Computational Universality in Symbolic Dynamical Systems ⋆

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Abstract. Many different definitions of computational universality for various types of systems have flourished since Turing's work. In this paper, we propose a general definition of universality that applies to arbitrary discrete time symbolic dynamical systems. For Turing machines and tag systems, our definition coincides with the usual notion of universality. It however yields a new definition for cellular automata and subshifts. Our definition is robust with respect to noise on the initial condition, which is a desirable feature for physical realizability.

We derive necessary conditions for universality. For instance, a universal system must have a sensitive point and a proper subsystem. We conjecture that universal systems have an infinite number of subsystems. We also discuss the thesis that computation should occur at the 'edge of chaos' and we exhibit a universal chaotic system.

1 Introduction

Computability is often defined via universal Turing machines. A Turing machine is a dynamical system, i.e., a set of configurations together with a transformation of this set. Here a configuration is composed of the state of the head and the whole content of the tape. Computation is done by observing the trajectory of an initial point under iterated transformation.

However there is no reason why Turing machines should be the only dynamical systems capable of universal computation, and indeed we know that many systems may perform universal computations.

Artificial neural networks [1], cellular automata [2], billiard balls on a pool table of some complicated form, or a ray of light between a set of mirrors [3] are such systems.

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For all these systems, many particular definitions of universality have been proposed. Most of them mimic the definition of computation for Turing machines: an initial point is chosen, then we observe the trajectory of this point and see whether it reaches some 'halting' set. See for instance [4] and [5].

However it has been shown that the computational capabilities of many of these systems are strongly affected by the presence of noise [6,7]; fault-tolerant cellular automata are built in [8]. See also [9,10,11] for some definitions of analog computation and issues relative to noise and physical realizability.

Moreover, many variants of these definitions exist and lead to different classes of universal dynamical systems. In particular, there is no consensus for what it means for a cellular automaton to be universal.

Another field of investigation is to make a link between the computational properties of a system and its dynamical properties. For instance, attempts have been made to relate 'universal' cellular automata to Wolfram's classification. It has also been suggested that a 'complex' system must be on the 'edge of chaos': this means that the dynamical behavior of such a system is neither simple (i.e., an attracting fixed point) nor chaotic; see [2,12,13,14]. Other authors nevertheless argue that a universal system may be chaotic: see [1].

The basic questions we would like to address are the following:

- What is a computationally universal dynamical system?
- What are the dynamical properties of a universal system?

A long-term motivation is to answer these questions from the point of view of physics. What natural systems are universal? Is the gravitational N-body problem universal [3]? Is the Navier-Stokes equation universal [15]?

However in this paper we especially focus on *symbolic* dynamical systems, i.e., systems defined on the Cantor set $\{0,1\}^{\mathbb{N}}$ or a subset of it. Some motivating examples of dynamical systems are Turing machines, cellular automata and subshifts. Let us briefly describe our ideas.

Extending Davis' definition of universal Turing machine, we say that a system is universal if some property of its trajectories, such as reachability of the halting set, is r.e.-complete.

However, rather than considering point-to-point or point-to-set properties, we consider set-to-set properties. Typically, given an initial set and a halting set, we look whether there is at least one configuration in the initial set whose trajectory eventually reaches the halting set.

We require the initial and halting sets to be closed open sets of the Cantor space endowed with the usual product topology, which are sets that can be described with a finite number of bits in a natural standard way.

Finally, we do not restrict ourselves to the sole property 'Is there a trajectory going from A to B?' (where A and B are closed open sets), but to any property of closed open sets that can be described in temporal logic.

This definition addresses the two issues raised above. Firstly, it is a general definition directly transposable to any symbolic system. Secondly, dealing with open sets rather than points takes into account some constraints of physical realizability, such as robustness to noise.

With this definition in mind, we prove necessary conditions for a symbolic system to be universal. In particular, we show that a universal symbolic system is not minimal, not equicontinuous and does not satisfy the effective shadowing property. This last property is a variant of the usual shadowing property. We conjecture that a universal system must have infinitely many subsystems, and we show that there is a chaotic system that is universal, contradicting the idea that computation can only happen on the 'edge of chaos'.

The paper is organized as follows: in Sections 2 and 3 we define effective symbolic systems; in Section 4 the syntax and semantics of temporal logic is exposed; in Section 5 the formal definition of universality is given, and simple examples are provided; this definition is discussed in Section 6; in Section 7 necessary conditions for a system to be universal are given, related to minimality, equicontinuity and effective shadowing property; in Section 8 we build a chaotic system that is universal, and briefly discuss the the existence of the 'edge of chaos'; Section 9 discusses possible directions for future work, including extension to analog systems.

2 Effective Symbolic Spaces

Effective symbolic dynamical systems are computable continuous transformations of a symbolic space. In this section and the next one, we provide a formal definition and elementary properties.

Definition 1 *A* symbolic space *is a compact metric space for which there is a countable basis of closed open sets (called* clopen sets*).*

The members of a symbolic space are called *points* or *configurations*. A typical example of a symbolic space is the Cantor set $\{0,1\}^{\mathbb{N}}$ endowed with the product topology. The topology is given by the metric $d(x,y)=0$ if $x=y$ and

$$d(x,y)=2^{-n}$$

where n is the index of the first bit on which x and y differ.

If $w\in\{0,1\}^*$ (the set of finite binary words), then $[w]$ denotes the set of all sequences beginning by w. In fact, sets of this form, usually called *cylinders*, are exactly the balls of the metric space. Any clopen set of $\{0,1\}^{\mathbb{N}}$ is a finite union of cylinders.

The same definition of distance almost immediately extends to the spaces $\{0,1\}^*\cup\{0,1\}^{\mathbb{N}}$, $A^{\mathbb{N}}$, $Q\times A^{\mathbb{Z}}$, $A^{\mathbb{Z}^d}$ where Q and A are finite and d is a positive integer.

Closed subsets of the Cantor space are themselves symbolic spaces. It is well-known that the converse is also true.

Proposition 1 *Every symbolic space is homeomorphic to a closed subset of the Cantor space. Every perfect symbolic space is homeomorphic to the Cantor space.*

For instance, $\{0,1\}^{\mathbb{Z}}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

In order to define computational universality, we need to make symbolic spaces effective: we would like to be able to perform boolean combinations on clopen sets.

Definition 2 *An* effective symbolic space *is a couple (X, P) where X is a symbolic space and $P : \mathbb{N} \to 2^X$ is an injective function ranging over all clopen sets of X, such that intersection and complementation of clopen sets are computable operations. This means that the index of $P_m \cap P_n$ is a computable function of m and n and the index of $X \setminus P_n$ is a computable function of n.*

The *index* of a clopen set P_k is the number k. Of course, union of clopen sets is then also computable.

