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ITERATIVE SYSTEMS OF REAL MÖBIUS TRANSFORMATIONS

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Abstract. We investigate iterative systems consisting of Möbius transformations on the extended real line. We characterize systems with unique attractor and give some sufficient conditions for minimality.

1. Introduction.

Iterative systems consisting of contractions have been used in Barnsley [1] or Edgar [2] to generate self-similar fractal sets. Iterative systems can be viewed as dynamical systems over a free semigroup and their theory can be generalized beyond the assumption of contractivity. Many concepts of topological dynamics can be generalized to this setting, for example attractors, minimality, transitivity or chain-transitivity.

In [4] we have used iterative Möbius systems to construct some symbolic representations of real numbers. In the present paper we investigate topological properties of iterative Möbius systems. We show that a Möbius system has a unique trivial attractor whenever it contains a parabolic or an elliptic transformation. In the case of hyperbolic Möbius systems, the existence of a nontrivial attractor depends on the values of quotients. We give a necessary and sufficient condition that a system of two hyperbolic transformations has only the trivial attractor. We show that some of these systems are not transitive. Finally we give two sufficient conditions for minimality.

2. Topological dynamics.

Given a finite alphabet A, we denote by $A^+ = \cup_{n\geq 1} A^n$ the set of words with letters from A. With the binary operation of concatenation, A^+ is the **free semigroup** over A. An **iterative system** over A^+ is a pair (X, F), where X is a compact metric space and $F : A^+ \times X \to X$ is a map such that for each $u \in A^+$, $F_u : X \to X$ is continuous, and $F_{uv}(x) = F_u(F_v(x))$. The system is generated by continuous maps $(F_a : X \to X)_{a\in A}$. A special case is an iterative system over $\mathbb{N} = \{1, 2, 3, \ldots\}$, which is the free semigroup over a one-letter alphabet. In this case the system is generated by a map $F_1 : X \to X$. A **morphism** $\Phi : (X, F) \to (Y, G)$ of two iterative systems over A^+ is a continuous map $\Phi : X \to Y$ such that $\Phi F_a = G_a \Phi$ for all $a \in A$. A **factor** is a surjective morphism, a **subsystem** is an injective morphism, and a **conjugacy** is a bijective morphism.

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Definition 1. Let (X,F) be an iterative system over A^+ and denote by d the distance on X. The **orbit relation** $\mathcal{O}$, the **recurrence relation** $\mathcal{R}$, the **nonwandering relation** $\mathcal{N}$, and the **chain relation** $\mathcal{C}$ are defined on X by

$$\begin{aligned}
(x,y)\in\mathcal{O} \quad&\Leftrightarrow\quad \exists u\in A^+, y=F_u(x)\\
(x,y)\in\mathcal{R} \quad&\Leftrightarrow\quad \forall\varepsilon>0, \exists u\in A^+, d(y,F_u(x))<\varepsilon\\
(x,y)\in\mathcal{N} \quad&\Leftrightarrow\quad \forall\varepsilon>0, \exists u\in A^+, \exists z, d(z,x)<\varepsilon, d(F_u(z),y)<\varepsilon\\
(x,y)\in\mathcal{C} \quad&\Leftrightarrow\quad \forall\varepsilon>0, \exists u\in A^+, \exists x_0,\dots,x_{|u|}, x=x_0, x_{|u|}=y,\ \&\\
&\qquad \forall i<n, d(F_{u_i}(x_i),x_{i+1})<\varepsilon
\end{aligned}$$

A system (X,F) is **minimal** if $\mathcal{R}=X\times X$, **transitive** if $\mathcal{N}=X\times X$ and **chain-transitive** if $\mathcal{C}=X\times X$. The orbit of a point is $\mathcal{O}(x)=\{F_u(x):\ u\in A^+\}$. The set of **periodic points** is $|\mathcal{O}|=\{x\in X:\ (x,x)\in\mathcal{O}\}$. The set of **transitive points** is $\mathcal{T}=\{x\in X:\ \overline{\mathcal{O}(x)}=X\}$. It is easy to see that a system is minimal iff it has no proper subsystems iff each its point is transitive (see Kůrka [3]).

