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ON THE MEASURE ATTRACTOR OF A CELLULAR AUTOMATON

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Abstract. Given a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, we define its small quasi-attractor $\mathcal{Q}_F$ as the nonempty intersection of all shift-invariant attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$. The measure attractor $\mathcal{M}_F$ is the closure of the supports of the members of the unique attractor of $F : \mathfrak{M}_\sigma(A^{\mathbb{Z}}) \to \mathfrak{M}_\sigma(A^{\mathbb{Z}})$ in the space of shift-invariant Borel probability measures.

1. **Introduction.** The dynamics of some cellular automata can be intuitively described as that of particles or signals moving in a neutral medium. When two or more signals meet they interact: some of them may disappear, some may be changed to signals of different kinds and new signals may be created (see e.g., Boccara, Naser and Roger [3]). Alternatively, the cellular space is regarded as decomposed into regular domains where the dynamics is simple, while the dynamics of boundaries between these domains is more complex (see e.g., Hanson and Crutchfield [5] or Wolfram [12]).

To describe these phenomena, the usual treatment of cellular automata as dynamical systems in the Cantor space is not very convenient. This approach captures well the point of view of an observer who sees a finite region of the cellular space. Accordingly, the dynamical properties of a cellular automaton change substantially when we compose a cellular automaton with a power of the shift map.

If we pretend to observe the whole cellular space, we do not see great difference between configurations which can be obtained from each other by applying the shift map. This point of view is adopted when we equip the configuration space with the Besicovitch or Weyl topologies as in Blanchard, Formenti and Kůrka [2]. This invariance with respect to the shift map is shared also by the dynamics of cellular automata in the space of shift-invariant Borel probability measures equipped with the weak* topology (see e.g., Kůrka and Maass [9], Pivato [11] or Kůrka [7]).

In the present paper we study two attracting constructs which are invariant under the composition with a power of the shift map. In the Cantor space we disregard attractors which are not subshifts. Given a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, we show that the intersection of all shift-invariant attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$, is nonempty. We call this intersection the **small quasi-attractor** and denote it by $\mathcal{Q}_F$.

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When a cellular automaton is initialized on a random configuration, we see sometimes even smaller limiting subset. This happens e.g., in the well-known elementary cellular automaton 184 which models the car traffic on a highway. In this case the small quasi-attractor coincides with the omega-limit set, and it consists in configurations containing both holes and jams. However, the limiting configurations usually contain only one kind of these signals: either holes or jams, depending on the initial probability distribution. This limiting behaviour is adequately described by the **measure attractor**. We consider a cellular automaton $F : \Sigma \to \Sigma$ on a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ as a dynamical system in the space $\mathfrak{M}_\sigma(\Sigma)$ of shift-invariant Borel probability measures and show that this dynamical system has always a unique attractor, whose support is a subshift $\mathcal{M}_F$. The **measure quasi-attractor** $\mathcal{MQ}_F$ of $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the support of the unique attractor of $F : \mathfrak{M}_\sigma(\mathcal{Q}_F) \to \mathfrak{M}_\sigma(\mathcal{Q}_F)$. While the measure attractor attracts all measures, the measure quasi-attractor attracts only ergodic measures from its neighbourhood.

In describing the measure attractor we use **weakly periodic** configurations which are fixed points of some $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$. A **signal subshift** is an infinite subshift consisting of all (p, q)-weakly periodic configurations. Some cellular automata have no signal subshifts at all, some have a finite number of them and some have an infinite number of them. The measure attractor can sometimes be constructed as a union of finitely many signal subshifts.

As pointed out by a referee, some of our theorems hold in a more general setting. Proposition 2 holds for any surjective map on a compact space. Proposition 12 holds for commuting maps $F, \sigma : X \to X$, where σ is a homeomorphism. Proposition 13 holds for any affine map on a compact convex space.

2. **Dynamical systems.** A dynamical system is a continuous map $F : X \to X$, where X is a compact metric space. It is **transitive** if for every nonempty open sets $U, V \subseteq X$ there exists $n > 0$ such that $F^n(U) \cap V \neq \emptyset$. Here, F^n is the n-th iteration of F. A set $Y \subseteq X$ is **invariant** if $F(Y) \subseteq Y$, **inversely invariant** if $F^{-1}(Y) \subseteq Y$, and **strongly invariant** if $F(Y) = Y$. The **omega-limit** of a closed invariant set Y is $\Omega_F(Y) := \bigcap_{n\geq 0} F^n(Y)$. A set $V \subseteq X$ is **inward** if F maps its closure into its interior, i.e., $F(\mathfrak{cl}(V)) \subseteq V^\circ$. A set $Y \subseteq X$ is an **attractor** if there exists a closed inward set V, such that $\Omega_F(V) = Y$. In every dynamical system, $\Omega_F(X)$ is the largest attractor. The **basin** of an attractor $Y \subseteq X$ is

$$\mathcal{B}_F(Y) := \{x \in X : \lim_{n\to\infty} d(F^n(x), Y) = 0\}.$$

The **chain relation** $\mathcal{C}_F \subseteq X \times X$ of $F : X \to X$ is defined by $(x, y) \in \mathcal{C}_F$ iff for every $\varepsilon > 0$ there exists an ε-chain from x to y, i.e., a sequence of states $x = x_0, \ldots, x_n = y$, such that $d(F(x_i), x_{i+1}) < \varepsilon$ for $i < n$. Then $\mathcal{C}_F \cap \mathcal{C}_F^{-1}$ is an equivalence on the set $|\mathcal{C}_F| = \{x \in X : (x, x) \in \mathcal{C}_F\}$ of the **chain-recurrent points**. The equivalence classes are called **chain components**. They are closed and invariant.

3. **Subshifts.** For a finite alphabet A, denote by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A. The length of a word $u = u_0 \ldots u_{n-1} \in A^n$ is denoted by $|u| := n$. The word of zero length is λ. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \ldots u_{j-1}$ and $u_{[i,j]} = u_i \ldots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{Z}}$ the

space of A-**configurations**, or doubly-infinite sequences of letters of A equipped with the metric $d(x,y) := 2^{-n}$, where $n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$.