Note for instance that the Cantor set $\{0,1\}^{\mathbb{N}}$, with any reasonable way to enumerate clopen sets, is an effective symbolic space. Some other effective spaces are for instance: $\{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$, $A^{\mathbb{N}^d}$, $A^{\mathbb{Z}^d}$, $Q \times A^{\mathbb{Z}}$, where Q and A are finite alphabets and d is a positive natural number.

Remark that we could ask intersections and complements to be primitive recursive rather than computable, without altering any of the examples and results of the text.

We now define natural maps between effective symbolic spaces.

Definition 3 *Let (X, P) and (Y, Q) two effective symbolic spaces. An* effective continuous map *is a continuous map $h : X \to Y$ such that $h^{-1}(Q_n)$ is a clopen set of X whose index according to P is computable as a function of n.*

If h is bijective then it is an effective homeomorphism*, and (X, P) is said to be* effectively homeomorphic *to (Y, Q).*

Note that the composition of effective continuous maps is an effective continuous map, the identity is an effective continuous map and the inverse map of an effective homeomorphism is also an effective homeomorphism. In particular, being effectively homeomorphic is an equivalence relation for effective symbolic spaces.

Given an effective symbolic space (X, P), a closed subset Y is said to be *effective* if the family of clopen sets intersecting Y is decidable.

Then the effective set Y can be endowed with the relative topology, generated by all intersections of clopen sets of X with Y. Thus, the enumeration $P_0, P_1, P_2, \ldots$ of clopen sets of X is also an enumeration of clopen sets of Y: $Y \cap P_0, Y \cap P_1, Y \cap P_2, \ldots$ This enumeration has repetitions, but we can detect them and renumber the sequence in an effective way. Hence we get an effective topology for the effective closed set Y. Equivalently, the inclusion $i : Y \hookrightarrow X$ is an effective continuous map.

We know that every symbolic space is a subspace of the Cantor space. The corresponding result also holds for effective symbolic spaces.

Proposition 2 *Every effective symbolic space is effectively homeomorphic to an effective subset of the Cantor space.*

Proof. Let X be a symbolic space, and $P_0, P_1, P_2, \ldots$ be an enumeration of all its clopen sets.

For every point $x \in X$, construct the infinite word $x_0x_1x_2x_3\ldots \in \{0,1\}^{\mathbb{N}}$, where $x_n = 1$ iff $x \in P_n$. Then the map $g : X \to \{0,1\}^{\mathbb{N}}$ is injective and continuous. As X is compact, $g(X)$ is closed. Moreover, as every step of the construction is effective, $g(X)$ is an effective closed set and the map g is effective.

If the space is perfect, then we construct another map h between X and $\{0,1\}^{\mathbb{N}}$. We may write X as a partition of two clopen sets $X = X_0 \cup X_1$, where X_0 is the first clopen set to be different from X and $\emptyset$; this is always possible thanks to perfectness.

Then for any word $w \in \{0,1\}^*$, we recursively split X_w into two clopen sets X_{w0} and X_{w1}. We choose X_{w0} to be the intersection of X_w and the first clopen set P_n such that $X_w \cap P_n$ is different from X_w and $\emptyset$.

Now we build a function h from X to $\{0,1\}^{\mathbb{N}}$ in the following way. Let x be in X and $(w_i)_{i\in\mathbb{N}}$ be the set of words such that $x \in X_{w_i}$ for all i. Then we set $h(x)$ to be the infinite word $\lim w_i$. It is easy to prove that this map is well-defined, injective, continuous and surjective. Thus X is homeomorphic to $h(X) = \{0,1\}^{\mathbb{N}}$. Moreover, as every step of the construction is effective, h is effective. $\square$

Thus we see that there is no loss of generality in supposing that in any effective symbolic space, for any rational ϵ there is a finite number of balls of radius ϵ and that we can compute all of them. Indeed, this is the case for all effective subsets of the Cantor space.

3 Effective Symbolic Systems

Now we would like to define dynamical systems. A (discrete-time) dynamical system is often defined as a continuous transformation of a compact metric space. Thus a symbolic dynamical system is a continuous transformation of a symbolic space. However we are naturally interested in *effective* symbolic dynamical systems.

Definition 4 *An* effective symbolic dynamical system *is an effective continuous map from an effective symbolic space to itself.*

In other words, an effective symbolic system is a symbolic space with a continuous map in which we can compute intersections, complements, and inverse images of clopen sets. This definition of effective function in a Cantor space is equivalent to classical definitions in computable analysis, for instance [16].

Extending Definition 3, we define a relation of equivalence between effective systems.

Definition 5 *Two effective symbolic systems (X, O, f) and (Y, P, g) are* effectively conjugated *if there is an effective homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. Then h is called an* effective conjugacy.

A cellular automaton is an example of effective symbolic system, acting on the space $A^{\mathbb{Z}^d}$, where A is the finite alphabet and d is the dimension. A Turing machine is an effective system acting on the space $Q \times A^{\mathbb{Z}}$, where Q is the finite set of states of the head and A is the finite tape alphabet.

Recall that a *shift* is a dynamical system on $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ (where A is a finite alphabet) with the map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}} : a_0a_1a_2a_3\ldots \mapsto a_1a_2a_3\ldots$ or $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}} : \ldots a_{-3}a_{-2}a_{-1}\underline{a_0}a_1a_2a_3\ldots \mapsto \ldots a_{-3}a_{-2}a_{-1}a_0\underline{a_1}a_2a_3\ldots$, where the symbol of index 0 is underlined. A *subshift* is a subsystem of the shift, i.e., a closed subset that is invariant under the shift map. A one-sided (two-sided) shift is an effective system.

An *effective subsystem* of an effective symbolic system is an effective closed subset that is invariant under the map. With the relative topology, it is itself an effective symbolic system.

If a subshift is an effective closed subset of the space, then it is again an effective symbolic system. The set of all finite words appearing at least once in at least one of the point of the subshift is called the *language* of the subshift. In fact it is easy to see that an effective subshift is exactly a subshift whose language is recursive.

From any dynamical system (effective or not), we can generate subshifts in a natural way. A *clopen partition* of a symbolic space is a partition of the space into clopen sets. Given a clopen partition $X = X_0 \cup \ldots \cup X_s$ of the space X, the subshift induced by this partition is the set of infinite words $a_0a_1a_2a_3\ldots \in \{X_0,\ldots,X_s\}^{\mathbb{N}}$ such that there is a point in a_0 whose trajectory goes successively through a_1, a_2, ...

Following this observation, we can characterize effective symbolic systems in terms of their induced subshifts.

