Manuscript submitted to
AIMS' Journals
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org

pp. **000–000**

MINIMALITY IN ITERATIVE SYSTEMS OF MÖBIUS TRANSFORMATIONS

Abstract. We study the parameter space of an iterative system consisting of two hyperbolic disc Möbius transformations. We identify several classes of parameters which yield discrete groups whose fundamental polygons have sides at the Euclidean boundary. It follows that these system are not minimal.

Petr Kůrka

Center for Theoretical Study,
Academy of Sciences and Charles University in Prague,
Jilská 1, CZ-11000 Praha 1, Czechia.

1. **Introduction.** Iterative systems consisiting of contractions have been used in Barnsley [1] or Edgar [3] to generate self-similar fractal sets. In Kůrka [5] we have investigated topological dynamics of iterative systems regarded as actions of a free semigroup. In particular we have studied iterative systems consisting of real Möbius transformations and we have characterized those which have non-trivial attractors. We have studied in detail the parameter space of an alternating iterative system consisting of two hyperbolic transformations whose stable fixed points are separated by their unstable fixed points. The eliptical region consists of parameters for which there exists an elliptic transformation. We have shown that outside of the elliptic region the systems have nontrivial attractors (and therefore are not minimal), while inside the elliptic region, the minimal systems are dense. In the present paper we show that the elliptic region contains many non-minimal systems as well. These systems are discrete groups whose fundamental polygons contain a side at the Euclidean boundary of the hyperbolic space.

2. **Topological dynamics.** Given a finite alphabet A, we denote by $A^+ = \cup_{n\geq 1} A^n$ the set of words with letters from A. The length of a word $u = u_0 \dots u_{n-1} \in A^n$ is denoted by $|u| := n$. Its prefix of length $m \leq n$ is denoted by $u_{[0,m)}$. With the binary operation of concatenation, A^+ is the **free semigroup** over A. An **iterative system** over A^+ is a pair (X, F), where X is a compact metric space and $F : A^+ \times X \to X$ is a map such that for each $u \in A^+$, $F_u : X \to X$ is continuous, and $F_{uv}(x) = F_u(F_v(x))$. The system is generated by continuous maps $(F_a : X \to X)_{a\in A}$. A **morphism** $\Phi : (X, F) \to (Y, G)$ of two iterative systems over A^+ is a continuous map $\Phi : X \to Y$ such that $\Phi F_a = G_a \Phi$ for each $a \in A$. A **factor** is a surjective morphism, a **subsystem** is an injective morphism, and a **conjugacy** is a bijective morphism. The **orbit** of $x \in X$ is $\mathcal{O}(x) := \{F_u(x) :\ u \in A^+\}$. The system (X, F) is **minimal** if $\mathcal{O}(x)$ is dense in X for every $x \in X$. The system (X, F) is **transitive** if for each nonempty open sets $U, V \subseteq X$ there exists $u \in A^+$ with $F_u(U) \cap V \neq \emptyset$. A subset $Y \subseteq X$ is **invariant**, if $F_a(Y) \subseteq Y$ for all $a \in A$. If Y

2000 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37F30.
Key words and phrases. topological dynamics, minimal systems, iterative Möbius systems.

is invariant and closed, then (Y, F) is a subsystem of (X, F). A closed set $W \subseteq X$ is **inward** if $F_a(\overline{W}) \subseteq \mathbf{int}(W)$ for each $a \in A$. A set $Y \subseteq X$ is an **attractor**, if there exists an inward set W such that Y is its omega-limit

$$Y = \Omega(W) = \bigcap_{n>0} \overline{\bigcup_{m>n} \bigcup_{u \in A^m} F_u(W)}.$$