The **cylinder** $[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l,l+|u|)} = u\}$ of a word $u \in A^*$ located at $l \in \mathbb{Z}$ is a clopen set (closed and open). The **shift map** $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i := x_{i+1}$. For any nonzero $u \in A^*$ we have a σ-periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{Z}}$, which is strongly σ-invariant and closed. For a subshift Σ there exists a set $D \subseteq A^*$ of forbidden words such that $\Sigma = \Sigma_D := \{x \in A^{\mathbb{Z}} : \forall u \sqsubseteq x, u \notin D\}$. A subshift Σ is of **finite type** (SFT), if $\Sigma = \Sigma_D$ for some finite set $D \subseteq A^*$ of forbidden words. A subshift is uniquely determined by its **language**

$$\mathcal{L}(\Sigma) := \bigcup_{n \geq 0} \mathcal{L}^n(\Sigma), \quad \text{where} \quad \mathcal{L}^n(\Sigma) := \{u \in A^n : \exists x \in \Sigma, u \sqsubseteq x\}.$$

A **labelled graph** over an alphabet A is a structure $G = (V, E, s, t, l)$, where V is a finite set of vertices, E is a finite set of edges, $s, t : E \to V$ are the **source** and **target maps**, and $l : E \to A$ is a labelling function. A finite or infinite word $w \in E^* \cup E^{\mathbb{Z}}$ is a path in G if $t(w_i) = s(w_{i+1})$ for all i. The source and target of a finite path $w \in E^n$ are $s(w) = s(w_0)$, $t(w) = t(w_{n-1})$. The label of a path is defined by $l(w)_i = l(w_i)$. A subshift Σ is sofic if there exists a labelled graph G and $\Sigma = \Sigma_G$ is the set of labels of all doubly infinite paths in G (see e.g. Lind and Marcus [10], or Kitchens [6]). A word $u \in A^*$ is **synchronizing**, if every path $w \in E^*$ with label $l(w) = u$ has the same target $t(u) := t(w)$. If Σ is a subshift of finite type, then there exists a labelled **synchronizing graph** G, such that $\Sigma = \Sigma_G$, and all sufficiently long words $u \in \mathcal{L}(\Sigma)$ are synchronizing (Lind and Marcus [10], page 85).

4. **Measures.** By a **measure** we mean a **Borel probability measure** on a Cantor space $A^{\mathbb{Z}}$. This is a countably additive function on the Borel sets of $A^{\mathbb{Z}}$ which assigns 1 to the full space. The (topological) **support** of a measure μ is

$$\mathcal{S}_\mu := \bigcap \{V \subseteq A^{\mathbb{Z}} : \mu(V) = 1 \text{ and } V \text{ is closed}\}.$$

If $\mathcal{S}_\mu = A^{\mathbb{Z}}$, we say that μ has **full support**. A measure on $A^{\mathbb{Z}}$ is determined by its values on centered cylinders $\mu(u) := \mu([u]_{-n})$ for $u \in A^{2n} \cup A^{2n+1}$. Thus a measure can be identified with a map $\mu : A^* \to [0,1]$ subject to **Kolmogorov compatibility conditions** $\mu(\lambda) = 1$, $\sum_{a \in A} \mu(ua) = \mu(u)$ for $|u|$ odd, and $\sum_{a \in A} \mu(au) = \mu(u)$ for $|u|$ even. The distance of two measures is

$$\varrho(\mu, \nu) := \sum_{n=0}^{\infty} \max\{|\mu(u) - \nu(u)| : u \in A^n\} \cdot 2^{-n}.$$

This metric gives the topology of weak* convergence on the compact space $\mathfrak{M}(A^{\mathbb{Z}})$ of Borel probability measures. If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a continuous map and $\mu \in \mathfrak{M}(A^{\mathbb{Z}})$, then $F\mu$ is a Borel probability measure defined by $(F\mu)(U) = \mu(F^{-1}(U))$ for any Borel set U. A measure μ is F-**invariant**, if $F\mu = \mu$. The map $\mu \mapsto F\mu$ is continuous and **affine**, i.e., it preserves convex combinations. Thus, we obtain a dynamical system $F : \mathfrak{M}(A^{\mathbb{Z}}) \to \mathfrak{M}(A^{\mathbb{Z}})$.

If a measure μ is σ-invariant, then the measure of a cylinder does not depend on its location, so $\mu(u) = \mu([u]_i)$ for every $u \in A^*$ and $i \in \mathbb{Z}$. Thus a σ-invariant measure is determined by a map $\mu : A^* \to [0,1]$ subject to **bilateral compatibility**

conditions

$$\mu(\lambda)=1,\quad \sum_{a\in A}\mu(ua)=\sum_{a\in A}\mu(au)=\mu(u),\quad u\in A^*.$$

Denote by $\mathfrak{M}_\sigma(A^{\mathbb{Z}}) := \{\mu\in\mathfrak{M}(A^{\mathbb{Z}}):\ \sigma\mu=\mu\}$ the closed subspace of σ-invariant measures. The support $\mathcal{S}_\mu := \{x\in A^{\mathbb{Z}}:\ \forall u\sqsubseteq x, \mu(u)>0\}$ of a σ-invariant measure μ is a subshift whose forbidden words are the words of zero measure. If $\Sigma\subseteq A^{\mathbb{Z}}$ is a subshift, then

$$\mathfrak{M}_\sigma(\Sigma) := \{\mu\in\mathfrak{M}_\sigma(A^{\mathbb{Z}}):\ \mathcal{S}_\mu\subseteq\Sigma\} = \{\mu\in\mathfrak{M}_\sigma(A^{\mathbb{Z}}):\ \forall u\in A^*\setminus\mathcal{L}(\Sigma), \mu(u)=0\}$$

is a closed subspace of $\mathfrak{M}_\sigma(A^{\mathbb{Z}})$. A measure $\mu\in\mathfrak{M}_\sigma(\Sigma)$ can be identified with a map $\mu:\mathcal{L}(\Sigma)\to[0,1]$ which satisfies the bilateral compatibility conditions. We say that $\mu\in\mathfrak{M}_\sigma(\Sigma)$ is **σ-ergodic**, if for every strongly σ-invariant set $Y\subseteq\Sigma$ either $\mu(Y)=0$ or $\mu(Y)=1$. If $\mu\in\mathfrak{M}_\sigma(\Sigma)$, then the support of μ is contained in the **densely recurrent subshift**

$$\mathcal{D}(\Sigma) := \mathfrak{cl}\{x\in\Sigma:\ \forall k>0, \limsup_{n\to\infty}\#\{i<n:\ x_{[i,i+k)}=x_{[0,k)}\}/n>0\}$$

of Σ, i.e., $\mathcal{S}_\mu\subseteq\mathcal{D}(\Sigma)$ (see Akin [1], Proposition 8.8, page 164). On the other hand, there exists $\mu\in\mathfrak{M}_\sigma(\Sigma)$ such that $\mathcal{S}_\mu=\mathcal{D}(\Sigma)$. If we define the support of a set $Y\subseteq\mathfrak{M}(A^{\mathbb{Z}})$ as $\mathcal{S}(Y) := \mathfrak{cl}(\bigcup_{\mu\in Y}\mathcal{S}_\mu)$, we get

Proposition 1. *Let $\Sigma\subseteq A^{\mathbb{Z}}$ be a subshift.*
(1) $\mathcal{S}(\mathfrak{M}_\sigma(\Sigma))=\mathcal{D}(\Sigma)$ and therefore, $\mathfrak{M}_\sigma(\Sigma)=\mathfrak{M}_\sigma(\mathcal{D}(\Sigma))$.
(2) Let G be a labelled graph and let $G_1,\dots,G_k$ be all its irreducible components. Then $\mathcal{D}(\Sigma_G)=\Sigma_{G_1}\cup\cdots\cup\Sigma_{G_k}$.

The proof is straightforward.