Proposition 3 *A continuous transformation of an effective symbolic space is effective if and only if there is an algorithm telling from a given clopen partition and a given finite word whether this word belongs to the language of the subshift induced by that partition.*

Proof. Let $X = X_0 \cup \ldots \cup X_s$ be a clopen partition. Then a word $a_0a_1\ldots a_{l-1}a_l \in \{X_0,\ldots,X_s\}^*$ is in the language of the subshift induced by the partition if and only if $f^{-l}(a_0) \cap f^{-(l-1)}(a_1) \cap \ldots \cap f^{-1}(a_{l-1}) \cap a_l$ is not empty. But we can check this.

Conversely, suppose that all induced subshifts have decidable languages, and that we can compute a decision algorithm from the partition. Let P_n be a clopen set of X. Consider the partition of at most four parts $X = X_0 \cup X_1 \cup X_2 \cup X_3$, where $X_0 = P_n \setminus f^{-1}(P_n)$, $X_1 = f^{-1}(P_n) \setminus P_n$, $X_2 = P_n \cap f^{-1}(P_n)$, $X_3 = X \setminus (P_n \cup f^{-1}(P_n))$. Some of these sets can be empty. This is the coarsest partition that is finer than both P_n and $f^{-1}(P_n)$. Now consider the language induced by this partition. Then X_0X_0, X_0X_2, X_3X_0, X_3X_2, X_1X_1, X_1X_3, X_2X_1,X_2X3 do not belong to the language. In fact, if we find a partition into four or less parts with this property, then we know that $f^{-1}(P_n) = X_1 \cup X_2$. Thus it is enough to enumerate all partitions of four elements or less, and check these conditions to compute $f^{-1}(P_n)$. □

If the subshifts have decidable languages, but decision algorithms are not computable, then the system may fail to be effective, as shown in the following example.

Example 1 *Assume k is an uncomputable total strictly increasing function on $\mathbb{N}$. We define a function f on the Cantor space $\{0,1\}^{\mathbb{N}}$ such that $f(x) = f_0(x)f_1(x)f_2(x)\ldots$, where f_i is the sum modulo two of the $k(i)$th first bits of x.*

Then it is not difficult to see that for every partition of $\{0,1\}^{\mathbb{N}}$, the subshift is of finite type (i.e., we can describe it as the infinite words avoiding a finite list of finite forbidden words); thus the language is decidable. However f is not effective, for otherwise we could compute k.

4 Temporal Logic

Our goal is to describe properties of trajectories that may be useful in defining universal computation, such as 'starting from here, the system eventually goes there'. Temporal logic, developed by Prior in 1953, is appropriate to express such properties. It was later used by computer scientists to express and prove sentences such as, typically, 'the program will not reach a forbidden state'; see [17] for a reference book on modal and temporal logic.

Formally, we suppose that we have a set $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, ...\}$ of proposition symbols indexed by $\mathbb{N}$ including two propositions that we will denote $\bot$ and $\top$, and we form all temporal formulae by composing the propositions symbols with the boolean operators $\vee$ and $\neg$, the temporal unary operator $\circ$ (read 'next') and the binary operator $\mathcal{U}$ ('until').

We can also add some usual abbreviations: $\phi \wedge \psi$ denotes $\neg(\neg\phi \vee \neg\psi)$, $\phi \Rightarrow \psi$ denotes $\neg\phi \vee \psi$, $\diamond\phi$ (read 'eventually ϕ') stands for $\top\mathcal{U}\phi$ and $\Box\phi$ (read 'always ϕ') for $\neg\diamond\neg\phi$.

We now give temporal formulae a semantics adapted to symbolic systems. Let (X, P, f) be an effective symbolic system. Recall that X is a symbolic space with clopen sets $P_0, P_1, P_2, \ldots$ and $f : X \to X$ is a continuous function. Then to each formula ϕ we assign a subset $|\phi|$ of X, called the *interpretation* of ϕ, in the following way. Then to each formula ϕ we assign a subset $|\phi|$ of X, called the *interpretation* of ϕ, in the following way.

- If ϕ is the proposition symbol $\mathcal{P}_n$, then $|\phi| = P_n$. Moreover we ask that $|\bot| = \emptyset$ and $|\top| = X$.
- If ϕ is $\phi_1 \vee \phi_2$ then $|\phi| = |\phi_1| \cup |\phi_2|$.
- If ϕ is $\neg\psi$ then $|\phi| = X \setminus |\psi|$.
- If ϕ is $\circ\psi$ then $|\phi| = f^{-1}(|\psi|)$.
- If ϕ is $\phi_1\mathcal{U}\phi_2$ then $|\phi| = \bigcup_{n\in\mathbb{N}} A_n$, where $A_0 = |\phi_2|$ and $A_{n+1} = f^{-1}(A_n) \cap |\phi_1|$ for all n.

In particular, if ϕ is $\diamond\psi$ then $|\phi| = \bigcup_{n\in\mathbb{N}} f^{-n}(|\psi|)$.

We say that a formula is *satisfiable* if $|\phi| \neq \emptyset$.

Intuitively, we may think that a formula ϕ represents a statement about a point of X, which is seen as 'the current configuration of the system'. This statement may be true for some points of X and false everywhere else. For example, $\diamond\mathcal{P}_n$ means 'when applying f iteratively the current configuration will eventually be in P_n'. The formula $\mathcal{P}_m\mathcal{U}\mathcal{P}_n$ means 'the configuration lies in P_m until it reaches P_n' or, in other words, 'the configuration will stay in P_m during a finite time and then get in P_n'.

Then $|\phi|$ is the set of points for which the assertion ϕ holds, and a satisfiable formula is verified by at least one configuration. Note that in the following, we will make no distinction between a proposition symbol $\mathcal{P}_n$ and the corresponding clopen set P_n.

5 Universal Systems

We are ready now to state the main definition. We define a universal system to be an effective system with some undecidable temporal property. Then we show that most usual ways to define computability are particular examples of this definition. Finally we briefly discuss the possible presence of different degrees of uncomputability in a system.

5.1 Main Definition

Davis [18] proposed the following definition: a Turing machine is universal if the relation 'x_n is in the orbit of x_m' is r.e.-complete, where x_m and x_n are arbitrary finite configurations. This definition bypasses the need for a description of a way to encode the input and decode the output of a computation. Here we modify Davis' definition in order to be applied to any effective symbolic system. Our choices are justified in Section 6.

Definition 6 *An effective dynamical system is* universal *if there is a recursive family of temporal formulae such that knowing whether a given formula of the family is satisfiable is an r.e.-complete problem.*

A *r.e.-complete* problem, or Σ_1-complete problem, is a recursively enumerable problem, to which any recursively enumerable problem is Turing-reducible.

