The traffic rule ECA184

The ECA184 has three infinite signal subshifts

$$\mathfrak{S}_{(1,1)}(F) = \mathcal{S}_{\{11\}} \cup \{1^\infty\}, \quad \mathfrak{S}_{(0,1)}(F) = \mathcal{S}_{\{10\}}, \quad \mathfrak{S}_{(-1,1)}(F) = \mathcal{S}_{\{00\}} \cup \{0^\infty\}$$

and a unique F-transitive attractor $\Omega_F = \mathfrak{S}_{(1,1)}(F) \overset{2}{\vee} \mathfrak{S}_{(-1,1)}(F) = \mathcal{S}_{\{1(10)^n0:\, n>0\}}$ which is sofic. The system has neither almost equicontinuous nor expansive directions. The directional entropy is continuous, but neither piecewise linear nor convex (Smillie [42]).

Example 13 (ECA62) $F(x)_i = x_{i-1}(x_i+1) + x_i x_{i+1}$.

000:0, 001:1, 010:1, 011:1, 100:1, 101:1, 110:0, 111:0.

There exists a spreading set $U = A^{\mathbb{Z}} \setminus ([0^6]_2 \cup [1^7]_1 \cup \bigcup_{v \in f^{-1}(1^7)} [v]_0)$ and $\Omega_F(U)$ is a subshift attractor which contains σ-transitive infinite signal subshifts $\mathfrak{S}_{(1,2)}(F)$ and $\mathfrak{S}_{(0,3)}(F)$ as well as their join. It follows $\mathcal{Q}_F = \Omega_F(U) = F^2(\mathfrak{S}_{(1,2)}(F) \overset{3}{\vee} \mathfrak{S}_{(0,3)}(F))$. The only other infinite signal subshifts are $\mathfrak{S}_{(4,4)}(F)$ and $\mathfrak{S}_{(-1,1)}(F)$ and

$$\Omega_F = F^2(\mathfrak{S}_{(4,4)}(F) \overset{3}{\vee} \mathfrak{S}_{(1,2)}(F) \overset{3}{\vee} \mathfrak{S}_{(0,3)}(F) \overset{3}{\vee} \mathfrak{S}_{(-1,1)}(F)).$$

In the space-time diagram in Figure 15, the words 00, 111 and 010, which do not occur in the intersection $\mathfrak{S}_{(1,2)}(F) \cap \mathfrak{S}_{(0,3)}(F) = \{(110)^\infty, (101)^\infty, (011)^\infty\}$ are displayed in grey (Kůrka [26]).

Example 14 (A multiplication rule) $(\mathbf{4}^{\mathbb{Z}}, F)$, *where*

$$F(x)_i = \left(2x_i + \left\lfloor \frac{x_{i+1}}{2} \right\rfloor\right) \bmod 4 = 2x_i + \left\lfloor \frac{x_{i+1}}{2} \right\rfloor - 4\left\lfloor \frac{x_i}{2} \right\rfloor.$$