5. **Cellular automata.** A cellular automaton is a continuous map $F:\Sigma\to\Sigma$ on a subshift $\Sigma\subseteq A^{\mathbb{Z}}$ which commutes with the shift map, i.e., $F\sigma=\sigma F$. For a cellular automaton F there exists a **local rule** $f:\mathcal{L}^{d+1}(\Sigma)\to A$ such that $F(x)_i=f(x_{[i-m,i-m+d]})$ for some **memory** $m\in\mathbb{Z}$ and **diameter** $d\geq 0$. A map $f:\mathcal{L}^{d+1}(\Sigma)\to A$ is a local rule of a cellular automaton iff it can be extended to a map $f:\mathcal{L}(\Sigma)\to\mathcal{L}(\Sigma)$ by $f(u)_i=f(u_{[i,i+d]})$ for $0\leq i<|u|-d$. If $F:\Sigma\to\Sigma$ is a cellular automaton and $Y\subseteq\Sigma$ is an attractor, then there exists an F-invariant set U clopen in Σ, such that $Y=\Omega_F(U)$. An attractor need not be a subshift. By a **subshift attractor** we mean an attractor which is a subshift. For example, the maximal attractor $\Omega_F(\Sigma)$, is a subshift attractor. If $\mu\in\mathfrak{M}_\sigma(\Sigma)$, then $(F\mu)(u)=\sum_{v\in f^{-1}(u)}\mu(v)$ is also a σ-invariant measure. Thus we get a dynamical system $F:\mathfrak{M}_\sigma(\Sigma)\to\mathfrak{M}_\sigma(\Sigma)$, which is a subsystem of $F:\mathfrak{M}(\Sigma)\to\mathfrak{M}(\Sigma)$. The properties of this dynamical system do not depend on the memory constant m. Cellular automata F and $F\sigma^p$ determine the same dynamical system on $\mathfrak{M}_\sigma(\Sigma)$.

Proposition 2. *If $F:\Sigma\to\Sigma$ is a surjective cellular automaton, then both maps $F:\mathfrak{M}(\Sigma)\to\mathfrak{M}(\Sigma)$ and $F:\mathfrak{M}_\sigma(\Sigma)\to\mathfrak{M}_\sigma(\Sigma)$ are surjective.*

Proof. If F is surjective, then the extension $f:\mathcal{L}(\Sigma)\to\mathcal{L}(\Sigma)$ of a local rule for F is surjective too. Given $\mu\in\mathfrak{M}(\Sigma)$ and $n>0$, there exists a map $\nu_n:\mathcal{L}^{n+d}(\Sigma)\to[0,1]$ such that $\sum_{v\in f^{-1}(u)}\nu_n(v)=\mu(u)$ for every $u\in\mathcal{L}^n(\Sigma)$. Then there exists a map $\nu'_n:\mathcal{L}^{n+d+1}(\Sigma)\to[0,1]$ such that $\sum_{a\in A}\nu'_n(ua)=\nu_n(u)$ for $u\in\mathcal{L}^{n+d}(\Sigma)$.

Continuing by induction, we extend ν_n to a measure $\nu_n : A^* \to [0,1]$, and we get $\sum_{v\in f^{-1}(u)} \nu_n(v) = \mu(u)$ whenever $|u| \le n$. Then

$$\varrho(F\nu_n, \mu) \le \sum_{i=0}^{\infty} \max\{|F\nu_n(u) - \mu(u)| :\ u \in A^i\} \cdot 2^{-i} \le \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n}.$$

Since $\mathfrak{M}(\Sigma)$ is compact, there exists a converging subsequence $\nu_{n_i} \to \nu \in \mathfrak{M}(\Sigma)$ and $F\nu = \mu$, so $F : \mathfrak{M}(\Sigma) \to \mathfrak{M}(\Sigma)$ is surjective. If $\mu \in \mathfrak{M}_\sigma(\Sigma)$ is σ-invariant, then set $\xi_n = (1/n)\sum_{i=0}^{n-1} \sigma^i\nu \in \mathfrak{M}(\Sigma)$. We have $F\xi_n = \mu$ and there exists a converging subsequence $\xi_{n_i} \to \xi \in \mathfrak{M}_\sigma(\Sigma)$ for which $F\xi = \mu$. Thus $F : \mathfrak{M}_\sigma(\Sigma) \to \mathfrak{M}_\sigma(\Sigma)$ is surjective. □

6. **Signal subshifts.**

Definition 1. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. A configuration $x \in A^{\mathbb{Z}}$ is **weakly periodic**, if $F^q\sigma^p(x) = x$ for some $q > 0$ and $p \in \mathbb{Z}$. We call (p,q) the **period** of x and p/q its speed. Denote by

$$\Sigma_{(p,q)} := \{x \in A^{\mathbb{Z}} :\ F^q\sigma^p(x) = x\}$$

the set of all weakly periodic configurations with period (p,q). A **signal subshift** is any infinite $\Sigma_{(p,q)}$.

Observe that $\Sigma_{(p,q)}$ is closed and σ-invariant, so it is a subshift provided it is nonempty. Moreover, $\Sigma_{(p,q)}$ is F-invariant and $F : \Sigma_{(p,q)} \to \Sigma_{(p,q)}$ is bijective. If $\Sigma_{(p,q)}$ is finite, it consists only of σ-periodic configurations.

Proposition 3. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with diameter d and memory m, so $F(x)_i = f(x_{[i-m,i-m+d]})$.*
(1) If $\Sigma_{(p,q)}$ is nonempty, then it is a subshift of finite type.
(2) If $\Sigma_{(p,q)}$ is infinite, then $m - d \le p/q \le m$.
(3) $\Sigma_{(p,q)} \subseteq \Sigma_{(np,nq)}$ for $n > 0$.
(4) If $p_0/q_0 \ne p_1/q_1$, then $\Sigma_{(p_0,q_0)} \cap \Sigma_{(p_1,q_1)}$ is finite.

Proof. (1) Set $D := \{u \in A^{dq+1} :\ f^q(u) \ne u_{mq-p}\}$. Then $\Sigma_{(p,q)} = \Sigma_D$.

(2) If $x \in \Sigma_{(p,q)}$, then $x_i = f^q(x_{[i+p-mq,i+p-mq+dq]})$. If $p-mq+dq < 0$, then there exists a function $g : A^{mq-p} \to A^{mq-p}$ such that $x_{[i+p-mq+1,i]} = g(x_{[i+p-mq,i-1]})$ for every $i \in \mathbb{Z}$. This is possible only if x is σ-periodic. Moreover, the period of x is bounded, so $\Sigma_{(p,q)}$ is finite. The proof is similar for $p > mq$.

(3) This is trivial.

(4) Let q be the least common multiple of q_0 and q_1 and set $p_2 = qp_0/q_0$, $p_3 = qp_1/q_1$. If $x \in \Sigma_{(p_0,q_0)} \cap \Sigma_{(p_1,q_1)}$, then $\sigma^{-p_2}(x) = F^q(x) = \sigma^{-p_3}(x)$, so $\sigma^{p_2-p_3}(x) = x$. Since $p_2 \ne p_3$, there are only a finite number of x which satisfy this equality. □

A cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is **positively expansive**, if there exists $\varepsilon > 0$ such that for any $x \ne y \in A^{\mathbb{Z}}$ there exists $k \ge 0$ such that $d(F^k(x), F^k(y)) > \varepsilon$.