Universality is obviously preserved by effective conjugacy.

Note that this may be interpreted as a non-deterministic scheme of computation. The computation succeeds iff at least one trajectory exhibits a given behavior.

For example, we may call *halting problem* for f, the satisfiability problem for formula:

$$(P_n \wedge \diamond P_m)_{n,m\in\mathbb{N}},$$

whose satisfiability problem reads: 'There is a configuration in the clopen set P_n that eventually reaches the clopen set P_m'. We may think of P_n as an initial configuration of which we know only the first digits and P_m as the halting set. The unspecified digits of the initial configuration may be seen as encoding the non-deterministic choices occurring during the computation.

5.2 Examples

Turing machines. Turing machines are often described as working only on finite configurations. A finite configuration is an element of $Q \times \{0,1\}^* \times \{0,1\}^*$, where Q denotes the set of states of the head, the first binary word is content of the tape to the left of the head and the second binary word is the right part of the tape. The rest of the tape is supposed to be entirely filled with blank symbols. Such a Turing machine is universal if given two finite configurations u and v, checking whether u is in the trajectory of v is an r.e.-complete problem.

This is a particular case of our definition. Indeed, let $W = \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ the set of finite and infinite binary words. Then the Turing machine transition function is also defined on $Q \times W \times W$, which is a compact space, whose isolated points are $Q \times \{0,1\}^* \times \{0,1\}^*$. Isolated points are in fact clopen sets of $Q \times W \times W$. So the problem of checking whether the formula $P_n \wedge \diamond P_m$ is satisfiable, given two clopen sets P_n and P_m, is r.e.-complete. Indeed, it is already r.e.-complete if we restrict ourselves to clopen sets that are isolated points, and it is recursively enumerable (although perhaps not r.e.-complete) on non-isolated clopen sets.

Tag systems. Tag systems were introduced by Post in 1920. A *tag system* is a transformation rule acting on finite binary words. At each step, a fixed number of bits is removed from the beginning of the word and, depending on the values of these bits, a finite word is appended at the end of the word. Minsky proved in 1961 that there is a so-called universal tag system, for which checking whether a given word will end up to the empty word when repeating the transformation is an r.e.-complete problem; see [2].

We can extend the rule of tag systems to infinite words, by just removing to them the fixed number of bits. Thus we have a dynamical system on the compact space $\{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ of finite and infinite words, in which finite words are clopen sets. Again, if the tag system is universal for the word-to-word definition, then it is universal for our definition with the formulae $P_n \wedge \diamond P_m$.

Collatz functions. We can also apply our definition to functions on integers. Let $\mathbb{N} \cup \{\infty\}$ be the topological space with the metric $d(n,m) = |\frac{1}{n+1} - \frac{1}{m+1}|$. This is effectively homeomorphic to the set $\{1^n 0^\infty | n \in \mathbb{N}\} \cup \{1^\infty\}$. Then a total computable map on $\mathbb{N}$ can be extended to an effective continuous map on $\mathbb{N} \cup \{\infty\}$ iff either it has a finite range and only one integer has an unbounded preimage set, or it has an infinite range and we can compute a (finite) bound on the largest preimage of every given integer.

For example, it is meaningful to ask whether the famous $3n+1$ function (which is effective) is universal. This is an unsettled question. But Conway [19] proved that similar functions, called Collatz functions, may be universal.

Transformations on countable sets. All preceding examples — Turing machines, tag systems, Collatz functions — fit into our definition of effective symbolic system, provided a minor modification: making the space of configurations compact.

More generally, let $f : X \to X$ be a transformation of a countable set $X = \{x_0, x_1, x_2, \ldots\}$. Then we may ask, given two points x_n and x_m, whether x_n is in the trajectory of x_m. If this problem is r.e.-complete, we would like to say that f is universal.

If we suppose that X is a discrete space (i.e., all points are isolated), then f is always continuous. However, this does not fit in our definition of dynamical system since X is not compact.

But we can sometimes embed X in a compact symbolic space $\overline{X}$ in which X is dense, and extend f to $\overline{X}$ in a continuous way. If $f : X \to X$ is universal according to point-to-point definition, then $f : \overline{X} \to \overline{X}$ is universal according to our definition with the formulae $(P_n \wedge \diamond P_m)_{n,m\in\mathbb{N}}$.

Turing machines without blank symbol. It is only slightly more complicated to build a universal Turing machine on $\{0,1\}^{\mathbb{N}}$. In such a Turing machine, there is no obvious notion of 'finite configuration'. The trick is basically to encode the initial data in a self-delimiting way.

Take a Turing machine that is universal for the usual, finite-configuration-to-set halting problem. Then add two new symbols L and R to the tape alphabet. On an initial configuration, put a L on the left and and a R on the right of the finite encoded data. When the head encounters an L, it pushes one cell to the left, leaving a blank symbol to make more space available for computation. It acts similarly for an R symbol. The working space is always delimited by a L and a R, and the symbols situated outside are considered as noise.

Cellular automata. We now give an example of a universal cellular automaton.

Let us take a universal Turing machine with a blank symbol. We suppose that when the halting state is reached, then the head comes back to the cell of index 0. We can simulate it in an almost classic way with a one-dimensional cellular automaton. The alphabet of the automaton is $A \cup (A \times Q) \cup \{L, R, Error\}$, where A is the tape alphabet (including the blank symbol) and Q the set of states.

Let us take a point in the cylinder $[L, \text{initial data of the Turing machine}, R]$, and observe its trajectory. The symbol L moves to the left at the speed of light, leaving behind blank symbols. The symbol R moves to the right in a similar way. Meanwhile, the space between L and R is used to simulate the Turing machine and is composed of symbols from A and exactly one symbol from $(A \times Q)$, which denotes the position of the head. When L or R symbols meet each other, then a spreading $Error$ symbol is produced, that erases everything.

This cellular automaton is universal for formulae $P_n \wedge \diamond P_m$. Indeed, there is an orbit from the cylinder $[L, \text{initial data of the Turing machine}, R]$ to the cylinder [halting state] (both cylinders centered at cell of index zero) if and only if the universal Turing machine halts on the initial data.

5.3 Decidable Systems

An effective system could fail to be universal for different reasons. There could exist families of formulae with an undecidable satisfiability problem, each of them being too easy (an intermediate r.e.-degree) or too difficult (higher than Σ_1 in the arithmetic hierarchy, for instance). The simplest case is when every temporal property is decidable.

Definition 7 *An effective dynamical system is said to be* decidable *if the satisfiability problem for the family of all temporal formulae is decidable.*

Decidability of an effective system is also preserved by effective conjugacy.

