Fast arithmetical algorithms in Möbius number systems

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***Abstract*—We analyze the time complexity of exact real arithmetical algorithms in Möbius number systems. Using the methods of Ergodic theory, we associate to any Möbius number system its transaction quotient $\mathbf{T}$ and show that the norm of the state matrix after n transactions is of the order $\mathbf{T}^n$. We argue that the Bimodular Möbius number system introduced in Kůrka [10] has transaction quotient less than 1.1, so that it computes the arithmetical operations faster than the standard positional r-ary systems.**

***Index Terms*—expansion subshifts; exact real arithmetical algorithms; emissions; absorptions; transactions.**

I. Introduction

While the floating-point system is still dominant in computer arithmetic, alternative systems which allow arbitrary precision and on-line algorithms have been considered as well. The classical ones are based on redundant positional systems (see e.g., Knuth [5]). In an unpublished but influential manuscript, Gosper [2] shows that continued fractions can be used for arithmetical algorithms, provided they are redundant. Based on these ideas, **exact real arithmetical algorithms** have been developped in Vuillemin [17], Kornerup and Matula [6], Potts [15], or Potts et al [16]. These algorithms perform a sequence of **input absorptions** and **output emissions** and update their inner state, which may be a (2×2)-matrix in the case of a Möbius transformation, a (2×4)-matrix in the case of binary operations like addition, multiplication or division, or an expression tree in the case of a transcendental function. Both emissions and absorptions are referred to as **transactions**.

Using the concepts and methods of symbolic dynamics, exact real arithmetic has been generalized in the theory of **Möbius number systems** (MNS) introduced in Kůrka [8] and [9]. Möbius number systems represent real numbers by infinite words from a one-sided subshift. The letters of the alphabet stand for real orientation-preserving Möbius transformations and the concatenation of letters corresponds to the composition of transformations. In Kůrka and Kazda [12] we have investigated interval MNS whose subshifts are determined by an interval cover or almost-cover indexed by the alphabet. Given a number x, we find an interval to which x belongs, take the inverse image of x by the corresponding transformation and repeat the procedure. The **expansion subshift** consists of all infinite words obtained. Using the concept of expansion quotient, we have given conditions which ensure that the extended real line is a factor of the expansion subshift. In Kůrka [10] we have investigated MNS in which rational numbers have periodic or preperiodic expansions. In Kůrka [11] we have characterized MNS whose expansion subshifts are of finite type or sofic and we have generalized the computation of the endpoints of cylinders by the Stern-Brocot graph.

The time complexity of exact real algorithms depends on the growth of their inner state matrices during computations. Heckmann [3] analyzes this process in positional number systems and proves the **Law of big numbers**, saying that the norm of the state matrix after n transactions is at least of the order $r^{n/2}$ for r-ary systems. This implies that the bit size of the state matrix grows at least linearly, and arithmetical operations have quadratic time complexity. In the present paper we show that the Law of big numbers does not apply to all Möbius number systems. Using methods of Ergodic theory we define the **transaction quotient** $\mathbf{T}$ of a MNS and argue that the average norm of the state matrix after n transactions is $\mathbf{T}^n$. We show that the **Nonredundant bimodular system** considered in Kůrka [10] has transaction quotient $\mathbf{T} < \sqrt{2}$ and outperforms the standard r-ary positional systems. In fact, numerical experiments suggest that the transaction quotient of the Nonredundant bimodular system is much closer to 1, at least $\mathbf{T} < 1.1$.

In redundant systems with sofic expansion subshift we consider the **Least norm algorithm** which minimizes the norm of the state matrices during the computation. In the **Redundant bimodular system** introduced in Kůrka [11], the algorithm gives good practical results. For the input length of several thousands, the norms of the state matrices remain most of the time bounded by 100, although fluctuations to much larger values occur sporadically. This suggests that the transaction quotient of the Bimodular system may be even equal to one, which would imply the existence of arithmetical algorithms with average linear time complexity.

II. Möbius transformations

The **extended real line** $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ can be regarded as a projective space, i.e., the space of one-dimensional subspaces of the two-dimensional vector space. On $\overline{\mathbb{R}}$ we have

homogenous coordinates $x = (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with equality $x = y$ iff $\det(x,y) = x_0 y_1 - x_1 y_0 = 0$. The norm of a vector $x \in \mathbb{R}^2$ is $||x|| = \sqrt{x_0^2 + x_1^2}$. We regard $x \in \overline{\mathbb{R}}$ as a column vector, and write it usually as $x = \frac{x_0}{x_1} = x_0/x_1$, for example $\infty = 1/0$. The **stereograhic projection** $\mathbf{h}(z) = (iz+1)/(z+i)$ maps $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} :\ |z| = 1\}$ in the complex plane, and the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} :\ \Im(z) > 0\}$ conformally to the unit disc $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$. Define the **circle distance** on $\overline{\mathbb{R}}$ by

$$\begin{aligned}
\varrho(x,y) &= \frac{1}{\pi} \arcsin \frac{|x-y|}{\sqrt{x^2+1} \cdot \sqrt{y^2+1}} \\
&= \frac{1}{\pi} \arcsin \frac{|x_0 y_1 - y_0 x_1|}{\sqrt{(x_0^2+x_1^2)(y_0^2+y_1^2)}} \\
&= \frac{1}{\pi} \arcsin \frac{|\det(x,y)|}{||x|| \cdot ||y||},
\end{aligned}$$

which is the length of the shortest arc in $\partial\mathbb{D}$ which joins $\mathbf{h}(x)$ and $\mathbf{h}(y)$ divided by 2π.

A real **orientation-preserving Möbius transformation** (MT) is a self-map of $\overline{\mathbb{R}}$ of the form

$$M_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d} = \frac{ax_0 + bx_1}{cx_0 + dx_1},$$

where $a,b,c,d \in \mathbb{R}$ and $\det(M_{(a,b,c,d)}) = ad - bc > 0$. MT acts also on the upper half-plane $\mathbb{U}$: If $z \in \mathbb{U}$ then $M(z) \in \mathbb{U}$ as well. On $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ we get **disc Möbius transformations** defined by

$$\begin{aligned}
\widehat{M}_{(a,b,c,d)}(z) &= \mathbf{h} \circ M_{(a,b,c,d)} \circ \mathbf{h}^{-1}(z) \\
&= (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha}),
\end{aligned}$$

where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. We have

$$|\widehat{M}(0)|^2 = \frac{||M||^2 - \det(M)}{||M||^2 + \det(M)},$$

where $||M_{(a,b,c,d)}|| = \sqrt{a^2+b^2+c^2+d^2}$ is the norm of the matrix M (see Kůrka [9]). The **circle derivation** of $M = M_{(a,b,c,d)}$ is defined by

$$\begin{aligned}
M^\bullet(x) &= \lim_{y \to x} \frac{\varrho(M(y), M(x))}{\varrho(y,x)} \\
&= \frac{(ad-bc) \cdot (x_0^2 + x_1^2)}{(ax_0 + bx_1)^2 + (cx_0 + dx_1)^2}, \\
&= \frac{\det(M) \cdot ||x||^2}{||M(x)||^2}.
\end{aligned}$$

The **expansion interval** of $M = M_{(a,b,c,d)}$ is

$$\mathbf{V}(M) = \{x \in \overline{\mathbb{R}} :\ (M^{-1})^\bullet(x) > 1\}.$$

If $M = R_\alpha = M_{(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, -\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2})}$ is a rotation, then $M^\bullet(x) = 1$ and $\mathbf{V}(M)$ is empty. Otherwise $\mathbf{V}(M)$ is a proper set interval, i.e., a nonempty open connected subset of $\overline{\mathbb{R}}$.

III. Intervals

A **set interval** is an open connected subset of $\overline{\mathbb{R}}$. A **proper interval** is a nonempty set interval properly included in $\overline{\mathbb{R}}$. We represent proper intervals by (2×2)-matrices whose columns are their left and right endpoints. The stereographic projection applied to $x = \frac{r \sin\alpha}{r \cos\alpha} \in \overline{\mathbb{R}}$ gives $\mathbf{h}(x) = \sin 2\alpha - i \cos 2\alpha = e^{i(2\alpha - \frac{\pi}{2})}$, so it duplicates the angles. Intervals with endpoints $x = \frac{r\sin\alpha}{r\cos\alpha}$, $y = \frac{s \sin\beta}{s\cos\beta}$ where $0 \le \alpha < 2\pi$, $\alpha < \beta < \alpha + \pi$ can therefore represent any proper interval. Since $\det(x,y) = rs\sin(\alpha - \beta) < 0$, we define matrix intervals as (2×2)-matrices with negative determinant, which we write as pairs $I = (\frac{x_0}{x_1}, \frac{y_0}{y_1})$ of their left and right endpoints $\mathfrak{l}(I) = \frac{x_0}{x_1}$, $\mathbf{r}(I) = \frac{y_0}{y_1}$. The set of **matrix intervals** is therefore

$$\mathbb{I}(\mathbb{R}) = \{(\tfrac{x_0}{x_1}, \tfrac{y_0}{y_1}) \in \mathrm{GL}(\mathbb{R},2) :\ x_0 y_1 - x_1 y_0 < 0\}$$

The length of an interval is defined by

$$\begin{aligned}
|(x,y)| &= \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x_0 y_0 + x_1 y_1}{x_0 y_1 - x_1 y_0} \\
&= \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x \cdot y}{\det(x,y)}
\end{aligned}$$

Then we get $|(\frac{r\sin\alpha}{r\cos\alpha}, \frac{s\sin\beta}{s\cos\beta})| = (\beta - \alpha)/\pi$, provided $0 < \beta - \alpha < \pi$. A matrix interval defines an open and closed set interval by

$$\begin{aligned}
z \in I &\iff \det(\mathfrak{l}(I), z) \cdot \det(z, \mathbf{r}(I)) > 0, \\
z \in \overline{I} &\iff \det(\mathfrak{l}(I), z) \cdot \det(z, \mathbf{r}(I)) \ge 0.
\end{aligned}$$

If $I = (\frac{r\sin\alpha}{r\cos\alpha}, \frac{s\sin\beta}{s\cos\beta})$, then $z = \frac{t\sin\gamma}{t\cos\gamma} \in I$ iff either $\alpha < \gamma < \beta$ or $\alpha + \pi < \gamma < \beta + \pi$. If $x, y \in \mathbb{R}$, then

$$(x,y) = \begin{cases} \{z \in \mathbb{R} :\ x < z < y\} & \text{if} \quad x < y, \\ \{z \in \mathbb{R} :\ x < z \text{ or } z < y\} \cup \{\infty\} & \text{if} \quad x > y. \end{cases}$$

When we transform intervals, we work with the matrix representations of MT rather than with the transformations themselves. Möbius transformations are represented by matrices

$$\mathbb{M}(\mathbb{R}) = \{M_{(a,b,c,d)} \in \mathrm{GL}(\mathbb{R},2) :\ ad - bc > 0\}$$

which act on vectors $x \in \mathbb{R}^2$ by multiplication $x \mapsto Mx$. Two matrices represent the same MT if one is a nonzero multiple of the other and the matrix multiplication corresponds to the composition of MT. If $M \in \mathbb{M}(\mathbb{R})$ and $I \in \mathbb{I}(\mathbb{R})$, then both MI and IM are intervals. While $MI = M(I)$ represents the M-image of the set interval of I, IM is the interval cut from I by M. This operation is used to obtain the Stern-Brocot graph of a MNS with expansion subshift of finite type (see Kůrka [11]).

IV. Subshifts

For a finite alphabet $\mathbb{A}$ denote by $\mathbb{A}^* := \bigcup_{m \ge 0} \mathbb{A}^m$ the set of finite words. The length of a word $u = u_0 \ldots u_{m-1} \in \mathbb{A}^m$ is $|u| = m$. We denote by $\mathbb{A}^{\mathbb{N}}$ the Cantor space of infinite words with the metric $d(u,v) = 2^{-k}$, where $k = \min\{i \ge 0 : u_i \ne v_i\}$. We say that $v \in \mathbb{A}^*$ is a subword of $u \in \mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ and write $v \sqsubseteq u$, if $v = u_{[i,j)} = u_i \ldots u_{j-1}$ for some $0 \le i \le j \le |u|$. The cylinder of $u \in \mathbb{A}^n$ is the set

$[u] = \{v \in \mathbb{A}^{\mathbb{N}} :\ v_{[0,n)} = u\}$. The **shift map** $\sigma : \mathbb{A}^{\mathbb{N}} \to \mathbb{A}^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty set $\Sigma \subseteq \mathbb{A}^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. If $D \subseteq \mathbb{A}^*$ then $\Sigma_D = \{x \in \mathbb{A}^{\mathbb{N}} :\ \forall u \sqsubseteq x, u \notin D\}$ is the subshift (provided it is nonempty) with **forbidden set** D. Any subshift can be obtained in this way. A subshift is uniquely determined by its **language**

$$\mathcal{L}(\Sigma) = \{u \in \mathbb{A}^* :\ \exists x \in \Sigma, u \sqsubseteq x\}.$$

Denote by $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap \mathbb{A}^n$.