Proposition 4. *A positively expansive cellular automaton has no signal subshifts.*

Proof. Suppose that F has memory m and there exists an infinite $\Sigma_{(p,q)}$ with $p \le 0$. There exist $x, y \in \Sigma_{(p,q)}$ with $x_{(-mq,\infty)} = y_{(-mq,\infty)}$ and $x_{-mq} \ne y_{-mq}$. Then for all $n \ge 0$, $F^n(x)_0 = F^n(y)_0$ and this is a contradiction. The proof is similar for $p \ge 0$. □

Example 1. The shift cellular automaton $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ has the unique signal subshift $\Sigma_{(-1,1)} = A^{\mathbb{Z}}$.

The uniqueness follows from Proposition 3(4). For every $p/q \neq -1$ we have $\Sigma_{(p,q)} \subseteq \Sigma_{(p,q)} \cap \Sigma_{(-1,1)}$, so $\Sigma_{(p,q)}$ is finite. Of course, $\Sigma_{(-q,q)} = \Sigma_{(-1,1)}$.

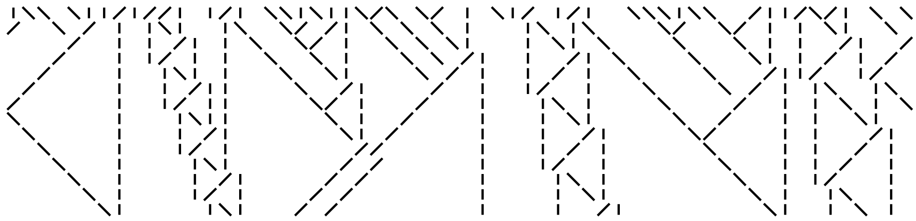

FIGURE 1. Infinitely many signal subshifts

Example 2. There exists a cellular automaton with infinitely many signal subshifts.

Consider a cellular automaton with alphabet $A = \{\texttt{b}, \texttt{s}, \texttt{r}, \texttt{l}\}$, where `b` is blank, `s` is a stationary particle, `r` is a right-going particle and `l` is a left-going particle. When either of these two particles meets `s`, it shifts it to the right and changes to the other going particle. If an `r` collides with an `l`, one of them annihilates. The transition table $f : A^4 \to A$ is given by

xxbl	xblx	xxry	xryx	xxsl	xslx	slxx	xxrs	xrsx	rsxx
l	b	b	r	b	s	r	b	l	s

,

where $\texttt{x} \in A$, $\texttt{y} \in \{\texttt{b}, \texttt{r}, \texttt{l}\}$, and the first applicable rule is used. We have $m = 2$ and $d = 3$, so $F(x)_i = f(x_{[i-2,i+1]})$. Then, for every $n \geq 0$, $\Sigma_{(1,2n+3)}$ is a signal subshift. In Figure 1 we can see signals with speeds 1/3, 1/5, and 1/7.

Example 3 (ECA 184)**.** The traffic rule $F(x)_i = 1$ iff $x_{[i-1,i]} = 10$ or $x_{[i,i+1]} = 11$.

We have two signal subshifts (see Figure 2) representing holes and jams

$$\Sigma_{(1,1)} = \{x \in A^{\mathbb{Z}} : \ 11 \not\sqsubseteq x\}, \quad \Sigma_{(-1,1)} = \{x \in A^{\mathbb{Z}} : \ 00 \not\sqsubseteq x\}.$$

Their intersection is the orbit $\mathfrak{o}((01)^\infty) = \{(01)^\infty, (10)^\infty\}$. There is one more signal subshift $\Sigma_{(0,1)} = \{x \in A^{\mathbb{Z}} : \ \forall i < j, x_i \leq x_j\}$.

Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and assume that $\Sigma_{(p_1,q_1)}, \ldots, \Sigma_{(p_n,q_n)}$ is a finite sequence of its signal subshifts with decreasing speeds, i.e., $p_i/q_i > p_j/q_j$ for $i < j$. We construct the **join** $\Sigma = \Sigma_{(p_1,q_1)} \vee \cdots \vee \Sigma_{(p_n,q_n)}$ as follows. A configuration $x \in A^{\mathbb{Z}}$ belongs to Σ iff there exist overlapping intervals $I_i = [a_i, b_i] \subseteq \mathbb{Z}$, such that $a_i < a_{i+1}$, $b_i < b_{i+1}$ and the restrictions $x|I_i$ belong to $\mathcal{L}(\Sigma_{(p_i,q_i)})$. A typical configuration looks like

$$\underbrace{\cdots \longrightarrow\longrightarrow \overbrace{u_1u_1u_1}^{}}_{\Sigma_{(p_1,q_1)}}\!\!\!\!\!\!\!\!\overbrace{\to\to \underbrace{u_2u_2u_2}_{}}^{\Sigma_{(p_2,q_2)}}\underbrace{\leftarrow\leftarrow \overbrace{u_3u_3u_3}^{}}_{\Sigma_{(p_3,q_3)}}\overbrace{\longleftarrow\longleftarrow \ldots}^{\Sigma_{(p_4,q_4)}}$$

where $(u_iu_iu_i)^\infty \in \Sigma_{(p_i,q_i)} \cap \Sigma_{(p_{i+1},q_{i+1})}$ and the arrows symbolize the speeds.

More formally, the join is a sofic subshift defined as follows: For each i, let $G_i = (V_i, E_i, s_i, t_i, l_i)$ be a synchronizing graph for $\Sigma_{(p_i,q_i)}$. If $u^\infty \in \Sigma_{(p_i,q_i)}$ is a σ-periodic configuration, denote by $t_i(u) \in V_i$ the unique target of all sufficiently

