STABILITY OF SUBSHIFTS IN CELLULAR AUTOMATA

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ABSTRACT. We show relations between several concepts of stability based on topological and measure-theoretical concepts in the Cantor and Besicovitch topological spaces. These concepts elucidate the behavior of cellular automata in which successively larger and larger regions are homogenized.

1. Introduction

Dynamics of many cellular automata (CA) is governed by a process of homogenization. At any time one can observe regions with certain structure separated by walls or defects with different structure. These defects move in an apparently random manner and when two neighboring defects meet, they annihilate one another. This kind of dynamics has been observed e.g., in elementary CA with rules 18, 184, 62, 54 (Boccara et al. [4]). The homogeneous configurations which do not contain defects form a subshift, usually a subshift of finite type or a sofic subshift. Intuitively, the subshift of homogeneous configurations behaves as an attractor, since the frequency of defects steadily decreases. Nevertheless the standard notion of attractor or μ-attractor as considered by Hurley [8, 9] does not apply here.

We believe that situations like this can be treated if the space of configurations is equipped with Besicovitch pseudometric. The behavior of CA in Besicovitch space has been recently investigated in Cattaneo et al. [5] and Blanchard et al. [2] with particular emphasis to chaotic systems. However, to consider attractors in the Besicovitch space, one has to surmount one difficulty. The standard concept of attractor has been developed mostly in the context of compact dynamics and does not work in the Besicovitch space, which is highly noncompact. Intuitively, an attractor should possess the following property: the trajectories of neighboring points stay close to it all the time and ultimately approach it. In compact spaces this property is equivalent to the standard notion of attractor, but not so in noncompact spaces.

We consider also the concept of μ-attracting set in the Besicovitch space. Here μ is a probability measure representing the distribution of the initial conditions. A closed subset of the state space is μ-attracting, if for μ-almost all points, the distance of their trajectory from the subset tends to zero. Using Cantor or Besicovitch topology, we obtain concepts of $C\mu$ or $B\mu$-attracting set. A mixing subshift is $B\mu$-attracting iff the frequency of defects tends to zero. In nonmixing subshifts, the latter condition is weaker and we call it $D\mu$-attractivity. We show that in the "Just gliders" CA of Milnor [13], the homogenous configuration is a $B\mu$-attracting mixing subshift, and in elementary CA 184, the homogenous configurations form a $D\mu$-attracting nonmixing subshift. We hypothesize that the same property has also the subshift of homogenous configurations in elementary CA 18.

2. Dynamical systems

A dynamical system is a couple (X,f) where X is a metric space (with metric d) and $f : X \to X$ is a continuous mapping. The distance of a point $x \in X$ from a set $Y \subseteq X$ is $d(x,Y) = \inf\{d(x,y) :\ y \in Y\}$. The open ball around Y of radius $r > 0$ is denoted by $B_r(Y) = \{x \in X :\ d(x,Y) < r\}$. The interior of a set Y is denoted by Y°.

Definition 1. *Let (X,f) be a dynamical system and let $Y \subseteq X$ be a nonempty closed invariant set, i.e., $f(Y) = Y$. We say that Y is stable if*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X, (d(x,Y) < \delta \ \Rightarrow\ \forall n \geq 0, d(f^n(x),Y) < \varepsilon)\,.$$

We say that Y is attracting if it is stable and moreover

$$\exists \delta > 0, \forall x \in X, \Big(d(x,Y) < \delta \ \Rightarrow\ \lim_{n\to\infty} d(f^n(x),Y) = 0\Big)\,.$$

A standard compactness argument shows that in compact spaces, attracting sets are exactly attractors: recall that a set $Y \subseteq X$ of a dynamical system (X,f) is an attractor, if there exists a closed set $V \subseteq X$ such that $f(V) \subseteq V^\circ$ and

$$Y = \omega(V)$$

Proposition 1. *Let Y be an attracting set in a compact dynamical system (X,f). Then for every $\varepsilon > 0$ there exists a closed set $V \subseteq B_\varepsilon(Y)$ such that $f(V) \subseteq V^\circ$ and $Y = \omega(V)$.*

Proof: For a given $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that for all $x \in X$,

$$d(x,Y) < \delta \ \Rightarrow\ \forall n \geq 0, d(f^n(x),Y) < \varepsilon, \text{ and } \lim_{n\to\infty} d(f^n(x),Y) = 0.$$

Choose $0 < \varepsilon_0 < \delta$ and $0 < \delta_0 < \varepsilon_0$ such that for all $x \in X$,

$$d(x,Y) < \delta_0 \ \Rightarrow\ \forall n \geq 0, d(f^n(x),Y) < \varepsilon_0.$$