3. **Möbius transformations.** A real **orientation-preserving Möbius transformation** (MT) is a self-map of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the form $\widehat{M}_{(a,b,c,d)}(x) = (ax+b)/(cx+d)$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. A real MT acts also on the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and on the **upper half-plane** $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$: if $z \in \mathbb{U}$ then $\widehat{M}(z) \in \mathbb{U}$. The map $\mathbf{d}(z) = (iz+1)/(z+i)$ maps $\mathbb{U}$ conformally to the **unit disc** $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. On the closed disc $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ we get **disc Möbius transformations** $M : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ defined by

$$M_{(\alpha,\beta)}(z) = \mathbf{d} \circ \widehat{M}_{(a,b,c,d)} \circ \mathbf{d}^{-1}(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. Conversely, each transformation $M_{(\alpha,\beta)}(z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$ with $|\alpha| > |\beta|$ is $\mathbf{d}$-conjugated to a real MT. Disc MT preserve the hyperbolic metric $ds = 2|dz|/(1 - |z|^2)$. The geodesics are arcs which are perpendicular to the unit circle.

Any disc MT can be expressed as a product of two reflections (see Beardon [2]). Given $w \in \mathbb{C}$, $r > 0$, the **reflection in the circle** $S(w, r) = \{z \in \mathbb{C} : |z - w| = r\}$ is $\varphi(z) = w + r^2/\overline{z - w}$. Given $w \in \mathbb{C}$ and $\alpha \in \partial\mathbb{D}$, the **reflection in the line** $P(w, \alpha) = \{w + t\alpha : t \in \mathbb{R}\}$ is $\varphi(z) = w + \alpha^2 \cdot \overline{z - w}$. Define the **contracting region**, **expansing region** and the **isometric circle** of a disc MT M by

$$\begin{aligned} U(M) &= \{z \in \mathbb{C} : |M'(z)| < 1\} \\ V(M) &= \{z \in \mathbb{C} : |(M^{-1})'(z)| > 1\} \\ I(M) &= \{z \in \mathbb{C} : |M'(z)| = 1\} \end{aligned}$$

If M is a rotation, i.e., if $M(z) = \alpha z$ for some $\alpha \in \partial\mathbb{D}$, then $U(M)$ and $V(M)$ are empty and $I(M) = \mathbb{C}$. Otherwise $I(M)$ is the circle $S(-\overline{\alpha}/\overline{\beta}, |\alpha/\beta|^2 - 1)$, $U(M)$ is its exterior, and $V(M)$ is the interior of the circle $S(\alpha/\overline{\beta}, |\alpha/\beta|^2 - 1)$. Moreover, we have $V(M^{-1}) = \mathbb{C} \setminus \overline{U(M)}$, $I(M) = \overline{U(M)} \cap \overline{V(M^{-1})}$, $M(U(M)) = V(M)$ and $M(I(M)) = I(M^{-1})$. For each disc MT M there exists a reflection $\varphi(z) = \alpha^2 \cdot \overline{z}$ in a diameter of the unit circle, such that $\varphi M \varphi = M^{-1}$. If M is a rotation, then φ can be the reflection in any diameter of the unit circle. Otherwise φ is the unique reflection which maps the center $\alpha/\overline{\beta}$ of $V(M)$ to the center $-\overline{\alpha}/\overline{\beta}$ of $V(M^{-1})$. In particular, $M(z) = \varphi(z)$ for any $z \in I(M)$.

The **square of the trace** is defined by $\mathbf{tr}^2(\widehat{M}_{(a,b,c,d)}) := (a+d)^2/(ad - bc)$, $\mathbf{tr}^2(M_{\alpha,\beta}) := (\alpha + \overline{\alpha})^2/(|\alpha|^2 - |\beta|^2)$. If $\mathbf{tr}^2(M) < 4$ then M is **elliptic**, if $\mathbf{tr}^2(M) > 4$ then M is **hyperbolic** and if $\mathbf{tr}^2(M) = 4$ then M is **parabolic** unless it is the identity. A hyperbolic transformation has two fixed points $s_M, u_M \in \partial\mathbb{D}$ with $|M'(s_M)| < 1$, $|M'(s_M)| \cdot |M'(u_M)| = 1$. A parabolic transformation has a unique fixed point $s_M \in \partial\mathbb{D}$ with $|M'(s_M)| = 1$. An elliptic transformation has no fixed point in $\partial\mathbb{D}$ but it has a unique fixed point $s_M \in \mathbb{D}$ with $M'(s_M) = e^{i \cdot \mathbf{rot}(M)}$, where the rotation angle $\mathbf{rot}(M) \in (0, 2\pi)$ satisfies $\mathbf{tr}^2(M) = 4\cos^2(\mathbf{rot}(M)/2)$. This formula does not determine the rotation angle uniquely. Such a formula can be obtained for a real MT. The rotation $R_\alpha(z) = e^{i\alpha} \cdot z$ is conjugated to the real MT $\widehat{R}_\alpha(x) = (x \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2})/(-x \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2})$ whose matrix has positive upper-right entry $\sin \frac{\alpha}{2}$. For any real orientation-preserving matrix $\widehat{M}$, the matrix $\widehat{M}\widehat{R}_\alpha\widehat{M}^{-1}$ has positive upper-right entry as well. Thus the **rotation angle** of a real elliptic MT is

