On dynamical continuum of Bolzano and Cauchy

PETR KŮRKA AND KATEŘINA TRLIFAJOVÁ

CENTER FOR THEORETICAL STUDY,
ACADEMY OF SCIENCES AND CHARLES UNIVERSITY IN PRAGUE,
JILSKÁ 1, CZ-11000 PRAHA 1, CZECHIA

We formalize a pre-Cantorian continuum on the base of Bolzano's theory of measurable numbers as exposed in his manuscript *Reine Zahlenlehre*. In doing so we use the insights of nonstandard analysis and computable analysis. We define dynamic real numbers as Bolzano-Cauchy sequences of rational numbers and real dynamic functions as limits of rational functions. The resulting structure contains infinitely small dynamic numbers which can be used in differential and integral calculus without the need for a nonconstructive free ultrafilter of nonstandard analysis nor for advanced concepts of mathematical logic.

03A05; 01A55

1 Introduction

One of the theorems of *Cours d'Analyse* of Cauchy (1825) [2] says that a limit of continuous functions is continuous. According to many historians of mathematics, Cauchy proved a wrong theorem, overlooking that the continuity is not sufficient and the uniform continuity is required in the assumptions. This reading of history is questioned and criticized by Lakatos(1980) [5] who points to several queer facts which are at odds with such an interpretation. In 1825, counterexamples to the theorem were well-known in the theory of Fourier's series. Moreover, even after the discovery of the concept of uniform convergence by Seidel in 1847, Cauchy presents his theorem in the same form with the same proof in 1853.

Lakatos asserts that Cauchy made no error in his proof. He just proved a different correct theorem because he worked with a different continuum concept. In contrast to the static Cantor-Weierstrassian continuum, the continuum of Cauchy was dynamic. The term variable (quantités variables) was not only a manner of speech but expressed the nature of real numbers. A real number might have been conceived as an approximation process (represented by a sequence of rational numbers) and not as a result of this process.

This conception has significant consequences for the convergence of functions. If a sequence of functions $\{f_n\}$ is applied to a sequence of rationals $\{x_n\}$, we have no limit $x = \lim_{n\to\infty} x_n$ to which f_n could be applied. Instead, we get a double sequence $\{f_n(x_m)\}$ and the result $f(x)$ should be some sequence $\{f_{n_k}(x_{m_k})\}_k$ which may depend on the subsequences $\{n_k\}$ and $\{m_k\}$. In a dynamical continuum, two equivalent approximating sequences with different speed of convergence would be regarded as different. Only when we conceive real numbers as equivalence classes of converging sequences of rational numbers, we get real numbers as static objects.

One more reason to rehabilitate the pre-Cantorian dynamic continuum comes from the computable analysis (see Pour-El and Richards(1989) [8]). A real number is computable if it is the limit of a computable convergent sequence of rational numbers with a computable modulus of convergence. One of the main insights of computable analysis is that the equivalence relation is undecidable for computable real numbers. There exists no algorithm which, given two computable convergent sequences and their computable moduli of convergence, would decide whether they converge to the same real number or not. Thus the standard concept of real number as an equivalence class of converging sequences of rational numbers is highly non-constructive since the equivalence relation is non-constructive. While this equivalence is quite useful in many contexts, we should treat it with caution.

Cauchy does not analyze the concept of real number (quantités) and relies rather on geometrical intuition. However, the dynamical nature of his continuum is sometimes hinted at:

> On dit qu'une quantité devient infiniment petite, lorsque sa valeur numérique décroit indéfiniment de manière à converger vers la limite zéro. Cauchy(1821) [2] p. 37.

An insight into the pre-Cantorian continuum can be obtained from the Bolzano's manuscript *Reine Zahlenlehre* (see Bolzano(1969-) [18] IIA8, partially translated in Russ(2004) [11] pp. 357 - 428), where he develops his theory of measurable numbers. As Lakatos(1980) [5] observes (referring to an earlier edition of *Reine Zahlenlehre* by Rychlík(1962) [12]),

> It is a most interesting historical fact that Bolzano, the best logical mind of the generation, made a real effort to clarify matters. He was possibly the only one to see the problems related to the difference between the two continuums: the rich Leibnizian continuum and, as he called it, its 'measurable' subset - the set of Weierstrassian real numbers. Bolzano makes it very clear that the field of 'measurable numbers' constitutes

> only an Archimedean subset of a continuum enriched by non-measurable - infinitely small or infinitely large - quantities. The editor makes a misguided attempt to reconstruct Bolzano's theory as a mere precursor of Cantor's theory of real numbers (cf. his dictionary of the two theories on p. 98); one wonders whether he has omitted some crucial passages from those parts of the manuscript which try to set up a consistent theory of the Leibnitz-Cauchy continuum[1]. No doubt, since Robinson has shed new light on the latter, historians will approach the Bolzano manuscript with new eyes and the relation between Bolzano's measurable and non-measurable quantities and Robinson's standard and non-standard numbers will be clarified.

As claimed by Robinson(1996) [9] (and qualified by Lakatos), the infinitesimal quantities of the pre-Cantorian continuum are captured in the non-standard analysis. Indeed nonstandard real numbers are sequences of real numbers which can be interpreted as approximating processes. The equality and all relations between nonstandard real numbers are defined in terms of a free ultrafilter. As a consequence, the embedding of real numbers into nonstandard real numbers satisfies the so called transfer principle. Every property which holds for the structure of the standard real numbers holds for the structure of the nonstandard real numbers as well (see e.g., Albeverio et al.(1986) [1] for a readable exposition). In particular, nonstandard real numbers are linearly ordered. However, the existence of a free ultrafilter cannot be proved constructively (axiom of choice must be used) and the logical formalism of nonstandard analysis was beyond the reach of pre-Cantorian mathematics.

Much closer to the spirit of dynamic continuum of Bolzano and Cauchy is the approach of Schmieden and Laugwitz (1958) [13] who work with the space of all rational sequences $\mathbb{Q}^{\mathbb{N}}$ called Ω-rational numbers. The subspace of real Ω-rational numbers is then defined by a condition which is equivalent to the Bolzano-Cauchy property. Thus any standard real number is an equivalence class of real Ω-rational numbers. Schmieden and Laugwitz then consider all functions defined on these real Ω-rational numbers and define continuity by a usual ε, δ-condition, where they admit as ε and δ also positive infinitesimals. This results in a much finer topology than the standard one, making the limit of functions $f_n(x) = x^n$ continuous from the left at $x = 1$.

In the present paper we propose a version of the pre-Cantorian dynamical continuum on the base of Bolzano's *Reine Zahlenlehre* using the insights of nonstandard analysis and computable analysis. Imitating Bolzano's measurable numbers, we define

[1] With the availability of the critical edition of the Bolzano's manuscripts in Bolzano(1969-) [18], it is now clear that the editor did omit some crucial passages.

a **dynamic real number** as a Bolzano-Cauchy (BC) sequence of rational numbers. This is equivalent to the concept of real Ω-number of Schmieden and Laugwitz. Two dynamic numbers are equivalent if their difference converges to zero. The standard real numbers are obtained by factorizing dynamic numbers by this equivalence. However, we refrain from this factorization, so each standard real number is represented by many dynamic numbers. None of these representations is distinguished, so dynamic numbers do not contain standard real numbers as a subset. For the definition of inequality we use the filter of cofinite sets instead of a free ultrafilter of nonstandard analysis. As a consequence, our continuum is not linearly ordered.

In the theory of functions we depart from both Bolzano's *Functionenlehre* [18] IIA10 and from Schmieden and Laugwitz (1958) [13] and conceive a **dynamic real function** as a sequence (or a limit) of rational functions. This is analogous to nonstandard analysis, where a nonstandard function is defined as a sequence of real functions. Dynamic real functions need not be defined everywhere since the resulting sequences need not be BC. This is reminiscent of computable analysis, where a real computable function is undefined when the algorithm which computes it fails to terminate.