A subset $Y\subseteq X$ is **invariant**, if $F_a(Y)\subseteq Y$ for all $a\in A$. If Y is invariant and closed, then (Y,F) is a subsystem of (X,F). A set $Y\subseteq X$ is **strongly invariant** if $Y=\bigcup_{a\in A}F_a(Y)$. A closed set $W\subseteq X$ is **inward** if $\bigcup_{a\in A}F_a(\overline{W})\subseteq W^\circ$. A set $Y\subseteq X$ is an **attractor**, if there exists an inward set W such that $Y=\Omega(W)$ is its omega-limit. A one-point attractor is called a **stable fixed point**. The **omega-limit** of a set $W\subseteq X$ is

$$\Omega(W)=\bigcap_{n\geq 0}\overline{\bigcup_{m\geq n}\bigcup_{u\in A^m}F_u(W)}.$$

Example 1. *Given an alphabet A, let $A^{\mathbb{N}}$ be the symbolic space of infinite words over A and define $S_a: A^{\mathbb{N}}\to A^{\mathbb{N}}$ by $S_a(x)=ax$. Then the* **shift iterative system** *$(A^{\mathbb{N}},S)$ over A^+ is minimal and has a dense set of periodic points.*

Proposition 2. *Assume that (X,F) is an iterative system over A^+ and $B\subset A$. If (X,F) is minimal over B^+, then (X,F) is minimal over A^+. Analogous statements hold for transitivity and chain-transitivity.*

Proposition 3. *Let (X,F) be an iterative system over A^+. The following conditions are equivalent.*
(1) (X,F) is transitive.
(2) For nonempty open sets $U,V\subseteq X$ there exists $u\in A^+$ with $F_u(U)\cap V\neq\emptyset$.
(3) The set of transitive points $\mathcal{T}$ is residual.
(4) The set of transitive points $\mathcal{T}$ is dense.

Proof. (1) $\Leftrightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3): Assume that (X,F) is transitive. Let $(U_n)_{n\geq 0}$ be a countable base of X. For $n\geq 0$ set $V_n:=\bigcup_{u\in A^+}F_u^{-1}(U_n)$. Then V_n are open and dense and their intersection $\mathcal{T}=\bigcap_{n\geq 0}V_n$ is residual.
(3) $\Rightarrow$ (4): By the Baire category theorem a residual set is dense.
(4) $\Rightarrow$ (2): If U,V are nonempty open sets, then there exists $x\in U\cap\mathcal{T}$ and therefore $F_u(x)\in V$ for some $u\in A^+$. □

Proposition 4. *Let (X,F) be an iterative system over A^+.*
(1) If $Y\subseteq X$ is an attractor, then Y is strongly invariant and $\mathcal{C}$-invariant, i.e., if $x\in Y$ and $(x,y)\in\mathcal{C}$, then $y\in Y$.
(2) If (X,F) is chain-transitive, then the only attractor is X.

The proof is analogous to the proof of Theorem 2.69 in Kůrka [3].

3. **Dynamics in the Hausdorff space.** The Hausdorff space $\mathcal{K}(X)$ of a metric space X is the space of compact subsets of X with metric

$$D(Y,Z) = \inf\{\varepsilon > 0 : \ Y \subseteq B_\varepsilon(Z), Z \subseteq B_\varepsilon(Y)\}.$$

Here $B_\varepsilon(Y) = \{x \in X : \ d(x,Y) < \varepsilon\}$, where $d(x,Y) = \inf\{d(x,y) : \ y \in Y\}$. If X is compact then so is $\mathcal{K}(X)$. If (X,F) is an iterative system over A^+, then $\widetilde{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\widetilde{F}(Y) = \bigcup_{a\in A} F_a(Y)$ is continuous, so $(\mathcal{K}(X), \widetilde{F})$ is an iterative system over $\mathbb{N}$. Recall that a map $F : X \to X$ is a **contraction**, if there exists a positive $q < 1$ such that for all $x_0, x_1 \in X$ we have $d(F(x_0), F(x_1)) \leq q \cdot d(x_0, x_1)$.