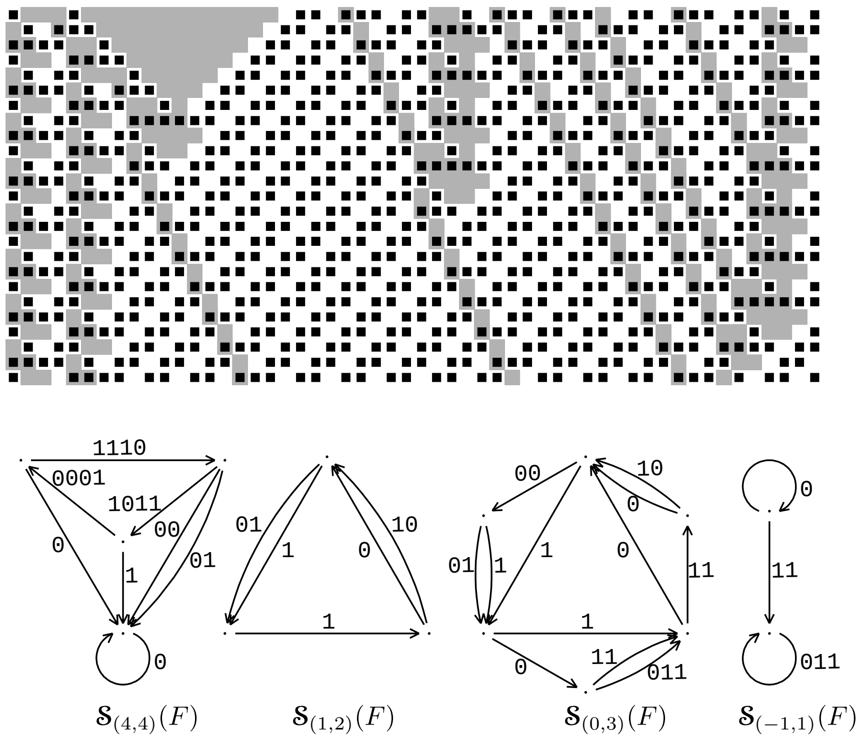


Figure 15: ECA 62 and its signal subshifts

00001000030000123000
00002000120000312000
00010000300001230000
00020001200003120000

Figure 16: The multiplication CA

00	01	02	03	10	11	12	13	20	21	22	23	30	31	32	33
0	0	1	1	2	2	3	3	0	0	1	1	2	2	3	3

We have

$$F^2(x)_i = \left(4x_i + 2 \cdot \left\lfloor \frac{x_{i+1}}{2} \right\rfloor + (x_{i+1}) \bmod 4\right) \bmod 2 = x_{i+1} = \sigma(x)_i.$$

Thus the CA is a "square root" of the shift map. It is bijective and expansive, and its entropy is $\ln 2$. The system expresses multiplication by two in base four. If $x \in A^{\mathbb{Z}}$ is left-asymptotic with 0^∞, then $\varphi(x) = \sum_{i=-\infty}^{i=\infty} x_i 4^{-i}$ is finite and $\varphi(F(x)) = 2\varphi(x)$.

Example 15 (A surjective rule) *($\mathbf{4}^{\mathbb{Z}}, F$), $m = 0$, $a = 1$, and the local rule is*

00	11	22	33	01	02	12	21	03	10	13	30	20	23	31	32
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3

The system is surjective but not closing. The first image automaton is in Figure 17 top. We see that $F(\mathbf{4}^{\mathbb{Z}})$ is the full shift, so F is surjective. The configuration $0^\infty.1^\infty$ has left-asymptotic preimages $0^\infty.(21)^\infty$ and $0^\infty.(12)^\infty$, so F is not right-closing. This configuration has also right-asymptotic preimages $0^\infty.(12)^\infty$ and $2^\infty.(12)^\infty$, so F is not left-closing (Figure 17 bottom). Therefore $\mathfrak{X}^-(F) = \mathfrak{X}^+(F) = \emptyset$.

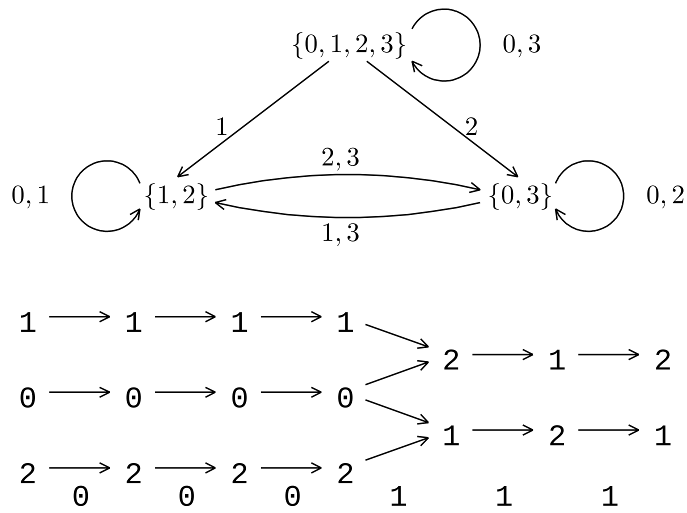


Figure 17: Asymptotic configurations

Example 16 $(\mathbf{2}^{\mathbb{Z}} \times \mathbf{2}^{\mathbb{Z}}, \mathrm{Id} \times \sigma)$, *i.e.,* $F(x,y)_i = (x_i, y_{i+1})$.