We say that a function f is full at some dynamic number x if it is defined for all y which are infinitely close to x. In Theorem 10 we prove that each full function is continuous. This may be regarded as a rather strong version of the Cauchy continuity theorem: The limit f of a sequence f_n of (rational) functions is continuous whenever the sequence $f_n(x_n)$ converges for every dynamic number x. We need not even assume that f_n are continuous. The class of dynamic functions is large enough to include most of the functions considered in analysis, in particular all continuous functions and all functions with a finite number of discontinuities. We show that the differential and integral calculus with infinitesimals is feasible in this setting. The derivative of a function at a point can be defined only from its values at infinitely close points and the differential calculus can be based on the calculus with the infinitesimals. We show that a function which is continuous on a compact interval has an indefinite integral and prove the fundamental theorem of calculus (Theorem 21). Finally we develop the theory of power series so that we obtain all analytic functions with their usual properties.

2 Bolzano's infinite number concepts

Let us briefly recall the main ideas of Bolzano's theory of measurable numbers as presented in his manuscript *Reine Zahlenlehre* (see Bolzano(1969-) [18] IIA8, partially translated in Russ(2004) [11] pp. 357 - 428.) We follow closely the exposition (in Czech) of Trlifajová(2006) [15]. Bolzano treats rational numbers as number concepts in which there is a finite multitude of arithmetic operations of addition, subtraction, multiplication and division. In **infinite number concepts**, an infinite multitude of arithmetic operations occur. An expression representing such a concept is an **infinite number expression**. Bolzano's examples of infinite number expressions are as follows:

$$1 + 2 + 3 + 4 + \cdots \textit{ in inf}.$$
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots \textit{ in inf}.$$
$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{16}\right) \cdots \textit{ in inf}.$$

An infinite number concept S is called a **measurable number** (§ 5) if for every positive integer q there are an integer p and two positive number expressions P and P^1, the former possibly being zero, such that the following equations are satisfied:

$$S = \frac{p}{q} + P \quad \text{and} \quad S = \frac{p+1}{q} - P^1.$$

A number expression is positive if it contains only positive numbers and no subtraction (see Bolzano(1969-) [18] IIA8, page 96). Infinitely small numbers are a special sort of measurable numbers. A positive number expression S is **infinitely small** (§ 22) if for any positive integer q there are positive number concepts P^1 and P^2 such that $S = P^1 = \frac{1}{q} - P^2$. An example is $S = \frac{1}{1+1+1+\ldots \textit{ in inf}.}$ A number expression S is **infinitely large** (§ 27) if for any positive q there is an integer p such that one of the equations $S = \frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$ is satisfied, but there is no p which satisfies them both at once.

Bolzano proves many theorems about number concepts, for example that the sum and the product of two measurable numbers is a measurable number. The theory is not entirely correct since an oscillating number expression $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots \textit{ in inf}.$ is not measurable in this sense but can be obtained as a sum of two measurable number concepts. As observed by Laugwitz(1962-1966) [6], the theory can be saved if we replace the condition of measurability by $S = \frac{p-1}{q} + P^1 = \frac{p+1}{q} - P^2$. In fact Bolzano seems to have been aware of the problem since the last note of *Reine Zahlenlehre* suggests this modification as well:

> Perhaps the theory of measurable numbers could be simplified if we formulated the definition of them so that A is called measurable if we have two equations of the form $A = \frac{p}{q} + P = \frac{p+n}{q} - P$, where for the identical n, q can be increased indefinitely (Russ(2004) [11] § 122, p. 428).

Obviously, the formula was intended in the form $A = \frac{p}{q} + P^1 = \frac{p+n}{q} - P^2$ with positive number expressions P^1, P^2. The omission of indexes could not be intentional as it makes no sense in the whole context. With this modification of the definition of measurability suggested by Laugwitz or by Bolzano himself, the measuring process is no more unique but the oscillating number concepts become measurable as well and measurable numbers become closed with respect to addition, subtraction and multiplication (see Sebestik(1992) [14] pp. 375-387 for a thorough discussion of this issue).

Starting in § 53 Bolzano discusses the concepts of order between number expressions and their equality or equivalence. Bolzano's first definition of equality was not correct but he adds a note where he revised it.

> If the pair of numbers A and B have a difference $A-B$ which, considered absolutely, has the same characteristics as zero itself in the process of measuring (i.e., it behaves like zero) in that for every denominator q, however large, the numerator of the measuring fraction is found to be $= 0$, and so it has only two equations $A - B = \frac{0}{q} + P^1 = \frac{1}{q} - P^2$, then we say that $A = B$. But if the difference has the characteristics of a number different from zero, and its true value is positive, then $A > B$, if it is negative , then $A < B$ (Russ(2004) [11] p. 391).

In § 107 Bolzano formulates the BC-condition and proves the theorem that every BC-sequence of measurable numbers has a limit that is a measurable number. Although his proof lacks some final demonstration, it is almost entirely correct (see Rusnock(2000) [10], pp. 186 - 188). If we wish to interpret Bolzano's theory in terms of contemporary mathematics, the easiest way is to interpret infinite number expression as sequences of rational numbers (see Rychlík(1962) [12]). Every number expression S can be described as a sequence of partial results of arithmetic operations $\{s_n\}$, and vice versa, every sequence $\{s_n\}$ can be described as the first term plus the infinite sum of differences between immediately following terms $s_1 + \sum(s_{n+1} - s_n)$. For instance the number expression $1+2+3+4+\ldots$ *in inf* corresponds to the sequence $\{\frac{1}{2}n(n+1)\} = \{1, 3, 6, 10, \ldots\}$ and $\frac{1}{1+1+1+\ldots \textit{ in inf}}$ corresponds to $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Then it is easy to see that a sequence $\{s_n\} \in \mathbb{Q}^N$ of partial results represents a measurable number if and only if it is a BC-sequence.

At first sight, in § 53 Bolzano abandons infinitesimals and treats equivalent number expressions as equal, which would be equivalent to the modern theory of real numbers. Nevertheless Bolzano inserts the following note where he clarifies the issue.

> Since the infinitely small numbers are not equivalent to zero in every respect, but only in respect of their process of measuring, it might be expedient, if we call such numbers zero, to call them a *relative* zero or *respective* zero [*relative oder beziehungsweise Null*]. In contrast to that, the concept of zero, which we already met in § 116, EG III might be called the absolute zero. (Russ(2004) [11] § 58, p. 395)

Even in later paragraphs of *Reine Zahlenlehre* Bolzano keeps speaking about infinitely small numbers. This would make no sense if his equivalence (denoted by $=$) were meant as equality or identity. Thus Bolzano's continuum as exposed in *Reine Zahlenlehre* is the rich continuum with infinitely small and infinitely large quantities and is not equivalent to the factorized Cantorian continuum. Nevertheless, in his later manuscript *Functionenlehre* (Bolzano(1969-) [18] IIA10), where continuity and differentiability issues are thoroughly discussed, infinitesimals are not used. In fact, they are never even mentioned. Only in his *Paradoxien der Unendlichen* (Bolzano(1969-) [18] IIA11, § 30) we find a short exposition how infinitesimals can be used in differential calculus.

3 Dynamic real numbers

Denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of nonnegative integers, by $\mathbb{Z}$ the set of integers, by $\mathbb{Q}$ the set of rational numbers, and by $\mathbb{Q}^+ = \{x \in \mathbb{Q} : \ x > 0\}$ the set of positive rational numbers. We say that a property of nonnegative integers holds **almost everywhere (a.e.)** if it holds for all but a finite number of integers, i.e., if the set of integers for which it holds belongs to the filter of cofinite sets.