$$\mathbf{rot}(\widehat{M}_{(a,b,c,d)}) = 2 \arccos \frac{\mathbf{sgn}(b) \cdot (a+d)}{2\sqrt{ad-bc}} \in (0, 2\pi)$$

The rotation angle of a disc MT is the rotation angle of the **d**-conjugated real MT. We say that M has irrational rotation angle, if $\mathbf{rot}(M)/2\pi$ is an irrational number. A disc MT M is elliptic iff $I(M) \cap I(M^{-1}) \neq \emptyset$.

Lemma 1. *If F, G are disc MT and $I(F) \cap I(G^{-1}) = I(F^{-1}) \cap I(G) = \emptyset$, then FG is not elliptic.*

Proof. The condition implies $V(F) \cap V(G^{-1}) = V(F^{-1}) \cap V(G) = \emptyset$. Assume by contradiction that FG is elliptic and $s \in \mathbb{D}$ is its fixed point, so $|F'(G(s)) \cdot G'(s)| = 1$. We distinguish three cases. 1. If $s \in V(G^{-1})$, then $|G'(s))| > 1$ and $|F'(G(s))| < 1$. Thus $G(s) \in U(F)$, and $s = FG(s) \in V(F)$ which is disjoint with $V(G^{-1})$ and this is a contradiction. 2. If $s \in U(G)$, then $G(s) \in V(G)$, so $G(s) \notin V(F^{-1})$ and $|G'(s)| < 1$, so $|F'(G(s))| < 1$, which is a contradiction. 3. If $s \in I(G)$ then $|G'(s)| = 1$, $G(s) \in I(G^{-1})$, so $G(s) \notin I(F)$ and $|F'(G(s))| \neq 1$, which is a contradiction. □

Lemma 2. *If F, G are disc MT with the same symmetry $\varphi(z) = \alpha^2 \overline{z}$ (i.e., $\varphi F \varphi = F^{-1}$, $\varphi G \varphi = G^{-1}$), and if $I(F) \cap I(G^{-1}) \neq \emptyset$, then FG is elliptic.*

Proof. If $I(F) \cap I(G^{-1})$ is nonempty, then there exists $s \in I(F) \cap I(G^{-1}) \cap \mathbb{D}$, and $t = F(s) = \varphi(s) = G^{-1}(s) \in \mathbb{D}$, so s is a fixed point of GF, and GF is elliptic. Since $\mathbf{tr}(FG) = \mathbf{tr}(GF)$, FG is elliptic as well. □

Lemma 3. *If M, F are disc MT, F is elliptic with $\mathbf{rot}(F) = \alpha \in (0, 2\pi)$ and $s_F \in I(M)$, then $s_F \in I(MF)$ and the angle between $I(MF)$ and $I(M)$ at s_F is $-\alpha/2$. It follows that the inner angle of $U(M) \cap U(MF)$ at s_F is either $\alpha/2$ or $\pi - \alpha/2$.*

Proof. Let L be the geodesic which passes through s_F and whose angle with $I(M)$ is $-\alpha/2$. Let σ_1 be the reflection in L and let σ_2 be the reflection in $I(M)$. Then $\sigma_2\sigma_1$ turns L by α, so $F = \sigma_2\sigma_1$, and $M = \sigma_3\sigma_2$, where σ_3 is the reflection in a diameter of the unit circle. Since $MF = \sigma_3\sigma_1$ and σ_3 is an isometry, we get $I(MF) = L$, □