A **labelled graph** over an alphabet $\mathbb{A}$ is a structure $\mathcal{G} = (V, E, s, t, \ell)$, where $V = |\mathcal{G}|$ is the set of vertices, E is the set of edges, $s, t : E \to V$ are the source and target maps, and $\ell : E \to \mathbb{A}$ is a labelling function. The subshift $\Sigma_{\mathcal{G}} \subseteq \mathbb{A}^{\mathbb{N}}$ of $\mathcal{G}$ consists of all labels of paths of $\mathcal{G}$. A subshift is **sofic**, if it is the subshift of a finite labelled graph. A subshift is of **finite type** (SFT) of order p, if its forbidden words have length at most p. In this case $u \in \mathbb{A}^{\mathbb{N}}$ belongs to Σ iff all subwords of u of length p belong to $\mathcal{L}(\Sigma)$ (see Lind and Marcus [13] or Kůrka [7]).

V. Möbius number systems

Definition 1: A **Möbius iterative system** over an alphabet $\mathbb{A}$ is a map $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ or a family of orientation-preserving Möbius transformations $(F_u : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{u \in \mathbb{A}^*}$ satisfying $F_{uv} = F_u \circ F_v$ and $F_\lambda = \mathrm{Id}$, where λ is the empty word. The **convergence space** $\mathbb{X}_F \subseteq \mathbb{A}^{\mathbb{N}}$ and the **symbolic representation** $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ are defined by

$$\begin{aligned} \mathbb{X}_F &:= \{u \in \mathbb{A}^{\mathbb{N}} :\ \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}, \\ \Phi(u) &:= \lim_{n\to\infty} F_{u_{[0,n)}}(i), \end{aligned}$$

where $i \in \mathbb{U}$ is the imaginary unit. If $\Sigma \subseteq \mathbb{X}_F$ is a subshift such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective, then we say that (F, Σ) is a **Möbius number system** (MNS). We say that a MNS (F, Σ) is **redundant**, if for every continuous map $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $\Phi f = g\Phi$.

Redundancy is necessary for the existence of exact arithmetical algorithms (see Weihrauch [18], Vuillemin [17], Kornerup and Matula [6], Potts [15] or Potts et al. [16]). If $u \in \mathbb{X}_F$ then $\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(z)$ for every $z \in \mathbb{U}$ (see Kazda [4]).

Definition 2: An **open almost-cover** for a Möbius iterative system $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a family of open intervals $\mathcal{W} = \{W_a :\ a \in \mathbb{A}\}$ such that $\bigcup_{a\in\mathbb{A}} \overline{W_a} = \overline{\mathbb{R}}$. We denote by $\mathcal{E}(\mathcal{W}) = \{\mathfrak{l}(W_a), \mathfrak{r}(W_a) :\ a \in \mathbb{A}\}$ the **set of endpoints** of $\mathcal{W}$. If $W_a \cap W_b = \emptyset$ for $a \neq b$, then $\mathcal{W}$ is an **open partition**. If $\bigcup_{a\in\mathbb{A}} W_a = \overline{\mathbb{R}}$ then $\mathcal{W}$ is a **cover**. The **interval cylinder** of $u \in \mathbb{A}^{n+1}$ is

$$W_u = W_{u_0} \cap F_{u_0} W_{u_1} \cap \cdots \cap F_{u_{[0,n)}} W_{u_n}.$$

The **expansion subshift** $\mathcal{S}_{\mathcal{W}}$ is defined by

$$\mathcal{S}_{\mathcal{W}} = \{u \in \mathbb{A}^{\mathbb{N}} :\ \forall k > 0, W_{u_{[0,k)}} \neq \emptyset\}.$$

We denote by $\mathcal{L}_{\mathcal{W}} = \mathcal{L}(\mathcal{S}_{\mathcal{W}})$ the language of $\mathcal{S}_{\mathcal{W}}$ and by $\mathcal{L}^n_{\mathcal{W}} = \mathcal{L}^n(\mathcal{S}_{\mathcal{W}})$.

For $uv \in \mathcal{L}_{\mathcal{W}}$ we have $W_{uv} = W_u \cap F_u W_v$. Given a cover $\mathcal{W}$, we construct nondeterministically the expansion $u \in \mathcal{S}_{\mathcal{W}}$ of $x = x_0 \in \overline{\mathbb{R}}$ as follows: Choose u_0 with $x \in W_{u_0}$, choose u_1 with $x_1 = F^{-1}_{u_0}(x_0) \in W_{u_1}$, choose u_2 with $x_2 = F^{-1}_{u_1}(x_1) \in W_{u_2}$, etc. Then $x \in W_{u_{[0,n)}}$ for each n, so W_u is the set of points which have expansion u.

Theorem 3 (Kůrka and Kazda [12]): Let $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and assume that $\mathcal{W}$ is an almost-cover of $\overline{\mathbb{R}}$ such that $W_a \subseteq \mathbf{V}(F_a)$ for each $a \in \mathbb{A}$. Then $(F, \mathcal{S}_{\mathcal{W}})$ is a Möbius number system. It is redundant provided $\mathcal{W}$ is a cover. For each $u \in \mathcal{S}_{\mathcal{W}}$ and $v \in \mathcal{L}_{\mathcal{W}}$,

$$\{\Phi(u)\} = \bigcap_{n\geq 0} \overline{W_{u_{[0,n)}}},\ \Phi([v] \cap \mathcal{S}_{\mathcal{W}}) = \overline{W_v}.$$

A stronger theorem which uses the concept of expansion quotient has been proved in Kůrka and Kazda [12]. Nevertheless our examples satisfy the condition of Theorem 3, so we adopt it as a definition.

Definition 4: An **interval Möbius number system** over an alphabet $\mathbb{A}$ is a pair $(F, \mathcal{W})$, where $F : \mathbb{A}^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Möbius iterative system and $\mathcal{W} = \{W_a :\ a \in \mathbb{A}\}$ is an almost-cover of $\overline{\mathbb{R}}$ such that $W_a \subseteq \mathbf{V}(F_a)$ for each $a \in \mathbb{A}$.

If $(F, \mathcal{W})$ is an interval MNS then

$$\lim_{n\to\infty} \max\{|W_u| :\ u \in \mathcal{L}^n_{\mathcal{W}}\} = 0.$$

This is an immediate consequence of the uniform continuity of $\Phi : \mathcal{S}_{\mathcal{W}} \to \overline{\mathbb{R}}$. Interval Möbius number systems whose expansion subshifts are of finite type have been characterized in Kůrka [11]:

Theorem 5 (Kůrka [11]): Assume that $(F, \mathcal{W})$ is an interval MNS. Then $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2 iff

$$\forall a, b \in \mathbb{A}, (F_a(W_b) \cap W_a \neq \emptyset\ \Rightarrow\ F_a(W_b) \subseteq W_a).$$

In this case $W_u = F_{u_{[0,n)}} W_{u_n}$ for each $u \in \mathcal{L}^{n+1}_{\mathcal{W}}$.

VI. Rational Möbius number systems

We say that an interval Möbius number system $(F, \mathcal{W})$ is **rational**, if its transformations have integer entries and its intervals have rational endpoints. We consider arithmetical algorithms in rational interval Möbius number systems. To analyze the cancellations which occur during transactions, we work with the matrices which represent the transformations, rather than with the transformations themselves. Denote by $\mathbb{Z}$ the set of integers. For $x \in \mathbb{R}^2 \setminus \{\binom{0}{0}\}$ denote by $\gcd(x)$ the greatest common divisor of x_0 and x_1. Denote by

$$\overline{\mathbb{Q}} = \{x \in \mathbb{Z}^2 \setminus \{\tbinom{0}{0}\} :\ \gcd(x) = 1\}$$

the set of (homogenous coordinates of) rational numbers which we understand as a subset of $\overline{\mathbb{R}}$. We have a map $\mathbf{d} : \mathbb{Z}^2 \setminus \{\binom{0}{0}\} \to \overline{\mathbb{Q}}$ given by $\mathbf{d}(x) = \frac{x_0/g}{x_1/g}$, where $g = \gcd(x)$. Denote by

$$\mathbb{M}(\mathbb{Z}) = \{M \in \mathrm{GL}^+(\mathbb{Z}, 2) :\ \gcd(M) = 1\},$$

where $\mathrm{GL}^+(\mathbb{Z}, 2)$ is the set of (2×2)-matrices with integers entries and positive determinant. For $M = M_{(a,b,c,d)} \in$

$\mathrm{GL}^+(\mathbb{Z},2)$ denote by $\mathfrak{d}(M) = M_{(a/g,b/g,c/g,d/g)}$, where $g = \gcd(M)$, so we have a mapping $\mathfrak{d} : \mathrm{GL}^+(\mathbb{Z},2) \to \mathbb{M}(\mathbb{Z})$. In $\mathbb{M}(\mathbb{Z})$ we have multiplication $MN = \mathfrak{d}(M \cdot N)$, where $M \cdot N$ is the matrix multiplication and a pseudo-inverse $M^{-1}_{(a,b,c,d)} = M_{(d,-b,-c,a)}$. Matrices of $\mathbb{M}(\mathbb{Z})$ act on $\overline{\mathbb{Q}}$ by $Mx = \mathbf{d}(M \cdot x)$.

We consider now the computation of an MT $M \in \mathbb{M}(\mathbb{Z})$ in an MNS. This means that we search for an algorithm which would compute a continuous function $\Psi_M : \mathcal{S}_{\mathcal{W}} \to \mathcal{S}_{\mathcal{W}}$ such that $\Phi\Psi_M(u) = M\Phi(u)$ for each $u \in \mathcal{S}_{\mathcal{W}}$. Such a mapping Ψ_M can exist only in redundant MNS. We consider first nonredundant systems in which Ψ_M is a partial mapping. If a MNS has the expansion subshift of finite type of order 2, then $W_{ua} = F_u W_a$ for each $ua \in \mathcal{L}_{\mathcal{W}}$ (see Theorem 5), which simplifies the algorithm considerably.

Definition 6: Let $(F, \mathcal{W})$ be a rational interval MNS with an open partition $\mathcal{W}$, whose expansion subshift is an SFT of order 2. The **unary graph** of $(F, \mathcal{W})$ is defined as follows: Its vertices are (X, a), where $X \in \mathbb{M}(\mathbb{Z})$ and $a \in \mathbb{A} \cup \{\lambda\}$. The labelled edges are

$$\begin{array}{rlll} \text{absorption:} & (X,a) \xrightarrow{b/\lambda} (XF_a, b) & \text{if} & ab \in \mathcal{L}^2_{\mathcal{W}} \\ \text{emission:} & (X,a) \xrightarrow{\lambda/c} (F_c^{-1}X, a) & \text{if} & XW_a \subseteq W_c. \end{array}$$

For the empty word λ we set $W_\lambda = \overline{\mathbb{R}}$. The labels of paths are concatenations of the labels of their edges. They have the form u/v, where $u \in \mathcal{L}_{\mathcal{W}}$ is the input word and $v \in \mathcal{L}_{\mathcal{W}}$ is the output word. Given $M \in \mathbb{M}(\mathbb{Z})$ and $u \in \mathcal{S}_{\mathcal{W}}$, the lazy algorithm which computes $v \in \mathcal{S}_{\mathcal{W}}$ with $\Phi(v) = M\Phi(u)$ starts at the vertex (M, λ), applies the emission action whenever possible and the absorption action otherwise. Since $\mathcal{W}$ is assumed to be an open partition, the algorithm is deterministic but partial. On some infinite input words the algorithms may give only a finite output word.

Proposition 7: If u/v is the label of an infinite path in the unary graph with source (M, λ), $u \in \mathcal{S}_{\mathcal{W}}$, and $w \in \mathbb{A}^{\mathbb{N}}$, then $w \in \mathcal{S}_{\mathcal{W}}$ and $\Phi(w) = M(\Phi(u))$. If u/v is the label of a finite path with source (M, λ), and $u \in \mathcal{L}_{\mathcal{W}}$, then $w \in \mathcal{L}_{\mathcal{W}}$ and $M(\Phi([u])) \subseteq \Phi([w])$.