Definition 1 *A* **dynamic (real) number** *is a BC-sequence of rational numbers, i.e., a mapping* $x : \mathbb{N} \to \mathbb{Q}$*, such that* $\forall \varepsilon \in \mathbb{Q}^+, \exists n \in \mathbb{N}, \forall i, j \geq n, |x_i - x_j| < \varepsilon$*. We denote by* $\mathbb{Q}^*$ *the set of dynamic numbers.*

The speed of convergence of a dynamic number x is given by its **modulus of convergence** $m_x : \mathbb{Q}^+ \to \mathbb{N}$ defined by $m_x(\varepsilon) = \min\{n \in \mathbb{N} : \ \forall i, j \geq n, |x_i - x_j| < \varepsilon\}$. Rational numbers are represented in $\mathbb{Q}^*$ as constant sequences, so we have an embedding $\mathbb{Q} \subset \mathbb{Q}^*$. We have some relations on $\mathbb{Q}^*$ defined by

$$\begin{aligned} x = y \quad &\Leftrightarrow \quad x_i = y_i \text{ a.e.} \\ x < y \quad &\Leftrightarrow \quad x_i < y_i \text{ a.e.} \\ x \leq y \quad &\Leftrightarrow \quad x_i \leq y_i \text{ a.e.} \end{aligned}$$

The relations $<, \leq$ are transitive but not linear. For the dynamic number $x = \{(-1)^n/n\}$ we have neither $x \leq 0$ nor $0 \leq x$. Define the equivalence relation $\approx$ by

$$x \approx y \;\Leftrightarrow\; \lim_{i\to\infty}(x_i - y_i) = 0$$

Thus $x \approx y$ if for every $\varepsilon \in \mathbb{Q}^+$ there exists $n \in \mathbb{N}$ such that for every $i \geq n$ we have $|x_i - y_i| < \varepsilon$. The relations $<, \approx, >$ are not exclusive. For $x = \{1/n\}$ we have both $0 \approx x$ and $0 < x$. We say that $x \in \mathbb{Q}^*$ is an **infinitesimal** number if $x \approx 0$. Arithmetical operations on $\mathbb{Q}^*$ are defined pointwise, in particular $(x+y)_n = x_n + y_n$, $(x-y)_n = x_n - y_n$, $(x \cdot y)_n = x_n \cdot y_n$. For $x > 0$ define the k-th root of x by $(\sqrt[k]{x})_n = max\{z \in \mathbb{N} : \; z^k < x_n \cdot n^k\}/n$. Denote by $\lfloor x \rceil \in \mathbb{Z}$ the closest integer to $x \in \mathbb{Z}$ which satisfies $\lfloor x \rceil - \frac{1}{2} \leq x < \lfloor x \rceil + \frac{1}{2}$. For $x \in \mathbb{Q}^*$ set $\lfloor x \rceil = \lfloor x_{m_x(1/2)} \rceil$.

Theorem 2 *If $x, y \in \mathbb{Q}^*$, then $x+y, x-y, x \cdot y \in \mathbb{Q}^*$, so $(\mathbb{Q}^*, 0, 1, +, -, \cdot)$ is a ring. If $x \not\approx 0$, then $1/x \in \mathbb{Q}^*$. If $x > 0$ then $(\sqrt[k]{x})^k \approx x$.*

Proposition 3

1. *For $x, y \in \mathbb{Q}^*$ we have $x \approx y$ iff $|x - y| < \varepsilon$ for each $\varepsilon \in \mathbb{Q}^+$.*
2. *If $x \in \mathbb{Q}^*$, then for each $\varepsilon \in \mathbb{Q}^+$ and $n \geq m_x(\varepsilon)$ we have $|x_n - x| < \varepsilon$.*
3. *If $x \not\approx y$ then either $x < y$ or $y < x$.*
4. *If $x \in \mathbb{Q}^*$ then $\lfloor x \rceil - 1 < x < \lfloor x \rceil + 1$.*

Proof 1. This is an immediate consequence of the definition.
2. For each n, $x_n = x_{n,p}$ is regarded here as a constant sequence, so for each $p \geq m_x(\varepsilon)$ we have $|x_{n,p} - x_p| = |x_n - x_p| < \varepsilon$ and therefore $|x_n - x| < \varepsilon$.
3. If $x \not\approx y$, then there exists $\varepsilon \in \mathbb{Q}^+$ such that for infinitely many n we have either $x_n + \varepsilon \leq y_n$ or $y_n + \varepsilon \leq x_n$. In the former case for each $p \geq \max\{m_x(\varepsilon/2), m_y(\varepsilon/2)\}$ there exists n such that $x_p < x_n + \frac{\varepsilon}{2} \leq y_n - \frac{\varepsilon}{2} < y_p$, so $x < y$.
4. For $n = m_x(1/2)$ we have $\lfloor x \rceil - \frac{1}{2} \leq x_n < \lfloor x \rceil + \frac{1}{2}$ and $x_n - \frac{1}{2} < x < x_n + \frac{1}{2}$. $\square$

A **sequence of dynamic numbers** is a map $x : \mathbb{N} \to \mathbb{Q}^*$, or a double sequence $x : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, $\{x_i\} = \{(x_i)_j\} = \{x_{ij}\}$. The convergent and BC sequences $\{x_n\}$ of dynamic numbers are defined in a standard way, and the equivalence of these concepts is easily proved:

$$\begin{array}{lll} \{x_n\} \text{ is convergent} & \text{iff} & \exists z \in \mathbb{Q}^*, \forall \varepsilon \in \mathbb{Q}^+, \exists n, \forall p \geq n, |x_p - z| < \varepsilon \\ \{x_n\} \text{ is BC} & \text{iff} & \forall \varepsilon \in \mathbb{Q}^+, \exists n, \forall p, q \geq n, |x_p - x_q| < \varepsilon \end{array}$$

Theorem 4 *A sequence $\{x_n\}$ of dynamic numbers is convergent iff it is a BC-sequence.*

Proof Let $x_n \in \mathbb{Q}^*$ be a BC sequence and denote by $x_{n,i} \in \mathbb{Q}$ the i-th member of x_n. For a given n there exists $k_n > n$ such that for each $p, q \geq k_n$ we have $|x_p - x_q| < 1/n$. For each n there exists $j_n = m_{x_n}(n)$ such that by Proposition 3(2), for each $i \geq j_n$ we have $|x_{n,i} - x_n| < 1/n$. Set $y_n = x_{n,j_n}$. For $p, q \geq k_n$ we have $|y_p - y_q| \leq |x_{p,j_p} - x_p| + |x_p - x_q| + |x_q - x_{q,j_q}| < \frac{1}{p} + \frac{1}{n} + \frac{1}{q} \leq \frac{3}{n}$, so $y \in \mathbb{Q}^*$. For $p, q \geq k_n$, $q \geq j_p$ we have $|x_{p,q} - y_q| \leq |x_{p,q} - x_p| + |x_p - x_q| + |x_q - x_{q,j_q}| < \frac{1}{p} + \frac{1}{n} + \frac{1}{q} \leq \frac{3}{n}$. so for each $p \geq k_n$ we have $|x_p - y| < 3/n$ and therefore $\lim_{n\to\infty} x_n = y$ and $\{x_n\}$ is convergent. The converse implication is straightforward. □

Proposition 5 *For a given a sequence $\{x_n\}$ of dynamic numbers, there exists $a = \liminf_{n\to\infty} x_n$, $b = \limsup_{n\to\infty} x_n$, with $a, b \in \mathbb{Q}^* \cup \{-\infty, +\infty\}$, such that for each $\varepsilon \in \mathbb{Q}^+$ we have*

$$\{n \in \mathbb{N} : x_n < a - \varepsilon\} \text{ is finite}, \quad \{n \in \mathbb{N} : x_n < a + \varepsilon\} \text{ is infinite},$$
$$\{n \in \mathbb{N} : x_n > b + \varepsilon\} \text{ is finite}, \quad \{n \in \mathbb{N} : x_n > b - \varepsilon\} \text{ is infinite}.$$