The system is bijective and sensitive but not transitive. $\mathfrak{A}(F) = \mathfrak{E}(F) = \emptyset$, $\mathfrak{X}^-(F) = (-\infty, 1)$, $\mathfrak{X}^+(F) = (0, \infty)$. There are infinite signal subshifts

$$\mathcal{S}_{(0,n)} = \mathbf{2}^{\mathbb{Z}} \times |\mathcal{O}_\sigma^n|, \quad \mathcal{S}_{(-n,n)} = |\mathcal{O}_\sigma^n| \times \mathbf{2}^{\mathbb{Z}}$$

where $|\mathcal{O}_\sigma^n| = \{x \in \mathbf{2}^{\mathbb{Z}} :\ \sigma^n(x) = x\}$, so the speed subshifts are $\mathcal{S}_0(F) = \mathcal{S}_{-1}(F) = A^{\mathbb{Z}}$.

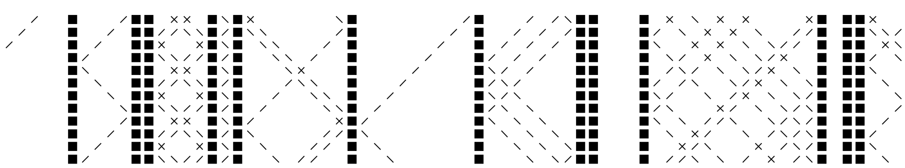


Figure 18: A bijective almost equicontinuous CA

Example 17 (A bijective CA) $(A^{\mathbb{Z}}, F)$, *where* $A = \{000, 001, 010, 011, 100\}$, *and*

$$F(x,y,z)_i = (x_i, (1+x_i)y_{i+1} + x_{i-1}z_i, (1+x_i)z_{i-1} + x_{i+1}y_i) \bmod 2$$

The dynamics is conveniently described as movement of three types of particles, $1 = 001$, $2 = 010$ and $4 = 100$. Letter $0 = 000$ corresponds to empty cell and $3 = 011$ corresponds to cell occupied by both $1 = 001$ and $2 = 010$. The particle $4 = 100$ is a wall which neither moves nor changes. Particle 1 goes to the left and when it hits a wall 4, it changes to 2. Particle 2 goes to the right and when it hits a wall, it changes to 1. Clearly 4 is a 1-blocking word, so the system is almost equicontinuous. It is bijective and its inverse is

$$F^{-1}(x,y,z)_i = (x_i, (1+x_i)y_{i-1} + x_{i+1}z_i, (1+x_i)z_{i+1} + x_{i-1}y_i) \bmod 2.$$

The first column subshift is $\Sigma_1(F) = \{0,1,2,3\}^{\mathbb{N}} \cup \{4^\infty\}$. We have infinite signal subshifts $\mathcal{S}_{(-1,1)}(F) = \{0,1\}^{\mathbb{Z}}$, $\mathcal{S}_{(1,1)}(F) = \{0,2\}^{\mathbb{Z}}$, $\mathcal{S}_{(0,1)}(F) = \{0,4\}^{\mathbb{Z}}$. For $q > 0$ we get

$$\mathcal{S}_{(0,q)}(F) = \{x \in A^{\mathbb{Z}} :\ \forall u \in \mathbf{4}^*, (4u4 \sqsubseteq x \implies (\exists m, 2m|u| = q) \text{ or } u \in \{0\}^*)\}$$

so the speed subshift is $\mathcal{S}_0(F) = A^{\mathbb{Z}}$. The equicontinuity directions are $\mathfrak{A}(F) = \{0\}$, $\mathfrak{E}(F) = \emptyset$. The expansivity directions are $\mathfrak{X}^-(F) = (-\infty, -1)$, $\mathfrak{X}^+(F) = (1, \infty)$.

