Languages, equicontinuity and attractors in cellular automata

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Abstract

We consider three related classifications of cellular automata. The first one is based on the complexity of languages generated by clopen partitions of the state space, i.e. on the complexity of the factor subshifts. The second one is based on the concept of equicontinuity and it is a modification of the classification introduced by Gilman [9]. The third one is based on the concept of attractor and it refines the classification introduced by Hurley [16]. We show relations between these classifications and give examples of cellular automata in the intersection classes. In particular, we show that every positively expansive cellular automaton is conjugate to a one-sided subshift of finite type and that every topologically transitive cellular automaton is sensitive to initial conditions. We also construct a cellular automaton with minimal quasi-attractor, whose basin has measure zero, answering a question raised in Hurley [16].

1 Introduction

Cellular automata are dynamical systems on the space of doubly infinite sequences of letters. As homomorphisms of the shift dynamical system, they have been of considerable interest in symbolic dynamics. In a classical paper, Hedlund [14] gave them a very thorough treatment and proved several deep theorems. New interest has been rekindled in the eighties, when Wolfram [23] studied cellular automata from the computational point of view. He classified them informally into four complexity classes, and several authors have offered a formalization of this classification. Thus Culik II et al. [7] classified cellular automata according to their behaviour on finite configurations.

In [17, 18, 19], we suggested a classification of general dynamical systems according to the complexity of the time evolution as described by languages generated by suitable partitions of the state space. The basic distinction is whether these languages are regular or not. A cellular automaton is regular, if the evolution at any specified positions as the time proceeds, gives rise to a regular language. An equivalent formulation is that a cellular automaton is regular, if all its one-sided factor subshifts are regular (i.e. sofic systems). In the present paper we also consider bounded periodic languages, which are a subclass of regular languages.

In [9, 10], Gilman introduced a classification based on the concept of equicontinuity. His classification depends on measure-theoretic concepts. He investigates the measure of set of points whose trajectories do not separate from a given trajectory. We modify this classification in a purely topological setting. A point is an equicontinuity point (or Lyapunov stable), if all neighbouring points remain close to it as the time proceeds. The equicontinuity set may be either the whole state space, or a nonempty proper subset of the state space, or it may be empty. In cellular automata, the last condition is equivalent to the sensitivity to initial conditions which is studied in chaotic dynamics (see e.g. Devaney [8], Glasner and Weiss [12] or Akin et al. [2]). There is also an important subclass of positively expansive cellular automata, which have been studied in depth (for the case of one-sided cellular automata) by Blanchard and Maass [3].

In [16] Hurley studied attractors in cellular automata. Besides attractors, he also defined quasi-attractors as countable intersections of attractors. This is a very useful concept which finds also applications in higher-dimensional dynamics. Thus for example, the Feigenbaum S-unimodal system on the real interval possesses a quasi-attractor. The Hurley classes distinguish the systems with two disjoint attractors, systems with a unique minimal quasi-attractor, and systems with a unique minimal attractor. (In contrast to Hurley, we do not include attractors among quasi-attractors.) We divide the last Hurley class into three more classes: systems with at least two attractors, systems with a unique attractor different from the entire state space and finally systems whose only attractor is the full state space. For an alternative approach to attractors in cellular automata see Hanson and Crutchfield [13].

We show the correlations between these classifications, give examples of cellular automata in the intersection classes, and prove that some intersection classes are empty. We relate these classifications to other properties of cellular automata like topological transitivity, surjectivity and openness. Every equicontinuous cellular automaton is bounded periodic, and has either disjoint attractors or a unique attractor which is a periodic orbit. Every positively expansive cellular automaton is conjugate to a topologically transitive subshift of finite type, and therefore regular. Every topologically transitive cellular automaton is sensitive. We consider also measure-theoretic ramifications of attractor classification and answer a question raised in Hurley [16]. We give an example of a cellular automaton with a unique minimal quasi-attractor, whose basin has measure zero.

Recently, Blanchard and Maass [4] proved that Coven cellular automata (see Coven [6]) have both equicontinuity and non-equicontinuity points, and their only attractor is the full space. This shows non-emptiness of one of the intersection classes, and answers a question left open in a previous version of the paper. Thus the only intersection class whose emptiness is still unclear is that of nonregular cellular automaton with both equicontinuity and non-equicontinuity points, whose only attractor is the full state space.

2 Cellular automata

Let A be a finite alphabet with at least two elements. We frequently use alphabet $2 = \{0,1\}$. Denote $\mathbb{N}$ the set of non-negative integers, $\mathbb{Z}$ the set of integers and $A^{\mathbb{Z}}$ the set of doubly infinite sequences of letters from A. For $n, m \in \mathbb{Z}$ denote $\langle n, m \rangle = \{i \in \mathbb{Z}; n \leq i \leq m\}$. For $x \in A^{\mathbb{Z}}$ denote $x_{\langle n,m \rangle} = x_n ... x_m \in A^{\langle n,m \rangle}$ the restriction of x to $\langle n, m \rangle$. For $u \in A^{\langle n,m \rangle}$ denote $\overline{u} \in A^{\mathbb{Z}}$ the point defined by $\overline{u}_{j(m-n+1)+i} = u_i$, $j \in \mathbb{Z}$, and $u^k \in A^{\langle n, n+(m-n+1)k-1 \rangle}$ the word defined by $u^k_{j(m-n+1)+i} = u_i$, $0 \leq j < k$. Denote $[u] = \{x \in A^{\mathbb{Z}}; x_{\langle n,m \rangle} = u\}$ the cylinder determined by u, so $\overline{u} \in [u]$. For $n \in \mathbb{N}$ define $A^n = A^{\langle 0, n-1 \rangle}$ and $A^* = \cup_{n \in \mathbb{N}} A^n$. For $u \in A^*$ denote $|u|$ its length. Define a metric d on $A^{\mathbb{Z}}$ by

$$d(x,y) = 2^{-n} \quad \text{where} \quad n = \inf\{i \geq 0; x_i \neq y_i \quad \text{or} \quad x_{-i} \neq y_{-i}\}$$

Topologically, $A^{\mathbb{Z}}$ is a countable product of discrete finite spaces, i.e. a compact, zero-dimensional, perfect metric space, and the cylinders are clopen (i.e. closed and open) sets. The two-sided shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ is continuous. A two-sided subshift is any closed σ-invariant subset of $A^{\mathbb{Z}}$. We consider also one-sided subshifts, which are closed σ-invariant subsets of $A^{\mathbb{N}}$, the set of one-way infinite sequences of letters from A. A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is of finite type iff there exists $k > 0$ such that for every $x \in A^{\mathbb{N}}$ we have $x \in \Sigma \Leftrightarrow (\forall n)([x_{\langle n, n+k-1 \rangle}] \cap \Sigma \neq 0)$. We refer to k as the order of Σ.

Definition 1 *A dynamical system (X, f) is a continuous map $f : X \to X$ of a compact metric space X to itself. A homomorphism $H : (X,f) \to (Y,g)$ of dynamical systems is a continuous map $H : X \to Y$ such that $Hf = gH$. A factorization is a homomorphism which is surjective. A conjugacy is a homomorphism which is bijective.*

Denote $f^n : X \to X$ the n-th iteration of f. A point $x \in X$ is periodic if $f^n(x) = x$ for some n. It is eventually periodic, if $f^m(x)$ is periodic for some m. A set $Y \subseteq X$ is invariant if $f(Y) \subseteq Y$. A dynamical system (X, f) is topologically transitive, if for every nonempty open sets $U, V \subseteq X$ there exists $k > 0$ with $f^k(U) \cap V \neq 0$. An ε-chain from $x_0 \in X$ to $x_n \in X$ is a sequence $x_0, ..., x_n$ such that $n > 0$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i < n$. A sequence x_i is ε-shadowed by a point $x \in X$, if $d(f^i(x), x_i) < \varepsilon$ for $0 \leq i \leq n$.