Proof If for all $a \in \mathbb{Z}$ the set $\{n \in \mathbb{N} : x_n > a\}$ is finite, then $\limsup_{n\to\infty} x_n = -\infty$. If for all $b \in \mathbb{Z}$ the set $\{n \in \mathbb{N} : x_n > b\}$ is infinite, then $\limsup_{n\to\infty} x_n = +\infty$. Otherwise let a_0 be the maximum of all $a \in \mathbb{Z}$ such that the set $\{n \in \mathbb{N} : x_n > a\}$ is infinite and let b_0 be the minimum of all $b \in \mathbb{Z}$ such that the set $\{n \in \mathbb{N} : x_n > b\}$ is finite. If a_i and b_i have been defined then set $d_i = (a_i + b_i)/2$ and

$$\begin{array}{lll} a_{i+1} = a_i, & b_{i+1} = d_i & \text{if } \{n \in \mathbb{N} : x_n > d_i\} \text{ is finite} \\ a_{i+1} = d_i, & b_{i+1} = b_i & \text{if } \{n \in \mathbb{N} : x_n > d_i\} \text{ is infinite} \end{array}$$

Then $a \approx b$ are BC sequences and we set $\limsup_{n\to\infty} x_n = b$. Limes inferior is defined analogously. □

Theorem 6 (Bolzano-Weierstrass) *Every bounded sequence of dynamic numbers has a convergent subsequence.*

Proof Assume that $a_0 \leq x_n \leq b_0$ for all n, where $a_0, b_0 \in \mathbb{Q}$ and set $n_0 = 0$. Assume that a_i, b_i, n_i have been defined and that the set $\{n \geq n_i : a_i \leq x_n \leq b_i\}$ is infinite. Set $d_i = (a_i + b_i)/2$ and

$$\begin{array}{ll} a_{i+1} = a_i,\ b_{i+1} = d_i & \text{if } \{n > n_i : a_i \leq x_n \leq d_i\} \text{ is infinite}, \\ a_{i+1} = d_i,\ b_{i+1} = b_i & \text{otherwise}. \\ n_{i+1} = \min\{n > n_i : a_{i+1} \leq x_n \leq b_{i+1}\}. & \end{array}$$

Then $a \approx b \in \mathbb{Q}^*$ and $\lim_{i\to\infty} x_{n_i} = a$. □

4 Dynamic real functions

For dynamic numbers $a < b$ define the open and closed intervals by

$$\begin{aligned}(a,b) &= \{x \in \mathbb{Q}^* : \exists \varepsilon \in \mathbb{Q}^+ : a + \varepsilon < x < b - \varepsilon\} \\ [a,b] &= \{x \in \mathbb{Q}^* : \forall \varepsilon \in \mathbb{Q}^+ : a - \varepsilon < x < b + \varepsilon\}\end{aligned}$$

Semi-open and infinite intervals $[a,b)$, (a,∞) are defined analogously. Note that the interval $(0,1)$ does not contain any infinitesimal number while the closed interval $[0,1]$ contains all infinitesimals. We now define dynamic functions as sequences (or limits) of rational functions.

Definition 7 *A* **dynamic (real) function** *is a partial mapping $f : \mathcal{D}(f) \to \mathbb{Q}^*$ whose domain $\mathcal{D}(f) \subseteq \mathbb{Q}^*$ is an interval, such that there exists a sequence of rational functions $f_n : \mathcal{D}(f) \cap \mathbb{Q} \to \mathbb{Q}$ and for $x \in \mathcal{D}(f)$ we have $f(x) = \{f_n(x_n)\}$ provided $\{f_n(x_n)\} \in \mathbb{Q}^*$ and $f(x)$ is undefined otherwise. We then say that f is the limit of f_n, $f = \lim_{n\to\infty} f_n$. Two dynamic functions are equivalent ($f \approx g$), if for every $x \in \mathbb{Q}^*$, $f(x)$ is defined iff $g(x)$ is defined and in this case $f(x) \approx g(x)$.*

The simplest case is a constant sequence of rational functions $f : \mathcal{D}(f) \cap \mathbb{Q} \to \mathbb{Q}$. Then f is extended to $\mathcal{D}(f)$ by $f(x)_n = f(x_n)$ if $\{f(x_n)\} \in \mathbb{Q}^*$ and $f(x)$ is undefined otherwise. In the following examples we show that whenever a dynamic function is discontinuous, it is not defined at some dynamic numbers.

Example 1 *The signum function* $\operatorname{sgn} : \mathbb{Q} \to \mathbb{Q}$ *is defined by* $\operatorname{sgn}(x) = -1, 0, 1$ *when* $x < 0, x = 0, x > 0$ *accordingly.*

Thus $\operatorname{sgn}(x) = 1$ for each positive infinitesimal but $\operatorname{sgn}(x)$ is undefined for oscillating infinitesimals like $x = \{(-1)^n/n\}$.

Example 2 *Define* $f = \lim_{n\to\infty} f_n$ *where* $f_n : \mathbb{Q} \to \mathbb{Q}$ *are given by* $f_n(x) = 1/(1 + n^2x^2)$.

Note that in the standard analysis this is an example of pointwise convergence which is not uniform. We have $f(0) = 1$ and $f(x) \approx 0$ whenever $x \in \mathbb{Q}^*$ is not infinitesimal. For $x = \{1/n^2\}$ we have $f(x) \approx 1$, for $x = \{1/n\}$ we get $f(x) = 1/2$ and for $x = \{\lfloor\sqrt{n}\rceil\}$ we get $f(x) \approx 0$. There exist many infinitesimals for which $f(x)$ is undefined, for example if $x_n = 1/n$ for n odd and $x_n = 1/n^2$ for n even.

Definition 8 *We say that a dynamic function $f : \mathcal{D}(f) \to \mathbb{Q}^*$ is* **full** *at $x \in \mathcal{D}(f)$, if it is defined for every $y \in \mathcal{D}(f)$ with $y \approx x$. We say that f is a* **full function**, *if it is full at every $x \in \mathcal{D}(f)$.*

Proposition 9 *If $f : \mathbb{Q} \to \mathbb{Q}$ is continuous then its extension $f : \mathbb{Q}^* \to \mathbb{Q}^*$ is a full function.*

The proof is straightforward. Thus every polynomial with rational coefficients is a full function. We now prove a version of Cauchy theorem saying that if a sequence of (rational) functions converges at every dynamical number, then the limiting function is continuous.

Theorem 10 *If a dynamic function f is full at $x \in \mathcal{D}(f)$, then it is continuous at x, i.e., for every $\varepsilon \in \mathbb{Q}^+$ there exists $\delta \in \mathbb{Q}^+$, such that for all $y \in \mathcal{D}(f)$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$ provided $f(y)$ is defined.*

Proof Assume by contradiction that a function $f = \lim_{n\to\infty} f_n$ is full but not continuous at $x \in \mathcal{D}(f)$. Thus there exists $\varepsilon \in \mathbb{Q}^+$ such that for each $n \in \mathbb{N}^+$ there exists $y_n \in \mathcal{D}(f)$ such that $|y_n - x| < 1/n$, f is defined at y_n and $|f(y_n) - f(x)| \not< \varepsilon$. It follows that there exists r_n with $|y_{nr_n} - x_{r_n}| < 1/n$ and $|f_{r_n}(y_{nr_n}) - f_{r_n}(x_{r_n})| \geq \varepsilon$. We can assume that $\{r_n\}$ is an increasing sequence. Define $z \in \mathbb{Q}^{\mathbb{N}}$ by

$$z_k = \begin{cases} x_{r_n} & \text{for} \quad r_n \leq k < r_{n+1},\ n \text{ even} \\ y_{nr_n} & \text{for} \quad r_n \leq k < r_{n+1},\ n \text{ odd} \end{cases}$$

