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Representing real numbers in Möbius number systems

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1. Introduction

The aim of this paper is to give the reader general idea of how can we represent real numbers using Möbius transformations in a way that encompasses several already established systems.

The theory of Möbius number systems was introduced in [3]. A Möbius number system assigns numbers to sequences of Möbius transformations obtained by composing a finite starting set of Möbius transformations. Möbius number systems have connections to other kinds of number representation systems. In particular, Möbius number systems can generalize continued fractions (see [4]).

A criterium for the existence of the number system is already known for a considerable class of sets of Möbius transformations (see [4], Theorem 9 and [2], Theorem 4.10), however there is still a gap between the sufficient and the necessary condition.

2. Preliminaries

Let $\mathbb{C}$ be the complex plane. Denote by $\mathbb{C} \cup \{\infty\}$ the *complex sphere* – the compact topological space obtained from $\mathbb{C}$ by adding the point in infinity. Likewise, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ will denote the compactification of the real line. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$ centered at 0. Note that $\overline{\mathbb{R}}$ is homeomorphic to $\mathbb{T}$; this will be important later. Finally, we let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ (the upper half plane) and by $\mathbb{D}$ we mean the *open* unit disc in the complex plane.

Our goal is to represent points in $\overline{\mathbb{R}}$ or, equivalently, in $\mathbb{T}$ using sequences of Möbius transformations. To represent $\overline{\mathbb{R}}$ we use transformations preserving $\mathbb{H}$ and to represent $\mathbb{T}$ we use transformations preserving $\mathbb{D}$. Through the paper, we distinguish half plane preserving Möbius transformations from disc preserving ones by placing a hat above the name of each half plane preserving transformation.

Definition 2.1. A Möbius transformation (MT) is any nonconstant function $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$M(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$.

As we have mentioned above, $\overline{\mathbb{R}}$ and $\mathbb{T}$ are homeomorphic. The stereographic projection $u : \mathbb{T} \to \overline{\mathbb{R}}$ and its inverse $d : \overline{\mathbb{R}} \to \mathbb{T}$ are actually Möbius transformations

$$u(z) = \frac{-iz + 1}{z - i}, \quad d(z) = \frac{iz + 1}{z + i}$$

that also act as homeomorphisms of $\mathbb{D}$ and $\mathbb{H}$. Therefore we can translate MTs that represent $\mathbb{T}$ to the ones that represent $\overline{\mathbb{R}}$ by composing the transformations with d and u in the appropriate order.

We will develop the theory of number representation only for $\mathbb{T}$, where things become easier to formulate. However, the example number systems represent the more practical set $\overline{\mathbb{R}}$.

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Möbius transformations have remarkable geometrical properties that we have mostly omitted in this paper; we recommend the book [1] to the curious reader.

3. Möbius number systems

Definition 3.1. A sequence $M_1, M_2, \ldots$ of disc preserving MTs *represents the number* $x \in \mathbb{T}$ if $M_n(0) \to x$ for $n \to \infty$.

While the above definition might look arbitrary at first glance, it is actually quite natural and has several different formulations (including the original definition by convergence of measures in [3]). In particular, Theorem 3.6 in [2] gives us that if $M_1, M_2, \ldots$ represents x then $M_n(K) \to \{x\}$ for any K compact subset of the open unit disc (where we take convergence in the Hausdorff metric).

The next step is to define what a *Möbius number system* is. Before we do that, let us recall several standard notations from symbolic dynamics.

Let A be a finite alphabet. A word over A is a (finite or infinite) sequence of symbols from A. By w_i we mean the i-th letter of the word w. For u, v finite words (or letters) we denote by uv the concatenation of u and v.

Denote by A^+ the set of all finite nonempty words over A and by A^ω the set of all one-sided infinite words over A.

A set $\Sigma \subset A^\omega$ is a *subshift* if there exists a set S of finite words such that $w \in \Sigma$ if w does not contain any $s \in S$ as a factor (i.e. there are no i, j such that $w_i w_{i+1} \ldots w_j = s$).

Assume that we have assigned to every letter $a \in A$ a corresponding MT F_a. We then define F_v for any finite word v as $F_v = F_{v_1} \circ F_{v_2} \circ \cdots \circ F_{v_n}$ (we compose mappings in the more usual way $(F \circ G)(z) = F(G(z))$).

Definition 3.2. Assume we are given a system of MTs $\{F_a : a \in A\}$. A subshift $\Sigma \subset A^\omega$ is a *Möbius number system* if:

(1) For every $w \in \Sigma$, the sequence $\{F_{w_0 w_1 \ldots w_n}\}_{n=0}^\infty$ represents some point $\Phi(w) \in \mathbb{T}$.
(2) The function $\Phi : \Sigma \to \mathbb{T}$ is continuous and surjective.