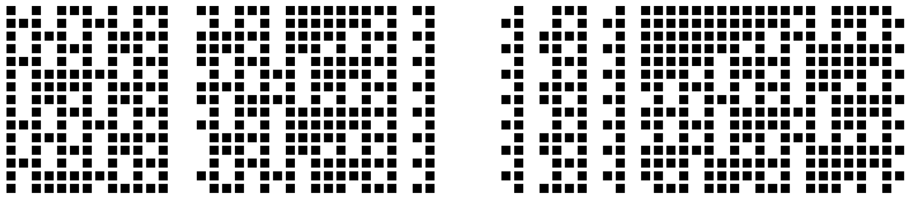


Figure 19: The Coven CA

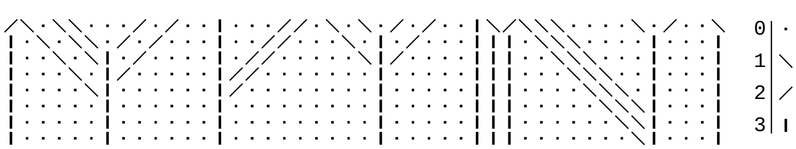
Example 18 (The Coven CA, Coven and Hedlund [13], Coven [12]) $(\mathbf{2}^{\mathbb{Z}}, F)$ *where* $F(x)_i = x_i + x_{i+1}(x_{i+2} + 1) \bmod 2$.

The CA is left-permutive with zero memory. It is not right-closing, since it has not constant number of preimages: $F^{-1}(0^\infty) = \{0^\infty\}$, $F^{-1}(1^\infty) = \{(01)^\infty, (10)^\infty\}$. It is almost equicontinuous with 2-blocking word 000 with offset 0. It is not transitive but it is chain-transitive and its unique attractor is the full space (Blanchard and Maass [4]). While $\Sigma_1(F) = \mathbf{2}^{\mathbb{Z}}$, the two-column factor subshift

$$\Sigma_2(F) = \{10, 11\}^{\mathbb{N}} \cup \{11, 01\}^{\mathbb{N}} \cup \{01, 00\}^{\mathbb{N}}$$

is sofic but not SFT and the entropy is $\mathbf{h}(A^{\mathbb{Z}}, F) = \ln 2$. For any $a, b \in \mathbf{2}$ we have $f(1a1b) = 1c$ where $c = a + b + 1$ (here f is the local rule and the addition is modulo 2). Define a CA $(\mathbf{2}^{\mathbb{Z}}, G)$ by $G(x)_i = (x_i + x_{i+1} + 1) \bmod 2$ and a map $\varphi : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ by $\varphi(x)_{2i} = 1$, $\varphi(x)_{2i+1} = x_i$. Then $\varphi : (\mathbf{2}^{\mathbb{Z}}, G) \to (\mathbf{2}^{\mathbb{Z}}, F)$ is an injective morphism and $(\mathbf{2}^{\mathbb{Z}}, G)$ is a transitive subsystem of $(\mathbf{2}^{\mathbb{Z}}, F)$. If $x_i = 0$ for all $i > 0$ and $x_{2i} = 1$ for all $i \leq 0$, then $(\overline{\mathcal{O}(x)}, F)$ is conjugated to the adding machine with periodic structure $\mathfrak{n} = (2, 2, 2, \ldots)$, and $I_n^+((10)^\infty) = 2$. We have $\mathfrak{E}(F) = \emptyset$, $\mathfrak{A}(F) = \{0\}$, $\mathfrak{X}^-(F) = (-\infty, 0)$ and $\mathfrak{X}^+(F) = \emptyset$. We have $\mathcal{S}_0(F) = \mathbf{2}^{\mathbb{Z}}$ and there exists an increasing sequence of non-transitive infinite signal subshifts $\mathcal{S}_{(0,2^n)}(F)$:

$$\mathcal{S}_{(0,1)}(F) = \mathcal{S}_{\{10\}} \subset \mathcal{S}_{(0,2)}(F) = \mathcal{S}_{\{1010,1110\}} \subset \mathcal{S}_{(0,4)}(F) \subset \cdots$$



$$\mathbf{h}_\alpha(A^{\mathbb{Z}}, F) = \begin{cases} (1 + |\alpha|) \ln 2 & \text{for} \quad \alpha \neq 0 \\ 2 \ln 2 & \text{for} \quad \alpha = 0 \end{cases}$$

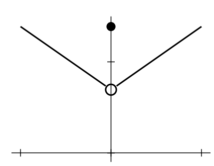


Figure 20: Directional entropy of Gliders and walls.