Actually, we have no example of a system that would be neither universal nor decidable. In a setting of point-to-point properties, it is proved in [20] that there exists cellular automata with an undecidable, but not r.e.-complete, halting problem.

6 Discussion on the Definition of Universality

Our definition of universality differs in several ways from what we could expect at first glance from a generalization of Turing machine universality. In this section we give various arguments to support the present definition against seemingly more obvious attempts. In particular, we justify the use of *set-to-set* properties, expressed in the formalism of *temporal logic*, on systems for which the transition function is *computable*.

6.1 Set-to-set Properties

Many definitions of universality for particular systems propose to observe point-to-point properties.

On the other hand, in [21] is proposed a definition for effective metric space; the basic idea is to endow a metric space with a countable dense set of points. Examples include the reals with rational points, the Cantor space with ultimately constant configurations, the Cantor space with ultimately periodic configurations.

So it could seem that it is possible to build a general definition of universality with point-to-point properties, but we show that it may lead to undesirable consequences. We then argue in favor of set-to-set properties.

The most natural idea would to say that a metric space with a dense set of points $(x_n)_{n\in\mathbb{N}}$ is universal if the property 'x_n is in the trajectory of x_m' is r.e.-complete.

However, as remarked in [22], this leads to conclude that the shift is universal; a consequence that is counter-intuitive. It sounds unreasonable to admit the shift as universal, because it does not treat any information, but just reads the memory.

Indeed if instead of ultimately periodic points we choose configurations with primitive recursive digits, then we take as initial configurations the sequence of states of the head of a universal Turing machine during a computation. And we just have to shift to know whether the halting state will appear.

The definition presented in this text overcomes this problem in a simple manner: the user needs only to give a finite number of bits as an initial condition. Instead of initial *configurations* we shall rather talk about initial *sets*, which may be seen as 'fuzzy points', points defined with finite accuracy.

This solution is also more satisfactory from the point of view of physical realizability. Indeed, we expect the set of configurations of a physical system to be uncountable in general, and specifying an initial point for the computation means *a priori* that we must give an infinite amount of information. Preparing a physical system to be in a very particular configuration is likely to be impossible, because of the noise or finite precision inherent to every measure.

6.2 Temporal Properties

What kind of property are we going to test on clopen sets (or, equivalently, on induced subshifts)? Here again, we must avoid trivialities. Suppose that we look at identity on the Cantor space. We now choose to observe the following property: a clopen set satisfies the property iff its index (i.e., the integer describing the clopen set) satisfies some r.e.-complete property on $\mathbb{N}$. Then we find again that identity is computationally universal, which is not a desirable property. The complexity of computation is artificially hidden in decoding of the answer.

On the other hand, we see no reason to restrict ourselves to the sole halting property: 'there is a trajectory from this clopen set to that clopen'. Any observable property could a priori be used as a basis for computation. For instance, the chaotic system built in Section 8 is universal but not for the halting property.

So we must precisely define a class of observable properties of clopen sets, not too large and not too restricted. Temporal logic, as defined above, has been widely used for decades to express expected properties of various transitions systems. We therefore say that a symbolic system is computationally universal iff there is a family of temporal formulae involving only clopen sets for which determining which formulae are satisfied is an r.e.-complete problem.

6.3 Effectiveness

Finally, the following example shows that it is useful to add an effectiveness structure on dynamical systems.

Fix an r.e.-complete set $H \subset \mathbb{N}$ of integers and consider the symbolic system $(\{0,1\}^{\mathbb{N}}, f)$ where

$$\begin{cases} f(1^n 0 x_0 x_1 x_2 \ldots) = 1^m 0 x_0 x_1 x_2 \ldots \\ \qquad f(1^\infty) \qquad = \qquad 1^\infty \end{cases}$$

and m is the largest integer strictly smaller then n such that $m \in H$ iff $n \in H$, or 0 if no such number exists. Just suppose that $13 \in H$. Then the relation 'the clopen set $[1^n 0]$ will eventually reach $[1^{13}0]$' is r.e.-complete, because H is. On the other hand, if we were provided an actual implementation of $(\{0,1\}^{\mathbb{N}}, f)$, we could decide an undecidable problem (namely, H) by observing the trajectories.

So there is a discrepancy between the computational complexity of properties of clopen sets and the actual possibilities of the machine. This is because we cannot compute even a single step of f: it is a 'non-simulable' system. We therefore restrict ourselves to systems such that the inverse image of a clopen set is computable. Note that for instance in [1] the author allows neural networks with non-recursive weights, leading to a non-computable transition function and to super-Turing capabilities.

7 Necessary Conditions for Universality

It has been highlighted in the Introduction that some attempts have been made to link computational capabilities of a system to its dynamical properties. This is also the purpose of this section.

All results proved in this section are in fact sufficient condition of decidability and can thus be interpreted as necessary conditions for universality. For instance, minimal systems are decidable, thus universal systems are not minimal.

For simplicity, we will write 'symbolic system' for 'effective symbolic dynamical system' — unless otherwise specified.

7.1 Minimality

A *minimal* dynamical system is a system with no subsystem (except the empty set and itself). It is characterized by the fact that all orbits are dense.

We prove that a minimal symbolic system is decidable. This is not surprising since in some way all trajectories of have the same behavior.

Proposition 4 *A minimal symbolic system is decidable. Hence it is not universal.*

Proof. Let $f : X \to X$ be an effective symbolic dynamical system. We shall prove that for the set of all temporal formulae, satisfiability is decidable. We use the recursive structure of formulae. Let ϕ be a formula; we prove recursively that its interpretation $[\phi]$ is always a clopen set, whose index may be computed.

- If ϕ is a proposition symbol, then $|\phi|$ is the clopen set indexed by the same integer.
- If ϕ is $\phi_1 \vee \phi_2$ then $|\phi|$ is $|\phi_1| \cup |\phi_2|$. If $|\phi_1|$ and $|\phi_2|$ are clopen sets then we may compute their union.
- If ϕ is $\neg\psi$ then $|\phi|$ is $X \setminus |\psi|$. If $|\psi|$ is a clopen set then we may compute its complement.
- If ϕ is $\phi_1 \mathcal{U} \phi_2$ then $|\phi|$ is $\bigcup_{n\in\mathbb{N}} B_n$, where $B_0 = |\phi_2|$ and $B_{n+1} = (f^{-1}(B_n) \cap |\phi_1|) \setminus B_n$ for all n. Clearly, if $|\phi_1|$ and $|\phi_2|$ are clopen sets then every B_n is a computable clopen set. But B_n is the set of points staying in $|\phi_1| \setminus |\phi_2|$ during exactly n steps before entering $|\phi_2|$. If all B_n are non-empty, then we can find a limit point, which stays in $|\phi_1| \setminus |\phi_2|$ for ever. Unless $|\phi_1| = X$ and $|\phi_2| = \emptyset$, this contradicts minimality. Hence in every case there is an empty

B_n. In fact, only a finite number of B_n are then non-empty. Therefore $|\phi|$ is a computable clopen set.