Then $z \in \mathbb{Q}^*$, $z \approx x$ and we show that $f(z)$ is undefined. Since $f(x) \in \mathbb{Q}^*$, there exists n_0 such that for every $n \geq n_0$ we have $|f_{r_{n+1}}(x_{r_{n+1}}) - f_{r_n}(x_{r_n})| < \varepsilon/2$. Then for $n \geq n_0$ even we get

$$\begin{aligned} |f_{r_{n+1}}(z_{r_{n+1}}) - f_{r_n}(z_{r_n})| \quad &\geq \quad |f_{r_{n+1}}(y_{n+1,r_{n+1}}) - f_{r_{n+1}}(x_{r_{n+1}})| - \\ & \quad\ |f_{r_{n+1}}(x_{r_{n+1}}) - f_{r_n}(x_{r_n})| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \end{aligned}$$

so $f(z)$ is not a BC-sequence, which is a contradiction. □

As an immediate consequence of Theorem 10 we get

Proposition 11 *If a dynamic function f is full at $x \in \mathcal{D}(f)$, then for all $y \in \mathcal{D}(f)$ with $x \approx y$ we have $f(x) \approx f(y)$.*

Thus we have an analogous principle as the nonstandard analysis where a function is continuous at x if $f(x+\varepsilon)-f(x)$ is infinitesimal for each infinitesimal ε. To clarify the relation of dynamic functions to standard real functions we show that any standard continuous function can be represented by a dynamic function. As shown by Examples 1 and 2, many noncontinuous real functions can be represented by dynamic functions as well, but these real functions must be left undefined at the points of discontinuity.

Proposition 12 *Define the set* $\mathbb{R} = \mathbb{Q}^*/\approx$ *of standard real numbers as the set of equivalence classes of dynamic numbers. Then for any real function* $g : \mathbb{R} \to \mathbb{R}$ *there exists a dynamic function* $f : \mathbb{Q}^* \to \mathbb{Q}^*$ *such that for any* $x \in \mathbb{Q}^*$*, if* g *is continuous at* $[x]$*, then* f *is full at* x *and in this case* $[f(x)] = g([x])$*. Here* $[x] \in \mathbb{R}$ *is the equivalence class of* x*.*

Proof Define $f_n : \mathbb{Q} \to \mathbb{Q}$ by $f_n(x) = \lfloor n \cdot g(x) \rceil / n$, where $\lfloor y \rceil \in \mathbb{Z}$ is the closest integer to $y \in \mathbb{R}$. Then set $f = \lim_{n\to\infty} f_n$. □

5 Full functions

In classical analysis, continuous functions on a compact interval are bounded, uniformly continuous, attain their extrema, satisfy the intermediate value theorem and are Riemann integrable. We get analogous properties in our setting.

Definition 13 *A full function* $f : [a,b] \to \mathbb{Q}^*$ *is* **uniformly continuous** *if*

$$\forall \varepsilon \in \mathbb{Q}^+, \exists \delta \in \mathbb{Q}^+, \forall x, y \in [a,b], (|x-y| < \delta \ \Rightarrow \ |f(x) - f(y)| < \varepsilon).$$

A full function $f : [a,b] \to \mathbb{Q}^*$ *is* **sequentially uniformly continuous** *if*

$$\forall \varepsilon \in \mathbb{Q}^+, \exists \delta \in \mathbb{Q}^+, \exists p \in \mathbb{N}, \forall n \geq p, \forall x, y \in [a,b], (|x-y| < \delta \ \Rightarrow \ |f_n(x)-f_n(y)| < \varepsilon).$$

A full function $f : [a,b] \to \mathbb{Q}^*$ *has a* **uniform modulus of convergence** $m_f : \mathbb{Q}^+ \to \mathbb{N}$ *if*

$$\forall \varepsilon > 0, \forall n, m \geq m_f(\varepsilon), \forall x \in \mathcal{D}(f) \cap \mathbb{Q}, |f_n(x) - f_m(x)| < \varepsilon.$$

Proposition 14 *Every full function* $f : [a,b] \to \mathbb{Q}^*$ *is uniformly continuous.*

Proof If not, there exists $\varepsilon \in \mathbb{Q}^+$ such that for every n there exist $x_n, y_n \in [a,b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| > \varepsilon$. By the Bolzano-Weierstrass theorem 6 there exist converging subsequences $x_{n_i} \to z$, $y_{n_i} \to z$ and f is not continuous at z which is a contradiction. □

Theorem 15 *If $f : [a,b] \to \mathbb{Q}^*$ is a full function then there exists an equivalent full function $g : [a,b] \to \mathbb{Q}^*$ which is sequentially uniformly continuous and has a uniform modulus of convergence.*

Proof Let $f = \lim_{n\to\infty} f_n$ where $f_n : [a_n, b_n] \cap \mathbb{Q} \to \mathbb{Q}$. Set $j_n = \lfloor na_n \rfloor$, $k_n = \lceil nb_n \rceil$. Define $g_n : [\frac{j_n}{n}, \frac{k_n}{n}] \to \mathbb{Q}$ as an approximation on each i/n and linearly between i/n and $(i+1)/n$.

$$g_n\left(\frac{i}{n}\right) = \begin{cases} \lfloor nf(a) \rceil / n & \text{for} \quad i = j_n \\ \lfloor nf(\frac{i}{n}) \rceil / n & \text{for} \quad j_n < i < k_n \\ \lfloor nf(b) \rceil / n & \text{for} \quad i = k_n \end{cases}$$

$$g_n(x) = g_n\left(\frac{i}{n}\right) + (nx - i)\left(g_n\left(\frac{i+1}{n}\right) - g_n\left(\frac{i}{n}\right)\right) \quad \text{for} \quad \frac{i}{n} \le x \le \frac{i+1}{n}.$$

For a given $\varepsilon \in \mathbb{Q}^+$ take $\delta \in \mathbb{Q}^+$ from the uniform continuity of f. Take p such that $\frac{1}{p} < \delta$ and assume $n \ge p$. Given $x \in [a,b]$, there exists i such that $x - \delta < i/n \le x < (i+1)/n < x + \delta$. We have $|f(a) - g_n(j_n/n)| < 1/n$, $|f(b) - g_n(k_n/n)| < 1/n$. For $l = i, i+1$ we have $|f(l/n) - g_n(l/n)| < 1/n$ and $|f(x) - f(l/n)| < \varepsilon$, so $|f(x) - g_n(l/n)| < \varepsilon + 1/n$. Since g_n is linear between i/n and $(i+1)/n$, we have also $|f(x) - g_n(x)| < \varepsilon + 1/n$ for all $x \in [\frac{i}{n}, \frac{i+1}{n}]$. Thus we have proved $\forall \varepsilon \in \mathbb{Q}^+, \exists p, \forall n \ge p, \forall x \in [a_n, g_n], |f(x) - g_n(x)| < \varepsilon + \frac{1}{n}$, so $f \approx g$. If $|x - y| < \delta$, then for all $n \ge p$ we have

$$|g_n(x) - g_n(y)| \le |g_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - g_n(y)| \le 3\varepsilon + \frac{2}{n}$$

so g is sequentially uniformly continuous. For $n, m \ge p$ we get

$$|g_n(x) - g_m(x)| \le |g_n(x) - f(x)| + |f(x) - g_m(x)| \le 2\varepsilon + \frac{1}{n} + \frac{1}{m}$$

so g has a uniform modulus of convergence. □

Theorem 16 (Intermediate value theorem) *If $f : [x,y] \to \mathbb{Q}^*$ is a full function and $f(x) < 0 < f(y)$, then there exists $z \in [x,y]$ such that $f(z) \approx 0$.*