Definition 2 *A dynamical system (X, f) has the shadowing property, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that every δ-chain is ε-shadowed by some point.*

A (one-dimensional) cellular automaton over an alphabet A is a dynamical system $(A^{\mathbb{Z}}, f)$ for which $\sigma f = f\sigma$. According to Curtis-Hedlund-Lyndon theorem (see Hedlund [14]), a dynamical system $(A^{\mathbb{Z}}, f)$ is a cellular automaton iff there exists a non-negative integer r (called radius) and a transition function $f_0 : A^{\langle -r,r\rangle} \to A$, such that

$$f(x)_i = f_0(x_{i-r}, ..., x_{i+r}), \quad x \in A^{\mathbb{Z}}, \ i \in \mathbb{Z}$$

In the sequel we identify f with f_0, and with maps $f : A^{\langle n-r,m+r\rangle} \to A^{\langle n,m\rangle}$. Every cellular automaton $(A^{\mathbb{Z}}, f)$ is conjugate to a cellular automaton $(B^{\mathbb{Z}}, g)$ with radius 1. It suffices to take the alphabet $B = A^r$, and define

$$g(u_0...u_{r-1}, u_r...u_{2r-1}, u_{2r}...u_{3r-1}) = f(u_0...u_{2r})...f(u_{r-1}...u_{3r-1})$$

3 Language classification

Given an alphabet A, a language over A is any subset $L \subseteq A^*$. A language is regular if there exists a finite automaton which recognizes it (see Hopcroft and Ullmann [15]). A language $L \subseteq A^*$ is right central, if for every $u \in L$ there exists $a \in A$ with $ua \in L$, and if v is a subword of u, then $v \in L$.

Definition 3 *A language $L \subseteq A^*$ is bounded periodic if there exist integers $m \geq 0$ and $n > 0$ such that for every $u \in L$ and $i \geq m$ we have $u_i = u_{i+n}$.*

Clearly every right central bounded periodic language is regular. Any $u \in L$ is determined by its initial segment of length $m + n$ and there is only a finite number of such strings.

A clopen partition of a zero-dimensional space X is a continuous function $\beta : X \to B$, where B is a finite set with the discrete topology. Thus $\{\beta^{-1}(b); b \in B\}$ is a collection of disjoint clopen sets. If (X, f) is a zero-dimensional dynamical system, denote $H_\beta : (X, f) \to (B^{\mathbb{N}}, \sigma)$ a homomorphism defined by $H_\beta(x)_i = \beta f^i(x)$. Denote $\Sigma_\beta(f) = H_\beta(X) \subseteq B^{\mathbb{N}}$ the subshift generated by β on X, so that $H_\beta : (X, f) \to (\Sigma_\beta(f), \sigma)$ is a factorization. Denote

$$\mathcal{L}_\beta(f) = \{u \in B^*; [u] \cap \Sigma_\beta(f) \neq 0\} \subseteq B^*$$

the language generated by f on β. Then $\mathcal{L}_\beta(f)$ is right central. In particular if $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift and $\alpha : A^{\mathbb{N}} \to A$ is the projection $\alpha(u) = u_0$, then $\mathcal{L}(\Sigma) = \mathcal{L}_\alpha(\Sigma) \subseteq A^*$ is the language of words occurring in the elements of Σ. Thus we have $\mathcal{L}(\Sigma_\beta(f)) = \mathcal{L}_\beta(f)$. Every factorization $H : (X, f) \to (\Sigma, \sigma)$, where $\Sigma \subseteq B^{\mathbb{N}}$ is a subshift, is generated by a clopen partition $\beta : X \to B$ defined by $\beta(x) = H(x)_0$. If $(A^{\mathbb{Z}}, f)$ is a cellular automaton, denote $H_{\langle n,m\rangle} : (X, f) \to (\Sigma_{\langle n,m\rangle}(f), \sigma)$ the factorization determined by $\beta : X \to A^{\langle n,m\rangle}$, defined by $\beta(x) = x_{\langle n,m\rangle}$. We call these column factorizations. Every one-sided factor subshift of a cellular automaton is clearly a factor of some column factorization.

Definition 4 *A zero-dimensional dynamical system (X, f) is regular (bounded periodic) if $\mathcal{L}_\beta(f)$ is regular (bounded periodic) for every clopen partition $\beta : X \to B$.*

Regular subshifts are called sofic systems and they are factors of subshifts of finite type (see e.g. Culik II et al. [7]). A subshift is bounded periodic iff it is finite, and it is a subshift of finite type in this case. A dynamical system (X, f) on a zero-dimensional space is regular (bounded periodic) iff every its one-sided factor subshift is regular (bounded periodic). Regularity in general dynamical systems has been considered by Kůrka [17, 18, 19]. Gilman [11] observes that every cellular automaton is context-sensitive and gives an example of a nonregular cellular automaton.

Corollary 1 (Language classification) *Every zero-dimensional dynamical system (X, f) falls in exactly one of the following classes.*
L1. *(X, f) is bounded periodic.*
L2. *(X, f) is regular but not bounded periodic.*
L3. *(X, f) is not regular.*

Proposition 1 *Every cellular automaton with the shadowing property is regular.*

Proof: Suppose $(A^{\mathbb{Z}}, f)$ is a cellular automaton with radius 1. We show that $\Sigma_{\langle -n,n\rangle}(f)$ is regular for every $n > 0$. By assumption, for $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$, such that every δ-chain is ε-shadowed by some point. Define $\Sigma \subseteq (A^{\langle -k-1,k+1\rangle})^{\mathbb{N}}$ by the condition $u = u^{(0)}u^{(1)}... \in \Sigma$ iff

$$u_i^{(j+1)} = f(u_{i-1}^{(j)}, u_i^{(j)} u_{i+1}^{(j)}) \quad \text{for} \quad -k \le i \le k, \;\; j \ge 0$$

Thus Σ is a subshift of finite type (of order 2). Define $H : \Sigma \to (A^{\langle -n,n\rangle})^{\mathbb{N}}$ by $H(u) = u_{\langle -n,n\rangle}^{(0)} u_{\langle -n,n\rangle}^{(1)}...$. We claim $H(\Sigma) = \Sigma_{\langle -n,n\rangle}(f)$. Indeed if $u \in \Sigma$, then any sequence $x^{(j)}$ with $x^{(j)} \in [u^{(j)}]$ is a δ-chain, so there exists $x \in A^{\mathbb{Z}}$ which ε-shadows $x^{(j)}$, i.e. $f^{(j)}(x)_{\langle -n,n\rangle} = x_{\langle -n,n\rangle}^{(j)}$. Thus $H(u) = H_{\langle -n,n\rangle}(x)$, so $H(\Sigma) \subseteq \Sigma_{\langle -n,n\rangle}(f)$. The other inclusion is clear. □

Proposition 2 *Every cellular automaton whose every column factorization is a subshift of finite type has the shadowing property.*

Proof: Suppose $(A^{\mathbb{Z}}, f)$ is a cellular automaton of radius 1, whose every column factorization is a subshift of finite type. Let $\varepsilon = 2^{-n}$, suppose that $\Sigma_{\langle -n,n\rangle}(f)$ has order m, and denote $\delta = 2^{-n-m+1}$. We show that every δ-chain is ε-shadowed by some point. Let $x^{(i)}$ be an infinite δ-chain. Then $f^j(x^{(i)})_{\langle -n,n\rangle} = x_{\langle -n,n\rangle}^{(i+j)}$ for $0 \le j < m$, so $x_{\langle -n,n\rangle}^{(i)}...x_{\langle -n,n\rangle}^{(i+m-1)} \in \mathcal{L}(\Sigma_{\langle -n,n\rangle}(f))$ for every i. Since $\Sigma_{\langle -n,n\rangle}(f)$ is of order m, the infinite sequence $(x_{\langle -n,n\rangle}^{(i)})_{i \ge 0}$

belongs to $\Sigma_{\langle -n,n\rangle}(f)$, so there exists $x \in A^{\mathbb{Z}}$ with $f^i(x)_{\langle -n,n\rangle} = x^{(i)}_{\langle -n,n\rangle}$ for every $i \geq 0$. This means that x ε-shadows $x^{(i)}$. □

Example 7 shows that the converse of Proposition 1 is not true. Example 17 shows that the converse of Proposition 2 is not true. For a more general treatment of the shadowing property see Botelho and Garzon [5].