Alternatively we could have proved that $\overline{|\phi|} \setminus |\phi|$ must be a subsystem (i.e., a closed invariant subset) and thus be empty. Then $|\phi|$ is a clopen set and from compactness is covered by a finite number of B_n.

□

Now suppose that the symbolic system is not minimal but consists of minimal subsystems attracting the whole space of configurations. In other words, the limit set is minimal. Recall that the limit set of a dynamical system $f : X \to X$ is the set $\bigcap_{n \geq 0} f^n(X)$.

Then such a system is again not universal.

Proposition 5 *A symbolic system with a minimal limit set is decidable.*

Proof. Like in the preceding proposition, we prove that the interpretation of every formula is a computable clopen set. The critical case is the 'until' connector.

Let a formula ϕ be of the form $\phi_1 \mathcal{U} \phi_2$. Then two possibilities may occur: either $|\phi_2|$ intersects the minimal subsystem, or not.

In the former case, then the argument of Proposition 4 still holds: if infinitely many B_n were non-empty, then there would be a closed orbit never crossing $|\phi_2|$, which is impossible.

In the latter case, there is a n such that $f^{-n}(|\phi_2|)$ is empty, because the minimal set attracts the whole space. Thus $B_n \subseteq f^{-n}(|\phi_2|)$ is also empty, and so are $B_{n+1}, B_{n+2}, \ldots$. □

For example, if all points uniformly converge to a periodic orbit, then the system is not universal.

This can be generalized:

Corollary 1 *If the limit set is the union of finitely many minimal subsystems, then the system is decidable.*

Proof. Every minimal system attracts an open part of the space. The basins of attraction must then cover the whole space of configurations. Thus their basins are clopen sets and from Proposition 5 the system is the disjoint union of decidable systems. Hence it is also decidable. □

A stronger statement is suggested by the intuition that a universal system is able to simulate many other systems.

Conjecture 1 *A universal symbolic system has infinitely many minimal subsystems.*

7.2 Equicontinuity

A system $f: X \to X$ is *equicontinuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^t(x), f^t(y)) < \epsilon$, for any points x, y and natural t. Note that equicontinuity in symbolic systems is rather a topological property than just a metric property, since instead of 'For every $\epsilon > 0$, there is a δ ...' we could say 'For every clopen partition, there is a clopen partition ...'

Proposition 6 *An equicontinuous symbolic system is decidable.*

Proof. Just like in the proof of Proposition 4, we show by induction on formulae that the interpretation of a formula is always a computable clopen set. The crucial case is the 'until' operator.

Let us suppose that that some formula ϕ is of the form $\phi_1 \mathcal{U} \phi_2$ and that $|\phi_1|$ and $|\phi_2|$ are clopen sets.

Then there is some $\epsilon > 0$ such that $|\phi_1|$ and $|\phi_2|$ are both finite unions of balls of radius ϵ.

We know from definition that $|\phi|$ is $\bigcup_{n \in \mathbb{N}} A_n$, where $A_0 = |\phi_2|$ and $A_{n+1} = f^{-1}(A_n) \cap |\phi_1|$ for all n. Thus every A_n is a computable clopen set and $|\phi|$ is open.

Now suppose that we can find an $x \in \overline{|\phi|} \setminus |\phi|$. Then for any δ there is a point $x' \in |\phi|$ such that $d(x, x') \leq \delta$. In particular, from equicontinuity we can choose δ such that for all $n \in \mathbb{N}$, $d(f^n(x), f^n(x')) \leq \epsilon$. But then $f^n(x)$ is in $|\phi_1|$ exactly when $f^n(x')$ is in $|\phi_1|$, and $f^n(x)$ is in $|\phi_2|$ exactly when $f^n(x')$ is in $|\phi_2|$. Hence $x \in |\phi_1 \mathcal{U} \phi_2|$: this is a contradiction. We conclude that $\overline{|\phi|} \setminus |\phi|$ is empty and that $|\phi|$ is a clopen set.

Moreover, from compactness, it is covered by a finite number of A_n: the process stops when $A_0 \cup \cdots \cup A_N = A_0 \cup \cdots \cup A_N \cup A_{N+1}$, for some N. As this is detectable, we can compute $|\phi|$. $\square$

We say that a point x of a dynamical system f is *sensitive* if there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a point y with $d(x, y) < \delta$ and a nonnegative time t such that $d(f^t(x), f^t(y)) > \epsilon$.

It is easy to show from compactness that an equicontinuous dynamical system is exactly a system with no sensitive point. Hence, we can deduce from the above result that a universal symbolic system must have a sensitive point.

Equicontinuity in the case of cellular automata has been given a combinatorial characterization in [23]. It is also proved that equicontinuous cellular automata are eventually periodic, thus confirming in this particular case that equicontinuity prevents computational universality from arising.

7.3 Shadowing Property

We now define the *effective shadowing property* for a dynamical system.

Definition 8 *Let (X, f) be a symbolic dynamical system. A* δ-pseudo-orbit *is a (finite or infinite) sequence of points $(x_n)_{n \geq 0}$ such that $d(f(x_n), x_{n+1}) < \delta$ for all n.*

A point x ϵ-shadows *a (finite or infinite) sequence $(x_n)_{n\geq 0}$ if $d(f^n(x), x_n) < \epsilon$ for all n.*

The dynamical system is said to have the shadowing property *if for all $\epsilon > 0$ there is a $\delta > 0$ such that any δ-pseudo-orbit is ϵ-shadowed by some point.*

If moreover such a δ can be effectively computed from ϵ then we say that the system has the effective shadowing property.

In the case of symbolic spaces, we can check that the effective shadowing property is invariant through effective conjugacies.

We can give the following interpretation to this property: suppose that we want to compute numerically the trajectory of x such that at every step numerical errors amount to δ. The resulting sequence of points is a δ-pseudo-orbit, and the shadowing property ensures that this pseudo-orbit is indeed ϵ-close to an actual trajectory of the system.

Proposition 7 *A symbolic system that has the effective shadowing property is decidable.*

Proof. Let us consider such a system (X, P, f) with the effective shadowing property, and a temporal formula ϕ.

We may effectively find a ϵ such that all clopen sets involved in ϕ are finite unions of balls of radius ϵ.

From the effective shadowing property, we effectively get some δ such that all δ-pseudo-orbits are ϵ-shadowed. We may suppose without loss of generality that $\delta \leq \epsilon$.