Theorem 5 (Barnsley [1]). *Let (X,F) be an iterative system over A^+ such that all F_a are contractions. Then (X,F) has a unique attractor Y which is the unique stable fixed point of $(\mathcal{K}(X), \widetilde{F})$. Moreover there exists a factor map $\Phi : (A^{\mathbb{N}}, S) \to (Y, F)$ defined by*

$$\Phi(u) = \bigcap_{n>0} F_{u_1\ldots u_n}(X).$$

Proposition 6. *Let (X,F) be an iterative system over A^+.*

(1) If Y is an attractor such that (Y,F) is minimal, then Y is a stable fixed point in $(\mathcal{K}(X), \widetilde{F})$.

(2) If Y is a stable fixed point in $(\mathcal{K}(X), \widetilde{F})$, then Y is an attractor in (X,F).

The proof is straightforward. Next examples show that the assumption of minimality in Proposition 6 can be relaxed, but transitivity is not strong enough.

Example 2. *If (X,F) is a disjoint union of finitely many minimal dynamical systems, then X is a stable fixed point in $(\mathcal{K}(X), \widetilde{F})$.*

Example 3. *Let (X,F) be a transitive system over $\mathbb{N}$ with a dense set of periodic points. Then X is not a stable fixed point in $(\mathcal{K}(X), \widetilde{F})$.*

Proof. For each $\varepsilon > 0$ there exists a finite invariant set $X_\varepsilon \subset X$ of periodic points such that $D(X, X_\varepsilon) < \varepsilon$ and $\widetilde{F}(X_\varepsilon) = X_\varepsilon$. □

4. **Möbius systems.** The extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is homeomorphic to the unit circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ which we parametrize by $\mathbb{T} = (-\pi, \pi]$. The stereographic projection yields mutually inverse transformations $\mathbf{x} : \mathbb{T} \to \overline{\mathbb{R}}$ and $\mathbf{t} : \overline{\mathbb{R}} \to \mathbb{T}$ given by $\mathbf{x}(t) = \tan\frac{t}{2}$, $\mathbf{t}(x) = 2\arctan x$. The circle distance on $\mathbb{T}$ yields a metric

$$d(x,y) := \min\{|\mathbf{t}(x) - \mathbf{t}(y)|, 2\pi - |\mathbf{t}(x) - \mathbf{t}(y)|\}$$

on $\overline{\mathbb{R}}$. Given $a, b \in \overline{\mathbb{R}}$, define the interval (a,b) as

$$(a,b) = \begin{cases} \{x \in \mathbb{R} : \ a < x < b\} & \text{if} \quad a < b \\ \{x \in \overline{\mathbb{R}} : \ a < x \text{ or } x < b\} & \text{if} \quad b < a \end{cases}$$

Closed intervals are defined analogously. We write $x_1 \prec x_2 \prec \cdots \prec x_n$ provided $n \geq 3$ and (x_i, x_{i+1}), (x_j, x_{j+1}) are disjoint intervals for $i \neq j$. If either $x_1 \prec x_2 \prec \cdots \prec x_n$ or $x_n \prec x_{n-1} \prec \cdots \prec x_1$, then we say that (x_n) is a monotone sequence.