Finally choose a closed set W with $B_{\varepsilon_0}(Y) \subset W \subset B_\delta(Y)$. If $x \in W$, then $d(f^n(x),Y) \to 0$ as $n \to \infty$, so there exists $n_x > 0$ with $f^{n_x}(x) \in B_{\delta_0}(Y)$. By continuity of f, there exists an open set $W_x \ni x$, such that $f^{n_x}(W_x) \subseteq B_{\delta_0}(Y)$, and for all $n \geq n_x$, $f^n(W_x) \subseteq B_{\varepsilon_0}(Y)$. The system $\{W_x :\ x \in W\}$ is an open cover of W, so it has a finite subcover $\{W_{x_1},\ldots,W_{x_m}\}$. Put $n = \max\{n_{x_1},\ldots,n_{x_m}\}$. Then we get $f^n(W) \subseteq B_{\varepsilon_0}(Y) \subseteq W^\circ$. Put $W_0 = W$. Since $f(f^{n-1}(W)) \subseteq W_0^\circ$ and $f^k(W) \subseteq B_\epsilon(Y)$ for every $k \geq 0$, there exists a closed set $W_1 \subseteq B_\epsilon(Y)$, for which $f^{n-1}(W) \subseteq W_1^\circ$, $f(W_1) \subseteq W_0^\circ$. Continuing in this way we obtain closed sets $W_0,\ldots,W_{n-1}$ in $B_\epsilon(Y)$ such that for $k \in \{0,...,n-1\}$

$$f^{n-k}(W) \subseteq W_k^\circ,\ \ f(W_k) \subseteq W_{k-1}^\circ.$$

For $V = W_0 \cup \cdots \cup W_{n-1}$ we get

$$f(V) \subseteq W_{n-1}^\circ \cup W_0^\circ \cup \cdots \cup W_{n-2}^\circ \subseteq V^\circ$$

and $V \subseteq B_\epsilon(Y)$. Since $W \subseteq B_\delta(Y)$ we get that for any $x \in V$, $\lim_{n\to\infty} d(f^n(x),Y) = 0$ and $w(V) = Y$. $\triangle$

Proposition 2. *Let (X,f) be a compact dynamical system and $V \subseteq X$ a closed subset for which $f(V) \subseteq V^\circ$. Then $Y = \omega(V)$ is an attracting set.*

Proof: 1. If $x \in Y$ then there exists a sequence of points $x_k \in V$, and an increasing sequence $(n_k)_{k\geq 0}$ such that $\lim_{k\to\infty} f^{n_k}(x_k) = x$. It follows $\lim_{k\to\infty} f^{n_k+1}(x_k) = f(x)$, so $f(x) \in Y$. On the other hand $f^{n_k-1}(x_k)$ has a subsequence which converges to some y. Then $y \in Y$ and $f(y) = x$. Thus we have proved $f(Y) = Y$.
2. Suppose that Y is not stable. Then there exists $\varepsilon > 0$, such that for every $\delta = \frac{1}{k}$ there exists $x_k \in X$ such that $d(x_k, Y) < \frac{1}{k}$, and $d(f^{n_k}(x_k), Y) \geq \varepsilon$ for some n_k. Since $f(V)$ is an invariant set, $Y \subseteq f(V) \subseteq V^\circ$, so the distance $d(Y, X \setminus V^\circ)$ is positive. It follows that there exists k_0 such that for all $k > k_0$, $x_k \in V$. (This argument does not work if $X = V^\circ$, but in this case $x_k \in V$ trivially.) The sequence $f^{n_k}(x_k)$ has a converging subsequence $f^{n_{k_i}}(x_{k_i}) \to x$ as $i \to \infty$, so we get $x \in \omega(V) = Y$. This is in contradiction with $d(x_k, Y) \geq \varepsilon$.
3. Since $Y \subseteq V^\circ$ there exists $\delta > 0$ such that $B_\delta(Y) = \{x \in X : d(x, Y) < \delta\} \subseteq V^\circ$. Suppose that $d(x, Y) < \delta$ and that $d(f^n(x), Y)$ does not converge to zero. Thus there exists $\varepsilon > 0$ such that for an infinite number of n, $d(f^n(x), Y) \geq \varepsilon$. There exists a converging sequence $f^{n_i}(x) \to y$ as $i \to \infty$, and $d(f^{n_i}(x), Y) \geq \varepsilon$. We get $y \in Y$ and $d(y, Y) \geq \varepsilon$ which is a contradiction. $\triangle$

In noncompact spaces the two properties differ. Let (X, f) be a dynamical system with $X = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $f(x, y) = (\frac{x}{2}, 2y)$. Then

$$V = \{(x, y) \in X : |y|x^2 \leq 1\}$$

is a closed set with $f(V) \subseteq V^\circ$, and $\omega(V) = \{(x, y) \in X : y = 0\}$. However, $\omega(V)$ is neither stable nor attracting.

3. Symbolic spaces

Let A be a finite alphabet with at least two letters. For $n \in \mathbb{N}$, denote by A^n the set of words over A of length n, by $A^* = \cup_{n\geq 0} A^n$ the set of finite words over A, and by $A^{\mathbb{Z}}$ the set of biinfinite sequences of letters of A. The i-th coordinate of a point $x \in A^{\mathbb{Z}}$ is denoted by x_i, and $x_{[j,k]} = x_j \dots x_k \in A^*$ is the segment of x between indices j and k. For $u \in A^n$, u^∞ is the infinite repetition of u, i.e. $(u^\infty)_{np+m} = u_m$ for every $p, m \in \mathbb{Z}$. We write $u \sqsubseteq v$ if $u = v_{[j,k]}$ for some $0 \leq j \leq k < |v|$. The cylinder of $u \in A^p$ at position $k \in \mathbb{Z}$ is the set

$$[u]_k = \{x \in A^{\mathbb{Z}} : x_{[k,k+p-1]} = u\}.$$

The Cantor metric on $A^{\mathbb{Z}}$ is defined by

$$d_C(x, y) = 2^{-k} \quad \text{where} \quad k = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

so $d_C(x, y) < 2^{-k}$ iff $x_{[-k,k]} = y_{[-k,k]}$. The cylinders are clopen (closed and open) sets for d_C. It is well known that all Cantor spaces (with different alphabets) are homeomorphic. The Cantor space is compact, totally disconnected and perfect.