Let us call δ-pseudo-interpretation of ϕ, the set of points from which there is a δ-pseudo-orbit that satisfies ϕ. More precisely, the δ-pseudo-interpretation is defined recursively on the structure of ϕ exactly like the interpretation defined in Section 4, except that every occurrence of f^{-1} is replaced by $f^{-1} \circ B_\delta$, where $B_\delta(A)$ is the union of all balls of radius δ intersecting the set A. Every time we must compute a preimage of a set, we first replace the set by an approximation of accuracy δ.

Note that the δ-pseudo-interpretation is computable. Indeed, the crucial case is to compute the pseudo-interpretation of $\phi_1 \mathcal{U} \phi_2$. Let us suppose that B is the pseudo-interpretation of ϕ_1 and A is the pseudo-interpretation of ϕ_2. Then the pseudo-intepretation of $\phi_1 \mathcal{U} \phi_2$ is $\bigcup_{n\in\mathbb{N}} A_n$ where $A_0 = A$ and $A_{n+1} = f^{-1}(B_\delta(A_n)) \cap B$ for all n. However there are only finitely many possible values for A_n, because there are only finitely many balls of radius δ. Hence the sequence $A_0, A_1, A_2, \ldots$ will loop at some point.

We claim now that ϕ has a non empty interpretation if and only if it has a non empty δ-pseudo-interpretation.

Indeed, if there is an orbit satisfying ϕ, then this orbit is also a pseudo-orbit satisfying ϕ. Conversely, if there is a pseudo-orbit satisfying ϕ, then this pseudo-orbit is ϵ-shadowed by some orbit. But as clopen sets involved in ϕ are union of balls of radius ϵ, then this orbit also satisfies ϕ. $\square$

In particular, the full shift is decidable.

The following proposition shows that we cannot lift effectiveness of the shadowing property in Proposition 7.

Proposition 8 *There is an undecidable symbolic system that has the shadowing property, but not the effective shadowing property.*

Proof. Let X_n be the subshift with forbidden words 0^t, where the universal Turing machine stops on data n in less than t steps. If the Turing machine eventually never halts on n, then X_n is the full shift; if it stops in k steps, then the forbidden word is 0^k. All these subshifts are effective, but we cannot compute a set of forbidden words. We can form the product of all X_n, and this an effective system.

Satisfiability for the family $(\Box\pi_n^{-1}[0])_{n\in\mathbb{N}}$, where π_n is nth coordinate projection, is Π_1-complete, so the system is undecidable.

It is known that a subshift of finite type has always the shadowing property, and even the effective shadowing property; see [24] for a proof. We define ϵ and δ as in Definition 8.

The product of subshifts that have the shadowing property has also the shadowing property. Basically, balls of radius ϵ in the system may be expressed as product of balls of radius ϵ' in a finite number of subshifts. Then we choose the smallest of the corresponding δ' in the subshifts. The product of balls of radius δ' may be expressed as union of balls of radius δ; this is the δ corresponding to ϵ.

Hence the system is effective, has the shadowing property but not the effective shadowing property, since it is undecidable. □

We don't know whether this system is universal, i.e., if there is a Σ_1-family of formulae.

Note also that Turing machines that satisfy the shadowing property have been given a combinatorial characterization in [25]; in particular, the proof shows that the link between ϵ and δ (see Definition 8) is linear. Hence the effective shadowing property is not stronger than the shadowing property in the case of Turing machines.

7.4 Sofic Systems

Kůrka, in [26], describes several kinds of 'simple' systems. Among them, *sofic* (or *regular*) systems are those systems whose all induced subshifts are sofic. A sofic subshift is a subshift whose language is regular. Can a sofic system be universal?

We consider first an easier question. Let an effective system be *effectively sofic* if it is sofic and there is an algorithm that builds from a given partition the finite automaton recognizing the language induced by the partition.

Then it is not hard to derive the following theorem.

Proposition 9 *An effectively sofic system is decidable.*

Proof. Given a formula, choose a partition of the space that is finer than the clopen sets appearing in the formula. Then we build the corresponding finite automaton. In this automaton, we must check whether there is an infinite path satisfying the formula. This is very close to a classical problem of model-checking. We first build a Büchi automaton labelled by the clopen sets of the partition and accepting exactly sequences of clopen sets verifying the formula. Then we compose it with the automaton recognizing the sofic subshift. If the resulting automaton accepts an infinite sequence, then the formula is satisfiable. See [27] for more details on this construction. □

However, if a system is sofic but not effectively, then the argument of the proof fails. Indeed:

Proposition 10 *There is a symbolic system that is sofic and universal.*

Proof. Let X_n be the subshift of $2^{\mathbb{N}}$ whose forbidden words are those of the form 10^t1, where the universal Turing machine does not stop on data n in less than t steps. If the Turing machine does not halt, then X_n is the sofic subshift $\{0^*10^\infty, 0^\infty\}$. If the Turing machine halts in k steps, then X_n is the subshift of finite type with forbidden words 11, 101, 1001, ..., $10^{k-1}1$. So all subshifts are sofic, but we cannot effectively build the automaton recognizing the language, for it would allow to solve the halting problem.

Now consider the product of all X_n. This product is again an effective symbolic system X, and all its induced subshifts are sofic, due to the fact that sofic subshifts are closed under finite products and factors. Thus the system is sofic, but not effectively sofic.

Finally, X is universal for the family $(\pi_n^{-1}([1]) \wedge \circ \diamond \pi_n^{-1}([1]))_n$, where π_n is the projection from X to X_n. □

8 A Universal Chaotic System and the Edge of Chaos

According to Devaney [28], a system is *chaotic* if it is infinite, topologically transitive and has a dense set of periodic points. We can prove that such a system is sensitive [29].

It is not difficult to prove the existence of a universal subshift. Indeed, consider all the forbidden words of the kind 01^n00^t1, where the universal Turing machine launched on data n does not halt in less than t steps. Then the subshift of all binary sequences avoiding this set of words is effective and universal.

Improving this construction, one gets the following result:

Proposition 11 *There exists an effective system on the Cantor space that is chaotic and universal.*

Proof. Take the universal one-sided subshift X on alphabet $\{0, 1\}$ defined just above and consider the language L on $\{0, 1, \S\}$, made of the words $w_1\S w_2\S \ldots \S w_n$, for any $w_1, w_2, \ldots, w_n$ in the language of X. In particular, L includes the language of X and the word $\S$.

Consider X^{chaos} the one-sided subshift whose language is L. We now prove that X^{chaos} is indeed a universal chaotic system.