Proof: We show by induction that when there is a path with source (M, λ) and label $ua/w \in \mathcal{L}_{\mathcal{W}} \times \mathbb{A}^*$, then $MW_{ua} \subseteq W_w$ and the target of the path is $(F_w^{-1}MF_u, a)$. Since $W_\lambda = \overline{\mathbb{R}}$, the first edge $(M, \lambda) \to (M, a)$ has label a/λ, so $MW_a \subseteq W_\lambda$ is satisfied. Suppose that the assumption holds for ua/w, and consider an edge $(F_w^{-1}MF_u, a) \to (F_w^{-1}MF_{ua}, b)$ with label b/λ. Then $MW_{uab} \subseteq MW_{ua} \subseteq W_w$, so the statement holds for the path label uab/w. Consider an edge $(F_w^{-1}MF_u, a) \to (F_{wc}^{-1}MF_u, a)$, with label λ/c, so $F_w^{-1}MF_uW_a \subseteq W_c$. Then $MW_{ua} = MF_uW_a \subseteq F_wW_c$. Since $MW_{ua} \subseteq W_w$, we get $MW_{ua} \subseteq W_w \cap F_wW_c = W_{wc}$, so the statement holds for the path label ua/wc. By Theorem 3 we get $M(\Phi([ua])) \subseteq \Phi([w])$. If u, v are infinite words, then for each n there exists k_n such that $MW_{u_{[0,k_n)}} \subseteq W_{w_{[0,n]}}$, so $M(\Phi(u)) \in M(\overline{W_{u_{[0,k_n)}}}) \subseteq \overline{W_{w_{[0,n]}}}$. It follows $M(\Phi(u)) = \Phi(w)$. ∎