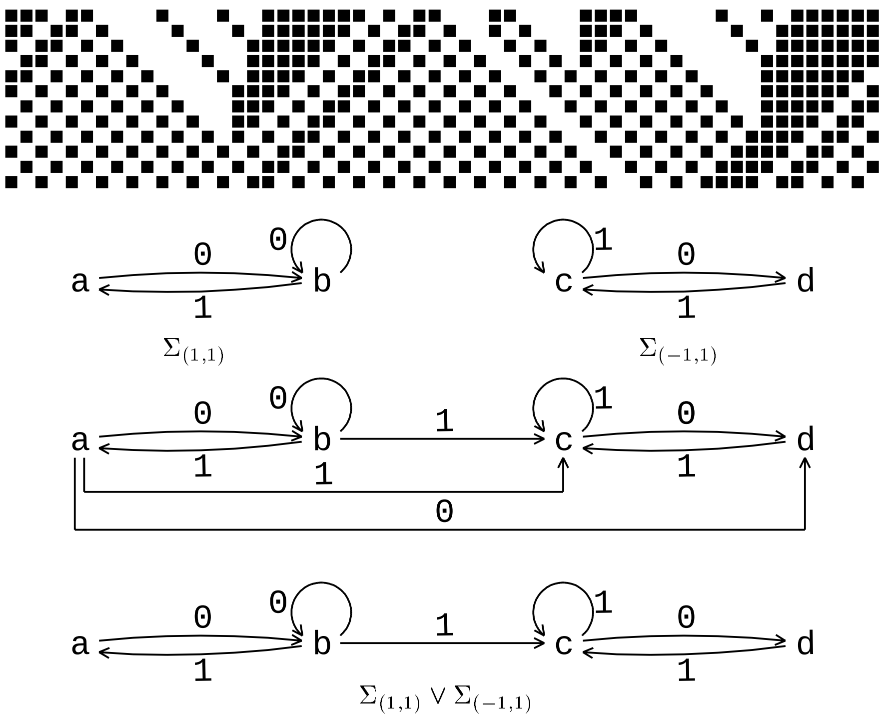

FIGURE 2. ECA 184 and its signal subshifts

long u^k in G_i. We take now the disjoint union of all graphs G_i and add some more edges. For each σ-periodic configuration $u^\infty \in \Sigma_{(p_i,q_i)} \cap \Sigma_{(p_j,q_j)}$, where $i < j$, and for each labelled edge $t_j(u) \xrightarrow{a} z$ we add an edge $t_i(u) \xrightarrow{a} z$. We repeat this operation finitely many times until no new edges can be added. The join $\Sigma = \Sigma_G$ is the sofic subshift of the resulting graph G.

The construction of the join $\Sigma_{(1,1)} \vee \Sigma_{(-1,1)}$ for ECA 184 is shown in Figure 2. In $\Sigma_{(-1,1)}$ we have labelled edges $c = t_2(01) \xrightarrow{0} d$, $c = t_2(01) \xrightarrow{1} c$, and $d = t_2(10) \xrightarrow{1} c$. In the join, the following edges are added (Figure 2 center):

$$a = t_1(01) \xrightarrow{0} d, \quad a = t_1(01) \xrightarrow{1} c, \quad b = t_1(10) \xrightarrow{1} c.$$

However, if we add to $\Sigma_{(1,1)} \cup \Sigma_{(-1,1)}$ only one edge $b = t_1(10) \xrightarrow{1} c$, we obtain a graph which generates the same subshift (Figure 2 bottom).

Proposition 5. *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, then $\Sigma_{(p,q)} \subseteq \Omega_F(A^{\mathbb{Z}})$ for every $p \in \mathbb{Z}$ and $q > 0$. If $\Sigma_1, \dots, \Sigma_n$ is a sequence of signal subshifts with decreasing speeds and if $\Sigma = \Sigma_1 \vee \cdots \vee \Sigma_n$ is their join, then*

$$\Sigma \subseteq F(\Sigma) \subseteq \cdots \subseteq F^k(\Sigma) \subseteq \cdots \subseteq \Omega_F(A^{\mathbb{Z}}).$$

Proof. Since $F : \Sigma_{(p,q)} \to \Sigma_{(p,q)}$ is bijective, we get $\Sigma_{(p,q)} \subseteq \Omega_F(A^{\mathbb{Z}})$. For the join Σ it suffices to prove that $\Sigma \subseteq F(\Sigma)$. Let $x \in \Sigma$ and let $w \in E^{\mathbb{Z}}$ be a path in G with label x. There exists a disjoint union of intervals $I_1 \cup \cdots \cup I_n = \mathbb{Z}$, such that $t(w_n) \in V_i$ for all $n \in I_i$ (some I_i may be empty). Suppose that I_i, I_j, I_k are three nonempty consecutive intervals and $I_j = [n, m]$. There exist σ-periodic configurations $u^\infty \in \Sigma_i \cap \Sigma_j$ and $v^\infty \in \Sigma_j \cap \Sigma_k$, such that $t_i(u) = s(w_n)$ and $t_j(v) = t(w_m)$.

$$\underbrace{\cdots \longrightarrow \cdot \xrightarrow{u} t_i(u)}_{I_i} \xrightarrow{l(w_n)} \underbrace{\cdot \longrightarrow \cdots \xrightarrow{l(w_m)} t_j(v)}_{I_j} \xrightarrow{v} \underbrace{\cdot \longrightarrow \cdots}_{I_k}$$

Then $x^{(j)} := u^\infty l(w_{[n,m]})v^\infty \in \Sigma_j$ and $y^{(j)} := F^{q_j-1}\sigma^{p_j}(x^{(j)}) \in \Sigma_j$ is its preimage with $F(y^{(j)}) = x^{(j)}$. There exist now consecutive disjoint intervals J_i, for all $i \in \{1,\dots,n\}$ whose union is $\mathbb{Z}$, such that if we define y so that it coincides with $y^{(i)}$ on J_i for all $i \in \{1,\dots,n\}$, then $F(y) = x$ and $y \in \Sigma$. Thus $x \in F(\Sigma)$, so $\Sigma \subseteq F(\Sigma)$. □

In many examples, we can find a join $\Sigma = \Sigma_1 \vee \cdots \vee \Sigma_n$ of some signal subshifts, such that $\Omega_F(A^{\mathbb{Z}}) = F^k(\Sigma)$ for some k. In this case, $\Omega_F(A^{\mathbb{Z}})$ is a sofic subshift. There do exist cellular automata with non sofic maximal attractors (see e.g., Culik, Hurd and Yu [4]) and our theory does not apply to them. We do not know whether every maximal attractor which is sofic can be represented as an F^k-image of a join of some signal subshifts.

7. The small quasi-attractor.

Proposition 6. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and U a clopen F-invariant set such that $\Omega_F(U)$ is a subshift attractor. Then for every $n > 0$ there exists $k > 0$ such that $F^k(U) \subseteq \bigcap_{i=-n}^{n} \sigma^i(U)$.*

Proof. We have $\Omega_F(U) = \Omega_F(\sigma^i(U)) \subseteq \sigma^i(U)$, and by compactness there exists $k > 0$ such that $F^k(U) \subseteq \sigma^i(U)$. □

Proposition 7. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. If $\Sigma_1,\dots,\Sigma_k \subseteq A^{\mathbb{Z}}$ are subshifts such that each Σ_i is an attractor of some $F^{q_i}\sigma^{p_i}$ where $q_i > 0$ and $p_i \in \mathbb{Z}$, then the intersection $\Sigma_1 \cap \cdots \cap \Sigma_k$ is nonempty.*

Proof. Let U_i be clopen sets invariant for $F^{q_i}\sigma^{p_i}$, such that $\Sigma_i = \Omega_{F^{q_i}\sigma^{p_i}}(U_i)$. Since a clopen set is a finite union of cylinders, there exists a configuration $x \in A^{\mathbb{Z}}$ such that the σ-orbit of x visits each U_i with bounded gaps, i.e.,

$$\exists n > 0, \forall i \leq k, \forall p \in \mathbb{Z}, \exists j < n, \sigma^{p+j}(x) \in U_i.$$

If q is the least common multiple of all q_i, then, by Proposition 6, there exists $m > 0$, such that $F^{qm}(x) \in U := \widetilde{\Omega}_\sigma(U_1) \cap \cdots \cap \widetilde{\Omega}_\sigma(U_k)$, where $\widetilde{\Omega}_\sigma(U_i) := \bigcap_{j\in\mathbb{Z}} \sigma^j(U_i)$. Thus $U \neq \emptyset$ is F^q-invariant. The nonempty set $\Omega_{F^q}(U)$ is contained in all Σ_i. □

Definition 2. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. The nonempty intersection of all subshift attractors of all $F^q\sigma^p$ where $q > 0$ and $p \in \mathbb{Z}$, is called the **small quasi-attractor** of F and denoted by $\mathcal{Q}_F$.