First note that X^{chaos} is a perfect subshift, so it is effectively conjugated to a system on the Cantor space. Then X^{chaos} has dense periodic points: indeed if w is a word of L then $(w\S)^\infty$ is in X^{chaos}. Finally X^{chaos} is topologically transitive: for any two finite words v, w of L, we can go from $[v]$ to $[w]$ with the sequence $v\S w\ldots$

According to Devaney's definition, X^{chaos} is thus chaotic.

Moreover, it is universal for the family $[01^n0] \wedge \circ^{n+1}(\neg[\S]\mathcal{U}[1])$. □

Note that the system built in the proof is also expansive, since it is a one-sided subshift.

The central idea of the 'edge of chaos' is that a system that has a complex behavior should be neither too simple nor chaotic. There are several ways to understand that.

Here we interpret 'complex system' by 'universal symbolic system'. Then 'too simple' could refer to the situation treated in Corollary 1: one or several attracting minimal subsystems. This includes of course the case of a globally attracting fixed point.

If we take 'chaotic' as meaning 'Devaney-chaotic', then computational universality need not be on the 'edge of chaos', since we have just provided a chaotic system that is universal.

However, many examples of chaotic systems (whatever the exact meaning given to 'chaotic', and for symbolic systems as well as for analog ones), although not all of them, have the shadowing property and even the effective shadowing property. For instance the shift and Smale's horseshoe (present in some physical systems), as well as all hyperbolic systems, satisfy the effective shadowing property with a linear relation between ϵ and δ (see Definition 8).

Thus we claim that the 'edge of chaos' could be replaced for general symbolic systems by the 'edge of effective shadowing property' — although this sounds less thrilling.

Note nevertheless the 'edge of chaos' has been intensively studied for cellular automata, and we don't know whether an example of chaotic universal cellular automaton exists.

9 Future Work

We formulated some open problems already. Is there a gap between decidability and universality? Is there a cellular automaton that is chaotic and universal? Must a universal system have infinitely many subsystems?

But many more questions are yet to solve. For instance, can we find a sufficient conditions of universality? What simplicity criteria proposed in [26] are sufficient conditions for decidability? Are the Game of Life and the automaton 110 universal? Can a linear cellular automaton be universal?

Let us conclude by looking at some more particular aspects in some depth.

9.1 Families of dynamical systems

Undecidability of the halting problem for Turing machines may be interpreted as the fact that we cannot decide, given a Turing machine and an initial data, whether the halting state will be reached.

However for some Turing machines the halting problem is easy to decide. Hence we would like to know for which subfamilies of Turing machines the halting problem is already undecidable. Proofs of the preceding section may be easily adapted and give us some insight about this.

For example, we can say:

Proposition 12 *The halting problem is decidable for any recursive family of Turing machines such that all of them are minimal.*

More generally, we can prove for instance the following proposition, again with the same arguments used in Section 7.

Proposition 13 *Given an effective family of equicontinuous symbolic dynamical systems, any property expressed by a family of temporal formulae is decidable.*

This can be put in contrast with the result in [30], where undecidability of properties for families of dynamical systems are studied, and sufficient conditions are given for a whole class of properties to be undecidable. Perhaps this way can be explored further.

9.2 Hierarchies

All formulae involved in examples of universal systems in the preceding sections were quite simple; in particular, they had no nested temporal operators, such as $\diamond\Box A$ for instance.

We can define Σ_n and Π_n formulae in a traditional way, where n is the number of alternance of 'until' operators and negations. In particular, Σ_1 formulae are those formulae where no 'until' is negated. We observe that all examples of universal formulae take place in Σ_1.

This hierarchy is to be put in parallel with the Borel hierarchy and the arithmetic hierarchy: Σ_1 formulae have interpretations that are Σ_1 Borel sets (i.e., open sets) and checking their satisfiability is a Σ_1 (i.e., recursively enumerable) problem.

We don't know much about the link between these hierarchies. In particular, we propose the following conjectures.

Conjecture 2 *If a symbolic system is universal for some family of temporal formulae, then it is universal for a family of Σ_1 formulae.*

Conjecture 3 *When there is a family of formulae whose satisfiability is Σ_1-complete, there is a family with a satisfiability problem in every level of the arithmetic hierarchy.*

9.3 Measure Theory

If we superimpose a 'reasonable' measure on symbolic systems then we can give a probabilistic definition, rather than non deterministic: instead of asking whether there is a trajectory from A to B, we would ask whether there is a positive probability to go from A to B. More generally, we check whether a formula has an interpretation of positive probability.

In some way, this probabilistic formulation is more physically relevant than the topological formulation we adopted in Section 5. Indeed, checking whether some temporal property is satisfiable might be practically impossible if the set of points verifying this property is non empty, but has probability zero.

However, if we restrict ourselves to Σ_1 formulae, as explained in the preceding subsection, then both definitions are equivalent. Indeed, a Σ_1 formula has an open interpretation; hence it is satisfiable if and only if it is satisfiable with positive probability.

The following conjecture, weaker than Conjecture 2, states that it is the case in general.

Conjecture 4 *A symbolic system is universal according to the topological definition if and only it is universal according to the measure-theoretic definition.*

This definition is an opportunity to investigate the link between universality and classical properties of measure-theoretic systems. For example, is it possible for an ergodic system to be universal? For a mixing system?

9.4 Analog Computation

It would be desirable to extend all definitions and results derived for symbolic systems to other kinds of systems, especially systems taking place in subsets of $\mathbb{R}^n$.

Let for instance $X = [0, 1]$ be the unit interval. We would like to make an effective space of it. For that purpose we endow it with a basis of open sets, indexed by the integers, stable under union and intersection, for which union, intersection and inclusion are computable. Then an effective system on X is a continuous transformation of X such that the inverse image of a basic open set is a computable basic open set.

Examples are given by the interval endowed with finite unions of open intervals of rational endpoints and any continuous piecewise affine map with rational coefficients.

However, the choice of a basis is at first glance arbitrary, and we could fear that the universality of a given map may depend on this choice. For example, is it possible for the logistic map $f : [0, 1] \to [0, 1] : x \mapsto 4x(1 - x)$, which is very similar to the shift, to be universal with respect to a clever choice of the basis? This would be similar to the phenomenon observed in Section 6.1.

It therefore remains to extend in a satisfactory way the definitions and results for systems in $\mathbb{R}^n$ in discrete time and even continuous time.

The resulting definition of universality could then be compared to existing definitions, for instance [4,5,9,11]. Then, results like those of Section 7 could hopefully be adapted. For instance, are minimal systems capable of universal computation? Such results could then be applied to physical systems. What systems that can be found in Nature are able to compute?

Hyperbolic dynamical systems are known to have the effective shadowing property. This would suggest that hyperbolic systems are not universal.

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