Proof Define sequences $a_i, b_i \in \mathbb{Q}^*$ by $a_0 = x$, $b_0 = y$, and if a_n and b_n have been defined then set $c_n = (a_n + b_n)/2$, and

$$\begin{array}{lll} a_{n+1} = c_n, & b_{n+1} = c_n & \text{if} \quad f(c_n) \approx 0 \\ a_{n+1} = a_n & b_{n+1} = c_n & \text{if} \quad f(c_n) > 0 \\ a_{n+1} = c_n, & b_{n+1} = b_n & \text{if} \quad f(c_n) < 0 \end{array}$$

If $f(c_n) \approx 0$ for some n, then $z = c_n$ is the required solution. If not, then both a and b are BC sequences of dynamic numbers, so they have a limit $z = \lim_{n\to\infty} a_n \approx$

$\lim_{n\to\infty} b_n$. If $f(z) > 0$ then $f(a_n) > 0$ for all sufficiently large n which is a contradiction. If $f(z) < 0$ then $f(b_n) > 0$ for all sufficiently large n which is a contradiction, so $f(z) \approx 0$. □

6 Derivatives

Infinitesimals can be used in differential calculus. For $x, y \in \mathbb{Q}^*$ define $x \ll y$ (x is **infinitely smaller** than y), if $|x| < |y| \cdot \varepsilon$ for every $\varepsilon \in \mathbb{Q}^+$. Thus $x \in \mathbb{Q}^*$ is infinitesimal iff $x \ll 1$. There are many obvious laws for the relation $\ll$. For example, if $x \ll z$ and $y \ll z$ then $x + y \ll z$ and $x \cdot y \ll z$.

Definition 17 *We say that a dynamic function f has a* **derivative** *$f'(x) \in \mathbb{Q}^*$ at $x \in \mathbb{Q}^*$, if f is full at x and for each infinitesimal ε we have $f(x+\varepsilon) - f(x) - \varepsilon \cdot f'(x) \ll \varepsilon$.*

The derivative is not unique, but any two derivatives of f are equivalent: From the inequalities $f(x+\varepsilon) - f(x) - \varepsilon \cdot f_0'(x) \ll \varepsilon$ and $f(x+\varepsilon) - f(x) - \varepsilon \cdot f_1'(x) \ll \varepsilon$, we get $\varepsilon \cdot |f_0'(x) - f_1'(x)| \ll \varepsilon$, so $f_0'(x) \approx f_1'(x)$.

Example 3 *Set $f_n(x) = (-1)^n \cdot x^2$.*

Then f is a dynamic function which is full at 0 and undefined everywhere else. At each infinitesimal x, f has zero derivative. Thus a function need not be defined in a neighborhood of a point to have a derivative. Nevertheless, if it is defined in a neighborhood of a point, then it satisfies the standard limit condition. This is proved in the next proposition which is an analogue of Theorem 10.

Proposition 18 *Suppose that f is a dynamic function with derivative $f'(x)$ at x. Then for any $\eta \in \mathbb{Q}^+$ there exists a $\delta \in \mathbb{Q}^+$ such that for all $y \in \mathbb{Q}^*$, such that $|y - x| < \delta$ we have $|f(y) - f(x) - f'(x)(y-x)| < \eta|y-x|$), provided $f(y)$ is defined..*

Proof Assume by contradiction that there exists $\eta \in \mathbb{Q}^+$ such that for $\delta = 1/n$ there exists $y_n \in \mathbb{Q}^*$ such that $f(y_n)$ is defined, $|y_n - x| < 1/n$, and $|f(y_n) - f(x) - f'(x)(y_n - x)| \not< \eta|y_n - x|$. Thus there exists r_n such that $|y_{nr_n} - x_{r_n}| < 1/n$, $|f_{r_n}(y_{nr_n}) - f_{r_n}(x_{r_n}) - f'(x)_{r_n}(y_{nr_n} - x_{r_n})| \geq \eta|y_{nr_n} - x_{r_n}|$. We can assume that r_n is an increasing sequence and set $\varepsilon_k = y_{nr_n} - x_{r_n}$ for all $r_n \leq k < r_{n+1}$. Then ε is an infinitesimal which violates the condition $f(x+\varepsilon) - f(x) - \varepsilon \cdot f'(x) \ll \varepsilon$. □

Theorem 19 *Let f, g be dynamic functions which have derivatives $f'(x), g'(x)$ at x. Then $f+g$, fg have at x derivatives $(f+g)'(x) = f'(x)+g'(x)$ and $(fg)'(x) = f'(x)g(x)+f(x)g'(x)$.*

Proof If ε is an infinitesimal, then by the assumption, $\alpha = f(x+\varepsilon)-f(x)-\varepsilon f'(x) \ll \varepsilon$ and $\beta = g(x+\varepsilon)-g(x)-\varepsilon g'(x) \ll \varepsilon$. Then we get

$$\begin{aligned} f(x+\varepsilon)g(x+\varepsilon) - f(x)g(x) - \varepsilon f'(x)g(x) - \varepsilon f(x)g'(x) \\ &= (f(x+\varepsilon)-f(x))(g(x+\varepsilon)-g(x)) + \\ &\quad f(x)(g(x+\varepsilon)-g(x)-\varepsilon g'(x)) + \\ &\quad (f(x+\varepsilon)-f(x)-\varepsilon f'(x))g(x) \\ &= (\varepsilon f'(x)+\alpha)(\varepsilon g'(x)+\beta) + f(x)\beta + g(x)\alpha \\ &\ll \varepsilon \end{aligned}$$

For the sum, the proof is similar. □

It follows that each polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ with $a_k \in \mathbb{Q}^*$ has derivative $P'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$.

7 Integration

Given a full function $f : [a,b] \to \mathbb{Q}^*$ define the **indefinite integral** $F_n : [a_n, b_n] \cap \mathbb{Q} \to \mathbb{Q}$ of f by

$$F_n(y) = \sum_{k=0}^{n-1} f_n\left(a_n + \frac{k(y-a_n)}{n}\right) \cdot \frac{y-a_n}{n}.$$

and $F(y) = \int_a^y f$ as the limit $F(y) = \{F_n(y_n)\}$ on $[a,b]$.

Theorem 20 *If $f : [a,b] \to \mathbb{Q}^*$ is a sequentially uniformly continuous function, then $F(y) = \int_a^y f$ is a full function on $[a,b]$.*

Proof Given $\varepsilon > 0$ there exists $\delta > 0$ and $q \in \mathbb{N}$ such that for every $n \geq q$ and $|x-y| < \delta$ we have $|f_n(x) - f_n(y)| < \varepsilon$. Given $y \in [a,b]$, take $m \geq q$ such that for all $n \geq m$ we have $|a_n - a_m| < \delta$, $|y_n - y_m| < \delta$ and $(b_n - a_n)/n < \delta$. Set $x_{n,i} = a_n + (y_n - a_n)i/n$ and let $(z_i)_{i=0,\ldots,r}$ be an increasing sequence which includes all $x_{n,i}$ with $i \le n$ and all $x_{m,i}$ with $i \le m$. There exists a sequence $(k_i)_{i=0,\dots,n}$ such that $k_0 = 0$, $k_n = r$, $x_{n,i} = z_{k_i}$, so

$$f(x_{n,i})(x_{n,i+1} - x_{n,i}) = \sum_{j=k_i}^{k_{i+1}-1} f(x_{n,i})(z_{j+1} - z_j)$$

For $k_i \le j < k_{i+1}$ we have $|z_j - x_{n,i}| < \delta$, so

$$\begin{aligned} &\left| \sum_{i=0}^{n-1} f_n(x_{n,i})(x_{n,i+1} - x_{n,i}) - \sum_{j=0}^{r-1} f_n(z_j) \cdot (z_{j+1} - z_j) \right| \\ &\quad \le \sum_{j=0}^{n-1} \sum_{j=k_i}^{k_{i+1}-1} |f_n(x_{n,i}) - f_n(z_j)|(z_{j+1} - z_j) \\ &\quad \le \varepsilon \sum_{j=0}^{r-1} (z_{j+1} - z_j) = \varepsilon(y_n - a_n) \end{aligned}$$