4. Examples

Attempting to come up with a Möbius number system, we will first try to emulate the standard binary representation of numbers.

Let us take the (half plane preserving) transformations $\hat{F}_0(x) = x/2$ and $\hat{F}_1(x) = (x+1)/2$. These transformations have their disc preserving counterparts F_0 and F_1. We want to see what happens if we take the system $\{F_0, F_1\}$ together with the full shift $\Sigma = \{0,1\}^\omega$. It can be shown that the function Φ maps Σ to an interval on $\mathbb{T}$ corresponding to $[0,1]$ under the stereographic projection.

Essentially, we have obtained the ordinary binary system, as $\Phi(w)$ corresponds to the number $0.w$. However, this is not a Möbius number system, as Φ is not surjective. We can fix this deficiency by introducing new transformations, but we will have to forbid some combinations of transformations to maintain convergence.

One way to do this is to take the following four transformations:

$$
\begin{aligned}
\hat{F}_{\overline{1}}(x) &= (x-1)/2 \\
\hat{F}_0(x) &= x/2 \\
\hat{F}_1(x) &= (x+1)/2 \\
\hat{F}_2(x) &= 2x
\end{aligned}
$$

and forbid the words $20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1$. The reason for disallowing 2 and 0 next to each other is that these transformations are inverse to each other. The first four forbidden pairs ensure that twos are going to be only at the beginning of the word. The last two forbidden words ensure continuity of the function Φ at 2^∞ and, as a side-effect, make the representation nicer (in cryptography, for example, we often wish to only deal with redundant representations of integers without 1 and $\overline{1}$ next to each other).

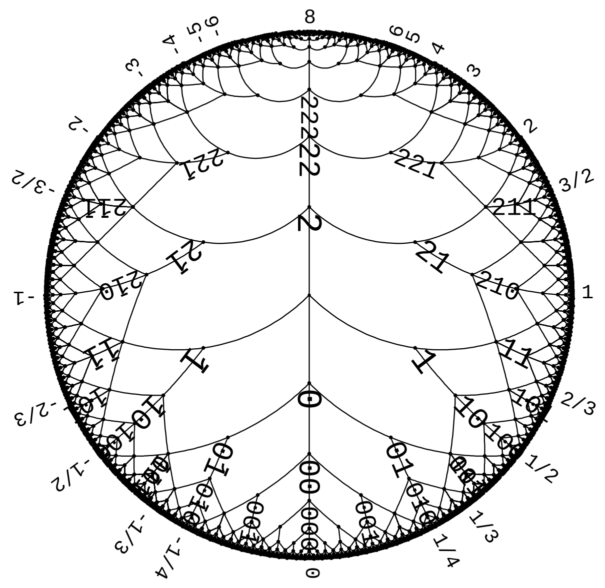

Figure 1: The Möbius binary signed system

The result is the Möbius number system depicted in Figure 1. The labelled points represent the images of the center of the circle under the corresponding sequence of transformations. Curves connect images of 0 that are next to each other in a given sequence. Observe that the images of 0 converge to the boundary of the disc.

The connection between MTs and continued fraction systems is well known. We show how to implement continued fractions as a Möbius number system. Let us take the following three transformations:

$$\begin{aligned}\hat{F}_{\overline{1}}(x) &= -1 + x \\ \hat{F}_0(x) &= -\frac{1}{x} \\ \hat{F}_1(x) &= 1 + x\end{aligned}$$

and forbid the words $00, 1\overline{1}, \overline{1}1, 101, \overline{1}0\overline{1}$ (the first three for obvious reasons, we are disallowing 101 and $\overline{1}0\overline{1}$ because the words $(01)^\infty$ and $(0\overline{1})^\infty$ do not represent any number).

It turns out that we obtain the regular continued fraction system, see Figure 2. The only complication is that we need to juggle with signs, using the transformation $-1/x$ instead of $1/x$ because the latter does not preserve the unit disc ($1/x$ preserves the unit circle but turns the disc inside out).

The function $\Phi : \Sigma \to \mathbb{T}$ mirrors the usual continued fraction numeration process which we describe here for real numbers. Let x be a real number, without loss of generality let $x \geq 1$. We take $x_{i+1} = x_i - 1 = F_1^{-1}(x_i)$ until $0 \leq x_{i+1} < 1$ then we take $x_{i+1} = -1/x_i = F_0^{-1}(x_i)$ and continue with $x_{i+1} = x_i + 1 = F_{\overline{1}}^{-1}(x_i)$ until $-1 < x_i \leq 0$, then we let $x_{i+1} = F_0^{-1}(x_i)$ and so on. Writing down the labels of MTs in the order they were used, we obtain the representation of x.