The Besicovitch pseudometric on $A^{\mathbb{Z}}$ is defined as follows:

$$d_B(x, y) = \limsup_{l\to\infty} \frac{\#\{j \in [-l, l] : x_j \neq y_j\}}{2l + 1}.$$

Here # denotes the number of elements of a finite set. We have $d_B(x, y) < \varepsilon$ if and only if

$$\exists l_0, \forall l \geq l_0, \#\{j \in [-l, l] : x_j \neq y_j\} < (2l + 1)\varepsilon.$$

A cellular automaton (CA) is any map $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which is defined by some local rule $f_0 : A^{2r+1} \to A$, where $r \geq 0$, by

$$f(x)_i = f_0(x_{i-r}, \ldots, x_{i+r}).$$

Every CA is continuous both with respect to Cantor metric and Besicovitch pseudometric.
A Borel probability measure in $A^{\mathbb{Z}}$ is a normalized nonnegative sigma additive function μ defined on Borel sets. It is uniquely given by its values on cylinders, satisfying $\mu(A^{\mathbb{Z}}) = 1$, and for every $u \in A^*$, $k \in \mathbb{Z}$, $\sum_{a\in A} \mu[ua]_k = \mu[u]_k$ and $\sum_{a\in A} \mu[au]_k = \mu[u]_{k+1}$. A measure μ is σ-invariant, if $\mu[u]_k = \mu[u]$ does not depend on k. A σ-invariant measure μ is σ-ergodic, if for every invariant measurable set ($\sigma(Y) \subseteq Y \subseteq A^{\mathbb{Z}}$) either $\mu(Y) = 0$ or $\mu(Y) = 1$. A Bernoulli measure is defined by a strictly positive probability vector $(p_a)_{a\in A}$ with $\sum_{a\in A} p_a = 1$ such that for $u = u_0 \ldots u_{n-1} \in A^n$ it is given by $\mu[u_0 \ldots u_{n-1}] = p_{u_o} \cdots p_{u_{n-1}}$. Every Bernoulli measure is σ-ergodic.

A subshift is any subset $\Sigma \subseteq A^{\mathbb{Z}}$, which is σ-invariant and closed in the Cantor metric. The language $L(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$ of a subshift $\Sigma \subseteq A^{\mathbb{Z}}$, is the set of words occurring in points of Σ. A subshift Σ is of finite type (SFT), if there exists a positive integer p called order, such that for $x \in A^{\mathbb{Z}}$,

$$x \in \Sigma \text{ iff } \forall i \in \mathbb{Z}, x_{[i,i+p-1]} \in L(\Sigma).$$

A subshift Σ is mixing, if

$$\forall u, v \in L(\Sigma), \exists q \geq 0, \forall m \geq q,\ \exists w \in A^m, uwv \in L(\Sigma).$$

If Σ is a SFT, q can be chosen independently of u and v.

For a SFT $\Sigma \subseteq A^{\mathbb{Z}}$ of order p and $x \in A^{\mathbb{Z}}$, define the density of Σ-defects in x by

$$d_D(x, \Sigma) = \limsup_{l\to\infty} \frac{\#\{i \in [-l, l] :\ x_{[i,i+p-1]} \notin L(\Sigma)\}}{2l + 1}.$$

The density of Σ-defects of $x \in A^{\mathbb{Z}}$ is related to the Besicovitch distance of x from Σ.

Proposition 3. *Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a SFT of order p. Then $d_D(x, \Sigma) \leq p \cdot d_B(x, \Sigma)$. If Σ is mixing with constant q, then $d_B(x, \Sigma) \leq q \cdot d_D(x, \Sigma)$.*

Proof: If $y \in \Sigma$, then

$$\begin{aligned} \#\{i \in [-l, l] :\ x_{[i,i+p-1]} \notin L(\Sigma)\} &\leq \#\{i \in [-l, l] :\ x_{[i,i+p-1]} \neq y_{[i,i+p-1]}\} \\ &\leq p \cdot \#\{i \in [-l, l] :\ x_i \neq y_i\} \end{aligned}$$

so $d_D(x, \Sigma) \leq p \cdot d_B(x, y)$ and therefore $d_D(x, \Sigma) \leq p \cdot d_B(x, \Sigma)$. Conversely, if Σ is mixing with constant q, then for a point $x \in A^{\mathbb{Z}}$ put

$$M(x) = \{i \in \mathbb{Z} :\ \exists j \in [0, q), x_{[i-j,i-j+p-1]} \notin L(\Sigma)\}.$$