Similarly we get $|\sum_{i=0}^{m-1} f_m(x_{m,i})(x_{m,i+1} - x_{m,i}) - \sum_{j=0}^{r-1} f_n(z_j)(z_{j+1} - z_j)| \le \varepsilon(y_m - a_m)$ so $F_n(y_n)$ is a BC sequence. □

Theorem 21 *If $f : [a,b] \to \mathbb{Q}^*$ is a sequentially uniformly continuous function, then f is a derivative of $F(y) = \int_a^y f$ on $[a,b]$.*

Proof For a given $\varepsilon > 0$ there exists positive $\delta < \varepsilon$ and $q \in \mathbb{N}$ such that for every $n \ge q$ and $|x - y| < \delta$ we have $|f_n(x) - f_n(y)| < \varepsilon$. Assume that $x, y \in [a,b]$, $x \approx y$ and take $m \ge q$ such that for all $n \ge m$ we have $(b_n - a_n)/n < \delta$, $|x_n - y_n| < \delta$. Set $z_{n,k} = a_k + (x_n - a_n)k/n$, $w_{n,k} = a_k + (y_n - a_n)k/n$. Since $w_{n,k+1} - w_{n,k} = (w_{n,k+1} - z_{n,k+1}) + (z_{n,k+1} - z_{n,k}) - (w_{n,k} - z_{n,k})$, $w_{n,0} = z_{n,0}$ and $w_{n,n} - z_{n,n} = y_n - x_n$, we get

$$\begin{aligned} F_n(y_n) - F_n(x_n) &= \sum_{k=0}^{n-1} (f_n(w_{n,k}) - f_n(z_{n,k})(z_{n,k+1} - z_{n,k}) \\ &\quad + \sum_{k=0}^{n-1} f_n(w_{n,k})(w_{n,k+1} - z_{n,k+1}) - \sum_{k=1}^{n-1} f_n(w_{n,k})(w_{n,k} - z_{n,k}) \\ &= \sum_{k=0}^{n-1} (f_n(w_{n,k}) - f_n(z_{n,k})(z_{n,k+1} - z_{n,k}) \\ &\quad + \sum_{k=0}^{n-2} (f_n(w_{n,k}) - f_n(w_{n,k+1})(w_{n,k+1} - z_{n,k+1}) \\ &\quad + f_n(w_{n,n-1})(w_{n,n} - z_{n,n}) \end{aligned}$$

and

$$
\begin{aligned}
|F_n(y_n) - F_n(x_n) - (y_n - x_n) f_n(x_n)| &\le \sum_{k=0}^{n-1} |f_n(w_{n,k}) - f_n(z_{n,k})| \frac{y_n - x_n}{n} \\
&+ \sum_{k=0}^{n-2} |f_n(w_{n,k}) - f_n(w_{n,k+1})| \cdot |w_{n,k+1} - z_{n,k+1}| \\
&+ |f_n(w_{n,n-1}) - f_n(x_n)| \cdot |y_n - x_n| \\
&\le \varepsilon |y_n - x_n|/n + \varepsilon |y_n - x_n| + \varepsilon |y_n - x_n|
\end{aligned}
$$

In the last inequality we have used $|w_{n,k} - z_{n,k}| < \delta$, $|w_{n,k} - w_{n,k+1}| < \delta$, $|w_{n,n-1} - x_n| < \delta$. It follows $|F(y) - F(x) - (y - x) \cdot f(x)| \ll |y - x|$ and f is a derivative of F. □

8 Power series

Definition 22 *A power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is a dynamic function of the form $f_n(x) = \sum_{k=0}^{n} a_k x^k$, where $a_k \in \mathbb{Q}$. Its* **radius of convergence** *is $R = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}$.*

The proof that a power series converges in its interval of convergence is a standard one.

Theorem 23 *Let $\{a_k\}$ be a sequence of rational numbers and R the radius of convergence of $f_n(x) = \sum_{k=0}^{n} a_k x^k$. Then $f : (-R, R) \to \mathbb{Q}^*$ is a full function. For each $x \in \mathbb{Q}^*$ with $|x| > R$, $f(x)$ is undefined.*

Proof Given $x \in (-R, R)$, we show that $f(x) \in \mathbb{Q}^*$. There exists $s, r \in \mathbb{Q}^+$ with $|x| < s < r < R$. There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $|x_n| < s$ and $|a_n| \le r^{-n}$. For $m > n \ge n_0$ there exists a constant C such that

$$
\begin{aligned}
|f_m(x_m) - f_n(x_n)| &\le \sum_{i=1}^{n} |a_i| \cdot |x_m^i - x_n^i| + \sum_{i=n+1}^{m} |a_i| \cdot |x_m^i| \\
&\le |x_m - x_n| \cdot \sum_{i=1}^{n} |a_i| \cdot \sum_{j=0}^{i-1} |x_m^j x_n^{i-1-j}| + \sum_{i=n+1}^{m} (s/r)^i \\
&\le \frac{|x_m - x_n|}{r} \cdot \left(C + \sum_{i=n_0}^{n} i (s/r)^{i-1} \right) + \sum_{i=n+1}^{m} (s/r)^i \\
&\le \frac{|x_m - x_n|}{r} \cdot \left(C + \frac{s/r}{(1 - s/r)^2} \right) + \frac{(s/r)^{n+1}}{1 - s/r}
\end{aligned}
$$

so $f(x) \in \mathbb{Q}^*$. If $|x| > R$ then there exists $s > 1$ such that $\{n \in \mathbb{N} : \ |a_n x^n| > s^n\}$ is infinite so $f_n(x_n)$ is not a BC sequence. □

Proposition 24 *The power series $f(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $F(x) = \sum_{k=0}^{\infty} a_k x_k$ have the same radius of convergence.*

Proof If $f(x) \in \mathbb{Q}^*$ then $F(x) \in \mathbb{Q}^*$ since $\sum_{k=0}^{n} |a_k x^k| \leq |a_0| + |x| \cdot \sum_{k=1}^{n} |k a_k x^{k-1}|$. Conversely, if $|x| < s$ and $f(s) \in \mathbb{Q}^*$, then the sequence $b_k = k(x/s)^{k-1}$ converges to zero, so there exists $b \in \mathbb{Q}^+$ such that $|b_k| < b$ for all k. Then

$$\sum_{k=1}^{n} |k a_k x^{k-1}| = \sum_{k=1}^{n} |a_k| s^{k-1} k |x/s|^{k-1} \leq (b/s) \sum_{k=1}^{n} |a_k| s^k$$

so $F(x) \in \mathbb{Q}^*$. □

Theorem 25 *The power series $f(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ is a derivative of the power series $F(x) = \sum_{k=0}^{\infty} a_k x_k$ in their common interval of convergence.*

Proof Let R be the radius of convergence of both $f(x)$ and $F(x)$. Then $\varphi(x) = \sum_{k=2} k(k-1) a_k x^{k-2}$ has the same radius of convergence R. Assume that $|x|, |y| < R$ so there exists $s < R$ with $|x|, |y| < s$, and

$$\begin{aligned}
|F_n(y_n) - F_n(x_n) - (y_n - x_n) f_n(x_n)| &\leq \sum_{k=1}^{n} |a_k| \cdot |y_n^k - x_n^k - k(y_n - x_n) x_n^{k-1}| \\
&\leq \ |y_n - x_n| \cdot \sum_{k=1}^{n} |a_k| \cdot |y_n^{k-1} + y_n^{k-2} x_n \cdots + y_n x_n^{k-2} + x_n^{k-1} - k x_n^{k-1}| \\
&\leq \ |y_n - x_n| \cdot \sum_{k=2}^{n} |a_k| \cdot (|y_n^{k-1} - x_n^{k-1}| + \cdots + |y_n - x_n| \cdot |x_n^{k-2}|) \\
&\leq \ |y_n - x_n|^2 \cdot \sum_{k=2}^{n} |a_k| \frac{k(k-1)}{2} s^{k-2} = |y_n - x_n|^2 \varphi(s)/2.
\end{aligned}$$

Thus if $y \approx x$ then $F_n(y_n) - F_n(x_n) - (y_n - x_n) f_n(x_n) \ll y_n - x_n$ and $f(x)$ is a derivative of $F(x)$. □

Thus for example, the exponential function $(e^x)_n = \sum_{k=0}^{n} x_n^k / k!$ is full and continuous in whole $\mathbb{Q}^*$. The standard identities are valid as equivalences. For example, a little of algebra shows that $e^{x+y} \approx e^x \cdot e^y$ for each $x, y \in \mathbb{Q}^*$.