4 Equicontinuity classification

For a dynamical system (X, f) define $Eq(f)$ (equicontinuity set) as the set of its Lyapunov stable points, i.e. points at which the family f^n is equicontinuous. Thus $x \in Eq(f)$ iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B_\delta(x))(\forall n \geq 0)(d(f^n(y), f^n(x)) < \varepsilon)$$

Here $B_\varepsilon(x) = \{y \in X; d(y,x) < \varepsilon\}$ is the ball with centre x and radius ε. Clearly the complement of $Eq(f)$ is invariant. We can classify dynamical systems according to the extent they are equicontinuous.

Definition 5 *Let (X, f) be a dynamical system.*
1. (X, f) is positively expansive if

$$(\exists \varepsilon > 0)(\forall x)(\forall y \neq x)(\exists n)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

2. (X, f) is sensitive (to initial conditions) if

$$(\exists \varepsilon > 0)(\forall x)(\forall \delta > 0)(\exists y \in B_\delta(x))(\exists n)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

3. $Eq(f) = \emptyset$ if

$$(\forall x)(\exists \varepsilon > 0)(\forall \delta > 0)(\exists y \in B_\delta(x))(\exists n)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

4. $Eq(f) \neq X$ if

$$(\exists x)(\exists \varepsilon > 0)(\forall \delta > 0)(\exists y \in B_\delta(x))(\exists n)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

5. (X, f) is not equicontinuous if

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x)(\exists y \in B_\delta(x))(\exists n)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

Since a compact metric space admits the unique uniformity compatible with the topology, all these properties are topological invariants. Clearly $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$, and since X is compact, $5 \Rightarrow 4$. If X has no isolated points, then $1 \Rightarrow 2$. In general, 3 does not imply 2. A counterexample is a cone connected at its vertex to a circle, i.e. $X = \{(r\cos\varphi, r\sin\varphi, r);\ 0 \leq r \leq 1\} \cup \{(-1 + \cos\varphi, \sin\varphi, 0)\}$ with dynamics of doubling the angle on every horizontal circle. However, if (X, f) is topologically transitive, then $3 \Rightarrow 2$ (see Glasner and Weiss [12]). In cellular automata we have even a stronger statement, which is a topological reformulation of Proposition 3.4 in Gilman [9].

Theorem 3 *Let $(A^{\mathbb{Z}}, f)$ be a cellular automaton which is not sensitive. Then $Eq(f)$ is a residual set.*

Proof: Suppose that the radius is $r = 1$. For $\varepsilon = 2^{-1}$ there exists $x \in A^{\mathbb{Z}}$ and $\delta = 2^{-m}$ such that for every $y \in A^{\mathbb{Z}}$, if $y_{\langle -m,m \rangle} = x_{\langle -m,m \rangle}$, then $f^n(x)_{\langle -1,1 \rangle} = f^n(y)_{\langle -1,1 \rangle}$ for all n. It follows that if $x_i = y_i$ for $i \leq m$, then $f^n(x)_i = f^n(y)_i$ for all $n \geq 0$ and $i \leq 1$. Similarly for the positive part. Thus $z = \overline{x_{\langle -m,m \rangle}}$ is a point of equicontinuity, so $Eq(f) \neq 0$. To show that $Eq(f)$ is residual, denote V_k the set of those points, which contain in both their positive and negative parts at least k occurrences of $x_{\langle -m,m \rangle}$. Clearly V_k is dense and open. It follows that $V = \cap_{k>0} V_k$ is a residual set and it is a subset of $Eq(f)$. □

Corollary 2 (Equicontinuity classification) *Every cellular automaton $(A^{\mathbb{Z}}, f)$ falls in exactly one of the following classes.*
E1. *$Eq(f) = A^{\mathbb{Z}}$, i.e. $(A^{\mathbb{Z}}, f)$ is equicontinuous.*
E2. *$\emptyset \neq Eq(f) \neq A^{\mathbb{Z}}$.*
E3. *$(A^{\mathbb{Z}}, f)$ is sensitive ($Eq(f) = \emptyset$) but not positively expansive.*
E4. *$(A^{\mathbb{Z}}, f)$ is positively expansive.*

Theorem 4 *For a cellular automaton $(A^{\mathbb{Z}}, f)$, the following three conditions are equivalent.*
1. $(A^{\mathbb{Z}}, f)$ is bounded periodic (i.e. it is in **L1***).*
2. $(A^{\mathbb{Z}}, f)$ is equicontinuous (i.e. it is in **E1***)*
3. There exist $m \geq 0$, $n > 0$ such that for every $x \in A^{\mathbb{Z}}$, and for every $i \geq m$ we have $f^{i+n}(x) = f^i(x)$.

Proof: $1 \Rightarrow 2$. There exist $m \geq 0$ and $n > 0$ such that for all $i \geq m$ $f^{i+n}(x)_0 = f^i(x)_0$. Thus the sequence $f^i(x)_0$ is determined by its first $n+m$ members, and these are determined by $x_{\langle -n-m,n+m \rangle}$. Similarly for $k > 0$, the sequence $f^i(x)_{\langle -k,k, \rangle}$ is determined by $x_{\langle -n-m-k,n+m+k \rangle}$, so $(A^{\mathbb{N}}, f)$ is equicontinuous. $2 \Rightarrow 3$. Suppose $(A^{\mathbb{Z}}, f)$ is an equicontinuous cellular automaton with radius 1. For $\varepsilon = 1$ there exists $\delta = 2^{-k}$ such that for every $x, y \in A^{\mathbb{Z}}$ with $x_{\langle -k,k \rangle} = y_{\langle -k,k \rangle}$ we have $f^i(x)_0 = f^i(y)_0$ for all i. Denote $u = x_{\langle -k,k \rangle}$. Then for $y \in [u]$ we have $f^i(y)_0 = f^i(\overline{u})_0$, and this is an eventually periodic trajectory, with some preperiod m_u and period n_u. Denote m the maximum of all m_u, and n the product of all n_u for $u \in A^{\langle -k,k \rangle}$. Then for every $i \geq m$ and every $x \in A^{\mathbb{Z}}$ we have $f^{i+n}(x)_0 = f^i(x)_0$. Since f commutes with the shift, we have $f^{i+n}(x) = f^i(x)$. $3 \Rightarrow 1$. This is obvious. □

Proposition 5 *Every equicontinuous dynamical system on $A^{\mathbb{Z}}$ has the shadowing property.*

Proof: For every $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that if $x_{\langle -k,k \rangle} = y_{\langle -k,k \rangle}$, then $f^i(x)_{\langle -n,n \rangle} = f^i(y)_{\langle -n,n \rangle}$ for all i. Let $x^{(0)}, x^{(1)}, \ldots$ be a δ-chain and $m > 0$.

Then $f(x^{(i)})_{\langle -k,k\rangle} = x^{(i+1)}_{\langle -k,k\rangle}$ for $0 \le i < m$. It follows $f^{m-i}(x^{(i)})_{\langle -n,n\rangle} = f^{m-i-1}(x^{(i+1)})_{\langle -n,n\rangle}$ for $0 \le i < m$, so $f^m(x^{(0)})_{\langle -n,n\rangle} = x^{(m)}_{\langle -n,n\rangle}$. Thus $x^{(0)}$ ε-shadows the δ-chain $x^{(i)}$. $\Box$

Note that Proposition 5 is not true for the unit interval.