A real, orientation preserving Möbius transformation is a self-map of the extended real line of the form $M_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d}$, where $ad-bc>0$. For a Möbius transformation $M = M_{(a,b,c,d)}$ we have

$$M^\bullet(x) := \lim_{y\to x} \frac{d(M(x),M(y))}{d(x,y)} = \frac{(ad-bc)(x^2+1)}{(ax+b)^2+(cx+d)^2}.$$

A Möbius transformation M is **hyperbolic**, if it has two fixed points in $\overline{\mathbb{R}}$, **parabolic** if it has a unique fixed point in $\overline{\mathbb{R}}$, and **elliptic**, if it has no fixed point in $\overline{\mathbb{R}}$. These cases are determined by the **trace** $\mathbf{tr}(M_{(a,b,c,d)}) := (a+d)/\sqrt{ad-bc}$. If $|\mathbf{tr}(M)| < 2$ then M is elliptic, if $|\mathbf{tr}(M)| > 2$ then M is hyperbolic and if $|\mathbf{tr}(M)| = 2$ then M is parabolic unless it is the identity. If s, r are fixed points of a hyperbolic transformation, then $M^\bullet(s)\cdot M^\bullet(r) = 1$. If s is stable, then $M^\bullet(s) < 1$. An elliptic transformation $M_{(a,b,c,d)}$ is conjugated to a self-map $z \mapsto z\cdot e^{i\alpha}$ on the unit complex circle, and α is called the **rotation angle** of $M_{(a,b,c,d)}$. If $ad-bc=1$ then

$$\cos\alpha = \frac{(a+d)^2-2}{2}, \quad \sin\alpha = \frac{(a+d)\sqrt{4-(a+d)^2}}{-2\cdot\operatorname{sgn}(c)}$$

We say that an iterative system $(\overline{\mathbb{R}}, F)$ over A^+ is a **Möbius system**, if all F_a are Möbius transformations. We say that a Möbius system is **hyperbolic**, if all F_a are hyperbolic. Since Möbius transformations are surjective, the maximal (**trivial**) attractor of a Möbius system is $\overline{\mathbb{R}}$. In some Möbius systems, **non-trivial attractors** exist as well.

Proposition 7. *Let $(\overline{\mathbb{R}}, F)$ be an iterative Möbius system. If F_u is either parabolic or elliptic for some $u \in A^+$, then $(\overline{\mathbb{R}}, F)$ has only the trivial attractor.*

Proof. Any Möbius system over $\mathbb{N}$ which consists of a parabolic or an elliptic transformation is chain-transitive. By Proposition 2, any A^+-system which contains a parabolic or elliptic transformation is chain-transitive and therefore has only the trivial attractor. □

Proposition 8. *Let $(\overline{\mathbb{R}}, F)$ be an iterative Möbius system. If for some $u \in A^+$, F_u is an elliptic transformation with rotation angle α such that $\alpha/2\pi$ is irrational, then $(\overline{\mathbb{R}}, F)$ is minimal.*

Proof. Any iterative system over $\mathbb{N}$ consisting of an elliptic transformation with irrational rotation angle is minimal. □

Proposition 9. *Let $(\overline{\mathbb{R}}, F)$ be a hyperbolic Möbius system such that there exists a closed interval $W \subset \overline{\mathbb{R}}$ which contains in its interior stable fixed points of all F_a and does not contain the unstable fixed point of any F_a. Then $(\overline{\mathbb{R}}, F)$ has a nontrivial attractor Y which is a factor of $(A^{\mathbb{N}}, S)$. Moreover, (Y, F) is minimal and has a dense set of periodic points.*

Proof. If the condition is satisfied, then W is an inward set. There exists n such that for any $u \in A^n$, F_u is a contraction on W. There exists a factor map $\Phi : (A^{\mathbb{N}}, S) \to (\Omega(W), F)$ defined by $\{\Phi(u)\} = \bigcap_{n>0} F_{u_1\dots u_n}(W)$. It follows that $(\Omega(W), F)$ is minimal and has a dense set of periodic points. □

A hyperbolic Möbius transformation F_a is determined by its stable fixed point s_a, its unstable fixed point r_a and its **quotient** $q_a < 1$, which satisfy

$$F_a(s_a) = s_a,\ F_a(r_a) = r_a,\ F_a^\bullet(s_a) = q_a,\ F_a^\bullet(r_a) = 1/q_a.$$