4. **Möbius iterative systems.**

Definition 1. *We say that $\widehat{F} : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, is a* **Möbius iterative system** *(MIS), if all $\widehat{F}_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are real orientation-preserving Möbius transformations. $\widehat{F}$ is conjugated to a disc MIS $F : A^+ \times \partial\mathbb{D} \to \partial\mathbb{D}$ by the stereographic projection* **d**. *The* **limit set** *Λ_F of F is defined by $\Lambda_F = \overline{\{F_u(0) : u \in A^+\}} \cap \partial\mathbb{D}$.*

If $\mathbb{G}(F) = \{F_u : u \in A^*\}$ is a discrete group, then Λ_F coincides with the classical concept (see Beardon [2] or Katok [4]).

Theorem 2 (Kůrka and Kazda [6]). *If $\{\overline{V(F_u)} : u \in A^+\}$ is a cover of $\partial\mathbb{D}$ then $\Lambda_F = \partial\mathbb{D}$.*

The proof is based on the expansion graph whose vertices are numbers $x \in \partial\mathbb{D}$ and whose labelled edges are $x \xrightarrow{u} F_u^{-1}(x)$ where $x \in \overline{V(F_u)}$. If $u \in A^{\mathbb{N}}$ is the label of an infinite path with source x, then $x = \lim_{n\to\infty} F_{u_{[0,n)}}(0)$. The limit set Λ_F is closed and F-invariant, so if $\Lambda_F \neq \partial\mathbb{D}$ then $(\partial\mathbb{D}, F)$ is not minimal. Conversely we have

Proposition 3. *If $\Lambda_F = \partial\mathbb{D}$ and if no $x \in \partial\mathbb{D}$ is fixed by all F_a, then $(\partial\mathbb{D}, F)$ is minimal.*

Proof. Given $x, y \in \partial\mathbb{D}$, we show that $y \in \overline{\mathcal{O}(x)}$. By the assumption there exists a sequence $u^{(n)} \in A^+$ with $\lim_{n\to\infty} F_{u^{(n)}}(0) = y$, so $\lim_{n\to\infty} \mathbf{diam}(V(F^{-1}_{u^{(n)}})) = \lim_{n\to\infty} \mathbf{diam}(V(F_{u^{(n)}})) = 0$. We have $F_a(x) \neq x$ for some $a \in A$, so there exists n_0 such that for all $n \geq n_0$ either $x \in U(F_{u^{(n)}})$ or $F_a(x) \in U(F_{u^{(n)}})$. Set $v^{(n)} = u^{(n)}$ in the former case and $v^{(n)} = u^{(n)}a$ in the latter case. Then $F_{v^{(n)}}(x) \in V(F_{u^{(n)}})$ for each $n \geq n_0$, so $\lim_{n\to\infty} F_{v^{(n)}}(x) = y$. □

Example 1. *There exists a MIS F with $\Lambda_F = \partial\mathbb{D}$ which is not minimal.*

Proof. Consider the alphabet $A = \{\overline{1}, 0, 1, \overline{0}\}$ and transformations $\widehat{F_{\overline{1}}}(x) = x - 1$, $\widehat{F_0}(x) = x/2$, $\widehat{F_1}(x) = x + 1$, $\widehat{F_{\overline{0}}}(x) = 2x$. All $\widehat{F_a}$ fix the point $\{\infty\}$, so all F_a fix the point i, but $\Lambda_F = \partial\mathbb{D}$ since $V(F_a)$ cover $\partial\mathbb{D}$. □

Proposition 4. *Let $(\partial\mathbb{D}, F)$ be an iterative Möbius system. If some F_u is elliptic with irrational rotation angle, then $(\partial\mathbb{D}, F)$ is minimal.*