Then $M(x)$ consists of intervals, each of which has length at least q (or infinite) and there exists $y \in \Sigma$, such that $y_i = x_i$ for $i \notin M(x)$. It follows

$$\#\{i \in [-l, l] :\ x_i \neq y_i\} \leq q \cdot \#\{i \in [-l, l] :\ x_{[i,i+p-1]} \notin L(\Sigma)\}$$

so $d_B(x, \Sigma) \leq d_B(x, y) \leq q \cdot d_D(x, \Sigma)$. △

4. Attracting and μ-attracting sets

Definition 2. *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA, $\Sigma \subseteq A^{\mathbb{Z}}$ a SFT with $f(\Sigma) = \Sigma$, μ a Bernoulli measure on $A^{\mathbb{Z}}$, and d either d_C or d_B or d_D. We say that Σ is μ-attracting (see Hurley* [8, 9]*), if*

$$\mu\{x \in A^{\mathbb{Z}} : \lim_{n\to\infty} d(f^n(x), \Sigma) = 0\} = 1.$$

Since the set $\{x \in A^{\mathbb{Z}} : \lim_{n\to\infty} d(f^n(x), \Sigma) = 0\}$ is σ-invariant, its measure is either zero or one. We write that Σ is $C\mu$, $B\mu$ or $D\mu$-attracting, if $d = d_C, d_B, d_D$ respectively (the definition uses only distance $d(x, \Sigma)$ of a point from a subshift, so it can be used with d_D, which is not a metric). In similar sense we say that Σ is C, B or D-attracting. From Proposition 3 we get immediately:

Proposition 4.
1. *If Σ is $B\mu$-attracting then it is $D\mu$-attracting.*
2. *If Σ is mixing and $D\mu$-attracting then it is $B\mu$-attracting.*

The following proof is due to F. Blanchard.

Proposition 5. *If Σ is a SFT which is C-attracting, then Σ is mixing.*

Proof: Without the loss of generality, we assume that the radius of the CA is one. Let p be the order of Σ. There exists m such that if $x_{[-m,m]} \in L(\Sigma)$, then $\lim_{n\to\infty} d_C(f^n(x), \Sigma) = 0$, so there exists n such that

$$x_{[-m,m]} \in L(\Sigma) \;\Rightarrow\; f^n(x)_{[-m-p,m+p]} \in L(\Sigma).$$

We can assume $p + 2n \geq 2m + 1$. Let $u, v \in L(\Sigma)$, $|u| = |v| = p$ and $k \geq 2n$. We construct $w \in A^k$ such that $uwv \in L(\Sigma)$ (see Fig. 1). Since $f(\Sigma) = \Sigma$, there exists $u', v' \in L(\Sigma)$, with $|u'| = |v'| = p+2n$, such that $f^n(u') = u$, $f^n(v') = v$. Construct a point $x \in A^{\mathbb{Z}}$, such that $x_{[0,p+2n-1]} = u'$, $x_{[p+k,2p+2n+k-1]} = v'$ and moreover $x_{[0,p+k-1]} \in L(\Sigma)$, so that $f^n(x)_{[n,n+p-1]} = u$, $f^n(x)_{[p+n+k,2p+n+k-1]} = v$. If either

$$0 \leq i < i + 2m \leq p + k - 1 \quad \text{or} \quad p + k \leq i < i + 2m \leq 2p + 2n + k - 1,$$

then $x_{[i,i+2m]} \in L(\Sigma)$, so $f^n(x)_{[i-p,i+2m+p]} \in L(\Sigma)$. In particular, $f^n(x)_{[i,i+p-1]} \in L(\Sigma)$ whenever $n \leq i < i + p \leq 2p + n + k$, so $f^n(x)_{[n,2p+n+k-1]} \in L(\Sigma)$. Thus for $w = f^n(x)_{[p+n,p+n+k-1]}$, $|w| = k$ and $uwv \in L(\Sigma)$. $\triangle$

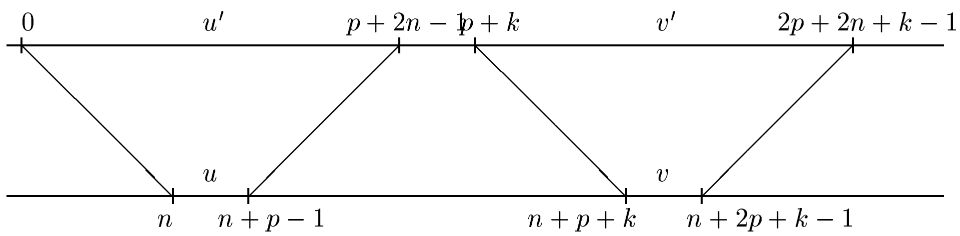
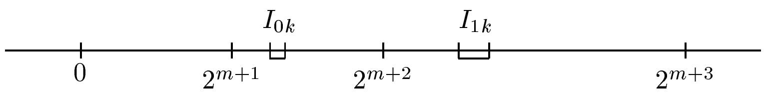


Figure 1. Mixing property.