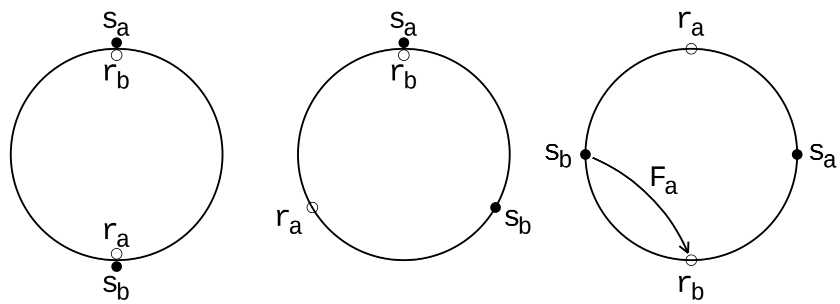

FIGURE 1. Hyperbolic Möbius systems

We consider now a Möbius system $(\overline{\mathbb{R}}, F_a, F_b)$ consistsing of two hyperbolic transformations. If the system does not satisfy the condition of Proposition 9, then either $s_a = r_b$ or $s_b = r_a$ or s_a, r_a, s_b, r_b is a monotone sequence. We are going to treat now these cases.

Proposition 10. *Let $(\overline{\mathbb{R}}, F_a, F_b)$ be a hyperbolic Möbius system.*

(1) If $s_a = r_b$ and $s_b = r_a$, then $(\overline{\mathbb{R}}, F_a, F_b)$ has only the trivial attractor and is not transitive. Both intervals $[s_a, s_b]$ and $[s_b, s_a]$ are invariant (Figure 1 left).

(2) If $s_a = r_b \prec r_a \prec s_b$, then $(\overline{\mathbb{R}}, F_a, F_b)$ has only the trivial attractor and is not transitive. The interval $[s_b, s_a]$ is invariant (Figure 1 center).

Proof. For each $x, y \in \overline{\mathbb{R}}$ we have $(x, s_a) \in \mathfrak{C}$, $(r_b, y) \in \mathfrak{C}$. Since $\mathfrak{C}$ is transitive, $(x, y) \in \mathfrak{C}$ and the system is chain-transitive. □

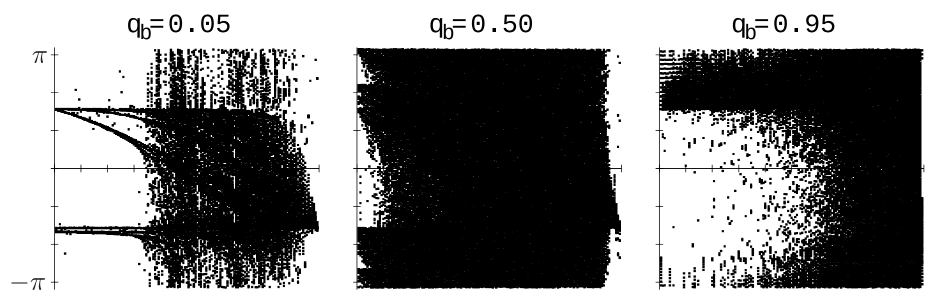

FIGURE 2. Bifurcation diagrams of alternating Möbius systems

5. **Alternating Möbius systems.** We say that a Möbius system $(\overline{\mathbb{R}}, F_a, F_b)$ is **alternating**, if $s_a \prec r_a \prec s_b \prec r_b \prec s_a$. In this case the existence of a nontrivial attractor depends on the quotients q_a, q_b. For fixed s_a, r_a, s_b, r_b denote by t_a the unique parameter q_a for which $F_a(s_b) = r_b$ (Figure 1 right), and by t_b the unique parameter q_b for which $F_b(s_a) = r_a$. In the examples and figures we use the **standard alternating system** given by

$$\begin{aligned} F_a(x) &= q_a x + 1 - q_a, \qquad s_a = 1,\ r_a = \infty,\ t_a = 0.5, \\ F_b(x) &= \frac{x}{-(1-q_b)x + q_b},\ s_b = -1,\ r_b = 0,\ t_b = 0.5. \end{aligned}$$

In Figure 2 we show bifurcation diagrams of standard alternating Möbius systems for fixed q_b and variable $q_a \in (0,1)$. For each parameter (q_a, q_b) a random orbit is displayed.