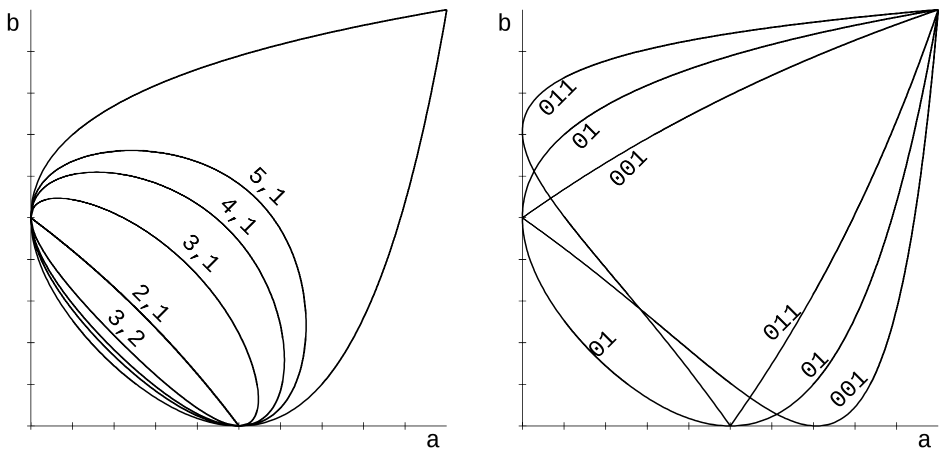

FIGURE 1. Parameter space of the standard alternating Möbius system: E_{01}, $R_{01}(n, n-1)$, $R_{01}(n, 1)$ for $n = 2, \ldots, 5$ (left), E_{01}, E_{001}, E_{011} (right).

5. **Alternating Möbius systems.** In Kůrka[5] we have shown that if $(\partial\mathbb{D}, F)$ is a Möbius iterative system with hyperbolic transformations F_a and if there exists a closed interval $W \subset \partial\mathbb{D}$ which contains in its interior stable fixed points of all F_a and does not contain the unstable fixed point of any F_a, then $(\partial\mathbb{D}, F)$ has a nontrivial attractor and therefore is not minimal. We have then considered alternating Möbius iterative system $(\partial\mathbb{D}, F_0, F_1)$, whose fixed points s_0, r_0, s_1, r_1, s_0 are

arranged counterclockwise on the circle. In this case the existence of a nontrivial attractor depends on the quotients of F_0, F_1.

Definition 5. *The* **standard alternating system** *with parameters* $0 < a, b < 1$ *is defined by*

$$\widehat{F}_0(x) = ax + 1 - a,\ \widehat{F}_1(x) = \frac{x}{-(1-b)x + b}$$

We have $s(F_0) = 1$, $r(F_0) = i$, $s(F_1) = -1$, $r(F_1) = -i$. Both F_0 and F_1 have the same symmetry $\varphi(z) = i\overline{z}$ with $\varphi F_i \varphi = F_i^{-1}$, so $(\partial\mathbb{D}, F_0, F_1)$ is conjugated to $(\partial\mathbb{D}, F_0^{-1}, F_1^{-1})$ by φ. If $a = b$, we have additional symmetry $\psi(z) = -i\overline{z}$ with $\psi F_0 \psi = F_1^{-1}$, $\psi F_1 \psi = F_0^{-1}$, so $(\partial\mathbb{D}, F_0, F_1)$ is conjugated to $(\partial\mathbb{D}, F_1^{-1}, F_0^{-1})$ by ψ and to $(\partial\mathbb{D}, F_0, F_1)$ by the rotation $\varphi\psi(z) = -z$.

Denote by $P = \{(a, b) :\ 0 < a, b < 1\}$ the parameter space of the standard alternating system. We have $\mathbf{tr}(\widehat{F}_{01}) = (a + b - (1-a)(1-b))/\sqrt{ab}$, and the upper-right entry $(1-b)$ of $\widehat{F}_{01}$ is positive. For $u \in A^+$ denote by

$$\begin{aligned}
E_u &= \{(a,b) \in P :\ |\mathbf{tr}(F_u)| < 2\} \\
E &= \bigcup_{n>0} (E_{0^n 1} \cup E_{01^n}) \\
R_u(n,k) &= \{(a,b) \in E_u :\ \mathbf{rot}(F_u) = \frac{2k\pi}{n}\} \\
R_{01}(n,k) &= \{(a,b) :\ a + b - (1-a)(1-b) = 2\sqrt{ab}\cos\frac{k\pi}{n}\}
\end{aligned}$$