Proposition 8. *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, then $\mathcal{Q}_F$ is F-invariant.*

Proof. If U is a clopen $F^q\sigma^p$-invariant set and $\Sigma_0 := \Omega_{F^q\sigma^p}(U)$ is a subshift, then $F^{-k}(U)$ is a clopen $F^q\sigma^p$-invariant set, and $\Sigma_k := \Omega_{F^q\sigma^p}(F^{-k}(U)) = F^{-k}(\Sigma_0)$ is a subshift attractor for $F^q\sigma^p$. Since Σ_0 is $F^q\sigma^p$-invariant, $F^q(\Sigma_0) \subseteq \Sigma_0$, thus $\Sigma_0 \subseteq F^{-q}(\Sigma_0)$, and $F(\Sigma_0) \subseteq F^{1-q}(\Sigma_0) = \Sigma_{q-1}$. Thus $\Sigma = \Sigma_0 \cap \cdots \cap \Sigma_{q-1}$ is nonempty and F-invariant. It follows that $\mathcal{Q}_F$ is F-invariant. □

Proposition 9. *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a surjective cellular automaton, then the small quasi-attractor is the full space: $\mathcal{Q}_F = A^{\mathbb{Z}}$.*

Proof. Assume that U is a clopen F-invariant set and $\Omega_F(U)$ is a subshift. By Proposition 6, $F^k(U) \subseteq \sigma^{-1}(U) \cap U$ for some $k > 0$. For the complement $V = A^{\mathbb{Z}} \setminus U$ we get $F^{-k}(V) \subseteq V \cap \sigma(V)$. By the Hedlund theorem, the uniform Bernoulli measure μ is F-invariant, so $\mu(V) = \mu(F^{-k}(V)) \leq \mu(V \cap \sigma(V))$, and it follows by σ-ergodicity that either $\mu(V) = 1$ or $\mu(V) = 0$. Since V is clopen, and $\mathcal{S}_\mu = A^{\mathbb{Z}}$, this means that either $V = A^{\mathbb{Z}}$ or $V = \emptyset$. But U is nonempty, so we must have $V = \emptyset$ and $U = A^{\mathbb{Z}}$. But then $\Omega_F(U) = A^{\mathbb{Z}}$ because F is surjective. Thus the only subshift attractor of F is the full space $A^{\mathbb{Z}}$. The same argument applies to each $F^q \sigma^p$. It follows that $\mathcal{Q}_F = A^{\mathbb{Z}}$. □

8. **The measure attractor.**

Definition 3. Given a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, we say that an F-invariant subshift $\Sigma \subseteq A^{\mathbb{Z}}$ **attracts ergodic measures** if there exists $\varepsilon > 0$, such that for every σ-ergodic measure $\mu \in \mathfrak{M}_\sigma(A^{\mathbb{Z}})$ such that $\varrho(\mu, \mathfrak{M}_\sigma(\Sigma)) < \varepsilon$, we have $\varrho(F^n\mu, \mathfrak{M}_\sigma(\Sigma)) \to 0$ as $n \to \infty$.

Proposition 10. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and let $D \subseteq A^*$ be a set of forbidden words. Then for every $\mu \in \mathfrak{M}_\sigma(A^{\mathbb{Z}})$ we have*

$$\lim_{n\to\infty} \varrho(F^n\mu, \mathfrak{M}_\sigma(\Sigma_D)) = 0 \iff \forall u \in D, \lim_{n\to\infty} (F^n\mu)(u) = 0$$

The proof is straightforward.

Proposition 11. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton, and $\Sigma \subseteq A^{\mathbb{Z}}$ a subshift attractor with basin $\mathcal{B}_F(\Sigma)$. If $\mu \in \mathfrak{M}_\sigma(A^{\mathbb{Z}})$ and $\mu(\mathcal{B}_F(\Sigma)) = 1$, then $\varrho(F^n\mu, \mathfrak{M}_\sigma(\Sigma)) \to 0$ as $n \to \infty$.*

In particular, for all $\mu \in \mathfrak{M}_\sigma(A^{\mathbb{Z}})$, $\varrho(F^n\mu, \mathfrak{M}_\sigma(\Omega_F(A^{\mathbb{Z}}))) \to 0$ as $n \to \infty$.

Proof. Given $u \in A^* \setminus \mathcal{L}(\Sigma)$ with $|u| = 2m+1$, then

$$\mu\{x \in A^{\mathbb{Z}} : \exists n_0, \forall n \geq n_0, d(F^n(x), \Sigma) < 2^{-m}\} = 1,$$

so $\mu\{x \in A^{\mathbb{Z}} : \exists n_0, \forall n \geq n_0, F^n(x)_{[-m,m]} \neq u\} = 1$. Thus for every $\delta > 0$ there exists n_0 such that $\mu\{x \in A^{\mathbb{Z}} : \forall n \geq n_0, F^n(x)_{[-m,m]} \neq u\} > 1 - \delta$. It follows that for every $\delta > 0$ there exists n_0 such that for every $n \geq n_0$,

$$(F^n\mu)(u) = \mu\{x \in A^{\mathbb{Z}} : F^n(x)_{[-m,m]} = u\} < \delta,$$

so $\lim_{n\to\infty} F^n\mu(u) = 0$. By Proposition 10, $\varrho(F^n\mu, \mathfrak{M}_\sigma(\Sigma)) \to 0$ as $n \to \infty$. □

Proposition 12. *A subshift attractor for F attracts ergodic measures.*

Proof. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and $U \subseteq A^{\mathbb{Z}}$ a clopen F-invariant set such that $\Sigma = \Omega_F(U)$ is a subshift attractor. Then $\Sigma \subseteq U \subseteq \mathcal{B}_F(\Sigma)$, and there exist $m \geq 0$, $k > 0$ and $u^{(1)}, \ldots, u^{(k)} \in A^{2m+1}$ such that $A^{\mathbb{Z}} \setminus U = [u^{(1)}]_{-m} \cup \cdots \cup [u^{(k)}]_{-m}$. Assume that $\varepsilon < 1/k2^{2m+1}$. If μ is a σ-ergodic measure and $\varrho(\mu, \mathfrak{M}_\sigma(\Sigma)) < \varepsilon$, then there exists $\nu \in \mathfrak{M}_\sigma(\Sigma)$ such that $\varrho(\nu, \mu) < \varepsilon$. Thus $\nu(U) = 1$, so $\nu(u^{(i)}) = 0$ and hence $\mu(u^{(i)}) < \varepsilon \cdot 2^{2m+1} < 1/k$ for $1 \leq i \leq k$. It follows that $\mu(A^{\mathbb{Z}} \setminus U) < 1$, so $\mu(\mathcal{B}_F(\Sigma)) > 0$. Since $\mathcal{B}_F(\Sigma)$ is σ-invariant, $\mu(\mathcal{B}_F(\Sigma)) = 1$. By Proposition 11, $\varrho(F^n\mu, \mathfrak{M}_\sigma(A^{\mathbb{Z}})) \to 0$ as $n \to \infty$. □