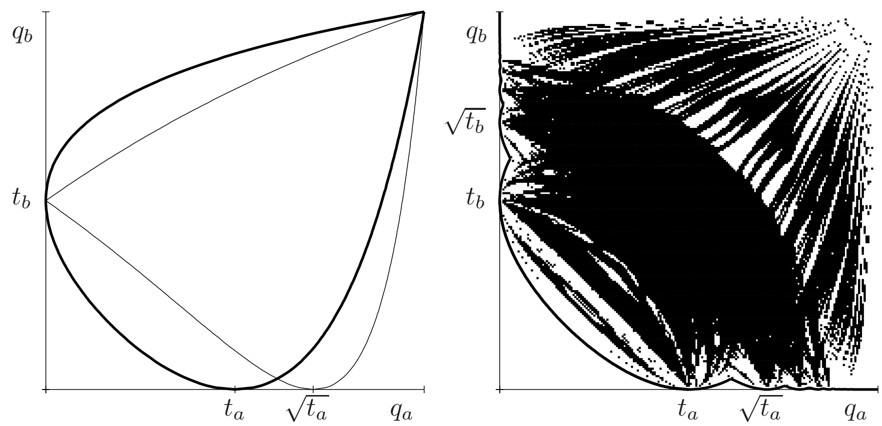

FIGURE 3. Parameter space of an alternating Möbius system

Denote by $P = (0,1) \times (0,1)$ the parameter space of an alternating Möbius system. We exclude the parameters with $q_a = 1$ or $q_b = 1$ whose transformations are identities. For $u \in A^+$ denote by $U_u := \{(q_a, q_b) \in P :\ F_u \text{ is hyperbolic}\}$. For the standard alternating system we get

$$\begin{aligned} \mathbf{tr}(F_{ab}) &= -(q_a q_b - 2q_a - 2q_b + 1)/\sqrt{q_a q_b}, \\ U_{ab} &= \{(q_a, q_b) \in P :\ (q_a q_b - 2q_a - 2q_b + 1)^2 > 4 q_a q_b\}. \end{aligned}$$

The set U_{ab} (Figure 3 left - thick line) consists of three connected components separated by points $(0, t_b)$ and $(t_a, 0)$. The set $U_{a^2b} = \{(q_a, q_b) \in P :\ (q_a^2, q_b) \in U_{ab}\}$ (Figure 3 left - thin line) consists of three connected components which are separated by points $(0, t_b)$ and $(\sqrt{t_a}, 0)$. The position of fixed points of F_{ab} and F_{ba} is different in each component of U_{ab} (see Figure 4). For $(q_a, q_b) \in U_{ab}$ we have

$$\begin{aligned} q_a < t_a \ \&\ q_b < t_b &\Rightarrow s_a \prec r_a \prec r_{ba} \prec s_{ba} \prec s_b \prec r_b \prec r_{ab} \prec s_{ab} \prec s_a \\ q_a > t_a &\Rightarrow s_a \prec r_a \prec s_b \prec s_{ba} \prec r_{ba} \prec s_{ab} \prec r_{ab} \prec r_b \prec s_a \\ q_b > t_b &\Rightarrow s_a \prec s_{ab} \prec r_{ab} \prec s_{ba} \prec r_{ba} \prec r_a \prec s_b \prec r_b \prec s_a \end{aligned}$$

Set

$$U := \bigcap_{n \geq 1} (U_{a^n b} \cap U_{ab^n}) = \bigcup_{n \in \mathbb{Z}} U_n$$

where $\mathbb{Z}$ is the set of integers, and the connected components U_n of of U are

$$\begin{aligned} U_0 &= U_{ab} \cap ((0, t_a) \times (0, t_b)) \\ U_n &= U_{a^n b} \cap U_{a^{n+1} b} \cap ((\sqrt[n]{t_a},\ \sqrt[n+1]{t_a}) \times (0, t_b)),\ n > 0 \\ U_{-n} &= U_{ab^n} \cap U_{ab^{n+1}} \cap ((0, t_a) \times (\sqrt[n]{t_b},\ \sqrt[n+1]{t_b})),\ n > 0 \end{aligned}$$

The boundary ∂U of U is shown in Figure 3 right (thick line).