Example 19 (Gliders and walls, Milnor [32]) *The alphabet is* $A = \{0, 1, 2, 3\}$, $F(x)_i = f(x_{[i-1,i+1]})$, *where the local rule is*

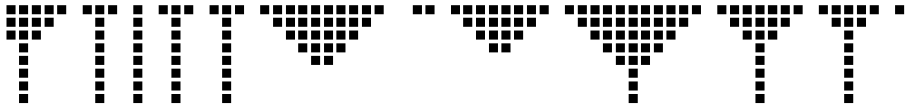


Figure 21: ECA132

x3x:3, 12x:3, 1x2:3, x12:3, 1xx:1, x1x:0, xx2:2, x2x0.

Directional entropy has discontinuity at $\alpha = 0$ (see Figure 20).

Example 20 (Golden mean subshift attractor: ECA132)

000:0, 001:0, 010:1, 011:0, 100:0, 101:0, 110:0, 111:1.

While the golden mean subshift $\mathcal{S}_{\{11\}}$ is the finite time maximal attractor of ECA12 (see Example 3), it is also an infinite time subshift attractor of ECA132. The clopen set $U := [00]_0 \cup [01]_0 \cup [10]_0$ is spreading and $\Omega_F = \mathcal{S}_{\{11\}} = \mathfrak{S}_{(0,1)}$. There exist infinite signal subshifts $\mathfrak{S}_{(1,1)}(F) = \mathcal{S}_{\{10\}}$, $\mathfrak{S}_{(-1,1)}(F) = \mathcal{S}_{\{01\}}$ and the maximal attractor is their join $\Omega_F = \mathfrak{S}_{(1,1)}(F) \overset{2}{\vee} \mathfrak{S}_{(0,1)}(F) \overset{2}{\vee} \mathfrak{S}_{(-1,1)}(F) = \mathcal{S}_{\{11\}} \cup (\mathfrak{S}_{(1,1)}(F) \overset{2}{\vee} \mathfrak{S}_{(-1,1)}(F))$.

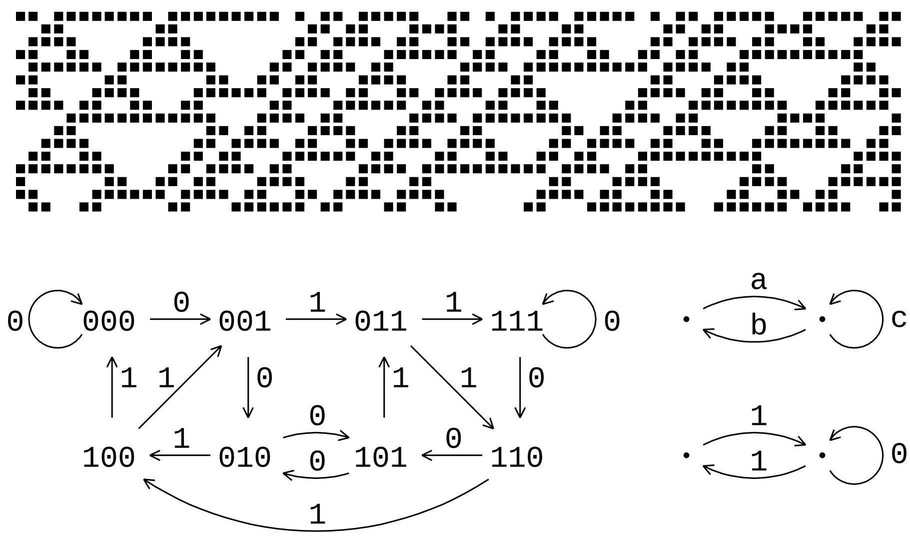


Figure 22: The first image graph and the even subshift

Example 21 (A finite time sofic maximal attractor) $(\mathbf{2}^{\mathbb{Z}}, F)$, *where* $m = -1$, $a = 2$ *and the local rule is*

0000:0, 0001:0, 0010:0, 0011:1, 0100:1, 0101:0, 0110:1, 0111:1,
1000:1, 1001:1, 1010:0, 1011:1, 1100:1, 1101:0, 1110:0, 1111:0.