We call E the **elliptic set** (see Figure 1 and Figure 5). In [5] we have proved that each system in $P \setminus \overline{E}$ has a non-trivial attractor and the systems at the boundary of E are not transitive. It follows that $E = \bigcup_{u \in A^+} E_u$ and that each minimal system is in the elliptic set. If there exists an irrational rotation F_u, then $(\partial\mathbb{D}, F)$ is minimal, so minimal systems are dense in E. We show that there exist non-minimal systems in E as well, disproving a conjecture in [5].

6. **Rational rotation angle.** We investigate alternating systems in which F_{01} is elliptic with rotation angle $\mathbf{rot}(F_{01}) = \pm 2\pi/m$. Then $F_{(01)^m} = \mathrm{Id}$ and $\mathbb{G}(F) = \{F_u :\ u \in A^*\}$ is a group. $\mathbb{G}(F)$ is a discrete group if Id is its isolated element. For a discrete group acting on $\mathbb{D}$ there exists a fundamental region, which is an open connected set P, such that $\{F_u(P) :\ u \in A^+\}$ tesselate $\mathbb{D}$. If $\mathbb{G}(F)$ is discrete and 0 is not fixed by any F_u, then the Ford region defined by $P = \bigcap_{u \in A^+} U(F_u)$ is a fundamental region. For $u \in A^+$ denote by $I_u = I(F_u)$ and $I_u^{-1} = I(F_u^{-1})$. In Figures 2, 3 we show the Ford fundamental polygons of various alternating systems whose locations in the parameter space are indicated in Figure 5 right. We write words u in the images $F_u(P)$ of the fundamental polygon around the fixed points s_{01} and s_{10}. The words u of the isometric circles I_u are written at their intersections with the unit circle.

Proposition 6. *If $\mathbb{G}(F)$ is a discrete group and there exits a fundamental polygon with a side at the Euclidean boundary (i.e., at the unit circle), then $(\partial\mathbb{D}, F)$ is not transitive.*

Proof. If $U, V \subset \partial\mathbb{D}$ are disjoint open intervals both at the Euclidean boundary of a fundamental polygon P, then $F_u(U) \cap V = \emptyset$ for every $u \in A^+$. □