Theorem 11. *An alternating Möbius system $(\overline{\mathbb{R}}, F_a, F_b)$ has a nontrivial attractor if and only if $(q_a, q_b) \in U$.*

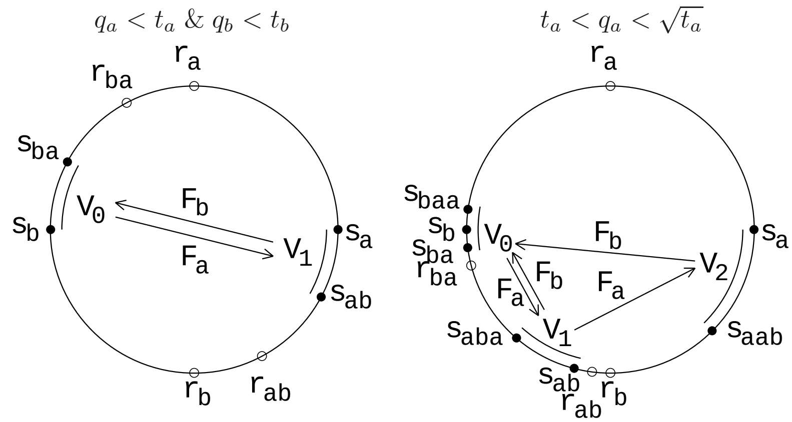

FIGURE 4. Fixed points of an alternating Möbius system

Proof. If $(q_a, q_b) \notin U$, then there exists $u \in A^+$ such that F_u is not hyperbolic, so (F_a, F_b) has a unique attractor by Proposition 7. Conversely assume that $(q_a, q_b) \in U$ and let n be the number of the connected component U_n of U to which (q_a, q_b) belongs. If $n = 0$ and $(q_a, q_b) \in U_0$, then $s_{ba} \prec s_b \prec s_{ab} \prec s_a \prec s_{ba}$ (Figure 4 left). Set $V_0 = [s_{ba}, s_b]$, $V_1 = [s_{ab}, s_a]$. Since $F_a(s_{ba}) = s_{ab}$ and $F_b(s_{ab}) = s_{ba}$, we get that $V_0 \cup V_1$ is an invariant set. Since the endpoints of V_i are stable periodic points, for each $\varepsilon > 0$ there exist closed intervals $V_i \subset W_i \subset B_\varepsilon(V_i)$, such that $W_0 \cup W_1$ is an inward set, so there exists a nontrivial attractor. If $n = 1$ and $(q_a, q_b) \in U_1$, then $s_{baa} \prec s_b \prec s_{ba} \prec s_{aba} \prec s_{ab} \prec s_{aab} \prec s_a$ (Figure 4 right). Set $V_0 = [s_{baa}, s_{ba}]$, $V_1 = [s_{aba}, s_{ab}]$, $V_2 = [s_{aab}, s_a]$. Then $F_a(V_i) \subseteq V_{i+1}$ for $i = 0, 1$, $F_a(V_2) \subseteq V_2$, and $F_b(V_i) \subseteq V_0$ for $i = 0, 1, 2$, so $V_0 \cup V_1 \cup V_2$ is an invariant set. Since the endpoints of V_i are stable periodic points, there exist closed intervals $V_i \subset W_i \subset B_\varepsilon(V_i)$ such that $W_0 \cup W_1 \cup W_2$ is an inward set. If $(q_a, q_b) \in U_2$, then we set $V_0 = [s_{baaa}, s_{baa}]$, $V_1 = [s_{abaa}, s_{aba}]$, $V_2 = [s_{aaba}, s_{aab}]$, $V_3 = [s_{aaab}, s_a]$. Anologously we proceed in each connected component U_n of U. $\square$