Proposition 6. *If a SFT Σ is D-attracting, then for every $x \in A^{\mathbb{Z}}$, $\lim_{n\to\infty} d_D(f^n(x), \Sigma) = 0$.*

Proof: Let p be the order of Σ and $m > 0$ be such that for every $z \in A^{\mathbb{Z}}$ with $d_D(z, \Sigma) < 2^{-m}$, $\lim_{n\to\infty} d_D(f^n(z), \Sigma) = 0$. For $j \geq 0$ and $0 \leq k < 2^{m+1}$ put

$$I_{jk} = (-2^{m+j+1} - (k+1)2^j, -2^{m+j+1} - k2^j] \cup [2^{m+j+1} + k2^j, 2^{m+j+1} + (k+1)2^j)$$



For a point $x \in A^{\mathbb{Z}}$ we show $\lim_{n\to\infty} d_D(f^n(x), \Sigma) = 0$. Let $y \in \Sigma$ be arbitrary. For $0 \leq k < 2^{m+1}$ construct a point x_k by

$$\begin{aligned} (x_k)_i &= x_i \quad \text{if } i \in I_{jk} \text{ for some } j \geq 0, \\ (x_k)_i &= y_i \quad \text{otherwise.} \end{aligned}$$

Then $d_D(x_k, \Sigma) < 2^{-m}$, so $\lim_{n\to\infty} d_D(f^n(x_k), x) = 0$ and

$$\forall \varepsilon > 0, \exists n_0, \forall n > n_0, \forall k \in \{0, ..., 2^{m+1} - 1\}, d_D(f^n(x_k), \Sigma) < \varepsilon 2^{-m-1},$$

so there exists l_0 such that for all $l \geq l_0$ and all k,

$$\{i \in [-l, l] : f^n(x_k)_{[i,i+p-1]} \notin L(\Sigma)\} \leq (2l+1)\varepsilon 2^{-m-1}.$$

Fix ε and $n > n_0$. For $l > l_0$, let j be such that $2^{m+j} \leq l < 2^{m+j+1}$. Then

$$\begin{aligned} &\#\{i \in [-l, l] :\ f^n(x)_{[i,i+p-1]} \notin L(\Sigma)\} \\ &\leq \sum_{k=0}^{2^{m+1}-1} \left(\#\{i \in [-l, l] :\ f^n(x_k)_{[i,i+p-1]} \notin L(\Sigma)\} + 4nj\right) \\ &\leq 2^{m+1}((2l+1)\varepsilon 2^{-m-1} + 4nj) \leq (2l+1)\left(\varepsilon + \frac{n2^{m+3}(\log_2 l - m)}{2l+1}\right) \end{aligned}$$

Here we count differences in every interval I_{jk} as well as differences at their borders, which propagate with time. It follows $d_D(f^n(x), \Sigma) < \varepsilon$, so $\lim_{n\to\infty} d_D(f^n(x), \Sigma) = 0$. $\triangle$

See Example 1 for a CA with D-attracting fixed point. In all the examples we know, if Σ is D-attracting, then $\Sigma = f^p(A^{\mathbb{Z}})$ for some $p > 0$.

Proposition 7. *Let f be a CA, Σ a SFT with $f(\Sigma) = \Sigma$ and μ any Bernoulli measure.*
1. If Σ is a C-attracting, then it is $C\mu$-attracting.
2. If Σ is a D-attracting, then it is $D\mu$-attracting.

Proof: 1. Let $\delta > 0$ be such that if $d(x, \Sigma) < \delta$, then $\lim_{n\to\infty} d(f^n(x), \Sigma) = 0$. Then

$$\mu\{x :\ d_C(x, \Sigma) < \delta\} > 0, \ \text{ and } \ \mu\{x :\ \lim_{n\to\infty} d_C(f^n(x), \Sigma) = 0\} > 0.$$

2. follows from Proposition 6. $\triangle$

Example 2 gives a CA with a $D\mu$-attracting fixed point which is not D-attracting.

Lemma 1. *Let μ be Bernoulli measure on $A^{\mathbb{Z}}$, let $U \subseteq A^{\mathbb{Z}}$ be a clopen set, $p = \mu(U)$, and $\varepsilon > 0$. Then there exist $a > 0$ and $0 < r < 1$ (which depend on ε), such that for all l,*

$$\mu\{x \in A^{\mathbb{Z}} : \#\{i \in [-l,l] : \sigma^i(x) \in U\} \geq (p+\varepsilon)(2l+1)\} \leq ar^{2l+1}.$$

Proof: There exists $k \geq 1$ and $B \subset A^k$ such that $U = \cup\{[u]_0 : u \in B\}$. Suppose first that $k = 1$. By a theorem of Bernstein (see Renyi, [14], p.389), we have

$$\mu\{x \in A^{\mathbb{Z}} : \#\{i \in [-l,l] : x_i \in B\} \geq (\varepsilon+p)(2l+1)\} \leq 2r^{2l+1}$$

where

$$r = \exp\left[-\frac{\varepsilon^2}{2p(1-p)(1+\frac{\varepsilon}{2p(1-p)})^2}\right].$$