Proposition 6 *Every topologically transitive cellular automaton is sensitive.*

Proof: Suppose $(A^{\mathbb{Z}}, f)$ is topologically transitive and not sensitive. By Lemma 1.2 of Glasner and Weiss [12], it is uniformly rigid, i.e. there exists a sequence n_k converging to infinity such that

$$(\forall \varepsilon > 0)(\exists k_0)(\forall k \ge k_0)(\forall x \in A^{\mathbb{Z}})(d(f^{n_k}(x), x) < \varepsilon)$$

Thus for $\varepsilon = 1$ there exists n such that $(\forall x \in A^{\mathbb{Z}})(d(f^n(x), x) < \varepsilon)$, i.e. $f^n(x)_0 = x_0$. Applying σ^i, we get $f^n(x)_i = x_i$, so f^n is the identity map. This is in contradiction with topological transitivity. $\Box$

We are going to prove that every positively expansive cellular automaton is conjugate to a one-sided subshift of finite type. The result has been independently obtained by Nasu [21].

Proposition 7 *Every positively expansive cellular automaton $(A^{\mathbb{Z}}, f)$ with radius 1 is surjective and conjugate to $(\Sigma_{\langle -1,1\rangle}(f), \sigma)$.*

Proof: Denote $B = A^{\langle -1,1\rangle}$ and $H : (A^{\mathbb{Z}}, f) \to (\Sigma_{\langle -1,1\rangle}(f), \sigma)$ the factorization. We show H is injective. Suppose by contradiction that $x, y \in A^{\mathbb{Z}}$, $x \ne y$, and $f^i(x)_{\langle -1,1\rangle} = f^i(y)_{\langle -1,1\rangle}$ for all $i \ge 0$. There exists j with $x_j \ne y_j$ and $|j| > 1$. Suppose $j > 1$ and define $z \in A^{\mathbb{Z}}$ by $z_j = x_j$ for $j \le 0$ and $z_j = y_j$ for $j \ge 0$. Then $x \ne z$, and for every $k > 0$ and every $i \ge 0$ we have $f^i(\sigma^{-k+1}(z))_{\langle -k,k\rangle} = f^i(\sigma^{-k+1}(x))_{\langle -k,k\rangle}$ and this is not possible since $(A^{\mathbb{Z}}, f)$ is positively expansive. Thus $(A^{\mathbb{Z}}, f)$ is conjugate to $(\Sigma_{\langle -1,1\rangle}(f), \sigma)$, and therefore every point $x \in A^{\mathbb{Z}}$ has under f a bounded number of preimages. By Theorem 5.12 of Hedlund [14], f is surjective. $\Box$

Lemma 8 *Let $(A^{\mathbb{Z}}, f)$ be a positively expansive cellular automaton with radius 1. Then there exists $n > 1$ such that for every $u \in A^{\langle -1,1\rangle}$, and for every $x, y \in [u]$, if $f(x)_{\langle 0,n-1\rangle} = f(y)_{\langle 0,n-1\rangle}$, then $x_2 = y_2$.*

Proof: Extend u arbitrarily for $i < -1$ and for $n > 1$ define $V_n \subseteq A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ by $(x, y) \in V_n$ iff $(\forall i < 2)(x_i = y_i = u_i)$, $x_2 \ne y_2$, and $f(x)_{\langle 0,n-1\rangle} = f(y)_{\langle 0,n-1\rangle}$ Then $V_{n+1} \subseteq V_n$ are closed sets. If they were all nonempty, we would have $(x, y) \in \cap_{n>1} V_n$, for which $x \ne y$, $f(x) = f(y)$, so $f^i(x)_{\langle -1,1\rangle} = f^i(y)_{\langle -1,1\rangle}$ for every $i \ge 0$. This is in contradiction with Proposition 7. $\Box$

Proposition 9 *Every positively expansive cellular automaton is open (i.e. the image of every open set is open).*

Let $(A^{\mathbb{Z}}, f)$ be a positively expansive cellular automaton with radius 1, and let n be as in Lemma 8. We define a set $B \subseteq A^{\langle -1,1\rangle} \times A^{\langle 0,n-1\rangle}$ by

$$(u_{-1}u_0u_1, w_0...w_{n-1}) \in B \text{ iff } (\exists u_2...u_n)(f(u_{-1}...u_n) = w_0...w_{n-1})$$

By Lemma 8 that there exists a function $g : B \to A$ defined by

$$g(u_{-1}u_0u_1, w_0...w_{n-1}) = u_2 \text{ iff } (\exists u_3...u_n)(f(u_{-1}...u_n) = w_0...w_{n-1})$$

Consider now a graph with vertices B and arrows labelled by A, defined by

$$(u_{-1}u_0u_1, w_0...w_{n-1}) \xrightarrow{w_n} (u_0u_1u_2, w_1...w_n) \text{ iff } u_2 = g(u_{-1}u_0u_1, w_0...w_{n-1})$$

It is easy to see that this graph is irreducible, and defines a subshift $\Sigma \subseteq B^{\mathbb{Z}}$ of order 2. We have a conjugacy $H : (A^{\mathbb{Z}}, \sigma) \to (\Sigma, \sigma)$ given by $H(u)_i = (u_{i-1}u_iu_{i+1}, f(u)_i...f(u)_{i+n-1})$, so topological entropy of (Σ, σ) is $\log \mathrm{card}(A)$. Since there is at most $\mathrm{card}(A)$ arrows leading from any vertex of B (labelled by different letters of A), there is exactly $\mathrm{card}(A)$ of them; otherwise the topological entropy of (Σ, σ) would be smaller than $\log \mathrm{card}(A)$. Similarly define a set $C \subseteq A^{\langle -1,1\rangle} \times A^{\langle -n+1,0\rangle}$ by $(u_{-1}u_0u_1, w_{-n+1}...w_0) \in C$ iff there exist $u_{-n}...u_{-2}$ with $f(u_{-n}...u_1) = w_{-n+1}...w_0$. Again we get that for every $(u_{-1}u_0u_1, w_{-n+1}...w_0) \in C$ and every $w_{-n} \in A$ there is exactly one $u_{-2} \in A$ with $(u_{-2}u_{-1}u_0, w_{-n}...w_{-1}) \in C$. It follows that for every $k > 0$ and $u \in A^{\langle -k,k\rangle}$, $f([u_{-k}...u_k])$ is the union of those cylinders $[w_{-n-k+2}...w_{n+k-2}]$, for which $(u_{-k}u_{-k+1}u_{-k+2}, w_{-n-k+2}...w_{-k+1}) \in C$, $f(u_{-k}...u_k) = w_{-k+1}...w_{k-1}$, and $(u_{k-2}u_{k-1}u_k, w_{k-1}...w_{n+k-2}) \in B$. Thus $f([u_{\langle -k,k\rangle}])$ is a clopen set, so f is an open map. □

The following theorem is an adaptation of Theorem 3.8 of Blanchard and Maass [3] to the case of two-sided cellular automata.

Theorem 10 *Every positively expansive cellular automaton is conjugate to a topologically transitive one-sided subshift of finite type.*

Proof: Let $(A^{\mathbb{Z}}, f)$ be an positively expansive cellular automaton. By Proposition 7, it is conjugate to $\Sigma = \Sigma_{\langle -1,1\rangle}(f) \subseteq B^{\mathbb{N}}$ where $B = A^{\langle -1,1\rangle}$. By Proposition 9, (Σ, σ) is open. By the Theorem 1 of Parry [22], a subshift is open iff it is a subshift of finite type. Let k be the order of Σ, so $x \in \Sigma \Leftrightarrow (\forall n)([x_{\langle n,n+k-1\rangle}] \cap \Sigma \neq 0)$. Let $n \geq 0$ and $w, w' \in B^n$. We prove that $f^j([w]) \cap [w'] \neq 0$ for some j. By compactness for every p there exists m such that for all $x, y \in A^{\mathbb{Z}}$ we have

$$(\forall i < m)(f^i(x)_{\langle -1,1\rangle} = f^i(y)_{\langle -1,1\rangle}) \quad \Rightarrow \quad x_{\langle -p,p\rangle} = y_{\langle -p,p\rangle}$$

Let m have this property for $p = 2n + k + 1$. We can assume $m > n + k - 1$. For $w \in B^n$ there exists $x \in A^{\mathbb{Z}}$ such that $w_i = f^i(x)_{\langle -1,1\rangle}$ for $0 \leq i < n$.