Theorem 12. *Let $(\overline{\mathbb{R}}, F_a, F_b)$ be an alternating Möbius system such that $(q_a, q_b) \in \partial U$ (the boundary of U). Then the system has a unique attractor but is not transitive.*

Proof. Since the system contains a parabolic transformation, it has a unique attractor by Proposition 7. If $(q_a, q_b) \in \partial U_0$, then $s_b \prec r_b \prec r_{ab} = s_{ab} \prec s_a \prec r_a \prec r_{ba} = s_{ba}$. The set $V_0 \cup V_1 = [s_{ba}, s_b] \cup [s_{ab}, s_a]$ is invariant, so the system is not transitive. Similarly if $(q_a, q_b) \in \partial U_1$, then the set $V_0 \cup V_1 \cup V_2$ is invariant. An analogous argument works in each connected component of ∂U. $\square$

Proposition 13. *The set $\{(q_a, q_b) \in P : (\overline{\mathbb{R}}, F_a, F_b)$ is minimal$\}$ is dense in $P \setminus U$.*

Proof. The trace and the rotation angle of F_{ab} are not constant in any open subset of $P \setminus U_{ab}$. If $W \subset P \setminus U$ is a nonempty open set, then there exists $n \geq 1$ such that either $W \cap U_{a^n b} \neq \emptyset$ or $W \cap U_{ab^n} \neq \emptyset$. In either case there exists $(q_a, q_b) \in W$ such that the rotation angle of $F_{a^n b}$ or F_{ab^n} is irrational, so the system is minimal by Proposition 8. $\square$

Proposition 14. *Let* $(\overline{\mathbb{R}}, F_a, F_b)$ *be an alternating Möbius system and let there exist* $u, v, w \in A^+$ *such that* F_u, F_v, F_w *are hyperbolic,* $s_u \prec F_v(s_u) \prec F_u(s_v) \prec s_v$, $s_u \prec r_w \prec s_v$, *and both* F_u *and* F_v *are contractions on* (s_u, s_v). *Then* $(\overline{\mathbb{R}}, F_a, F_b)$ *is a minimal system (Figure 5).*

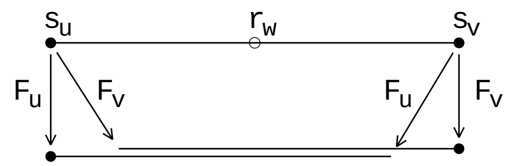

FIGURE 5. Minimal alternating Möbius systems

Proof. We show that for any nonempty open interval V there exist words $y, z \in A^+$ such that $F_y^{-1}(V) \cup F_z^{-1}(V) = \overline{\mathbb{R}}$. We consider three cases.
1. If $s_u \in V$, then $F_{u^n}^{-1}(V)$ converges to $\overline{\mathbb{R}} \setminus \{r_u\}$, and for some $n > 0$ we get $F_{u^n}^{-1}(V) \cup F_{u^n v}^{-1}(V) = \overline{\mathbb{R}}$. An analogous argument works if $s_v \in V$.
2. If $V \subseteq Y = (s_u, s_v)$, then since F_u, F_v are contractions on Y and $F_u(Y) \cup F_v(Y) = Y$, there exists $z \in \{u, v\}^+$ such that $F_z^{-1}(V) \cap \{s_u, s_v\} \neq \emptyset$, and we apply case 1.
3. If V is arbitrary, then for some n, $F_{w^n}^{-1}(V) \cap Y \neq \emptyset$, and we apply case 2. □

The values of parameters which satisfy the condition of Proposition 14 (up to the length 7 of words u, v and w) are shown in Figure 3 right. Note that the set of such parameters is open.

Conjecture. *For each* $(q_a, q_b) \in P \setminus \overline{U}$, *the alternating Möbius system is minimal.*

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