If $k > 1$, we use the fact that $x_{[j+1,j+k]}, x_{[j+k+1,j+2k]}, \ldots$, are independent, so

$$\begin{aligned}
&\mu\{x \in A^{\mathbb{Z}} : \#\{i \in [-l,l] : x_{[i+1,i+k]} \in B\} \geq (p+\varepsilon)(2l+1)\} \\
&\leq \quad \mu\{x \in A^{\mathbb{Z}} : \exists j < k, \#\{i \in [\tfrac{-l}{k}, \tfrac{l}{k} - 1] : x_{[j+ik+1,j+ik+k]} \in B\} \geq \tfrac{(p+\varepsilon)(2l+1)}{k}\} \\
&\leq \quad 2k \cdot r^{\frac{2l+1}{k}}
\end{aligned}$$

where r is computed with respect to the Bernoulli measure induced by μ on $(B^k)^{\mathbb{Z}}$. $\triangle$

Proposition 8. *Let μ be a Bernoulli measure and Σ a $C\mu$-attracting SFT of order p. Then*

$$\lim_{t\to\infty} \mu\{x \in A^{\mathbb{Z}} : f^t(x)_{[0,p-1]} \notin L(\Sigma)\} = 0.$$

Proof: By the assumption

$$\mu\{x \in A^{\mathbb{Z}} : \exists t_0, \forall t \geq t_0, f^t(x)_{[0,p-1]} \in L(\Sigma)\} = 1,$$

so for every $\varepsilon > 0$ there exists t_0 such that

$$\mu\{x \in A^{\mathbb{Z}} : \forall t \geq t_0, f^t(x)_{[0,p-1]} \in L(\Sigma)\} > 1 - \varepsilon.$$

It follows that for every $\varepsilon > 0$ there exists t_0 such that for every $t \geq t_0$,

$$\mu\{x \in A^{\mathbb{Z}} : f^t(x)_{[0,p-1]} \notin L(\Sigma)\} < \varepsilon$$

which proves our claim. $\triangle$

Proposition 9. *Let μ be a Bernoulli measure and Σ a SFT of order p such that*

$$\lim_{t\to\infty} \mu\{x \in A^{\mathbb{Z}} : f^t(x)_{[0,p-1]} \notin L(\Sigma)\} = 0.$$

Then Σ is $D\mu$-attracting.

Proof: By the assumption for every $\varepsilon > 0$ there exists t_0 such that for every $t \geq t_0$,

$$\mu\{x \in A^{\mathbb{Z}} : f^t(x)_{[0,p-1]} \notin L(\Sigma)\} < \varepsilon.$$

Since this is a clopen set, by Lemma 1 there exists $a > 0$ and $0 < r < 1$ such that

$$\mu\{x \in A^{\mathbb{Z}} : \#\{i \in [-l,l] : f^t(x)_{[i,i+p-1]} \notin L(\Sigma)\} \geq (2l+1)\varepsilon\} \leq ar^{2l+1}.$$

By Borel-Cantelli lemma

$$\mu\{x \in A^{\mathbb{Z}} : \forall l_0, \exists l \geq l_0, \#\{i \in [-l,l] : f^t(x)_{[i,i+p-1]} \notin L(\Sigma)\} \geq (2l+1)\varepsilon\} = 0.$$

We conclude that for every $\varepsilon > 0$ there exists t_0 such that for all $t \geq t_0$,

$$\mu\{x \in A^{\mathbb{Z}} : d_D(f^t(x), \Sigma) < \varepsilon\} = 1,$$

and therefore

$$\mu\left\{x \in A^{\mathbb{Z}} : \forall \varepsilon > 0, \exists t_0, \forall t \geq t_0, d_D(f^t(x), \Sigma) < \varepsilon\right\} = 1$$

so Σ is $D\mu$-attracting. $\triangle$

Example 4 shows a CA with a $C\mu$-attracting fixed point which is not $D\mu$-attracting.

5. Examples

Example 1. *$A = \{0,1\}$, $f(x) = 0^\infty$. The point 0^∞ is D-attracting.*

Example 2. *$A = \{0,1\}$, $f(x)_i = x_{i-1}x_ix_{i+1}$. The point 0^∞ is C-attracting, and $D\mu$-attracting but not D-attracting, since it does not attract the fixed point 1^∞.*

Example 3 (Adding machine). $\begin{array}{c|c|c|c|c} *2 & 0x & 1x & 2x & 2 \\ \hline * & 1 & 2 & 1 & 0 \end{array}$ *for $x \in \{*, 2\}$, with alphabet $A = \{0,1,2,*\}$. Here the first applicable rule is used and if there are no applicable rule the symbol is not changed.*

A finite string $u \in \{0,1,2\}^*$ surrounded by stars represents a number in binary expansion (in which two's are allowed as well). Every time step this number grows by one, but, if its first digit is 2, it is erased (see Fig. 2).

```
 *0*00*11*22*000*111*222*
**1*01*12**1*001*112**11*
**2*02*21**2*002*121**12*
****11**2****011*202**21*
****12*******012**11***2*
****21*******021**12*****
*****2*******102**21*****
*************111***2*****
*************112*********
```

Figure 2. Adding machine

Proposition 10. *For any Bernoulli measure μ, the fixed point $*^\infty$ is $C\mu$-attracting but not C-attracting in the Adding machine CA of Example 3.*