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nonredundant unary procedure;
input matrix M ∈ 𝕄(ℤ);
input number u ∈ L_W ∪ S_W;
output number v ∈ L_W ∪ S_W;
variables X ∈ 𝕄(ℤ) (state matrix), n, m ∈ ℕ (input and output pointers)
begin
  X := M; n := 0; m := 0;
  while n < |u| repeat
    if ∀b ∈ 𝔸, XW_{u_n} ⊈ W_b then begin
      X := XF_{u_n};
      n := n + 1;
      end;
    else begin
      v_m := b; where XW_{u_n} ⊆ W_b
      X := F_b^{-1} X;
      m := m + 1;
      end;
    end;
  end;
```

TABLE I
THE NONREDUNDANT UNARY ALGORITHM

Assume that $(F, \mathcal{W})$ is an integer MNS such that $\mathcal{W}$ is an open partition and $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2. We consider the **Nonredundant unary algorithm** (see Table I) which computes M on symbolic representations of real numbers with the use of the unary graph. The input for the algorithm is either a finite word $u \in \mathcal{L}_{\mathcal{W}}$ or an infinite word $u \in \mathcal{S}_{\mathcal{W}}$. Since $\mathcal{W}$ is an open partition, there exists at each step at most one $b \in \mathbb{A}$ with $MW_{u_n} \subseteq W_b$, so the algorithm is deterministic. If the input $u \in \mathcal{L}_{\mathcal{W}}$ is finite then the algorithm halts in a finite time, its output $v \in \mathcal{L}_{\mathcal{W}}$ is finite, its length is stored in the variable m and $\Phi(\overline{W_u}) \subseteq \overline{W_v}$. If the input $u \in \mathcal{S}_{\mathcal{W}}$ is infinite, the algorithm never stops and during infinite time produces either infinite or finite output v. The latter possibility occurs (with probability zero) if $M(\Phi(u_{[n,\infty)})) \in \mathcal{E}(\mathcal{W})$ is an endpoint of $\mathcal{W}$ for some n.

If the entries of the state matrix X are represented in a positional binary system, then the length of this representation (the bit length of X) is of the order $\log_2 ||X||$. A multiplication of X with a matrix F_a then requires $\log_2 ||X|| \cdot \log_2 ||F_a||$ elementary operations on their binary representations. The comparison $I \subseteq W_b$ requires $\log_2 ||I|| \cdot \log_2 ||W_b||$ elementary operations. Thus there exists a constant $c > 0$ such that each step of the algorithm requires at most $c \cdot \log_2 ||X||$ elementary operations. We argue in next sections that the norm of the state matrices X_n at time n is of the order $\mathbf{T}^n$, where $\mathbf{T} \geq 1$ is a transaction quotient. For the bit size we get $\log_2 ||X_n|| \approx n \log_2 \mathbf{T}$. It follows that the time complexity of the algorithm (number of elementary operations with the bit representations of the matrices) is quadratic of the order $\log_2 \mathbf{T} \cdot n^2/2$. If $\mathbf{T} = 1$, then the matrices remain bounded and the time complexity of the algorithm is linear in the input length u.

VII. Singular transformations

Besides orientation-preserving MT with positive determinant, we consider orientation-reversing MT with negative determinant, **singular** MT with $\det(M) = 0$ and $||M|| > 0$, and the **zero** MT $M_{(0,0,0,0)} = 0$. Each MT defines a closed graph (relation)

$$\widetilde{M} = \{(x,y) \in \overline{\mathbb{R}}^2 : \ (ax_0 + bx_1)y_1 = (cx_0 + dx_1)y_0\}.$$

An MT $M = M_{(a,b,c,d)}$ is singular iff $\forall z \in \mathbb{U}, M(z) \in \overline{\mathbb{R}}$ iff $M(i) \in \overline{\mathbb{R}}$ where $i \in \mathbb{U}$ is the imaginary unit. In this case we call $\mathbf{s}(M) = M(i)$ the stable point of M. If $\frac{a}{c} \neq \frac{0}{0}$ then $M(i) = \frac{a}{c}$, and if $\frac{b}{d} \neq \frac{0}{0}$ then $M(i) = \frac{b}{d}$. Thus $\mathbf{s}(M) \in \{\frac{a}{c}, \frac{b}{d}\} \cap \overline{\mathbb{R}}$. Similarly the unstable point of M is defined by $\mathbf{u}(M) = M^{-1}(i) \in \{-\frac{b}{a}, -\frac{d}{c}\} \cap \overline{\mathbb{R}}$. A singular MT yields the graph $\widetilde{M} = (\overline{\mathbb{R}} \times \{s_M\}) \cup (\{u_M\} \times \overline{\mathbb{R}})$, and the zero MT yields the full graph $\widetilde{M} = \overline{\mathbb{R}}^2$. The space of all nonzero MT can be identified with the projective linear space $\mathrm{PL}(\mathbb{R}, 4)$ of one-dimensional subspaces of the Euclidean space $\mathbb{R}^4$ with metric

$$d_4(X,Y) = \min\left\{\left|\left|\frac{X}{||X||} - \frac{Y}{||Y||}\right|\right|, \left|\left|\frac{X}{||X||} + \frac{Y}{||Y||}\right|\right|\right\}.$$

Assume that $(F, \mathcal{W})$ is a MNS such that $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2. If X is an orientation preserving MT and $u \in \mathcal{S}_{\mathcal{W}}$ then $\lim_{n\to\infty} XF_{u_{[0,n)}}(0) \in \overline{\mathbb{R}}$ and $\lim_{n\to\infty} |XF_{u_{[0,n)}} W_{u_n}| = 0$. This means that the sequence $MF_{u_{[0,n)}}$ of MT converges to the subspace of singular MT: There exists a sequence H_n of singular MT such that $\lim_{n\to\infty} d_4(XF_{u_{[0,n)}}, H_n) = 0$.

We modify the unary algorithm from Table I so that the absorption step is performed whenever the length of the interval XW_a is greater than some small fixed $\varepsilon > 0$. Then the state matrices remain in vicinity of singular matrices. The growth of the norm of the state matrices during the computation can then be approximated by the growth of the norm of singular matrices subjected to the unary algorithm. To test the length of an interval $I = (x, y)$ we need not evaluate its actual length. Instead we perform the test

$$\frac{x \cdot y}{\det(x,y)} < \delta = \tan(\pi\varepsilon - \pi/2),$$

where we choose a rational $\delta \in (-\infty, +\infty)$.

Next proposition shows that the absorption and emission actions on singular matrices are independent. Emission acts on the columns of singular matrices while absorption acts on the rows of the singular matrices. Thus the absorption and emission processes can be studied separately.

Proposition 8: If X is a singular matrix, then for each matrix F with positive determinant we have

$$\mathbf{s}(XF) = \mathbf{s}(X), \ \mathbf{s}(FX) = F(\mathbf{s}(X)),$$

$$\mathbf{u}(FX) = \mathbf{u}(X), \ \mathbf{u}(XF) = F^{-1}(\mathbf{u}(X))$$

This follows from $XF(i) = X(i)$.

VIII. Invariant emission measure

Denote by $\mathcal{M}(X)$ the space of Borel probability measures on a compact metric space X with the Hutchinson metric (see Barnsley [1]). A continuous mapping $F : X \to Y$ between compact metric spaces can be extended to a continuous mapping $F : \mathcal{M}(X) \to \mathcal{M}(Y)$ by $(F\mu)(U) = \mu(F^{-1}(U))$. For an integrable function $\varphi : Y \to \mathbb{R}$ we have

$$\int_Y \varphi \, d(F\mu) = \int_X \varphi \circ F \, d\mu$$

The circle length of intervals determines the Cauchy measure $\gamma \in \mathcal{M}(\overline{\mathbb{R}})$ by $\gamma(I) = |I|$ for each interval I. If $\mu \in \mathcal{M}(\overline{\mathbb{R}})$ is absolutely continuous with respect to γ then it has a density $h_\mu : \overline{\mathbb{R}} \to [0, \infty)$ given by

$$h_\mu(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon(x)|},$$

where $B_\varepsilon(x) = \{y \in \overline{\mathbb{R}} : \ \varrho(y,x) < \varepsilon\}$. Thus $d\mu(x) = h_\mu(x)\, d\gamma(x)$ and

$$\mu(I) = \int_I h_\mu(x)\, d\gamma(x) = \int_I \frac{h_\mu(x)\, dx}{\pi(1+x^2)}$$

for each interval I. If M is a MT and μ has a density, then

$$h_{M\mu}(x) = h_\mu(M^{-1}(x)) \cdot (M^{-1})^\bullet(x),$$

in particular $h_{M\gamma}(x) = (M^{-1})^\bullet(x)$. For an integrable function φ and an interval I, the substitution $y = M(x)$ gives

$$\int_{M(I)} \varphi(y)\, d\gamma(y) = \int_I \varphi(M(x)) \cdot M^\bullet(x) \cdot d\gamma(x).$$

Given an interval MNS $(F, \mathcal{W})$, denote by $\bar{W}_a = W_a \cup \{\mathfrak{l}(W_a)\}$ the semiclosed interval with its left endpoint. Denote by $\mathcal{C}(\mathcal{W})$ the set of all functions $h : \overline{\mathbb{R}} \to [0, \infty)$ which are continuous on each $\bar{W}_a$ and have limits from the left at each $\mathfrak{r}(W_a)$. With the supremum distance $d(h,g) = \sup_{x\in\overline{\mathbb{R}}} |h(x) - g(x)|$, $\mathcal{C}(\mathcal{W})$ is a compact metric space. A **partition of unity** for a MNS $(F, \mathcal{W})$ over an alphabet $\mathbb{A}$ is a system of nonnegative functions $w_a \in \mathcal{C}(\mathcal{W})$ indexed by the alphabet $\mathbb{A}$ such that $\sum_{a\in A} w_a(x) = 1$ for each $x \in \overline{\mathbb{R}}$ and $\mathrm{supp}(w_a) = \{x \in \overline{\mathbb{R}} : \ w_a(x) > 0\} \subseteq \bar{W}_a$. A partition of unity determines the emission Markov process $(X_n)_{n\geq 0}$ over $\overline{\mathbb{R}}$ with transition probabilities

$$\mathbb{P}[X_{n+1} = F_a^{-1}(x) | X_n = x] = w_a(x) :$$

The emission $x \xrightarrow{a} F_a^{-1}(x)$ happens with probability $w_a(x)$. The path with source $x \in \overline{\mathbb{R}}$ and label $u \in \mathcal{L}_{\mathcal{W}}^{k+1}$ has probability

$$w_u(x) = w_{u_0}(x) \cdot w_{u_1}(F_{u_0}^{-1}(x)) \cdots w_{u_k}(F_{u_{[0,k)}}^{-1}(x)).$$

We have $w_{uv}(x) = w_u(x) \cdot w_v(F_u^{-1}(x))$ for each $uv \in \mathcal{L}_{\mathcal{W}}$. The process X determines the emission map $\mathcal{E} : \mathcal{M}(\overline{\mathbb{R}}) \to \mathcal{M}(\overline{\mathbb{R}})$ given by

$$d(\mathcal{E}\mu)(x) = \sum_{a\in\mathbb{A}} w_a(F_a(x)) \cdot d(F_a^{-1}\mu)(x).$$

This means that

$$\int \varphi(x)\, d(\mathcal{E}\mu)(x) = \sum_{a\in\mathbb{A}} \int \varphi(F_a(y)) \cdot w_a(y)\, d\mu(y)$$

for each continuous function $\varphi : \overline{\mathbb{R}} \to \mathbb{R}$. If μ has density h_μ, then $\mathcal{E}\mu$ has density

$$\begin{aligned} h_{\mathcal{E}\mu}(x) &= \lim_{\varepsilon\to 0} \sum_{a\in\mathbb{A}} \frac{\mu(F_a(B_\varepsilon(x))) \cdot w_a(F_a(x))}{|B_\varepsilon(x)|} \\ &= \lim_{\varepsilon\to 0} \sum_{a\in\mathbb{A}} \frac{|F_a(B_\varepsilon(x))| \cdot h_\mu(F_a(x)) \cdot w_a(F_a(x))}{|B_\varepsilon(x)|} \\ &= \sum_{a\in\mathbb{A}} F_a^\bullet(x) \cdot h_\mu(F_a(x)) \cdot w_a(F_a(x)) \end{aligned}$$