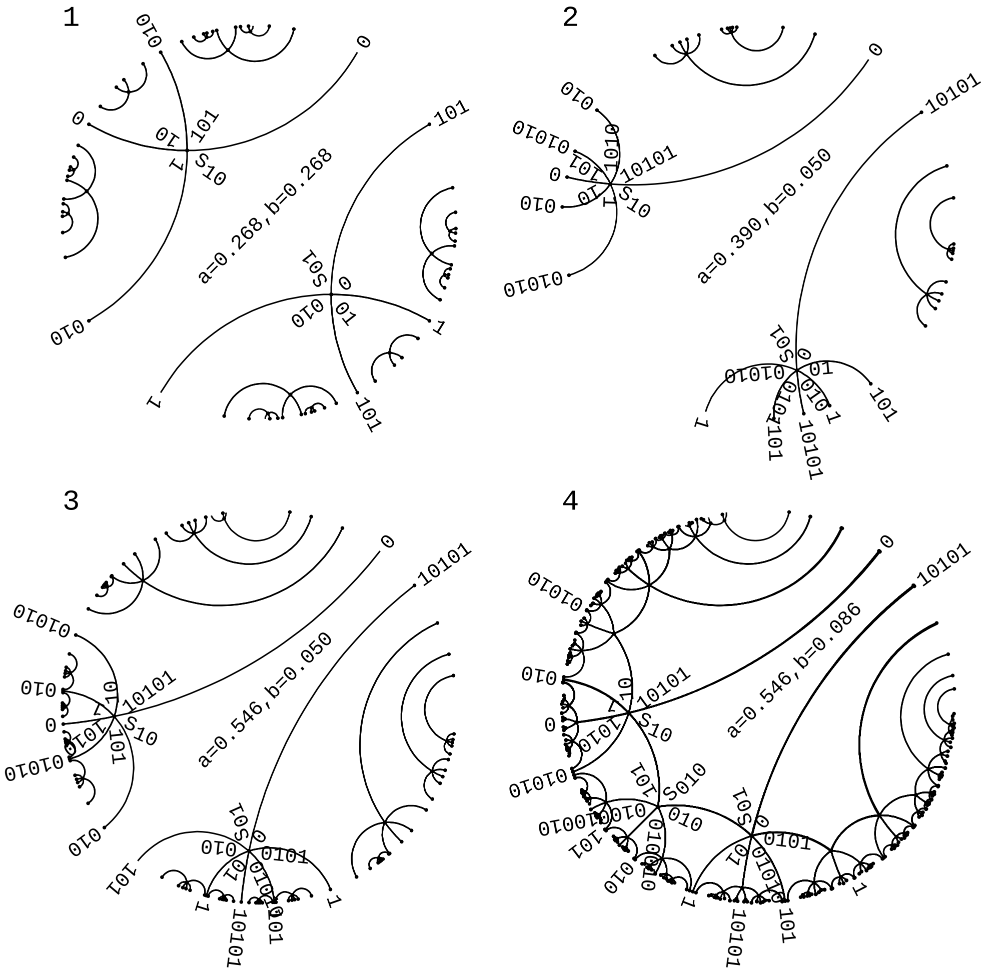

FIGURE 2. Isometric circles and Ford regions. 1(top left): $\mathbf{rot}(F_{01}) = \pi$, 2(top right): $\mathbf{rot}(F_{01}) = 2\pi \cdot 2/3$, 3(bottom left): $\mathbf{rot}(F_{01}) = 2\pi/3$, 4(bottom right): $\mathbf{rot}(F_{01}) = 2\pi/3$, $\mathbf{rot}(F_{010}) = 2\pi \cdot 4/5$.

Proposition 7. *If $\mathbf{rot}(F_{01}) = 2\pi(m-1)/m$ for some $m \geq 2$, then $\mathbb{G}(F)$ is a discrete group whose fundamental region has a side at the Euclidean boundary, so $(\partial\mathbb{D}, F)$ is not minimal (see Figure 2(1,2) and Figure 3(7)).*

Proof. The isometric circles $I_0, I_{010}, \dots, I_{(01)^{m-1}0} = I_1^{-1}$ all pas through the fixed point s_{10} of F_{10}. By Lemma 3, the angle between $I_{(01)^k0}$ and $I_{(01)^{k+1}0}$ is $-2\pi(m-1)/2m = \pi/m$, so the inner angle of $U(F_{(01)^k0}) \cap U(F_{(01)^{k+1}0})$ at s_{10} is either π/m or $\pi - \pi/m$. Consider first the case with $a = b$, when we have the symmetry $\psi(z) = -i\overline{z}$ with $\psi F_i \psi = F_{1-i}$ (see Figure 2(1) with $m = 2$ and $a = b = 2 - \sqrt{3}$). Then $\psi(U(F_0)) = U(F_1^{-1})$ and this implies that the inner angle of $U(F_0) \cap U(F_1^{-1})$ is π/m. As the system moves along the curve $\{(a,b) \in P : \ \mathbf{rot}(F_{01}) = 2\pi(m-1)/m\}$, this angle cannot change, because both F_0 and F_1^{-1} are hyperbolic, so I_0, I_1^{-1} cannot cross 0. Thus for all parameters a, b with $\mathbf{rot}(F_{01}) = 2\pi(m-1)/m$ the inner angle of $U(F_0) \cap U(F_1^{-1})$ is π/m. Denote by $P = U(F_0) \cap U(F_0^{-1}) \cap U(F_1) \cap U(F_1^{-1})$ and