Proof: For every $k > 0$ construct a point $x \in A^{\mathbb{Z}}$ by $x_{[0,k-1]} = *^k$ and $x_i = 0$ for $i \notin [0, k-1]$. Then $f^n(x)_i \neq *$ for every $n \geq 0$ and $i \notin [0, k-1]$, so $*^\infty$ is not C-attracting. If μ is a Bernoulli measure on $A^{\mathbb{Z}}$, then with probability one, there exists no infinite string of $\{0,1,2\}$, and every finite string $u \in \{0,1,2\}^*$ surrounded by stars disappears, so $*^\infty$ is $C\mu$-attracting. $\triangle$

In the next example we consider some random walks and Markov chains. Let I be a countable set of states. A sequence $(X_i)_{i\geq 0}$ of I-valued random variables is a Markov chain, if

$$\Pr[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = \Pr[X_{n+1} = j | X_n = i] = P_{ij};$$

P_{ij} is the transition probability from i to j in one step and $P_{ij}^n = \sum_{k\in I} P_{ik}^{n-1} P_{kj}$ is the probability of transition from i to j in n steps. A Markov chain is aperiodic and irreducible, if there exists n_0 such that for every $i, j \in I$, for all $n \geq n_0$,

$P_{ij}^n > 0$. A state $i \in I$ is recurrent, if $\Pr[\exists n > 0, X_n = i | X_0 = i] = 1$ and this happens iff $\sum_{n\geq 0} P_{ii}^n = \infty$. A $\mathbb{Z}^n$-valued Markov chain is a random walk, if $(X_{n+1} - X_n)_{n\geq 0}$ are independent, identically distributed random variables. An irreducible and aperiodic random walk in $\mathbb{Z}$ and $\mathbb{Z}^2$ is recurrent iff $E(X_{n+1} - X_n) = 0$ (see Karlin [10]).

Example 4 (Just gliders). *$A = \{-1, 0, 1\}$, for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$,*

$$\begin{array}{ll} f(x)_i = 1 & \text{if } x_{i-1} = 1,\ x_i \neq -1,\ x_{i+1} \neq -1 \\ f(x)_i = -1 & \text{if } x_{i+1} = -1,\ x_i \neq 1,\ x_{i-1} \neq 1 \\ f(x)_i = 0 & \text{otherwise} \end{array}$$

Thus 1 is a particle (glider) which moves to the right with velocity 1 and -1 moves to the left with velocity -1. When two particles cross, they both annihilate. Gilman [6] shows, that if μ is a Bernoulli measure and $\mu([1]) > \mu([-1])$, then $\mathbf{2}^{\mathbb{Z}}$ is $C\mu$-attracting.

Proposition 11. *Let $(A^{\mathbb{Z}}, f)$ be the "Just gliders" CA from Example 4, and μ a Bernoulli measure with $\mu([1]) = \mu([-1])$. Then $\{0^\infty\}$ is $D\mu$-attracting and $B\mu$-attracting but not $C\mu$-attracting.*

Proof: If $f^n(x)_0 = 1$, there is a particle 1 at site $-n$, which is not annihilated during time interval $[0, n]$, so whenever $x_j = -1$ for some $-n < j \leq n$, then $x_i = 1$ for some $-n < i < j$ and particles at i and j annihilate. Thus

$$f^n(x)_0 = 1 \Leftrightarrow \forall j \in [-n, n], \sum_{i=-n}^{j} x_i > 0.$$

Since μ is σ-invariant,

$$\mu\{x \in A^{\mathbb{Z}} : f^n(x)_0 = 1\} = \mu\{x \in A^{\mathbb{Z}} : \forall t \in [0, 2n], \sum_{i=0}^{t} x_i > 0\}.$$

Since $(\sum_{i=0}^t x_i)_{t\geq 0}$ is a recurrent random walk, this probability tends to zero. Similarly, $\lim_{n\to\infty} \mu\{x \in A^{\mathbb{Z}} : f^n(x)_0 = -1\} = 0$. By Proposition 9, $\{0^\infty\}$ is $D\mu$-attracting.

We show now that $\{0^\infty\}$ is not a $C\mu$-attracting. Put $S_0 = T_0 = 0$, $S_1 = -x_1$, $T_1 = x_0$,

$$S_n = -x_1 - \ldots - x_n,\ T_n = x_0 + x_{-1} + \ldots + x_{-n+1},\ n \geq 1.$$

Then $(S_n, T_n)_{n\geq 0}$ is an irreducible, aperiodic, recurrent random walk in $\mathbb{Z}^2$. If $j \in [-n, 0]$, then $\sum_{i=-n}^{j} x_i = T_{n+1} - T_{|j|}$ and if $j \in [0, n]$, then $\sum_{i=-n}^{j} x_i = T_{n+1} - S_j$. We get

$$f^n(x)_0 = 1 \Leftrightarrow \forall j \in [0, n], T_j < T_{n+1}\ \&\ S_j < T_{n+1}.$$