Define $v \in B^m$ by $v_i = f^i(x)_{\langle n+k, n+k+2\rangle}$. If we modify the sequence $f^{m-k}(x)$ in the interval $\langle -n, n\rangle$, the word v is not affected. Thus there exists $x' \in A^{\mathbb{Z}}$ such that $f^i(x')_{\langle -1,1\rangle} = w'_i$ for $0 \le i < n$, and $f^i(x')_{\langle n+k,n+k+2\rangle} = v_{m-k+i}$ for $0 \le i < k$. Extend v to a word in B^{2m-k} by $v_i = f^{i-m+k}(x')_{\langle n+k,n+k+2\rangle}$ for $m \le i < 2m-k$. Then $v_{\langle 0,m-1\rangle}, v_{\langle m-k,2m-k-1\rangle} \in \mathcal{L}(\Sigma)$, since they are generated by x and x' respectively. Since Σ is of order k, we get $v \in \mathcal{L}(\Sigma)$ too. Thus there exists $y \in A^{\mathbb{Z}}$ for which $f^i(y)_{\langle n+k,n+k+2\rangle} = v_i$ for $0 \le i < 2m-k$, $f^i(y)_{\langle -1,1\rangle} = w_i$ for $0 \le i < n$, and $f^{m-k+i}(y)_{\langle -1,1\rangle} = w'_i$ for $0 \le i < n$. Thus $f^{m-k}([w]) \cap [w'] \neq 0$. □

Theorem 11 **E4** $\subseteq$ **L2**.

Proof: If $(A^{\mathbb{Z}}, f)$ is positively expansive, then it is regular by Theorem 10. If $\Sigma_{\langle -1,1\rangle}(f)$ were bounded periodic, it would be finite, and this is a contradiction. □

5 Attractor classification

Definition 6 *Let (X,f) be a dynamical system. The ω-limit of a set $Y \subseteq X$ is $\omega(Y) = \cap_{n>0}\overline{\cup_{m>n} f^m(Y)}$. A nonempty set $Y \subseteq X$ is an attractor if there exists an open set U such that $f(\overline{U}) \subseteq U$ and $Y = \omega(U)$. A nonempty set is a quasi-attractor, if it is a countable intersection of attractors, but itself is not an attractor. Y is a minimal attractor (minimal quasi-attractor) if it does not contain any attractor (quasi-attractor) as a proper subset.*

In a zero-dimensional space a set Y is an attractor iff there exists a clopen invariant set C with $Y = \omega(C)$. The set of attractors is at most countable. The largest attractor is $\omega(X)$. If Y, W are attractors, then $Y \cup W$ is again an attractor. $Y \cap W$ need not be an attractor, but if it is nonempty, then $\omega(Y \cap W)$ is an attractor. The basin of an invariant set Y is $\mathcal{B}(Y) = \{x \in X; \omega(x) \subseteq Y\}$. The basin of an attractor is an open invariant set. The following classification is a refinement of a classification introduced by Hurley in [16]. It can be formulated for arbitrary dynamical systems.

Corollary 3 (Attractor classification) *Every dynamical system (X,f) falls in exactly one of the following classes.*
A1. *There exists a pair of disjoint attractors.*
A2. *There exists a unique minimal quasi-attractor*
A3. *There exists a unique minimal attractor different from $\omega(X)$*
A4. *There exists a unique attractor $\omega(X) \neq X$.*
A5. *There exists a unique attractor X*

Proof: Suppose that every two attractors have a non-empty intersection. Let Y_i be the sequence of all attractors, and define $W_0 = Y_0$, $W_{i+1} = \omega(Y_{i+1} \cap W_i)$.

Then W_i is a decreasing sequence of attractors, and $W = \cap_i W_i$ is either a minimal attractor or a minimal quasi-attractor, which is unique. (If there were two different minimal attractors, then their intersection would contain an attractor, which is a contradiction. The same argument applies to quasi-attractors.) If $\omega(X)$ is the minimal attractor, then it is the only attractor. $\square$

If a cellular automaton has a pair of disjoint attractors, then every attractor contains a pair of disjoint attractors, and therefore there are uncountably many quasi-attractors. If there exists a unique minimal attractor, its basin is a dense open set (see Hurley [16]).

Define $\mathcal{C}f \subseteq X \times X$ by the property that $(x, y) \in \mathcal{C}f$ iff for any $\varepsilon > 0$ there exists an ε-chain from x to y. Denote by $|\mathcal{C}f| = \{x \in X; (x, x) \in \mathcal{C}f\}$ the set of chain-recurrent points. Then $(\forall x \in X)(\exists y \in |\mathcal{C}f|)((x, y) \in \mathcal{C}f)$, and $\mathcal{C}f \cap (\mathcal{C}f)^{-1}$ is an equivalence relation on $|\mathcal{C}f|$. The equivalence classes are invariant. They are called chain components in Hurley [16] and basic sets in Akin [1]. Any attractor Y is determined by its chain recurrent points via $Y = \mathcal{C}f(Y \cap |\mathcal{C}f|)$ (see Akin [1], Proposition 3.8; here for $W \subseteq X$, $\mathcal{C}f(W) = \{y \in X; (\exists x \in X)((x, y) \in \mathcal{C}f)\}$). If $x \in |\mathcal{C}f|$, and $y \in X$, then

$$(x, y) \in \mathcal{C}f \text{ iff for every attractor } Y,\ x \in Y \text{ implies } y \in Y$$

(see Akin [1], Proposition 3.11). Thus two chain recurrent points lie in the same chain component iff they are contained in exactly the same attractors. If Y is a minimal attractor or a minimal quasi-attractor, it is a chain component. Indeed, if $x, y \in Y \cap |\mathcal{C}f|$, then $(x, y) \in \mathcal{C}f$, since they are contained in the same attractors. Thus $Y \cap |\mathcal{C}f|$ is a chain component. If $x \in Y \cap |\mathcal{C}f|$, and $(x, y) \in \mathcal{C}f$, then there exists $z \in |\mathcal{C}f|$ with $(y, z) \in \mathcal{C}f$. It follows $y, z \in Y \cap |\mathcal{C}f|$, so $Y = \mathcal{C}f(Y \cap |\mathcal{C}f|) \subseteq Y \cap |\mathcal{C}f|$. Thus Y is a chain component.

A closed invariant set $Y \subseteq X$ is chain transitive, if $Y \times Y \subseteq \mathcal{C}f$. The set $\omega(X)$ is chain transitive iff $\omega(X)$ is the unique attractor. Clearly every topologically transitive set is chain transitive. Thus we have

Proposition 12 *If $(\omega(X), f)$ is topologically transitive, then (X, f) is either in* **A4** *or in* **A5**. *If (X, f) is topologically transitive, it is in* **A5**.