The system has finite time maximal attractor $\Omega_F = F(\mathbf{2}^{\mathbb{Z}}) = \mathcal{S}_{\{01^{2n+1}0:\, n \geq 0\}}$ (Figure 22 left). This is the **even subshift** whose minimal right-resolving presentation is in Figure 22 right. We have $E = \{a, b, c\}$, $l(a) = l(b) = 1$, $l(c) = 0$. A word is synchronizing in $\mathcal{G}$ (and intrinsically synchronizing) iff it contains 0. The factor map ℓ is right-resolving and also left-resolving. Thus ℓ is left-closing and the even subshift is AFT. We have $\ell^{-1}(1^\infty) = \{(ab)^\infty, (ba)^\infty\}$, and $|\ell^{-1}(x)| = 1$ for each $x \neq 1^\infty$. Thus the even subshift is near-Markov, and it cannot be an infinite time maximal attractor.

```
3 0 3 0 0 3 0 0 0 3 0 0 0 0 3 1 1 0 2 3 0 0 0 0 0 3 0 1 1 0 2 3 1 1 0 1 2 3
3 1 3 0 1 3 0 0 1 3 0 0 0 1 3 1 1 1 1 3 0 0 0 0 1 3 0 1 1 1 1 3 1 1 0 2 1 3
3 2 3 0 2 3 0 0 2 3 0 0 0 2 3 1 1 1 2 3 0 0 0 0 2 3 0 1 1 1 2 3 1 1 1 0 2 3
3 1 3 1 1 3 0 1 1 3 0 0 1 1 3 1 1 2 1 3 0 0 0 1 1 3 0 1 1 2 1 3 1 1 1 1 1 3
3 2 3 1 2 3 0 1 2 3 0 0 1 2 3 1 2 0 2 3 0 0 0 1 2 3 0 1 2 0 2 3 1 1 1 1 2 3
3 1 3 2 1 3 0 2 1 3 0 0 2 1 3 2 0 1 1 3 0 0 0 2 1 3 0 2 0 1 1 3 1 1 1 2 1 3
3 2 3 0 2 3 1 0 2 3 0 1 0 2 3 0 0 1 2 3 0 0 1 0 2 3 1 0 0 1 2 3 1 1 2 0 2 3
3 1 3 1 1 3 1 1 1 3 0 1 1 1 3 0 0 2 1 3 0 0 1 1 1 3 1 0 0 2 1 3 1 2 0 1 1 3
3 2 3 1 2 3 1 1 2 3 0 1 1 2 3 0 1 0 2 3 0 0 1 1 2 3 1 0 1 0 2 3 2 0 0 1 2 3
3 1 3 2 1 3 1 2 1 3 0 1 2 1 3 0 1 1 1 3 0 0 1 2 1 3 1 0 1 1 1 3 0 0 0 2 1 3
3 2 3 0 2 3 2 0 2 3 0 2 0 2 3 0 1 1 2 3 0 0 2 0 2 3 1 0 1 1 2 3 0 0 1 0 2 3
3 1 3 1 1 3 0 1 1 3 1 0 1 1 3 0 1 2 1 3 0 1 0 1 1 3 1 0 1 2 1 3 0 0 1 1 1 3
3 2 3 1 2 3 0 1 2 3 1 0 1 2 3 0 2 0 2 3 0 1 0 1 2 3 1 0 2 0 2 3 0 0 1 1 2 3
3 1 3 2 1 3 0 2 1 3 1 0 2 1 3 1 0 1 1 3 0 1 0 2 1 3 1 1 0 1 1 3 0 0 1 2 1 3
3 2 3 0 2 3 1 0 2 3 1 1 0 2 3 1 0 1 2 3 0 1 1 0 2 3 1 1 0 1 2 3 0 0 2 0 2 3
```