Proposition 13. *If $F : \Sigma \to \Sigma$ is a cellular automaton, then $F : \mathfrak{M}_\sigma(\Sigma) \to \mathfrak{M}_\sigma(\Sigma)$ has a unique attractor $\Omega_F(\mathfrak{M}_\sigma(\Sigma)) = \mathfrak{M}_\sigma(\mathcal{D}(\Omega_F(\Sigma)))$.*

Proof. Suppose that there exists an attractor Y which is strictly smaller than the maximal attractor $\Omega_F(\mathfrak{M}_\sigma(\Sigma))$. Then there exists a chain component $W \subseteq \Omega_F(\mathfrak{M}_\sigma(\Sigma)) \setminus Y$ and W is closed and invariant (see Akin [1] or Kůrka [8]). Choose $\mu \in Y$, $\nu \in W$, and consider $\mu_a = (1-a)\mu + a\nu$, where $0 \le a \le 1$. Set

$$c := \inf\{a > 0 : \forall n \ge 0, F^n\mu_a \notin Y\}.$$

Since the distance $\varrho(W, Y)$ is positive, $c < 1$ and $F^n\mu_c$ gets arbitrarily close to Y. It follows that for every $\varepsilon > 0$ there exist $a > c$ and $n > 0$ such that $\varrho(F^n\mu_a, Y) < \varepsilon$, but $F^n\mu_a$ is never attracted to Y. This is a contradiction. Thus, $\Omega_F(\mathfrak{M}_\sigma(\Sigma))$ is the unique attractor. By Proposition 2, $\mathfrak{M}_\sigma(\Omega_F(\Sigma)) \subseteq \Omega_F(\mathfrak{M}_\sigma(\Sigma))$, while the inverse inclusion is trivial. By Proposition 1 we get $\mathfrak{M}_\sigma(\Omega_F(\Sigma)) = \mathfrak{M}_\sigma(\mathcal{D}(\Omega_F(\Sigma)))$. □

Definition 4. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton.
(1) The **measure attractor** of F is $\mathcal{M}_F := \mathcal{S}(\Omega_F(\mathfrak{M}_\sigma(A^{\mathbb{Z}}))) = \mathcal{D}(\Omega_F(A^{\mathbb{Z}}))$, where the second equality is by Proposition 13.
(2) The **measure quasi-attractor** of F is $\mathcal{MQ}_F := \mathcal{D}(\Omega_F(\mathcal{Q}_F))$.

Given a shift-invariant measure μ, the μ-limit $\Lambda_\mu(F)$ of F has been defined in Kůrka and Maass [9] as the subshift whose forbidden words $u \in A^*$ are those for which $F^n\mu(u) \to 0$ as $n \to \infty$. Proposition 13 implies that $\Lambda_\mu(F) \subseteq \mathcal{M}_F$ for every $\mu \in \mathfrak{M}_F$.

Corollary 1. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton.*
(1) If $\mu \in \mathfrak{M}_\sigma(A^{\mathbb{Z}})$, then $\varrho(F^n\mu, \mathfrak{M}_\sigma(\mathcal{M}_F)) \to 0$ as $n \to \infty$.
(2) $\mathcal{MQ}_F$ attracts ergodic measures.

Proof. (1) This follows from Proposition 11 and Proposition 1(1).

(2) Assume that U is a clopen $F^q\sigma^p$-invariant set. As in the proof of Proposition 8, we get subshift attractors $\Sigma_k = \Omega_{F^q\sigma^p}(F^{-k}(U))$ which, by Proposition 12, attract ergodic measures under F^q. Then $\Sigma = \Sigma_0 \cap \cdots \cap \Sigma_{q-1}$ is an F-invariant subshift which attracts ergodic measures under F. It follows that $\mathcal{MQ}_F$ attracts ergodic measures. □

9. **Examples.**

Example 4 (ECA 136)**.** The product rule $F(x)_i = x_i x_{i+1}$ in the binary alphabet.

Every $[0]_i$ is a clopen invariant set but its omega-limit is not a subshift. The only signal subshifts are

$$\Sigma_{(0,1)} = \{x \in A^{\mathbb{Z}} : \forall i < j, x_i \le x_j\}, \quad \Sigma_{(-1,1)} = \{x \in A^{\mathbb{Z}} : \forall i < j, x_i \ge x_j\}.$$

The unique subshift attractor

$$\Omega_F(A^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}} : \forall k > 0, 10^k1 \not\sqsubseteq x\} = \Sigma_{(0,1)} \vee \Sigma_{(-1,1)}$$

is their join (Figure 3). This is a sofic subshift and $\mathcal{M}_F = \{0^\infty, 1^\infty\}$. Each $F\sigma^n$ has the same unique subshift attractor $\Omega_{F\sigma^n}(A^{\mathbb{Z}}) = \Omega_F(A^{\mathbb{Z}})$. However, $G = F^2\sigma^{-1}$ is the ECA 128 defined by $G(x)_i = x_{i-1}x_i x_{i+1}$, and it has the minimal subshift attractor $\{0^\infty\}$. Thus the small quasi-attractor is $\mathcal{Q}_F = \{0^\infty\}$ and the measure quasi-attractor is also $\mathcal{MQ}_F = \{0^\infty\}$.

Example 5 (ECA 184)**.** The traffic rule $F(x)_i = 1$ iff $x_{[i-1,i]} = 10$ or $x_{[i,i+1]} = 11$.

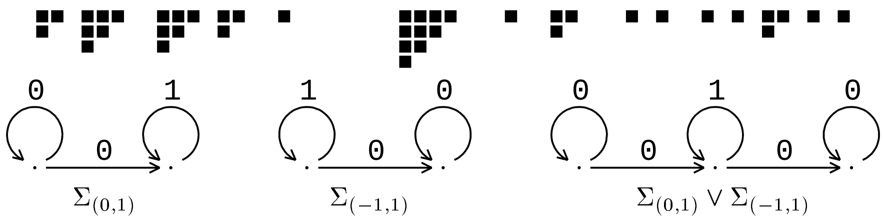

FIGURE 3. ECA 136 and its signal subshifts

The omega-limit is the join (Figure 2)

$$\Omega_F(A^{\mathbb{Z}}) = \{x \in A^{\mathbb{Z}} : \forall k > 0, 1(10)^k 0 \not\sqsubseteq x\} = \Sigma_{(1,1)} \vee \Sigma_{(-1,1)}.$$