Proposition 13 *If (X, f) is a chain transitive system with the shadowing property, then it is topologically transitive.*

Proof: Let $U, V \subseteq X$ be nonempty open sets. There exists $\varepsilon > 0$ such that $x \in B_\varepsilon(x) \subseteq U$, $y \in B_\varepsilon(y) \subseteq V$. Let $\delta > 0$ be such that every δ-chain is ε-shadowed by some orbit. There exists a δ-chain $x = x_0, ..., x_m = y$, so there exists z with $d(f^i(z), x_i) < \varepsilon$. It follows $f^m(z) \in f^m(U) \cap V$. $\square$

Theorem 14 $\mathbf{E1} \subseteq \mathbf{A1} \cup \mathbf{A4}$

Proof: If $(A^{\mathbb{Z}}, f)$ is equicontinuous, then by Theorem 4, every point is eventually periodic. By Proposition 5, $(A^{\mathbb{Z}}, f)$ has the shadowing property. It follows that $(x, y) \in \mathcal{C}f$ iff there exists $n > 0$ such that $f^n(x) = y$, and the only chain components are the periodic orbits. If $(A^{\mathbb{Z}}, f)$ were in **A5**, then the full space would be a chain component, and this is a contradiction. Suppose that $(A^{\mathbb{Z}}, f)$ belongs either to **A2** or **A3**. Denote Q the minimal attractor or minimal quasi-attractor. Then Q is a periodic orbit since it is a chain component, and there exists an attractor $Y \supset Q$ different from Q. It follows there exists a chain component $P \subseteq Y - Q$, and $P \times Q \subseteq \mathcal{C}f$. This is again a contradiction. $\Box$

Theorem 15 **E4** $\subseteq$ **A5**

Proof: By Theorem 10, every positively expansive cellular automaton is topologically transitive, so its unique attractor is the full space. $\Box$

Proposition 16 *Cellular automata in class* **E2**$\cap$**A5** *do not have the shadowing property.*

The proof follows from Propositions 5 and 13.

6 Intersection classes

Example 1 **L1** $\cap$ **E1** $\cap$ **A1** $\neq \emptyset$.

Proof: Take $A = 2$, $f(x)_i = x_i$ (identity map). Then $(A^{\mathbb{Z}}, f)$ is equicontinuous, and the cylinders $[0]$, $[1]$ are disjoint attractors.

Example 2 **L1** $\cap$ **E1** $\cap$ **A4** $\neq \emptyset$.

Proof: Take $A = 2$, $f(x)_i = 0$. Then $(A^{\mathbb{Z}}, f)$ is equicontinuous, and $\omega(A^{\mathbb{Z}}) = \overline{0}$ is the unique attractor different from $A^{\mathbb{Z}}$.

Example 3 **L2** $\cap$ **E2** $\cap$ **A1** $\neq \emptyset$.

Proof: Take $A = 2$, $f(x)_i = 0$ iff $x_{i-1} + x_i + x_{i+1} \leq 1$. Then $\overline{0}, \overline{1} \in Eq(A^{\mathbb{Z}}, f)$, $\overline{01} \notin Eq(A^{\mathbb{Z}}, f)$, and $\omega([0011])$, $\omega([1100])$ are disjoint attractors. The language is clearly regular. While every sequence $f^i(x)_0$ is eventually periodic, the length of the preperiod is not bounded, so the system is not bounded periodic.

Example 4 **L2** $\cap$ **E2** $\cap$ **A2** $\neq \emptyset$.

Proof: Take $A = 2$, $f(x)_i = x_i x_{i+1}$ (multiplication). Then $\overline{0} \in Eq(A^{\mathbb{Z}}, f)$, $\overline{1} \notin Eq(A^{\mathbb{Z}}, f)$, and $\overline{0} = \cap_k \omega([0^k])$ is the unique minimal quasi-attractor. It has a residual basin $\mathcal{B}(\overline{0}) = \{x \in A^{\mathbb{Z}}; (\forall n \in \mathbb{Z})(\exists m > n)(x_m = 0)\}$

Example 5 **L2** $\cap$ **E2** $\cap$ **A3** $\neq \emptyset$.

Proof: Take $A = 2$, $f(x)_i = x_{i-1}x_i x_{i+1}$ (multiplication). Then $\overline{0} \in Eq(A^{\mathbb{Z}}, f)$, $\overline{1} \notin Eq(A^{\mathbb{Z}}, f)$, and $\overline{0} = \omega([0])$ is the unique minimal attractor different from $\omega(A^{\mathbb{Z}})$. (We have for example $\overline{1} \in \omega(A^{\mathbb{Z}})$).

Example 6 $\mathbf{L2} \cap \mathbf{E2} \cap \mathbf{A4} \neq \emptyset$.

Take $A = \{0, 1, 2, 3\}$, with transition function given by

XY0	X2Y	2ZX	X03	03X	X13	X3Y
0	1	2	1	2	3	1

where $X, Y \in \{0, 1, 2, 3\}$, and $Z \in \{1, 3\}$. Here, for each triple of letters, the first applicable rule is used. If there is no applicable rule, the letter is left unchanged. We see that 0s propagate to the left erasing everything. In a row of 1s, a 3 travels to the left till it encounters a 0. It rewrites it to a 1, changes itself to a 2, and travels through the field of 1s to the right, erasing any 3 which it may encounter. Thus sequences $\overline{0}$, $\overline{1}$ and ...000111... are fixed points. In a sequence ...000111..., only the rightmost 0 can be rewritten to 1. This may be done only by a 3 arriving from the right, which would be changed to a 2, preventing the arrival of any other 3. Thus $\overline{0} \in Eq(f)$, while $\overline{1} \notin Eq(f)$. $\omega(A^{\mathbb{Z}})$ consists of those sequences, which do not contain a string of the form 22, 33, 02, 31^n2, 01^n21^m2, where $m, n \geq 0$. We have $(\overline{0}, ...000111...) \in \mathcal{C}f$, and $(\overline{0}, \overline{1}) \in \mathcal{C}f$. This might be seen in the right part of the ε-chain

0 12001311131133331000 3
0 1000311131131110001 3
0 0001211311311100003 1
0 0001123113111000012 1
0 0001112131110000011 3
0 0001111211100000013 1
0 0001111121000000031 1
0 0001111110000000121 1
0 0001111100000000112 1
0 0001111000000000111 3
0 0001110000000000113 1

For every $x \in \omega(A^{\mathbb{Z}})$ we have $(\overline{1}, x) \in \mathcal{C}f$, so $(\omega(A^{\mathbb{Z}}), f)$ is chain transitive. Moreover the system is regular. For example $u \in \Sigma_{\langle 0,0 \rangle}(f)$ iff it does not contain any of the strings: 01^n2, 01^n3, 21^n2, 21^n3, 33, $X0Y$, where $n \geq 0$, $X \neq 0 \neq Y$. This is a regular language.

Example 7 $\mathbf{L2} \cap \mathbf{E2} \cap \mathbf{A5} \neq \emptyset$.

Blanchard and Maass [4] prove that all Coven automata (see Coven [6]) belong to this class. They are automata over alphabet $A = 2$, the simplest example being $f(x)_i = x_i + x_{i+1}(x_{i+2} + 1)$ mod 2. Using this we may construct another cellular automaton in the class $\mathbf{L2} \cap \mathbf{E2} \cap \mathbf{A4}$ using alphabet $A = 3$, and taking the arguments in the above formula modulo 2.

Example 8 $\mathbf{L2} \cap \mathbf{E3} \cap \mathbf{A1} \neq \emptyset$.

Put $A = 2 \times 2$, $f(x, y)_i = (x_i, y_{i+1})$. Then $(A^{\mathbb{Z}}, f)$ is sensitive and not positively expansive. $[0] \times 2^{\mathbb{Z}}$ and $[1] \times 2^{\mathbb{Z}}$ are disjoint attractors.

Example 9 $\mathbf{L2} \cap \mathbf{E3} \cap \mathbf{A2} \neq \emptyset$.

Proof: Take $A = 2 \times 2$, $f(x, y)_i = (x_i x_{i+1}, y_{i+1})$. Then $(A^{\mathbb{Z}}, f)$ is sensitive, not positively expansive, $Q = \overline{0} \times 2^{\mathbb{Z}}$ is the unique minimal quasi-attractor. Its basin $\mathcal{B}(\overline{0} \times 2^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}}; (\forall n \in \mathbb{Z})(\exists m > n)(x_m = 0)\} \times 2^{\mathbb{Z}}$ is residual.