We say that $\mu \in \mathcal{M}(\overline{\mathbb{R}})$ is an $(F, \boldsymbol{w})$-invariant emission measure if $\mathcal{E}\mu = \mu$. In this case

$$d\mu(x) = \sum_{u\in\mathcal{L}^n_{\mathcal{W}}} w_u(F_u(x)) \cdot d(F_u^{-1}\mu)(x)$$

for each $n > 0$. If $\mathcal{W}$ is a partition, then $w_a(x) = 1$ iff $x \in \bar{W}_a$ and $w_a(x) = 0$ otherwise. The emission process is then a deterministic function $\mathcal{E} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by $\mathcal{E}(x) = F_a^{-1}(x)$ for $x \in \bar{W}_a$. We say that $\mathcal{E}$ is **expanding**, if there exists $r > 1$ such that $(F_a^{-1})^\bullet(x) \geq r$ for each $x \in W_a$. We say that $\mathcal{E}$ is **transitive**, if for each nonempty open sets $U, V \subseteq \overline{\mathbb{R}}$ there exists n such that $\mathcal{E}^n(U) \cap V \neq \emptyset$.

Theorem 9: Assume that $(F, \mathcal{W})$ is an interval MNS such that $\mathcal{W}$ is an open partition, $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2, $\mathcal{E}$ is transitive and $\overline{W_a} \subset \mathbf{V}(F_a)$ for each $a \in A$. Then there exists a unique $\mathcal{E}$-invariant ergodic Borel probability measure μ on $\overline{\mathbb{R}}$ which is absolutely continuous with respect to the Cauchy measure.

Proof: Since $\overline{W_a} \subset \mathbf{V}(F_a)$, $\mathcal{E}$ is expanding, and the theorem is a consequence of Theorem 1.2 in Mañé [14], p. 168. The existence and unicity of the density can be also proved directly. The map

$$(\mathcal{E}h)(x) = \sum_{a\in\mathbb{A}} F_a^\bullet(x) \cdot h(F_a(x)) \cdot w_a(F_a(x))$$

is a contraction on $\mathcal{C}(\mathcal{W})$, so it has a unique fixed point. ■

Since $||M(x)||/||x|| = \sqrt{\det(M)/M^\bullet(x)}$, we define the n-th emission quotient $\mathbf{e}_n$ by

$$\mathbf{e}_n = \sum_{u\in\mathcal{L}^n_{\mathcal{W}}} \int \ln \sqrt{\frac{\det(F_u)}{(F_u^{-1})^\bullet(x)}} \cdot w_u(x)\, d\mu(x).$$

Proposition 10: $\mathbf{e}_{n+m} \leq \mathbf{e}_n + \mathbf{e}_m$ for each $n, m > 0$.

Proof: Since $F_{uv} = \mathfrak{d}(F_u \cdot F_v)$, we have $\det(F_{uv}) \leq \det(F_u) \cdot \det(F_v)$ for each $uv \in \mathcal{L}^{n+m}_{\mathcal{W}}$. For $u \in \mathcal{L}_{\mathcal{W}}$ and $x \in \overline{\mathbb{R}}$ set $g_u(x) = \frac{1}{2}\ln(\det(F_u)/(F_u^{-1})^\bullet(x))$. Then

$$\begin{aligned} g_{uv}(x) &\leq \frac{1}{2} \ln \frac{\det(F_u) \cdot \det(F_v)}{(F_v^{-1})^\bullet(F_u^{-1}(x)) \cdot (F_u^{-1})^\bullet(x)} \\ &= g_u(x) + g_v((F_u^{-1}(x)). \end{aligned}$$

Using the substitution $x = F_u(y)$ we get

$$\begin{aligned} \mathbf{e}_{n+m} &\leq \sum_{uv\in\mathcal{L}^{n+m}_{\mathcal{W}}} \int g_u(x) \cdot w_{uv}(x)\, d\mu(x) + \\ &\sum_{uv\in\mathcal{L}^{n+m}_{\mathcal{W}}} \int g_v(y) \cdot w_u(F_u(y)) \cdot w_v(y)\, d(F_u^{-1}\mu)(y) \\ \mathbf{e}_{n+m} &\leq \mathbf{e}_n + \sum_{v\in\mathcal{L}^m_{\mathcal{W}}} \int g_v(y) \cdot w_v(y)\, d\mu(y) \end{aligned}$$

and $\mathbf{e}_{n+m} \leq \mathbf{e}_n + \mathbf{e}_m$ ■

Thus there exists the limit $\mathbf{e} = \lim_{n\to\infty} \mathbf{e}_n/n$, and $\mathbf{e} \leq \mathbf{e}_n$ for each n. We call $\mathbf{E}_n = \exp(\mathbf{e}_n/n)$ and $\mathbf{E} = \exp(\mathbf{e})$ the **emission quotients**. If $\mathcal{W}$ is a partition then

$$\mathbf{E} = \lim_{n\to\infty} \frac{1}{n} \sum_{u\in\mathcal{L}^n_{\mathcal{W}}} \ln \sqrt{\frac{\det(F_u)}{(F_u^{-1})^\bullet(\mathfrak{l}(W_u))}} \cdot \mu(W_u)$$

Since the invariant measure is ergodic, the norm of the state matrices in the emission process grows as $\mathbf{E}^n$:

Theorem 11: Assume that $(F, \mathcal{W})$ is a MNS such that $\mathcal{W}$ is a partition, $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2, $\mathcal{E}$ is transitive and $\overline{W_a} \subset \mathbf{V}(F_a)$ for each $a \in A$. Then

$$\lim_{n\to\infty} \frac{||F^{-1}_{\Phi^{-1}(x)_{[0,n)}}(x)||}{||x|| \cdot \mathbf{E}^n} = 1 \quad \text{almost surely.}$$

IX. Invariant symbolic measure

If μ is an $(F, \boldsymbol{w})$-invariant measure on $\overline{\mathbb{R}}$, then the probability that the expansion of $x \in \overline{\mathbb{R}}$ is $u \in \mathcal{L}_{\mathcal{W}}$ is

$$P_u = \int w_u(x)\, d\mu(x).$$

This formula gives a measure $P \in \mathcal{M}(\mathcal{S}_{\mathcal{W}})$ which we denote by $P = \Phi^{-1}_{\boldsymbol{w}}\mu$. If $\mathcal{W}$ is a partition then $P_u = \mu(\bar{W}_u)$.

Proposition 12: If μ is an $(F, \boldsymbol{w})$-invariant measure, then the measure $P = \Phi^{-1}_{\boldsymbol{w}}\mu$ is σ-invariant and $\Phi P = \mu$.

Proof: For an interval $I \subset \overline{\mathbb{R}}$ we have

$$\begin{aligned} (\Phi P)(I) &= P(\Phi^{-1}(I)) \\ &= \lim_{n\to\infty} \sum \{P_u :\ u \in \mathcal{L}^n_{\mathcal{W}}, W_u \cap I \neq \emptyset\} \\ &= \lim_{n\to\infty} \sum_{u\in\mathcal{L}^n_{\mathcal{W}}} \int_I w_u(x)\, d\mu(x) \\ &= \int_I d\mu(x) = \mu(I), \end{aligned}$$

so $\Phi P = \mu$. We show that P is σ-invariant. Using substitutions $x = F_a(y)$ we get

$$\begin{aligned} (\sigma P)_u &= \sum_{a\in\mathbb{A}} \int w_a(x) \cdot w_u(F_a^{-1}(x)) \cdot h_\mu(x)\, d\gamma(x) \\ &= \int \left(\sum_{a\in\mathbb{A}} w_a(F_a(y)) \cdot h_\mu(F_a(y)) \cdot F_a^\bullet(y) \right) \\ &\qquad \cdot w_u(y)\, d\gamma(y) \\ &= \int h_\mu(y) \cdot w_u(y)\, d\gamma(y) = P_u \end{aligned}$$

■

X. Invariant absorption measure

We assume that the input for the absorption process are words $u \in \mathcal{S}_{\mathcal{W}}$ distributed according to the measure $P = \Phi^{-1}_{\boldsymbol{w}}\mu$, where μ is the invariant emission measure. Given a matrix $M = M_{(a,b,c,d)}$, denote by ${}^tM = M_{(a,c,b,d)}$ its transposed matrix and by ${}^tM^{-1} = M_{(d,-c,-b,a)}$ the inverse of its transposed matrix. A matrix M acts on the rows of matrices (by multiplication from the right) in the same way as the transposed matrix ${}^tM$ acts on the columns of matrices or on homogenous coordinates of real numbers. The absorbtion process is a mapping $\mathcal{A}$ on the space $\overline{\mathbb{R}}\times\mathcal{S}_{\mathcal{W}}$ given by $\mathcal{A}(x,u) = ({}^tF_{u_0}(x), \sigma(u))$. The map extends to a self-map $\mathcal{A}$ of $\mathcal{M}(\overline{\mathbb{R}}\times\mathcal{S}_{\mathcal{W}})$ by

$$(\mathcal{A}\nu)(I,u) = \sum_{a\in\mathbb{A}} \nu({}^tF_a^{-1}(I), au).$$

If $\nu(\overline{\mathbb{R}},u) = P_u$ then $(\mathcal{A}\nu)(\overline{\mathbb{R}},u) = P_u$ as well. In this case we say that P is the $\mathcal{S}_{\mathcal{W}}$-projection of ν. Each $u \in \mathcal{L}_{\mathcal{W}}$ determines the conditional measure $\nu_u \in \mathcal{M}(\overline{\mathbb{R}})$ by $\nu_u(I) = \nu(I,u)/P_u$. Note that $\sum_{v\in\mathcal{L}_{\mathcal{W}}^m} \nu_{uv}\cdot P_{uv} = \nu_u \cdot P_u$ for each $u \in \mathcal{L}_{\mathcal{W}}$ and $m > 0$. For the densities we get

$$\begin{aligned} h_{(\mathcal{A}\nu)_u}(x) &= \lim_{\varepsilon\to 0} \frac{(\mathcal{A}\nu)(B_\varepsilon(x),u)}{|B_\varepsilon(x)|\cdot P_u} \\ &= \lim_{\varepsilon\to 0} \sum_{a\in\mathbb{A}} \frac{(\mathcal{A}\nu)_{au}({}^tF_a^{-1}(B_\varepsilon(x))\cdot P_{au}}{|B_\varepsilon(x)|\cdot P_u} \\ &= \sum_{a\in\mathbb{A}} h_{\nu_{au}}({}^tF_a^{-1}(x))\cdot({}^tF_a^{-1})^\bullet(x)\cdot P_{au}/P_u \end{aligned}$$

and

$$h_{(\mathcal{A}^k\nu)_u}(x) = \sum_{v\in\mathbb{A}^k} h_{\nu_{vu}}({}^tF_v^{-1}(x))\cdot({}^tF_v^{-1})^\bullet(x)\cdot P_{vu}/P_u$$

We can prove the existence of an invariant absorption measure for a class of MNS:

Definition 13: We say that $(F,\mathcal{W})$ is a **MNS with transpositions**, if for each $a \in \mathbb{A}$ there exists $t(a) \in \mathbb{A}$ with $F_{t(a)} = {}^tF_a$ and

$$u \in \mathcal{L}_{\mathcal{W}}^n \text{ iff } t(u) = t(u_{n-1})\cdots t(u_1)t(u_0) \in \mathcal{L}_{\mathcal{W}}^n.$$

Theorem 14: Assume that $(F,\mathcal{W})$ is a MNS with transpositions such that $\mathcal{W}$ is an open partition, $\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2, $\mathcal{E}$ is transitive and $\overline{W_a} \subset \mathbf{V}(F_a)$ for each $a \in A$. Then there exists a stable $\mathcal{A}$-invariant absorption measure ν whose $\mathcal{S}_{\mathcal{W}}$-projection is $P = \Phi_{\mathbf{w}}^{-1}\mu$.

Proof: Denote by $V_a = F_{t(a)}^{-1}(\overline{W_{t(a)}})$. The mappings $F_{t(a)} : V_a \to \overline{W_{t(a)}}$ are contractions since $\overline{W_a} \subseteq \mathbf{V}(F_a)$. If $ab \in \mathcal{L}_{\mathcal{W}}^2$ then $t(b)t(a) \in \mathcal{L}_{\mathcal{W}}^2$, $\overline{W_{t(a)}} \subseteq V_b$. Denote by

$$X = \bigcup_{a\in\mathbb{A}} (V_a \times [a]) \subset \overline{\mathbb{R}}\times\mathcal{S}_{\mathcal{W}}$$

If $(x,u) \in X$, then $F_{t(u_0)}(x) \in \overline{W_{t(u_0)}} \subseteq V_{u_1}$, so $\mathcal{A}(x,u) \in X$. Thus X is $\mathcal{A}$-invariant. Since $F_{t(a)}$ are contractions also on a sufficiently small neighbourhood of V_a, the set X is actually an attractor of $\mathcal{A}$. Consider the space

$$\mathcal{X} = \{\nu \in \mathcal{M}(X) :\ \mathrm{supp}(\nu_u) \subseteq V_{u_0},\ \nu(\overline{\mathbb{R}},u) = P_u\}$$

Then $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ is a contraction and therefore has a unique fixed point $\nu \in \mathcal{X} \subseteq \mathcal{M}(\mathbb{R}\times\mathcal{S}_{\mathcal{W}})$. ∎

If there exists an $\mathcal{A}$-invariant measure of ν, then we define the n-th absorption quotient by

$$\begin{aligned} \mathbf{a}_n &= \int_{\overline{\mathbb{R}}\times\mathcal{S}_{\mathcal{W}}} \ln\sqrt{\frac{\det(F_{u_{[0,n)}})}{({}^tF_{u_{[0,n)}})^\bullet(x)}}\, d\nu(x,u) \\ &= \sum_{u\in\mathcal{L}_{\mathcal{W}}^n} P_u \int \ln\sqrt{\frac{\det(F_u)}{({}^tF_u)^\bullet(x)}}\, d\nu_u(x) \end{aligned}$$