Thus $f^n(x)_0 = 1$, if there exists $k > 0$ such that $(S_n, T_n)_{n\geq 0}$ enters into the set

$$\{(S, T) \in \mathbb{Z} : \max(S, T) = k\}$$

for the first time at time $n+1$, and when it enters, $T_{n+1} = k$. Put $Y_n = \max(S_n, T_n)$, $Z_n = T_n - S_n$. Then $(Y_n, Z_n)_{n\geq 0}$ is an aperiodic, irreducible and recurrent Markov chain in $\mathbb{Z}^2$, and $(Z_n)_{n\geq 0}$ is an aperiodic, irreducible and recurrent random walk in $\mathbb{Z}$. For every $k > 0$, there exists (with probability one) the first entrance time $t_k = \min\{n > 0 : Y_{n+1} = k\}$. Suppose that

$$q_k = \mu\{x \in A^{\mathbb{Z}} : \exists k_0, \forall k \geq k_0, Z_{t_k+1} < 0\} = 1.$$

Since $(Y_n, -Z_n)_{n\geq 0}$ has the same distribution as $(Y_n, Z_n)_{n\geq 0}$, we get

$$q_k = \mu\{x \in A^{\mathbb{Z}} : \exists k_0, \forall k \geq k_0, -Z_{t_k+1} < 0\} = 1$$

and this is a contradiction, so $q_k < 1$, and

$$\mu\{x \in A^{\mathbb{Z}} : \forall t_0, \exists t \geq t_0, F^t(x)_0 = 1\} > 0.$$

Since this is the probability of a σ-invariant set, it is actually equal to 1, and

$$\mu\{x \in A^{\mathbb{Z}} : \forall t_0, \exists t \geq t_0, d(F^t(x), 0^\infty) = 1\} = 1$$
$$\mu\{x \in A^{\mathbb{Z}} : \lim_{t\to\infty} F^t(x) = 0^\infty\} = 0$$

so $\{0^\infty\}$ is not $C\mu$-attracting. $\triangle$

Example 5.

$+++$	$++-$	$+-+$	$+--$	$-++$	$-+-$	$--+$	$---$
$+$	$+$	$+$	$-$	$-$	$-$	$+$	$-$

This is the elementary CA with rule 184 in alphabet $B = \{-1, +1\}$.

This is a model of car trafic. A car represented by -1 (written as $-$ in the table) moves to the right, provided its right neighbour field is free, i.e., contains $+1$ (written as $+$). This means that every word $-+$ is rewritten by $+-$. Call a hole any string of two or more $+$ and jam a string of two or more $-$. Holes move to the right and jams move to the left until they collide. On collision, holes and jams of equal length annihilate. This is the same dynamics as "Just gliders" CA $(A^{\mathbb{Z}}, f)$ of Example 4 in the alphabet $A = \{-1, 0, 1\}$.

Proposition 12. *Let $(B^{\mathbb{Z}}, g)$ be rule 184 CA and μ the uniform Bernoulli measure, i.e., $\mu([-1]) = \mu([1]) = \frac{1}{2}$. Then $\Sigma = \{(-11)^\infty, (1-1)^\infty\}$ is $D\mu$-attracting but not $B\mu$-attracting.*

Proof: We have

$$g^n(x)_{[0,1]} = ++ \quad \Leftrightarrow \quad \forall t \in [-n, n+1], \sum_{i=-n}^{t} x_i > 0,$$
$$g^n(x)_{[0,1]} = -- \quad \Leftrightarrow \quad \forall t \in [-n, n+1], \sum_{i=t}^{n+1} x_i < 0.$$

Using the same argument as in the Example 4 we show that Σ is $D\mu$-attracting.

If $d_B(g^n(x), \Sigma)$ converges to 0, then there is n_0 such that for all $n \geq n_0$, $d_B(g^n(x), \Sigma) < \frac{1}{2}$. For every $n \geq n_0$, either $d_B(g^n(x), (-11)^\infty) < \frac{1}{2}$ and $d_B(g^{n+1}(x), (1-1)^\infty) < \frac{1}{2}$ or vice versa. Assume that Σ is $B\mu$-attracting and put

$$Y_0 = \{x \in B^{\mathbb{Z}} : \lim_{n\to\infty} d_B(g^{2n}(x), (-11)^\infty) = 0\},$$
$$Y_1 = \{x \in B^{\mathbb{Z}} : \lim_{n\to\infty} d_B(g^{2n}(x), (1-1)^\infty) = 0\}.$$

Then $\sigma(Y_0) = Y_1$ and $\sigma(Y_1) = Y_0$. Since σ^2 is μ-mixing and both Y_0 and Y_1 are σ^2-invariant, they have either measure 0 or 1. Since they are disjoint, they have measure 0 and $Y_0 \cup Y_1$ has measure 0 too. This is a contradiction. $\triangle$.

6. Concluding remarks

From all the concepts introduced, the D-attractivity seems to be the strongest because of Proposition 6. We conjecture that if Σ is D-attracting, then $\Sigma = f^k(A^{\mathbb{Z}})$ for some $k \in \mathbb{N}$. This would also imply that Σ is C-attracting. On the other hand, $D\mu$-attractivity seems to be the weakest concept as it is impled by both $C\mu$-attractivity and $B\mu$-attractivity. The overall scheme of our concepts could be as follows (unproved implications are with a question mark).

$$D \overset{?}{\to} C \to C\mu \overset{?}{\to} B\mu \to D\mu$$

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