Example 10 $\mathbf{L2} \cap \mathbf{E3} \cap \mathbf{A3} \neq \emptyset$.

Proof: Take $A = 2 \times 2$, $f(x, y)_i = (x_{i-1} x_i x_{i+1}, y_{i+1})$. Then $(A^{\mathbb{Z}}, f)$ is sensitive, not positively expansive, and its minimal attractor $\overline{0} \times 2^{\mathbb{Z}}$ is different from $\omega(A^{\mathbb{Z}})$.

Example 11 $\mathbf{L2} \cap \mathbf{E3} \cap \mathbf{A4} \neq \emptyset$.

Proof: Take $A = \{0, 1, 2\}$ and $f(x)_i = x_{i-1} + x_i + x_{i+1} \bmod 2$. Then $(A^{\mathbb{Z}}, f)$ is sensitive, not positively expansive, and its unique attractor $\{0, 1\}^{\mathbb{Z}}$ is different from $A^{\mathbb{Z}}$.

Example 12 $\mathbf{L2} \cap \mathbf{E3} \cap \mathbf{A5} \neq \emptyset$.

Proof: Take $A = 2$, and $f(x)_i = x_{i+1}$ (the shift map). Then $(A^{\mathbb{Z}}, f)$ is sensitive, not positively expansive, open, surjective, transitive, and therefore its unique attractor is $A^{\mathbb{Z}}$.

Example 13 $\mathbf{L2} \cap \mathbf{E4} \cap \mathbf{A5} \neq \emptyset$.

Proof: Take $A = 2$ and $f(x)_i = x_{i-1} + x_i + x_{i+1} \bmod 2$. Then $(A^{\mathbb{Z}}, f)$ is positively expansive, and its unique attractor is $A^{\mathbb{Z}}$.

Example 14 $\mathbf{L3} \cap \mathbf{E3} \cap \mathbf{A4} \neq \emptyset$ *(cf. Gilman [11])*

Proof: Take $A = 2$, $f(x)_i = x_{i+1} x_{i+2}$. Then $(A^{\mathbb{Z}}, f)$ is sensitive and not positively expansive. We have $\omega(A^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}}; 10^k 1 \not\sqsubseteq x; k > 0\}$. $\omega(A^{\mathbb{Z}})$ is not topologically transitive since $f^k([100] \cap \omega(A^{\mathbb{Z}})) \cap [111] = 0$ for every k. However, $\omega(A^{\mathbb{Z}})$ is clearly chain transitive. It is not regular since a string $1^n 0^m 1$ occurs in $f^i(x)_0$ iff $m > n$.

Proposition 17 *Let $(A^{\mathbb{Z}}, f)$, $(B^{\mathbb{Z}}, g)$ be cellular automata with disjoint alphabets, and let $k : A \cup B \to A$ be a map which is identity on A. Call the annexation of $(B^{\mathbb{Z}}, g)$ to $(A^{\mathbb{Z}}, f)$ the cellular automaton $(C^{\mathbb{Z}}, h)$, where $C = A \cup B$,*
$h(u)_i = g(u_{i-1}, u_i, u_{i+1})$ if $u_{i-1}, u_i, u_{i+1} \in B$, and
$h(u)_i = f(k(u_{i-1}), k(u_i), k(u_{i+1}))$ otherwise. Then
1. If $(C^{\mathbb{Z}}, h)$ is regular, then so are $(A^{\mathbb{Z}}, f)$ and $(B^{\mathbb{Z}}, g)$.
*2. If $(A^{\mathbb{Z}}, f)$ is in **E2** or **E3**, then so is $(C^{\mathbb{Z}}, h)$.*
*3. If $(A^{\mathbb{Z}}, f)$ is in **A1** or **A2** or **A3**, then so is $(C^{\mathbb{Z}}, h)$.*

Proof: For every clopen partition β, $\mathcal{L}_\beta(f)$, $\mathcal{L}_\beta(g)$ are subsets of $\mathcal{L}_\beta(h)$. If $x \in A^{\mathbb{Z}}$, then $x \in Eq(f)$ iff $x \in Eq(h)$. $(C^{\mathbb{Z}}, h)$ keeps all the attractors of $(A^{\mathbb{Z}}, f)$. $\square$

Proposition 18 *The intersection classes* $\mathbf{L3} \cap \mathbf{E2} \cap \mathbf{A1}$, $\mathbf{L3} \cap \mathbf{E2} \cap \mathbf{A2}$, $\mathbf{L3} \cap \mathbf{E2} \cap \mathbf{A3}$, $\mathbf{L3} \cap \mathbf{E3} \cap \mathbf{A1}$, $\mathbf{L3} \cap \mathbf{E3} \cap \mathbf{A2}$, $\mathbf{L3} \cap \mathbf{E3} \cap \mathbf{A3}$ *are nonempty.*

Proof: Construct the annexation of the cellular automaton from Example 14 to cellular automata from the corresponding equicontinuity and attractor classes.

Example 15 $\mathbf{L3} \cap \mathbf{E2} \cap \mathbf{A4} \neq \emptyset$.

Here the argument in the proof of Proposition 17 does not work. We modify Example 6 so that a row of 3s in a field of 1s moves one field per step in front, and two fields per step in rear. We have again $A = \{0, 1, 2, 3\}$, with transition function given by

XY0	X2Y	2ZX	X03	03X	X13	X3W	XY33W
0	1	2	1	2	3	1	1

where $X, Y \in \{0, 1, 2, 3\}$, $Z \in \{1, 3\}$, and $W \in \{1, 2\}$. We show an example of an ε-chain;

0 2111111113331111333333331100 3
0 1211111133111113333333111001 3
0 1121111311111133333111110003 1
0 1112113111111333311111100012 1
0 1111231111113331111111000011 3
0 1111121111133111111110000013 1
2 1111112111311111111100000031 1
2 2111111213111111111000000121 1

Example 16 $\mathbf{L3} \cap \mathbf{E3} \cap \mathbf{A5} \neq \emptyset$.

Take $A = 2^3$ and $f : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ defined by

$$f(x, y, z)_i = (x_{i+1}, y_{i-1}\hat{x}_{i-1} + z_{i-1}x_{i-1}, z_{i+2}\hat{x}_{i+2} + y_{i+2}x_{i+2})$$

Here $\hat{0} = 1$, $\hat{1} = 0$. Then f is bijective as its inverse is

$$f^{-1}(x, y, z)_i = (x_{i-1}, y_{i+1}\hat{x}_{i-1} + z_{i-2}x_{i-1}, z_{i-2}\hat{x}_{i-1} + y_{i+1}x_{i-1})$$

There are three "pure particles" $4 = 100$, $2 = 010$, $1 = 001$, and their combinations. 4 travels to the left with speed 1 and never changes. 2 travels to the right with speed 1. It can cross 4 without change if their distance is odd. When their distance is even, they meet at the same field and 2 is changed to 1, which travels to the left with speed 2. When 1 encounters 4, it changes again to 2.