Proposition 15: $\mathbf{a}_{n+m} \leq \mathbf{a}_n + \mathbf{a}_m$

Proof: For $(x,u) \in \overline{\mathbb{R}}\times\mathcal{S}_{\mathcal{W}}$ set

$$g_n(x,u) = \frac{1}{2}\ln\frac{\det(F_{u_{[0,n)}})}{({}^tF_{u_{[0,n)}})^\bullet(x)}.$$

We get $g_{n+m}(x,u) \leq g_n(x,u) + g_m(\mathcal{A}^n(x,u))$ and

$$\begin{aligned} \mathbf{e}_{n+m} &= \int g_{n+m}(x,u)\, d\nu(x,u) \\ &\leq \int g_n\, d\nu(x,u) + \int g_m \circ \mathcal{A}^n\, d\nu(x,u) \\ &= \mathbf{e}_n + \int g_m\, d(\mathcal{A}^n\nu), \end{aligned}$$

so $\mathbf{e}_{n+m} \leq \mathbf{e}_n + \mathbf{e}_m$. ∎

The invariant absorption measure need not be absolutely continuous with respect to the Cauchy measure, so we cannot always compute with densities. The absorption quotient si defined as limit $\mathbf{a} = \lim_{n\to\infty} \mathbf{a}_n/n$, $\mathbf{A}_n = \exp(\mathbf{a}_n/n)$, $\mathbf{A} = \exp(\mathbf{a})$. If the absorption process is ergodic, and M is a singular matrix, then the norm of $MF_{u_{[0,n)}}$ grows as $\mathbf{A}^n$. If emissions alternate with absorption so that the ratio of their numbers converges to 1 then the norm $F_{v_{[0,n)}}^{-1} MF_{u_{[0,n)}}$ grows as $\mathbf{E}^n\mathbf{A}^n$. Thus we define the transaction quotients by

$$\mathbf{t}_n = \mathbf{a}_n + \mathbf{e}_n,\ \mathbf{t} = \mathbf{a} + \mathbf{e},\ \mathbf{T}_n = \mathbf{A}_n\cdot\mathbf{E}_n,\ \mathbf{T} = \mathbf{A}\cdot\mathbf{E}.$$

We have $\mathbf{T} \geq 1$, since the norm of the state matrices cannot be smaller than 1.

XI. BINARY SYSTEM

The essential feature of positional number systems is that they consist of linear transformations of the form $M(x) = ax + b$ which have fixed point ∞.

Example 1: The nonredundant binary system in the alphabet $\mathbb{A} = \{\overline{2},\overline{1},1,2\}$ is given by the following transformations and open intervals:

a	F_a	W_a	$F_a^{-1}(W_a)$
$\overline{2}$	$[2,-1,0,1]$	$(\infty,-1)$	$(\infty,0)$
$\overline{1}$	$[1,-1,0,2]$	$(-1,0)$	$(-1,1)$
1	$[1,1,0,2]$	$(0,1)$	$(-1,1)$
2	$[2,1,0,1]$	$(1,\infty)$	$(0,\infty)$

The values of the disc Möbius transformations $\widehat{F}_u(0) \in \mathbb{D}$ are shown in Figure 1 top. The curves between the values $\widehat{F}_u(0)$ are constructed as follows. For each MT M there exists a family $(M^r)_{r\in\mathbb{R}}$ of MT such that $M^0 = \mathrm{Id}$, $M^1 = M$, and $M^{r+s} = M^rM^s$. Each value $\widehat{F}_u(0)$ in the diagram is joined to $\widehat{F}_{ua}(0)$ by the curve $(\widehat{F}_u\widehat{F}_a^r(0))_{0\leq r\leq 1}$. The labels $u \in \mathbb{A}^*$ at $\widehat{F}_u(0)$ are written in the direction

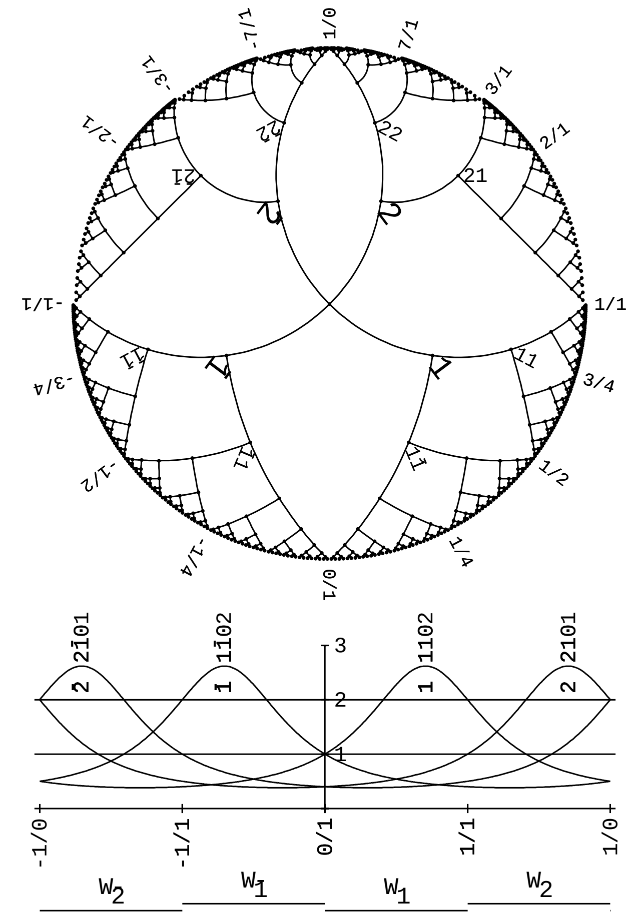

Fig. 1. A nonredundant binary system: Means $\widehat{F}_u(0)$ (top), expansion intervals W_a and the circle derivations $(F_a^{-1})^\bullet(x)$ (bottom)

of the tangent vectors $\widehat{F}'_u(0)$. Figure 1 bottom shows the intervals W_a and the circle derivations $(F_a^{-1})^\bullet(x)$. We can see that $W_a \subseteq \mathbf{V}(F_a)$ and that all these circle derivations have the same shape.

The expansion subshift $\mathcal{S}_\mathcal{W}$ is a SFT of order 2 with forbidden words $\overline{2}1, \overline{2}2, \overline{1}\overline{2}, \overline{1}2, 12, 1\overline{2}, 21, 2\overline{1}$. Each word $u \in \mathcal{S}_\mathcal{W}$ can be written as $u = a^n v$, where $a \in \{\overline{2}, 2\}$, $0 \le n \le \infty$ and $v \in \{\overline{1}, 1\}^\mathbb{N}$. The emision transformation has the attractor $[-1, 1]$ where it is transitive and has the unique absolutely continuous invariant measure μ with Lebesgue density $h(x) = 1/2$, so $h_\mu(x) = \pi(1 + x^2)/2$. Since $(F_a^{-1})^\bullet(x) = \frac{2(x^2+1)}{(2x-a)^2+1}$ for $a \in \{\overline{1}, 1\} = \{-1, 1\}$, we get

$$\mathbf{e}_1 = \frac{1}{4}\int_{-1}^{0} \ln \frac{(2x+1)^2+1}{x^2+1}\,dx + \frac{1}{4}\int_{0}^{1} \ln \frac{(2x-1)^2+1}{x^2+1}\,dx = 0.$$

Since both the transformations ${}^tF_{\overline{1}}(x) = \frac{x}{-x+2}$, ${}^tF_1(x) = \frac{x}{x+2}$ have the stable fixed point 0, the unique invariant measure of the absorption process is the point measure concentrated at 0. Since $({}^tF_{\overline{1}})^\bullet(0) = {}^tF_1^\bullet(0) = 1/2$ and $P_{ab} = 1/4$ for $a, b \in \{\overline{1}, 1\}$, we get

$$\mathbf{a}_1 = \frac{1}{4}\left(\ln \frac{\det(F_0)}{({}^tF_0)^\bullet(0)} + \ln \frac{\det(F_3)}{({}^tF_3)^\bullet(0)}\right) = \ln 2.$$

Thus $\mathbf{E}_1 = 1$, $\mathbf{A}_1 = 2$ and the first transaction quotient is $\mathbf{T}_1 = 2$. For $u \in \{\overline{1}, 1\}^n$ we get

$$F_u = \begin{bmatrix} 1 & 2^{n-1}u_0 + \cdots + 2u_{n-2} + u_{n-1} \\ 0 & 2^n \end{bmatrix}$$

so $\det(F_u) = 2^n$, and $\det(F_{uv}) = \det(F_u) \cdot \det(F_v)$ for each $uv \in \{\overline{1}, 1\}^*$. In the proofs of Proposition 10 and Proposition 15 we have equalities in this case, so $\mathbf{e}_n = n\mathbf{e}_1$, $\mathbf{a}_n = n\mathbf{a}_1$, and $\mathbf{T}_n = \mathbf{T}_1 = 2$ for each n. This corresponds with the results of Heckmann [3], whose Law of big numbers can be interpreted in our setting as $\mathbf{T} \ge \sqrt{2}$ for binary positional systems. Note however, that Heckmann uses the norm $||M_{(a,b,c,d)}|| = \max\{|a|, |b|, |c|, |d|\}$ which is smaller than our Euclidean norm.

XII. A NONREDUNDANT BIMODULAR SYSTEM

Bimodular systems have been studied in Kůrka [10] and [11] because of their high symmetry and nice properties. The rational numbers have preperiodic expansions in these systems, and there exist several almost-covers whose expansion subshifts are of finite type or sofic. The system consists of the only eight transformations with norm $\sqrt{6}$, the trace (the sum of the entries on the diagonal) 3 and the determinant 2. Its transformations generate whole **bimodular group** which consists of all MT with integer entries whose determinant is a power of 2.

Example 2: A nonredundant bimodular system in alphabet $\mathbb{A} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is given by the following transformations and open intervals:

a	F_a	W_a	$F_a^{-1}(W_a)$	$t(a) : V_a$
0	$[1, 0, 1, 2]$	$(0, \frac{1}{2})$	$(0, 2)$	$1 : [0, 1]$
1	$[1, 1, 0, 2]$	$(\frac{1}{2}, 1)$	$(0, 1)$	$0 : [0, 2]$
2	$[2, 0, 1, 1]$	$(1, 2)$	$(1, \infty)$	$3 : [\frac{1}{2}, \infty]$
3	$[2, 1, 0, 1]$	$(2, \infty)$	$(\frac{1}{2}, \infty)$	$2 : [1, \infty]$
4	$[2, -1, 0, 1]$	$(\infty, -2)$	$(\infty, -\frac{1}{2})$	$5 : [\infty, -1]$
5	$[2, 0, -1, 1]$	$(-2, -1)$	$(\infty, -1)$	$4 : [\infty, -\frac{1}{2}]$
6	$[1, -1, 0, 2]$	$(-1, -\frac{1}{2})$	$(-1, 0)$	$7 : [-2, 0]$
7	$[1, 0, -1, 2]$	$(-\frac{1}{2}, 0)$	$(-2, 0)$	$6 : [-1, 0]$

Means, circle derivations and intervals of the system can be seen in Figure 2. The expansion subshift is an SFT of order 2 with transitions 00, 01, 02, 10, 11, 22, 23, 31, 32, 33, 44, 45, 46, 54, 55, 66, 67, 75, 76, 77. The emission process is not transitive, but it has two transitive subsystems. F_0, F_1, F_2, F_3 are transitive on $[0, \infty]$ and F_4, F_5, F_6, F_7 are transitive on $[\infty, 0]$. On each interval $[0, \infty]$ and $[\infty, 0]$ there exists a unique invariant absolutely continuous measure and any convex combination of these two measures is invariant on $\overline{\mathbb{R}}$. The density of the $(\frac{1}{2}, \frac{1}{2})$-convex combination can be seen in Figure 3. The system has transpositions (see Definition 13) and the assumptions of Theorem 14 are satisfied, so there exists a unique invariant absorbtion measure ν.

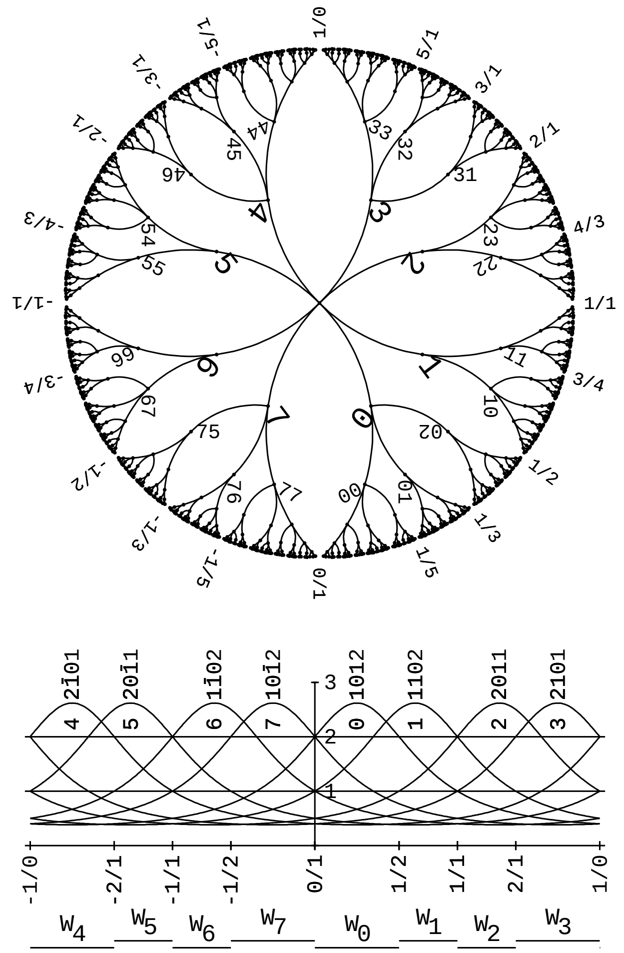

Fig. 2. Means, circle derivations and intervals of the nonredundant bimodular system,

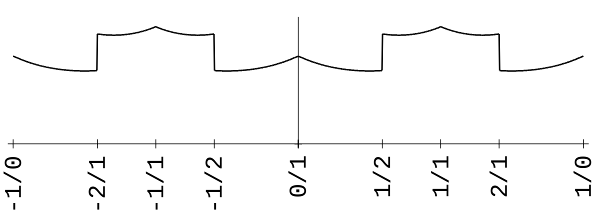

Fig. 3. The density of the invariant emission measure of the nonredundant bimodular system.

Numerical approximations suggest that the conditional measures ν_u do not have densities. Figure 4 shows the densities of the fifth iteration $\mathcal{A}^P(\gamma, P)$ of the uniform Cauchy measure. For the computation of the absorption quotients we therefore approximate ν_u by the uniform measures on the intervals $V_a = F^{-1}_{t(a)}(\overline{W_{t(a)}})$. We get:

n	1	2	3	$\cdots$	7	8	9
$\mathbf{A}_n$	2.20	1.90	1.88	$\cdots$	1.77	1.76	1.76
$\mathbf{E}_n$	0.91	0.79	0.78	$\cdots$	0.74	0.73	0.73
$\mathbf{T}_n$	2.00	1.50	1.47	$\cdots$	1.31	1.30	1.29

In contrast to the binary system the higher transaction quotients steadily decrease. We see that $\mathbf{T} < \sqrt{2}$, so arithmetical algorithms are faster than in the binary system. Numerical simulations suggest that the transaction quotient is less than 1.2.

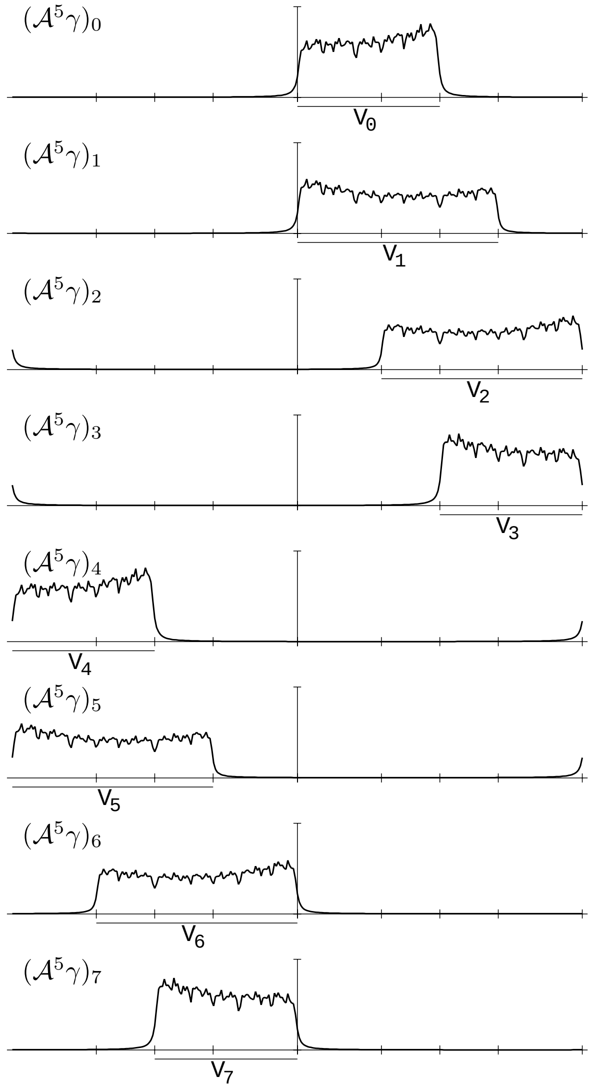

Fig. 4. Approximations of the invariant absorption measures of the nonredundant bimodular system: densities of $(\mathcal{A}^5\gamma)_a$.

XIII. Sofic redundant systems

Recall that an interval MNS $(F, \mathcal{W})$ is redundant if its expansion intervals overlap, i.e., if $\mathcal{W}$ is a cover (see Theorem 3). In nonredundant systems, the computations of the unary algorithm is not guaranteed to produce an infinite output. Unfortunately, as shown in Kůrka [11], an MNS whose interval cylinders form a cover cannot have the expansion subshift of finite type. A more general class are sofic subshifts which are factors of subshifts of finite type. A MNS with sofic expansion subshift has a presentation whose vertices are intervals of an open partition which refines $\mathcal{W}$:

Definition 16: Let $(F, \mathcal{W})$ be an interval MNS over an alphabet $\mathbb{A}$. An open interval partition $\mathcal{V} = \{V_p : p \in \mathbb{B}\}$ is an **SFT refinement** of $\mathcal{W}$, if the following two conditions

are satisfied for each $a \in \mathbb{A}$, $p, q \in \mathbb{B}$:
1. If $V_p \cap W_a \neq \emptyset$ then $V_p \subseteq W_a$,
2. If $V_p \subseteq W_a$ and $V_q \cap F_a^{-1} V_p \neq \emptyset$ then $V_q \subseteq F_a^{-1} V_p$.