9 Discussion

While we do not claim to have reconstructed the continuum as it was understood by Bolzano and Cauchy, we have shown that a theory of a dynamical continuum with infinitesimal quantities is feasible and can be formulated with concepts which were well-known and current at the middle of the 19th century. Advanced concepts of formal logic and set theory like ultrafilters are not necessary. The Dedekind-Cantor-Weierstrass formalism of standard real numbers was not a historical necessity but one of possible alternatives.

Rather than technical difficulties, there are two major conceptual issues which have to be overcome to get a dynamical continuum. First, we should abandon the effort to construct the continuum to the image of the structure of rational numbers. We cannot do with real numbers everything that we are used to do with rational numbers. One of the most significant results of the computable analysis is that any computable full (everywhere defined) real function is continuous (see Pour-El and Richards(1989) [8]). The characteristic functions of inequalities are not defined everywhere, since the algorithm which may try to compute them would fail to terminate. We obtain similar results when we define dynamic real functions as limits of rational functions. The fact that the dynamical continuum of dynamic numbers is not linearly ordered may seem paradoxical and counterintuitive but in fact the linearity is not essential and can be very well disposed of.

The other preconception which should be abandoned, is the requirement that each quantity is measurable by a unique number. This desire motivated Bolzano to make his measuring process unique and this desire is also behind the definition of standard real numbers as equivalence classes of BC-sequences. This desire for uniqueness has been also dominant in representations of real numbers in positional number systems or continued fractions. Each irrational number can be expressed by a unique infinite simple continued fractions and each rational number can be expressed by two finite simple continued fractions (see e.g., Perron(1913) [7]). In fact the theory of continued fractions may be regarded as an alternative foundation for real numbers which preceded not only Bolzano and Cauchy but, according to some speculations, even Eudoxos (see Fowler(1987) [3]). Similarly in the binary positional system, a number has two representations iff it is a rational number of the form $p/2^n$, otherwise it has a unique representation. In both cases the uniqueness was desired but could not be attained for topological reasons[2]. But this near-uniqueness is highly undesirable when we wish to compute with these representations. Arithmetical algorithms can work only

[2]While the real line is a connected topological space, the symbolic space is totally discon-

with number systems which are **redundant**, i.e., in which any real number has many symbolic representations (see Weihrauch(2000) [17], Vuillemin(1990) [16] or Kůrka and Kazda(2010) [4]). And the representation of real numbers by BC-sequences of rational numbers is one of the most redundant representations possible. This is why the arithmetical operations with them are so simple - it suffices to define them pointwise.

While we have refrained from adopting the intuitionistic constructive positions and we have admitted also noncomputable BC-sequences as dynamic numbers, our approach opens the way for computable analysis with infinitesimals. This would be, we believe, an even more adequate version of a pre-Cantorian continuum. While Bolzano and Cauchy did not have the concept of computability, Bolzano speaks about rules or ideas which form his infinite number expressions and it is tempting to interpret such rules or ideas as algorithms. The following short note suggests that Bolzano was not totally unaware of the issues of computability.

> It may be worth remarking, for some readers, that in this definition I am only saying that, in an infinite number concept, an infinite multitude of operations of addition, subtraction, multiplication or division is *required*. I am not saying that it contains the idea of each one of these operations individually, and thereby that it contains an infinite multitude of ideas as its characteristic components. In this latter case such a concept would be composed of infinitely many parts, and therefore it would be inconceivable for a finite mind such as ours. (Russ(2004) [11] § 3 p. 358).

Acknowledgments

The research was supported by the Research Program CTS MSM 0021620845 and by the Czech Science Foundation research project GAČR 201/09/0854. We thank Jan Sebestik for inspiring discussions.

References

[1] **S Albeverio**, **R Høegh-Krohn**, **J E Fenstad**, **TLindstrøm**, *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, New York (1986)

nected, so they cannot be homeomorphic and some real numbers must have more than one symbolic representations.

[2] **A L Cauchy**, *Cours d'Analyse de l'École Royal Polytechnique*, Debure, Paris (1821)

[3] **D H Fowler**, *The Mathematics of Plato's Academy. A new reconstruction*, Oxford University Press, Oxford (1987)

[4] **P Kůrka**, **A Kazda**, *Möbius number systems based on interval covers*, Nonlinearity 23 (2010) 1031–1046

[5] **I Lakatos**, *Cauchy and the continuum: the significance of non-standrad analysis for the history of mathematics (edited by J.P.Cleave)*, from: "Mathematics, Science and Epistemology", (J Worall, G Currie, editors), Philosophical papers 2, Cambridge University Press, Cambridge (1980) 43–60

[6] **D Laugwitz**, *Bemerkungen zu Bolzanos Größenlehre*, Archive for History of Exact Sciences 2 (1962-1966) 398–409

[7] **O Perron**, *Die Lehre von Kettenbrüchen*, Teubner, Leipzig (1913)

[8] **M B Pour-El**, **J I Richards**, *Computability in Analysis and Physics*, Springer-Verlag, Berlin (1989)

[9] **A Robinson**, *Non-standard analysis*, Princeton University Press (1996)

[10] **P Rusnock**, *Bolzano's Philosophy and the Emergence of Modern Mathematics*, Rodopi, Amsterdam (2000)

[11] **S Russ**, *The Mathematical Works of Bernard Bolzano*, Oxford University Press, Oxford (2004)

[12] **K Rychlík** (editor), *Theorie der realen Zahlen im Bolzanos handschriftlichen Nachlasse*, Academia, Praha (1962)

[13] **C Schmieden**, **D Laugwitz**, *Eine Erweiterung der Infitesimalrechnung*, Math. Zeitschr. 69 (1958) 1–39

[14] **J Sebestik**, *Logique et mathématique chez Bernard Bolzano*, Librarie Philosophique J.Vrin, Paris (1992)

[15] **K Trlifajová**, *Bolzanova měřitelná a dnešní reálná čísla (Bolzano's measurable and contemporary real numbers)*, from: "Osamělý myslitel Bernard Bolzano (A lonely thinker Bernard Bolzano)", (K Trlifajová, editor), Filosofia, Praha (2006) 75–100

[16] **J E Vuillemin**, *Exact real computer arithmetic with continued fractions*, IEEE Transactions on Computers 39 (August 1990) 1087–1105

[17] **K Weihrauch**, *Computable analysis. An introduction*, EATCS Monographs on Theoretical Computer Science, Springer-Verlag, Berlin (2000)

[18] **E Winter**, **J Berg**, **F Kambartel**, **J Loužil**, **E Morscher**, **B van Rootselaar** (editors), *Bernard Bolzano Gesamtausgabe*, Frommann-Holzboog, Stuttgart (1969-)

kurka@cts.cuni.cz

http://www.cts.cuni.cz/~kurka