The language $L = \mathcal{L}_\beta(f)$, where $\beta(x) = x_0$ has following property: Suppose $n \geq 0$, $u \in A^{4n+5}$, and $40^n 10^{3n+2}$ is the initial string of u. Then $u \in L$ iff $u_{4n+4} \in \{2,3,6,7\}$. Indeed suppose $x_0 = 4$, $f^i(x)_0 = 0$ for $1 \leq i \leq n$, and $n+2 \leq i \leq 4n+3$, and $f^{n+1}(x)_0 = 1$. Then $f^{2n+2}(x)_{-2n-2} \in \{5,7\}$ (another 2 might come from the left), and $f^{2n+2+i}(x)_{-2n-2+i} = 2$ for $1 \leq i \leq 2n+1$. It follows $f^{4n+4}(x)_0 \in \{2,3,6,7\}$ (another 4 and/or 1 may come from the right). It is easy to see that the language L with this property is not regular. We show that $(A^{\mathbb{Z}}, f)$ is topologically transitive. Suppose $n > 0$, and $u, v \in A^n$. Define $x \in A^{\mathbb{Z}}$ by $x_i = u_i$ for $0 \leq i < n$, $x_i = 4$ otherwise. Then $f^j(x)_i = 4$ whenever either $2i + j \geq 2n + 3$ or $2i + j \leq -5$. Analogously define $y \in A^{\mathbb{Z}}$ by $y_i = v_i$ for $0 \leq i < n$, $y_i = 4$ otherwise. Finally construct $z \in A^{\mathbb{Z}}$ by $z_i = f^{2n+3}(x)_i$ for $i < n$, and $z_i = f^{-2n-3}(y)_i$ for $i \geq 0$ (this coincides for $0 \leq i < n$). Then $z \in f^{2n+3}([u]) \cap f^{-2n-3}([v])$. It is obvious that $(A^{\mathbb{Z}}, f)$ is sensitive and not positively expansive.

$$\begin{array}{c}
\ldots\ 020000000002444444444000040000100\ \ldots \\
\ldots\ 002000000004644444440004001000 0\ \ldots \\
\ldots\ 000200000054444444400040100000 0\ \ldots \\
\ldots\ 000020004464444440004100000000 0\ \ldots \\
\ldots\ 000002045444444400050000000000 0\ \ldots \\
\ldots\ 000000644644444000402000000000 0\ \ldots \\
\ldots\ 000014454444400040002000000000\ \ldots \\
\ldots\ 001044446444000400000200000000\ \ldots \\
\ldots\ 100444544440004000000002000000\ \ldots
\end{array}$$

Example 17 *There exists a cellular automaton with the shadowing property, whose column factorizations are not subshifts of finite type.*

Proof: Take $A = \{0,1,2,3,4\}$, with transition function given by

02	12	13	14	2	34	3
1	1	1	0	3	4	2

where again the first applicable rule is used, and if there is no applicable rule, the letter is left unchanged. Denote $\Sigma = \Sigma_{\langle 0,0 \rangle}(f)$. Then $\mathcal{L}(\Sigma) \subseteq \{0,1\}^* \cup \{2,3,4\}^*$, and $\Sigma \cap \{2,3,4\}^{\mathbb{N}}$ is a subshift of finite type. If $x \in A^{\mathbb{Z}}$, and $x_{i+1} \in \{0,1\}$ for some i, then $f^j(x)_i = x_i$ for all j. It follows that if $\varepsilon = 2^{-n}$, and $\delta = 2^{-n-1}$, then every δ-chain is ε-shadowed by some point. On the other hand, $01^n1 \in \mathcal{L}(\Sigma)$ iff n is even, so Σ is not a subshift of finite type. $\square$

Consider the uniform product measure on $A^{\mathbb{Z}}$. If Y is an attractor, then $\sigma(Y)$ is an attractor too. If Y is a minimal attractor or minimal quasi-attractor, it is σ-invariant, and its basin $\mathcal{B}(Y)$ is σ-invariant too. Thus minimal attractors, minimal quasi-attractors and their basins have either measure zero or one. In class **A2** Hurley [16] distinguishes cases of basin with measure zero and one. In Examples 4 and 9, the basin of the minimal quasi-attractor has measure one.

Example 18 *There exists a sensitive cellular automaton with unique minimal quasi-attractor, whose basin has measure zero.*

Put $A = 3 \times 2$, $f(u,v)_i = (u_i - v_i, \hat{u}_{i+1} v_{i+1})$, where $0 - 1 = 0$, $\hat{0} = 1$, $\hat{1} = \hat{2} = 0$. Thus the first component decreases when the second component is 1, and 1s in the second components are used up in the process. Unused 1s in the second components travel to the left until they encounter nonzero first component. $(A^{\mathbb{Z}}, f)$ is sensitive and not positively expansive. Its minimal quasi-attractor is $Q = \overline{0} \times 2^{\mathbb{Z}} = \cap_k \omega([0^k] \times 2^{\mathbb{Z}})$. For a given $(u,v) \in 3^{\mathbb{Z}} \times 2^{\mathbb{Z}}$, and $n \in \mathbb{Z}$, the first component of $\omega(u,v)_n$ is zero iff there exists $m \geq n$ such that $w_m = u_n - v_n + u_{n+1} - v_{n+1} + \ldots + u_m - v_m \leq 0$ (here the operations are carried from the left). If (u,v) is chosen randomly with uniform distribution, then $u_i - v_i \in \{-1, 0, 1, 2\}$ is a random variable with distribution $(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6})$. The sequence of random variables w_m is a Markov chain with initial distribution of $\mu[u_n - v_n = 0] = \frac{1}{2}$, $\mu[u_n - v_n = 1] = \frac{1}{3}$, $\mu[u_n - v_n = 2] = \frac{1}{6}$, and transition probabilities $P_{0,0} = 1$, and $P_{m,m-1} = \frac{1}{6}$, $P_{m,m} = \frac{1}{3}$, $P_{m,m+1} = \frac{1}{3}$, $P_{m,m+2} = \frac{1}{6}$ for $m > 0$. There is a positive probability that this chain does not ever visit zero. Thus the probability that the first component of $\omega(u,v)_n$ is zero is less than one and therefore the measure of $\mathcal{B}(Q)$ is less than 1. Since $\mathcal{B}(Q)$ is shift-invariant, this probability is zero.

Conjecture 1 *If a cellular automaton has a unique minimal quasi-attracttor, then this quasi-attractor has a residual basin.*

7 Conclusion

Corollary 4 *The relation between language, equicontinuity, and attractor classes is given by*

		A1	**A2**	**A3**	**A4**	**A5**
L1	**E1**	*1*	*0*	*0*	*1*	*0*
	E2	*0*	*0*	*0*	*0*	*0*
	E3	*0*	*0*	*0*	*0*	*0*
	E4	*0*	*0*	*0*	*0*	*0*
L2	**E1**	*0*	*0*	*0*	*0*	*0*
	E2	*1*	*1*	*1*	*1*	*1*
	E3	*1*	*1*	*1*	*1*	*1*
	E4	*0*	*0*	*0*	*0*	*1*
L3	**E1**	*0*	*0*	*0*	*0*	*0*
	E2	*1*	*1*	*1*	*1*	*?*
	E3	*1*	*1*	*1*	*1*	*1*
	E4	*0*	*0*	*0*	*0*	*0*

	A1	**A2**	**A3**	**A4**	**A5**
E1	*1*	*0*	*0*	*1*	*0*
E2	*1*	*1*	*1*	*1*	*1*
E3	*1*	*1*	*1*	*1*	*1*
E4	*0*	*0*	*0*	*0*	*1*

	A1	**A2**	**A3**	**A4**	**A5**
L1	*1*	*0*	*0*	*1*	*0*
L2	*1*	*1*	*1*	*1*	*1*
L3	*1*	*1*	*1*	*1*	*1*

	E1	**E2**	**E3**	**E4**
L1	*1*	*0*	*0*	*0*
L2	*0*	*1*	*1*	*1*
L3	*0*	*1*	*1*	*0*

We have seen several correlations between language, equicontinuity and attractor classifications. There is also a correspondence to the Wolfram classification introduced in [23]. Systems of class I and II belong to **L1** $\cup$ **L2**, **E1** $\cup$ **E2**, and **A2** $\cup$ **A3**. Systems of class III belong to **L2** $\cup$ **L3**, **E3** $\cup$ **E4**, and **A4** $\cup$ **A5**. Finally systems of class IV belong to **L3**, **E2** $\cup$ **E3**, and **A1**. We leave open the question of existence of systems in class **L3** $\cap$ **E2** $\cap$ **A5**.

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