The labelled graph $\mathcal{G}$ of $(F, \mathcal{W}, \mathcal{V})$ has vertices $|\mathcal{G}| = \mathbb{B}$ and labelled edges

$$p \xrightarrow{a} q \text{ if } V_p \subseteq W_a \text{ and } V_q \subseteq F_a^{-1} V_p$$

For $u \in \mathcal{L}_{\mathcal{W}} \cup \mathcal{S}_{\mathcal{W}}$ denote by $\mathcal{P}(u) \subseteq \mathbb{B}^* \cup \mathbb{B}^{\mathbb{N}}$ the set of paths with label u.

Theorem 17 (Kůrka [11]): Assume that $(F, \mathcal{W})$ is an interval MNS over an alphabet $\mathbb{A}$. Then $\mathcal{S}_{\mathcal{W}}$ is a sofic subshift iff there exists an SFT refinement $\mathcal{V}$ of $\mathcal{W}$.

Theorem 18: Assume that $(F, \mathcal{W})$ is an interval MNS over $\mathbb{A}$ with an SFT refinement $\mathcal{V} = \{V_p : \ p \in \mathbb{B}\}$ and let $\mathcal{G}$ be the labelled graph of $(F, \mathcal{W}, \mathcal{V})$. Then $\mathcal{S}_{\mathcal{W}}$ is the language of $\mathcal{G}$. For each infinite word $u \in \mathcal{S}_{\mathcal{W}}$ there exist at most two paths in $\mathcal{G}$ with label u. There exists $r > 0$ such that the set $\{p_{[0,n-r)} : \ p \in \mathcal{P}(u)\}$ has at most two elements for each finite word $u \in \mathcal{L}_{\mathcal{W}}^n$. There exists $s > 0$ such that $\mathcal{P}(u)$ has at most s elements for each $u \in \mathcal{L}_{\mathcal{W}}$.

Proof: Assume that $p_0 \xrightarrow{u_0} p_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} p_n$ is a labelled path, so $V_{p_i} \subseteq W_{u_i}$ and $F_{u_i} V_{p_{i+1}} \subseteq V_{p_i}$. Then

$$\begin{aligned}
F_{u_{[0,n)}} V_{p_n} &\subseteq F_{u_{[0,n-1)}} V_{p_{n-1}} \subseteq \cdots \subseteq F_{u_0} V_{p_1} \subseteq V_{p_0}, \\
F_{u_{[0,n)}} V_{p_n} &\subseteq F_{u_{[0,n-1)}} W_{u_{n-1}} \cap \cdots \cap F_{u_0} W_{u_1} \cap W_{u_0} \\
&\subseteq W_{u_{[0,n)}},
\end{aligned}$$

so $W_{u_{[0,n)}} \neq \emptyset$ and $u_{[0,n)} \in \mathcal{L}_{\mathcal{W}}$. Conversely assume that $u \in \mathcal{L}_{\mathcal{W}}^n$ so $W_u \neq \emptyset$. There exists $p_0 \in \mathbb{B}$ such that

$$\emptyset \neq V_{p_0} \cap W_u \subseteq V_{p_0} \cap W_{u_0},$$

so $V_{p_0} \subseteq W_{u_0}$. There exists p_1 such that

$$\begin{aligned}
\emptyset &\neq V_{p_1} \cap F_{u_0}^{-1}(V_{p_0} \cap W_u) \\
&\subseteq V_{p_1} \cap F_{u_0}^{-1} V_{p_0} \cap W_{u_1}
\end{aligned}$$

so $V_{p_1} \subseteq W_{u_1}$, $V_{p_1} \subseteq F_{u_0}^{-1} V_{p_0}$, and $V_{p_1} \cap F_{u_0}^{-1} W_u \neq \emptyset$. We continue by induction. If we have constructed p_k with $V_{p_k} \cap F_{u_{[0,k)}}^{-1} W_u \neq \emptyset$, there exists p_{k+1} with

$$\begin{aligned}
\emptyset &\neq V_{p_{k+1}} \cap F_{u_k}^{-1}(V_{p_k} \cap F_{u_{[0,k)}}^{-1} W_u) \\
&\subseteq V_{p_{k+1}} \cap F_{u_k}^{-1} V_{p_k} \cap W_{u_{k+1}}
\end{aligned}$$

so $V_{p_{k+1}} \subseteq W_{u_{k+1}}$, $V_{p_{k+1}} \subseteq F_{u_k}^{-1} V_{p_k}$, and $V_{p_{k+1}} \cap F_{u_{[0,k+1)}}^{-1} W_u \neq \emptyset$. In this way we construct the whole path p for u. Thus $u \in \mathcal{L}_{\mathcal{W}}$ iff it is the label of a path in $\mathcal{G}$. Let r be the smallest integer such that for all $u \in \mathcal{L}_{\mathcal{W}}^r$ and for all $p \in \mathbb{B}$ we have $|W_u| < |V_p|$. Let $u \in \mathcal{L}_{\mathcal{W}}^n$ with $|u| = n > r$ and let $p \in \mathbb{B}^{n+1}$ be a path with label u. Then

$$\begin{aligned}
F_{u_{[0,n)}} V_{p_n} &\subseteq F_{u_{[0,n-r)}} V_{p_{n-r}} \cap W_u \\
&\subseteq F_{u_{[0,n-r)}} (V_{p_{n-r}} \cap W_{u_{[n-r,n)}})
\end{aligned}$$

Thus $V_{p_{n-r}} \cap W_{u_{[n-r,n)}} \neq \emptyset$ and there exist at most two letters p_{n-r} with this property. Since $F_{u_{n-r-1}} V_{p_{n-r}} \subseteq V_{p_{n-r-1}}$, the letter p_{n-r-1} is uniquely determined by p_{n-r}. Similarly, all letters p_i with $i < n - r$ are uniquely determined by p_{n-r-1}, so the set $\{p_{[0,n-r)} : \ p \in \mathcal{P}(u)\}$ has at most two elements. It follows that for each infinite $u \in \mathcal{S}_{\mathcal{W}}$ there exist at most two infinite paths with label u and that there exists an integer s such that $\mathcal{P}(u)$ has at most s elements for each $u \in \mathcal{L}_{\mathcal{W}}$. ■

The algorithm for computing $\mathcal{P}(u)$ of $u \in \mathcal{S}_{\mathcal{W}}$ is based on a simple recursive fromula

$$\mathcal{P}(ua) = \{pb \in \mathbb{B}^{n+1} : \ p \in \mathcal{P}(u) \ \& \ u_{n-1} \xrightarrow{a} b\}$$

Since the size of $\mathcal{P}(u)$ is bounded, the algorithm has linear time complexity. The number of elementary operations to compute $\mathcal{P}(u)$ is bounded by a linear function of the length u. Given a word $u \in \mathcal{S}_{\mathcal{W}}$ and its path $p \in \mathcal{P}(u)$, we can compute $v \in \mathcal{S}_{\mathcal{W}}$ with $M(\Phi(u)) = \Phi(v)$ for any MT M:

Definition 19: Let $(F, \mathcal{W}, \mathcal{V})$ be an integer MNS with sofic expansion subshift and the refinement partition $\mathcal{V}$. The **redundant unary graph** has vertices (X, p) where $X \in \mathbb{M}(\mathbb{Z})$ and $p \in \mathbb{B}$. Its labelled edges are

$$\begin{array}{llll}
(X,p) & \xrightarrow{a/\lambda} & (XF_a, q) & \text{if } \ V_p \subseteq W_a, \ F_a V_q \subseteq V_p \\
(X,p) & \xrightarrow{\lambda/b} & (F_b^{-1} X, p) & \text{if } \ X V_p \subseteq W_b
\end{array}$$

Theorem 20: Let $(F, \mathcal{W}, \mathcal{V})$ be an integer MNS such that $\mathcal{W}$ is a cover with sofic expansion subshift $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{V}$ is an SFT refinement partition $\mathcal{V}$.
1. If $u/v \in \mathbb{A}^{\mathbb{N}} \times \mathbb{A}^{\mathbb{N}}$ is the label of an inifinite path with source (M, p_0), then $u, v \in \mathcal{S}_{\mathcal{W}}$ and $M(\Phi(u)) = \Phi(v)$.
2. If $u \in \mathcal{S}_{\mathcal{W}}$ then there exists $v \in \mathcal{S}_{\mathcal{W}}$ and an infinite path with label u/v.

Proof: First note that if

$$(M,p) \xrightarrow{\lambda/b} (F_b^{-1} M, p) \xrightarrow{a/\lambda} (F_b^{-1} M F_a, q)$$

is a path in the graph, then $M F_a V_q \subseteq M V_p \subseteq W_b$, so

$$(M,p) \xrightarrow{a/\lambda} (M F_a, q) \xrightarrow{\lambda/b} (F_b^{-1} M F_a, q)$$

is a path as well. If $u/v \in A^n \times A^m$ is the label of a finite path, then it is the label of a path

$$\begin{array}{lll}
(M, p_0) & \xrightarrow{u_0/\lambda} & (M F_{u_0}, p_1) \xrightarrow{u_1/\lambda} \cdots \\
& \xrightarrow{u_{n-1}/\lambda} & (M F_u, p_n) \xrightarrow{\lambda/v_0} (F_{v_0}^{-1} M F_u, p_n) \xrightarrow{\lambda/v_1} \cdots \\
& \xrightarrow{\lambda/v_{m-1}} & (F_v^{-1} M F_u, p_n)
\end{array}$$

We get $V_{p_i} \subseteq W_{u_i}$, $F_{u_i} V_{p_{i+1}} \subseteq V_{p_i}$ and

$$F_u V_{p_n} \subseteq \cdots \subseteq F_{u_{01}} V_{p_2} \subseteq F_{u_0} V_{p_1} \subseteq V_{p_0}.$$

Since $F_u V_{p_n} \subseteq F_{u_{[0,i)}} V_{p_i} \subseteq F_{u_{[0,i)}} W_{u_i}$ for each $i < n$, we get $F_u V_{p_n} \subseteq W_u$. Since $M F_u V_{p_n} \subseteq F_{v_{[0,j)}} W_{v_j}$ for each $j < m$, we get $M F_u V_{p_n} \subseteq W_v$. If $u/v \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ is the label of an infinite path, then for each m there exists n such that $u_{[0,n)}/v_{[0,m)}$ is the label of a finite path and

$$\begin{aligned}
\emptyset &\neq F_{u_{[0,n)}} P_{p_n} \subseteq W_{u_{[0,n)}} \\
\emptyset &\neq M F_{u_{[0,n)}} P_{p_n} \subseteq W_{v_{[0,m)}}
\end{aligned}$$

so $u, v \in \mathcal{S}_{\mathcal{W}}$. The intersection

$$\bigcap_n F_{u_{[0,n)}} \overline{V_{p_n}} \subseteq \bigcap_n \overline{W_{u_{[0,n)}}}$$

redundant unary procedure;
input matrix: $M \in \mathbb{M}(\mathbb{Z})$;
input number: $u \in \mathcal{L}_{\mathcal{W}} \cup \mathcal{S}_{\mathcal{W}}$;
input path: $p \in \mathbb{B}^* \cup \mathbb{B}^{\mathbb{N}}$ with label u;
input threshold $0 < \varepsilon < L(\mathcal{W})$;
output number: $v \in \mathcal{S}_{\mathcal{W}}$;
variables $X \in \mathbb{M}(\mathbb{Z})$ (state), $n, m \in \mathbb{N}$ (input and output pointers)
begin
  $X := M$; $n := 0$; $m := 0$;
  while $n < |u|$ repeat
    if $|XV_{p_n}| \geq \varepsilon$ then begin
      $X := XF_{u_n}$;
      $n := n + 1$;
      end;
    else begin
      $v_m := b$, where $XV_{p_n} \subseteq W_b$ and
        $||F_c^{-1}X|| \geq ||F_b^{-1}X||$ for each c with $XV_{p_n} \subseteq W_c$
      $X := F_b^{-1}X$;
      $m := m + 1$;
      end;
    end;

TABLE II
REDUNDANT UNARY LEAST NORM ALGORITHM

is nonempty by compactness and has zero diameter, so it contains the unique point $\Phi(u)$. The intersection

$$\bigcap_n MF_{u_{[0,n)}}\overline{V_{p_n}} \subseteq \bigcap_m \overline{W_{v_{[0,m)}}}$$

is a nonempty singleton which contains both $M(\Phi(u))$ and $\Phi(v)$, so $M(\Phi(u)) = \Phi(v)$. If $u \in \mathcal{S}_{\mathcal{W}}$ then the diameter of $F_{u_{[0,n)}}V_{p_n}$ converges to zero as n goes to infinity. Since $\mathcal{W}$ is a cover, there exists its Lebesgue number $L(\mathcal{W}) > 0$ (the length of overlaps) such that for each interval I with length $|I| < L(\mathcal{W})$ there exists $a \in \mathbb{A}$ with $I \subseteq W_a$. Thus for each m there exists $v \in \mathbb{A}^m$ and n such that $(u_{[0,n)}, v)$ is the label of a path. It follows that there exists $v \in \mathbb{A}^{\mathbb{N}}$ such that u/v is the label of a path and therefore $M\Phi(u) = \Phi(v)$. ■