Figure 23: Logarithmic perturbation speeds

Example 22 (Logarithmic perturbation speeds) $(\mathbf{4}^{\mathbb{Z}}, F)$ *where* $m = 0$, $a = 1$, *and the local rule is*

$$\begin{array}{cccccccc} 00{:}0, & 01{:}0, & 02{:}1, & 03{:}1, & 10{:}1, & 11{:}1, & 12{:}2, & 13{:}2, \\ 20{:}0, & 21{:}0, & 22{:}1, & 23{:}1, & 30{:}3, & 31{:}3, & 32{:}3, & 33{:}3. \end{array}$$

The letter 3 is a 1-blocking word. Assume $x_i = 3$ for $i > 0$ and $x_i \neq 3$ for $i \leq 0$. If $\varphi(x) = \sum_{i=1}^{\infty} x_{-i} \cdot 2^i$ is finite, then $\varphi(F(x)) = \varphi(x) + 1$. Thus $(\overline{\mathcal{O}(x)}, F)$ is conjugated to the adding machine with periodic structure $\mathbf{n} = (2, 2, 2, \ldots)$, although the system is not left-permutive. If $x = 0^\infty.3^\infty$, then for any $i < 0$, $(F^n(x)_i)_{n \geq 0}$ has preperiod $-i$ and period 2^{-i}. For the zero configuration we have

$$\begin{array}{rcl} n < 2^s + s & \Longrightarrow & F^n(W_s^-(0^\infty)) \subseteq W_0^-(0^\infty) \\ 2^{s-1} + s - 1 \leq n < 2^s + s & \Longrightarrow & I_n^-(0^\infty) = s \end{array}$$

and therefore $\log_2 n - 1 < I_n^-(0^\infty) < \log_2 n + 1$. More generally, for any $x \in \{0,1,2\}^{\mathbb{Z}}$ we have $\lim_{n\to\infty} I_n^-(x)/\log_2 n = 1$.

```
3 0 3 0 0 3 0 0 0 3 0 0 0 0 3 1 1 0 2 3 0 0 0 0 0 3 0 0 0 0 0 3 0 3
3 1 3 0 1 3 0 0 1 3 0 0 0 1 3 1 1 1 1 3 0 0 0 0 1 3 0 0 0 0 1 3 1 3
3 2 3 0 2 3 0 0 2 3 0 0 0 2 3 1 1 1 2 3 0 0 0 0 2 3 0 0 0 0 2 3 2 3
3 1 3 1 1 3 0 1 1 3 0 0 1 1 3 1 1 2 1 3 0 0 0 1 1 3 0 0 0 1 0 3 1 3
3 2 3 1 2 3 0 1 2 3 0 0 1 2 3 1 2 0 2 3 0 0 0 1 2 3 0 0 0 1 1 3 2 3
3 1 3 2 1 3 0 2 1 3 0 0 2 1 3 2 0 1 1 3 0 0 0 2 1 3 0 0 0 1 1 3 1 3
3 1 3 0 2 3 1 0 2 3 0 1 0 1 3 0 0 1 2 3 0 0 1 0 2 3 0 0 0 1 2 3 2 3
3 2 3 1 1 3 1 1 1 3 0 1 0 2 3 0 0 2 1 3 0 0 1 1 1 3 0 0 0 2 0 3 1 3
3 1 3 1 2 3 1 1 2 3 0 1 1 1 3 0 1 0 2 3 0 0 1 1 2 3 0 0 1 0 1 3 2 3
3 2 3 2 1 3 1 2 1 3 0 1 1 2 3 0 1 1 1 3 0 0 1 2 1 3 0 0 1 0 1 3 1 3
3 0 3 0 2 3 2 0 2 3 0 1 2 1 3 0 1 1 2 3 0 0 2 0 2 3 0 0 1 0 2 3 2 3
3 1 3 1 0 3 0 1 1 3 0 2 0 2 3 0 1 2 1 3 0 1 0 1 1 3 0 0 1 1 0 3 1 3
3 2 3 1 1 3 0 1 2 3 1 0 1 1 3 0 2 0 2 3 0 1 0 1 2 3 0 0 1 1 1 3 2 3
3 1 3 1 2 3 0 2 1 3 1 0 1 2 3 1 0 1 1 3 0 1 0 2 1 3 0 0 1 1 1 3 1 3
3 2 3 2 1 3 1 0 2 3 1 0 2 1 3 1 0 1 2 3 0 1 1 0 2 3 0 0 1 1 2 3 2 3
3 0 3 0 2 3 1 1 1 3 1 1 0 2 3 1 0 2 1 3 0 1 1 1 1 3 0 0 1 2 0 3 1 3
```