Even a quick glance on Figure 2 reveals that parts of the circle are missing. While Φ is indeed surjective, the convergence of the images of 0 is sometimes quite slow in this system and so the depth used in our computer graphics was not enough to get near certain points. We can improve the speed of convergence by adding more transformations like in [4].

5. Existence theorem

In this section we touch on the problem, how to decide, given a system of MTs $\{F_a : a \in A\}$ if there exists a Möbius number system using these transformations. Our goal is to come up with a sufficient and necessary condition on $\{F_a : a \in A\}$ to admit a number system. So far, we only have a partial characterization.

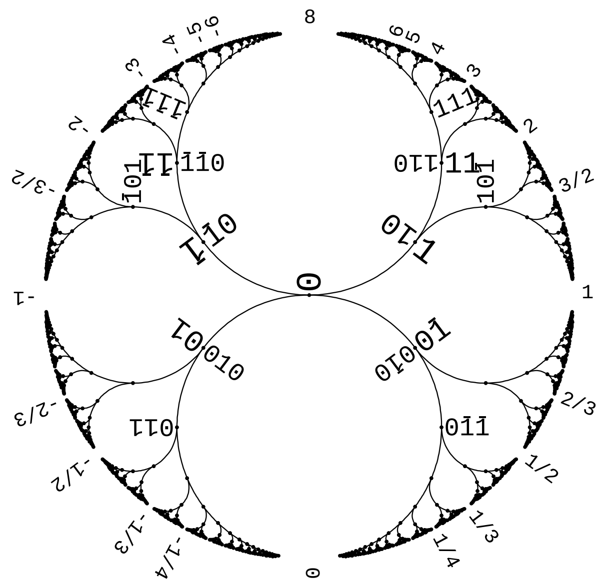

Figure 2: The regular continued fractions

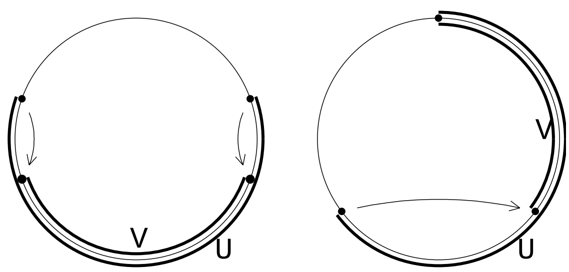

Figure 3: The expansion and contraction intervals of $\hat{M}_0(x) = x/2$ and $\hat{M}_1(x) = x + 1$

Definition 5.1. Given a system of MTs $\{F_a : a \in A\}$, let $U_u = \{z \in \mathbb{T} : \; |F_u'(z)| < 1\}$ resp. $V_u = \{z \in \mathbb{T} : \; |(F_u^{-1})'(z)| > 1\}$ be the *contraction interval of* F_u resp. the *expansion interval of* F_u^{-1} (see Figure 3). Note that the sets U_n, V_u are either intervals on the unit circle or (if F_u is a rotation) empty sets.

Observe that it is $F_u(U_u) = V_u$. Intuitively, we expect that if a word w begins with the finite sequence u then $\Phi(w)$ lies somewhere close to V_u, because that is where all of the points of U_u get mapped by F_u. While in general $\Phi(w)$ and V_u need not posses any sensible relation (consider the case of $F_b = F_a^{-1}$ when F_{ab} is the identity map), we can use this idea to obtain some insight in the existence problem if we consider *all* the intervals V_u:

Theorem 5.2. *Let $\{F_a : a \in A\}$ be MTs.*

(1) If $\overline{\bigcup\{V_u : u \in A^+\}} \neq \mathbb{T}$, then there does not exist any Möbius number system.
(2) If there exists a finite $B \subset A^+$ such that $\{\overline{V}_u : u \in B\}$ cover $\mathbb{T}$, then there exists a Möbius number system.

For proof, see Theorem 9 in [4] and Theorem 4.10 in [2].

6. Conclusions

In this paper, we have shown how sequences of MTs can represent real numbers in a rather natural way. The notion of representation allows us to restate several known number systems in the language of Möbius number systems. In particular, we can view regular continued fractions as a Möbius number system. The only drawback of Möbius number systems compared to commonly used numeration systems is that Möbius system typically does not use the full shift and so we have to make sure we avoid forbidden words.

We have some sufficient and some necessary existence conditions for a Möbius number systems, however there is still a gap in the characterization that we hope to close in the future. A similar question, which sets of MTs admit a number system with only finitely many forbidden words (i.e. Σ is a shift of finite type) remains wide open.

7. Acknowledgements

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