If $k \notin \{-1, 1\}$, then $\Omega_F(A^{\mathbb{Z}})$ is chain-transitive for $F\sigma^k$, so it is the unique attractor. For $k = 1$, $F\sigma$ has a clopen invariant set $U = A^{\mathbb{Z}} \setminus [00]_0$, but $\Omega_{F\sigma}(U)$ is not a subshift. All attractors for $F\sigma$ can be obtained from the set of chain-recurrent configurations

$$|\mathcal{C}_{F\sigma}| = \Sigma_{(0,1)}(F\sigma) \cup \Sigma_{(-2,1)}(F\sigma) = \Sigma_{(1,1)}(F) \cup \Sigma_{(-1,1)}(F).$$

For $x, y \in |\mathcal{C}_{F\sigma}|$ we have $(x, y) \in \mathcal{C}_{F\sigma}$ iff either $x = y \in \Sigma_{(0,1)}(F\sigma)$ or $y \in \Sigma_{(-2,1)}(F\sigma)$. Thus every configuration of $\Sigma_{(0,1)}(F\sigma)$ is a single chain-component and the only other chain component is all of $\Sigma_{(-2,1)}(F\sigma)$.

We show that the only subshift attractor of $F\sigma$ is $\Omega_F(A^{\mathbb{Z}})$. Assume that U is a clopen $F\sigma$-invariant set and $\Omega_{F\sigma}(U)$ is a subshift. Since each $x \in \Sigma_{(0,1)}(F\sigma)$ is a fixed point for $F\sigma$, $\Omega_{F\sigma}(U) \cap \Sigma_{(0,1)}(F\sigma) = U \cap \Sigma_{(0,1)}(F\sigma) = U_0$ is a clopen invariant set in $\Sigma_{(0,1)}(F\sigma)$. Since $\Sigma_{(0,1)}(F\sigma)$ is σ-transitive, $U_0 = \Sigma_{(0,1)}(F\sigma)$. Since distinct attractors have distinct chain-recurrent points, the only subshift attractor of $F\sigma$ is $\Omega_F(A^{\mathbb{Z}})$. A similar analysis applies to $F\sigma^{-1}$ and to all $F^q\sigma^{\pm q}$. Thus the small quasi-attractor and the measure attractor are

$$\mathcal{Q}_F = \Omega_F(A^{\mathbb{Z}}), \quad \mathcal{M}_F = \mathcal{M}\mathcal{Q}_F = \Sigma_{(1,1)} \cup \Sigma_{(-1,1)}.$$

Example 6 (ECA 62)**.** The rule is $F(x)_i = 0$ iff $x_{[i-1,i+1]} \in \{000, 110, 111\}$.

The only preimages of 0^6 are 0^8, 01^7 and 1^8. However 1^7 is not in $\mathcal{L}(\Omega_F(A^{\mathbb{Z}}))$, so the only preimage of 0^6 in the omega-limit is 0^8. Thus there exists an attractor which does not contain 0^∞. Indeed, the set $V = [0^6]_2 \cup [1^7]_1 \cup \bigcup_{v \in f^{-1}(1^7)} [v]_0$ is inversely invariant, so $U = A^{\mathbb{Z}} \setminus V$ is a clopen invariant set and $\Omega_F(U)$ is a subshift attractor. It contains σ-transitive signal subshifts $\Sigma_{(1,2)}$ and $\Sigma_{(0,3)}$, whose intersection is $\mathfrak{o}((110)^\infty)$, as well as their join. Similarly as in Example 5 we show

$$\mathcal{Q}_F = \Omega_F(U) = F^2(\Sigma_{(1,2)} \vee \Sigma_{(0,3)}), \quad \mathcal{M}\mathcal{Q}_F = \Sigma_{(1,2)} \cup \Sigma_{(0,3)}.$$

The only other signal subshifts are $\Sigma_{(4,4)}$ and $\Sigma_{(-1,1)}$ with intersections $\Sigma_{(4,4)} \cap \Sigma_{(0,3)} = \mathfrak{o}((111010110001)^\infty)$, $\Sigma_{(4,4)} \cap \Sigma_{(-1,1)} = \{0^\infty\}$, and $\Sigma_{(1,2)} \cap \Sigma_{(0,3)} \cap \Sigma_{(-1,1)} = \mathfrak{o}((110)^\infty)$. We get

$$\begin{aligned} \Omega_F(A^{\mathbb{Z}}) &= F^2(\Sigma_{(4,4)} \vee \Sigma_{(1,2)} \vee \Sigma_{(0,3)} \vee \Sigma_{(-1,1)}), \\ \mathcal{M}_F &= \{0^\infty\} \cup \Sigma_{(1,2)} \cup \Sigma_{(0,3)}. \end{aligned}$$

In Figure 4 we can see a configuration whose left part belongs to $\Sigma_{(4,4)} \vee \Sigma_{(-1,1)}$ and whose right part belongs to $\Sigma_{(1,2)} \vee \Sigma_{(0,3)}$ In the space-time diagram, the words 00, 111 and 010, which do not occur in $(110)^\infty$, are displayed in grey.

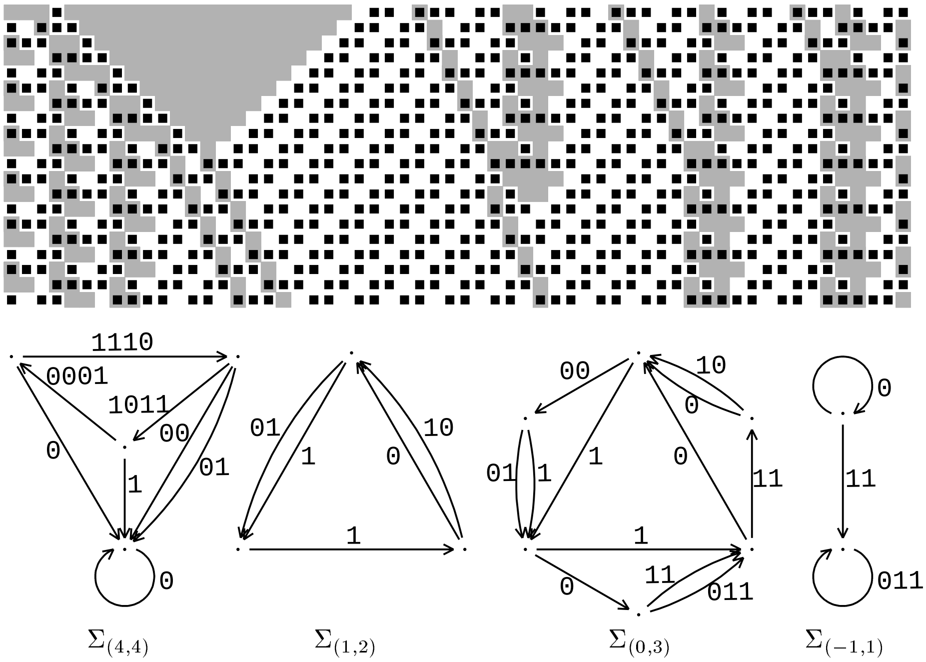

FIGURE 4. ECA 62 and its signal subshifts

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