Figure 24: Sensitive system with logarithmic perturbation speeds

Example 23 (A sensitive CA with logarithmic perturbation speeds) $(\mathbf{4}^{\mathbb{Z}}, F)$ *where* $m = 0$, $a = 2$ *and the local rule is*

$$\begin{array}{cccccc} 33x{:}0, & 032{:}0, & 132{:}1, & 232{:}0, & 02x{:}1, & 03x{:}1, \\ 12x{:}2, & 13x{:}2, & 20x{:}0, & 21x{:}0, & 22x{:}1, & 23x{:}1. \end{array}$$

A similar system is constructed in Bressaud and Tisseur [9]. If $i < j$ are consecutive sites with $x_i = x_j = 3$, then $F^n(x)_{(i,j)}$ acts as a counter machine whose binary value is increased by one unless $x_{j+1} = 2$:

$$\begin{aligned} B_{ij}(x) &= \sum_{k=1}^{j-i-1} x_{i+k} \cdot 2^{j-i-1-k} \\ B_{ij}(F(x)) &= B_{ij}(x) + 1 - \xi_2(x_{j+1}) - 2^{j-i-1} \cdot \xi_2(x_{i+1}) \end{aligned}$$

Here $\xi_2(2) = 1$ and $\xi_2(x) = 0$ for $x \neq 2$. If $x \in \{0, 1, 2\}$, then $\lim_{n\to\infty} I_n^-(x)/\log_2(n) = 1$. For periodic configurations which contain 3, Lyapunov exponents are positive. We have $\lambda^-((30^n)^\infty) \approx 2^{-n}$.

13 Future directions

There are two long-standing problems in topological dynamics of cellular automata. The first is concerned with expansivity. A positively expansive CA is conjugated to a one-sided full shift (Theorem 29). Thus the dynamics of this class is completely understood. An analogous assertion claims that bijective expansive CA are conjugated to two-sided full shifts or at least to two-sided subshifts of finite type (Conjecture 30). Some partial results have been obtained in Nasu [37].

Another open problem is concerned with chaotic systems. A dynamical system is called chaotic, if it is topologically transitive, sensitive, and has a dense set of periodic points. Banks et al., [3] proved that topological transitivity and density of periodic points imply sensitivity, provided the state space is infinite. In the case of cellular automata, transitivity alone implies sensitivity (Codenotti and Margara [10] or a stronger result in Moothathu [33]). By Conjecture 15, every transitive CA (or even every surjective CA) has a dense set of periodic points. Partial results have been obtained by Boyle and Kitchens [6], Blanchard and Tisseur [5] and Acerbi et al., [1]. See Boyle and Lee [7] for further discussion.

Interesting open problems are concerned with topological entropy. For CA with non-negative memory, the entropy of a CA can be obtained as the entropy of the column subshift whose width is the radius (Proposition 54). For the case of negative memory and positive anticipation, an analogous assertion would say that the entropy of the CA is the entropy of the column subshift of width $2r+1$ (Conjecture 55). Some partial results have been obtained in Di Lena [29]. Another aspect of information flow in CA is provided by Lyapunov exponents which have many connections with both topological and measure-theoretical entropies (see Bressaud and Tisseur [5] or Pivato [38]). Conjecture 11 states that each sensitive CA has a configuration with a positive Lyapunov exponent.

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