Assume that $(F, \mathcal{W}, \mathcal{V})$ is an integer MNS such that $\mathcal{W}$ is a cover, $\mathcal{S}_{\mathcal{W}}$ is sofic and $\mathcal{V}$ is its SFT refinement. Given a constant $\varepsilon > 0$ smaller than the Lebesgue number $L(\mathcal{W})$, we consider the **Redundant unary least norm algorithm** with input matrix $M \in \mathbb{M}(\mathbb{Z})$, input word $u \in \mathcal{L}_{\mathcal{W}} \cup \mathcal{S}_{\mathcal{W}}$ and input path p with label u (see Table II). The algorithm performs an absorption whenever the length of the interval XV_p is greater than ε. If $|XV_p| < \varepsilon$ then at least one letter b with $XV_p \subseteq W_b$ exists and the algorithm chooses the letter with the smallest norm of $F_b^{-1}X$. If the input $p_0 \xrightarrow{u_0} p_1 \cdots p_{n-1} \xrightarrow{u_{n-1}} p_n$ is finite then the output $v \in \mathcal{L}_{\mathcal{W}}$ is finite too and $MF_u(\overline{V_p}) \subseteq M(\overline{W_u}) \subseteq \overline{W_v}$. If the input path $p \in \mathbb{B}^{\mathbb{N}}$ with label $u \in \mathcal{S}_{\mathcal{W}}$ is infinite, then the output $v \in \mathcal{S}_{\mathcal{W}}$ is infinite and $M(\Phi(u)) = \Phi(v)$.

The time complexity of the Least norm algorithm depends on the norm of the state matrices X. There exists a constant $c > 0$ such that the n-th step of the Least norm algorithm requires at most $c \cdot \ln ||X_n||$ elementary operations. If the state matrices remain bounded during the computation, then the algorithm would have linear time complexity in the length of the input number u.

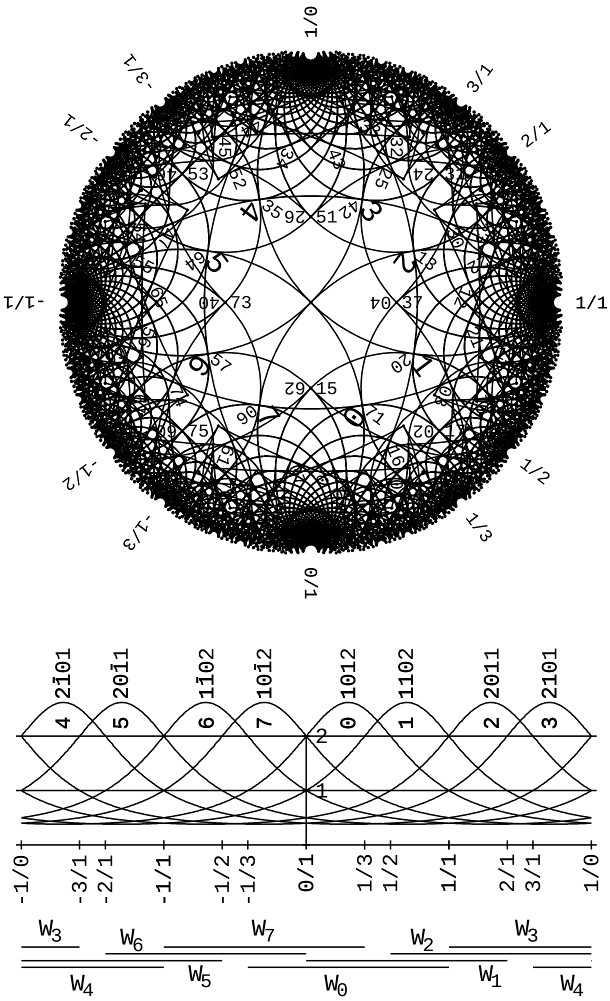

Fig. 5. Means, circle derivations and intervals of the redundant bimodular system.

XIV. Redundant bimodular system

Example 3: The redundant bimodular system in the alphabet $\mathbb{A} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is given by the following transformations and expansion intervals:

a	F_a	W_a	$F_a^{-1}(W_a)$
0	$[1, 0, 1, 2]$	$(-\frac{1}{3}, 1)$	$(-\frac{1}{2}, \infty)$
1	$[1, 1, 0, 2]$	$(0, 2)$	$(-1, 3)$
2	$[2, 0, 1, 1]$	$(\frac{1}{2}, \infty)$	$(\frac{1}{3}, -1)$
3	$[2, 1, 0, 1]$	$(1, -3)$	$(0, -2)$
4	$[2, -1, 0, 1]$	$(3, -1)$	$(2, 0)$
5	$[2, 0, -1, 1]$	$(\infty, -\frac{1}{2})$	$(1, -\frac{1}{3})$
6	$[1, -1, 0, 2]$	$(-2, 0)$	$(-3, 1)$
7	$[1, 0, -1, 2]$	$(-1, \frac{1}{3})$	$(\infty, \frac{1}{2})$

Means, circle derivations and intervals of the system can be seen in Figure 5. The cover $\mathcal{W}$ has been obtained by

$$W_a = \mathbf{V}(F_a) = \{x \in \overline{\mathbb{R}} : (F_a^{-1})^{\bullet}(x) > 1\}.$$

The system is sofic and its SFT-refinement partition over alphabet $\mathbb{B} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B\}$ has endpoints $0, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \infty, -3, -2, -1, -\frac{1}{2}, -\frac{1}{3}$. Table III shows for each $(p, a) \in \mathbb{B} \times \mathbb{A}$ with $V_p \subseteq W_a$ the intervals V_p, transformations F_a, preimages $F_a^{-1}(V_p)$ and the followers of p with label a. Figure 6 shows the labelled graph $\mathcal{G}$ of

pa	V_p	F_a	$F_a^{-1}(V_p)$	followers
00	$(0, \frac{1}{3})$	$[1, 0, 1, 2]$	$(0, 1)$	$0, 1, 2$
01	$(0, \frac{1}{3})$	$[1, 1, 0, 2]$	$(-1, -\frac{1}{3})$	$9, A$
07	$(0, \frac{1}{3})$	$[1, 0, -1, 2]$	$(0, \frac{1}{2})$	$0, 1$
10	$(\frac{1}{3}, \frac{1}{2})$	$[1, 0, 1, 2]$	$(1, 2)$	3
11	$(\frac{1}{3}, \frac{1}{2})$	$[1, 1, 0, 2]$	$(-\frac{1}{3}, 0)$	B
20	$(\frac{1}{2}, 1)$	$[1, 0, 1, 2]$	$(2, \infty)$	$4, 5$
21	$(\frac{1}{2}, 1)$	$[1, 1, 0, 2]$	$(0, 1)$	$0, 1, 2$
22	$(\frac{1}{2}, 1)$	$[2, 0, 1, 1]$	$(\frac{1}{3}, 1)$	$1, 2$
31	$(1, 2)$	$[1, 1, 0, 2]$	$(1, 3)$	$3, 4$
32	$(1, 2)$	$[2, 0, 1, 1]$	$(1, \infty)$	$3, 4, 5$
33	$(1, 2)$	$[2, 1, 0, 1]$	$(0, \frac{1}{2})$	$0, 1$
42	$(2, 3)$	$[2, 0, 1, 1]$	$(\infty, -3)$	6
43	$(2, 3)$	$[2, 1, 0, 1]$	$(\frac{1}{2}, 1)$	2
52	$(3, \infty)$	$[2, 0, 1, 1]$	$(-3, -1)$	$7, 8$
53	$(3, \infty)$	$[2, 1, 0, 1]$	$(1, \infty)$	$3, 4, 5$
54	$(3, \infty)$	$[2, -1, 0, 1]$	$(2, \infty)$	$4, 5$
63	$(\infty, -3)$	$[2, 1, 0, 1]$	$(\infty, -2)$	$6, 7$
64	$(\infty, -3)$	$[2, -1, 0, 1]$	$(\infty, -1)$	$6, 7, 8$
65	$(\infty, -3)$	$[2, 0, -1, 1]$	$(1, 3)$	$3, 4$
74	$(-3, -2)$	$[2, -1, 0, 1]$	$(-1, -\frac{1}{2})$	9
75	$(-3, -2)$	$[2, 0, -1, 1]$	$(3, \infty)$	$5,$
84	$(-2, -1)$	$[2, -1, 0, 1]$	$(-\frac{1}{2}, 0)$	A, B
85	$(-2, -1)$	$[2, 0, -1, 1]$	$(\infty, -1)$	$6, 7, 8$
86	$(-2, -1)$	$[1, -1, 0, 2]$	$(-3, -1)$	$7, 8$
95	$(-1, -\frac{1}{2})$	$[2, 0, -1, 1]$	$(-1, -\frac{1}{3})$	$9, A$
96	$(-1, -\frac{1}{2})$	$[1, -1, 0, 2]$	$(-1, 0)$	$9, A, B$
97	$(-1, -\frac{1}{2})$	$[1, 0, -1, 2]$	$(\infty, -2)$	$6, 7$
$A6$	$(-\frac{1}{2}, -\frac{1}{3})$	$[1, -1, 0, 2]$	$(0, \frac{1}{3})$	0
$A7$	$(-\frac{1}{2}, -\frac{1}{3})$	$[1, 0, -1, 2]$	$(-2, -1)$	8
$B0$	$(-\frac{1}{3}, 0)$	$[1, 0, 1, 2]$	$(-\frac{1}{2}, 0)$	A, B
$B6$	$(-\frac{1}{3}, 0)$	$[1, -1, 0, 2]$	$(\frac{1}{3}, 1)$	$1, 2$
$B7$	$(-\frac{1}{3}, 0)$	$[1, 0, -1, 2]$	$(-1, 0)$	$9, A, B$

TABLE III
THE SFT PARTITION OF THE REDUNDANT BIMODULAR SYSTEM

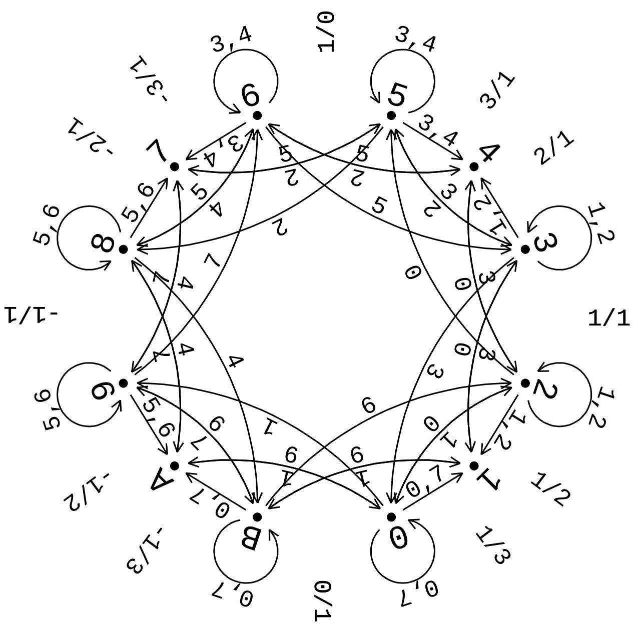

Fig. 6. The labelled graph of the redundant bimodular system

$(F, \mathcal{W}, \mathcal{V})$. Table IV shows a computation of the set $\mathcal{P}(u)$ of paths of a word $u \in \mathcal{L}_{\mathcal{W}}$.

An example of the run of the least norm algorithm can be seen in Table V The first column gives the input pointer n to

input number: $u = 524600160$
0: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B$
1: $63, 64, 75, 86, 87, 88, 99, 9A$
2: $633, 634, 635, 646, 757, 758$
3: $6354, 6355, 6466, 6467, 6468, 7579, 758A, 758B$
4: $64687, 64688, 75799, 7579A, 7579B, 758A0, 758B1, 758B2$
5: $7579BA, 7579BB, 758A00, 758A01, 758A02, 758B13, 758B24, 758B25$
6: $7579BBA, 7579BBB, 758A000, 758A001, 758A002, 758A013, 758A024, 758A025$
7: $758A0009, 758A000A, 758A001B, 758A0020, 758A0021, 758A0022, 758A0133, 758A0134$
8: $758A00099, 758A0009A, 758A0009B, 758A000A0, 758A001B1, 758A001B2$
9: $758A0009BA, 758A0009BB, 758A000A00, 758A000A01, 758A000A02, 758A001B13, 758A001B24, 758A001B25$

TABLE IV
PATHS WITH A GIVEN LABEL

the input vertex p_n and input letter u_n. The second column gives the output pointer m to the output letter v_m whixh is in the third column, so that the whole output v can be read in the third column from top to bottom. The fourth column gives the state matrix X and the last column gives a part of the input path $p_n \xrightarrow{a_n} p_{n+1} \cdots p_{n+k} \xrightarrow{a_{n+k}} p_{n+k+1}$ in the form $p_n^{a_n} p_{n+1} \cdots p_{n+k}^{a_{n+k}} p_{n+k+1}$. The algorithm gives good practical results. For input numbers of several thousands letters, the norms of the state matrices remain most of the time below 100. This suggests that the algorithm may have (at least statistically) linear time complexity.

Binary operations like addition, subtraction, multiplication and division can be computed in MNS with the use of **bilinear fractional transformations** of the form

$$P(x, y) = \frac{axy + bx + cy + d}{exy + fx + gy + h},$$

which are MT in both variables x and y. If $P(x, y)$ is a biliner fractional transformation and $M(x)$ is an MT, then $P(M(x), y)$, $P(x, M(y))$, as well as $M(P(x, y))$ are again biliner fractional transformations. For redundant sofic subshifts, the least norm algorithm can be easily adapted for bilinear fractional transformations. However, in the redundant bimodular system, this binary algorithm does not perform so efficiently as its unary version: the norm of the state (2×4)-matrices steadily grows. Perhaps a more sofisticated algorithm is necessary to keep the state vector bounded. Other possibility is that the algorithm may work in another sofic MNS with high redundancy.

Conjecture 1: There exists a redundant sofic Möbius number system in which the multiplication and division algorithms have average linear time complexity.

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input matrix: $M = [2, 7, 1, 5]$
input number: $u = 5246001601235234544434252253324646675353$
input path: $758A001B1346646675555546357554687 99A86630$
$\Phi(u) \in (-2.6340791995200, -2.6340791995174)$
$M(\Phi(u)) \in (0.7319947483486, 0.7319947483500)$

n	m	out	state matrix X	input path
0	0		$[2, 7, 1, 5]$	$7^5 5^2 8^4 A^6 0^0 0^0 1$
5	0		$[3, 5, 3, 7]$	$0^0 1^1 B^6 1^0 3^1 4^2 6$
5	1	0	$[3, 5, 0, 1]$	$0^0 1^1 B^6 1^0 3^1 4^2 6$
6	2	3	$[7, 8, 2, 4]$	$1^1 B^6 1^0 3^1 4^2 6^3 6$
6	3	3	$[5, 4, 4, 8]$	$1^1 B^6 1^0 3^1 4^2 6^3 6$
8	4	1	$[3, 3, 2, 18]$	$1^0 3^1 4^2 6^3 6^5 4^2 6$
8	5	1	$[2, -6, 1, 9]$	$1^0 3^1 4^2 6^3 6^5 4^2 6$
8	6	5	$[1, -3, 2, 6]$	$1^0 3^1 4^2 6^3 6^5 4^2 6$
10	7	6	$[1, 1, 2, 8]$	$4^2 6^3 6^5 4^2 6^3 6^4 7$
12	8	1	$[-3, -3, 6, 5]$	$6^5 4^2 6^3 6^4 7^5 5^4 5$
13	9	6	$[1, -1, 7, 5]$	$4^2 6^3 6^4 7^5 5^4 5^4 5$
15	10	0	$[1, 0, 9, 6]$	$6^4 7^5 5^4 5^4 5^4 5^3 5$
16	11	0	$[2, -1, 8, -1]$	$7^5 5^4 5^4 5^4 5^3 5^4 4$
16	12	0	$[2, -1, 3, 0]$	$7^5 5^4 5^4 5^4 5^3 5^4 4$
18	13	1	$[4, -3, 6, -3]$	$5^4 5^4 5^3 5^4 4^2 6^5 3$
19	14	0	$[8, -7, 2, -1]$	$5^4 5^3 5^4 4^2 6^5 3^2 5$
21	15	3	$[12, 0, 8, 1]$	$5^4 4^2 6^5 3^2 5^2 7^5 5$
21	16	2	$[6, 0, 2, 1]$	$5^4 4^2 6^5 3^2 5^2 7^5 5$
22	17	2	$[6, -3, -2, 2]$	$4^2 6^5 3^2 5^2 7^5 5^3 5$
25	18	4	$[29, -1, -20, 4]$	$5^2 7^5 5^3 5^3 4^2 6^4 8$
25	19	6	$[19, 1, -10, 2]$	$5^2 7^5 5^3 5^3 4^2 6^4 8$
26	20	5	$[39, 1, 3, 5]$	$7^5 5^3 5^3 4^2 6^4 8^6 7$
26	21	3	$[18, -2, 3, 5]$	$7^5 5^3 5^3 4^2 6^4 8^6 7$
26	22	2	$[9, -1, -6, 6]$	$7^5 5^3 5^3 4^2 6^4 8^6 7$
28	23	6	$[10, 6, -9, -3]$	$5^3 4^2 6^4 8^6 7^4 9^6 9$
29	24	5	$[5, 4, -4, -2]$	$4^2 6^4 8^6 7^4 9^6 9^6 A$
29	25	6	$[3, 3, -2, -1]$	$4^2 6^4 8^6 7^4 9^6 9^6 A$
31	26	5	$[9, -3, -1, 1]$	$8^6 7^4 9^6 9^6 A^7 8^5 6$
32	27	4	$[4, -6, -1, 3]$	$7^4 9^6 9^6 A^7 8^5 6^3 6$
32	28	5	$[2, -3, 1, 0]$	$7^4 9^6 9^6 A^7 8^5 6^3 6$
34	29	3	$[1, -5, 2, -4]$	$9^6 A^7 8^5 6^3 6^5 3^3 0$
35	30	1	$[0, -6, 1, -5]$	$A^7 8^5 6^3 6^5 3^3 0$
35	31	2	$[0, -3, 1, -2]$	$A^7 8^5 6^3 6^5 3^3 0$
37	32	2	$[6, -3, 4, -1]$	$6^3 6^5 3^3 0$
38	33	3	$[2, 0, 8, 3]$	$6^5 3^3 0$
40	34	0	$[8, 4, 9, 6]$	0
40	35	2	$[4, 2, 5, 4]$	0
40	36	2	$[2, 1, 3, 3]$	0
40	37	1	$[1, -1, 3, 3]$	0
40	38	0	$[1, -1, 1, 2]$	0
40	39	7	$[2, -2, 2, 1]$	0
40	40	5	$[1, -1, 3, 0]$	0
40	41	5	$[1, -1, 7, -1]$	0

output number: $v = 0331156160001032246532656545312230221075 5$
$\Phi(v) \in (0.7319947483426, 0.7319947483451)$

TABLE V
THE UNARY LEAST NORM ALGORITHM

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