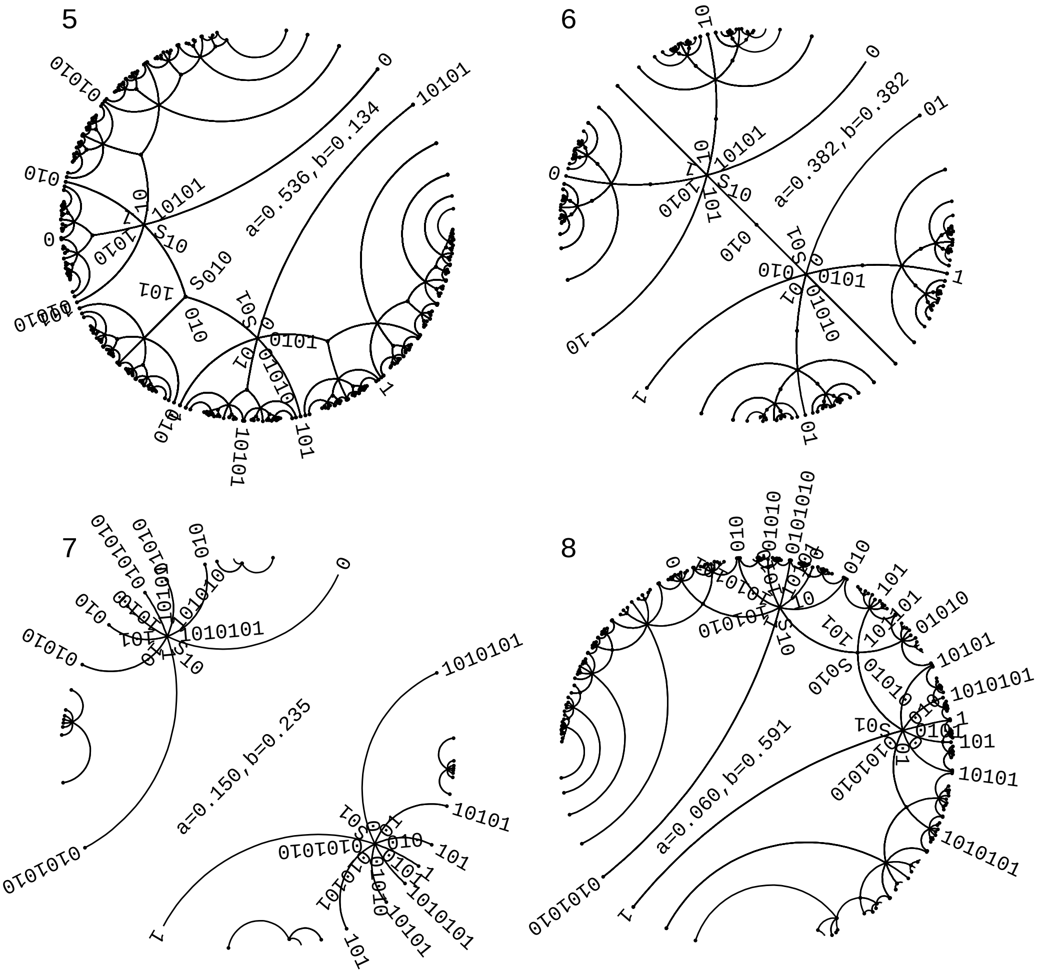

FIGURE 3. Isometric circles and Ford regions. 5(top left): $\mathbf{rot}(F_{01}) = 2\pi/3$, $\mathbf{rot}(F_{010}) = 2\pi \cdot 2/3$, 6(top right): $\mathbf{rot}(F_{01}) = 2\pi/3$, $\mathbf{rot}(F_{010}) = \pi$, 7(bottom left): $\mathbf{rot}(F_{01}) = 2\pi \cdot 3/4$, 8(bottom right): $\mathbf{rot}(F_{01}) = 2\pi/4$, $\mathbf{rot}(F_{101}) = 2\pi \cdot 3/4$.

by $\overline{P}$ its Euclidean closure. Then $\overline{P}$ is a hexagon with two sides at the Euclidean boundary. We have fixed points $s_{01} \in I_0^{-1} \cap I_1$, $s_{10} \in I_1^{-1} \cap I_0$ with $s_{10} = \varphi(s_{01}) = F_1(s_{01}) = F_0^{-1}(s_{01})$, which are the only vertices of P and form a cycle of length 2 and order m. We get

$$\begin{array}{rclrcl}
\overline{P} \cap F_0(\overline{P}) & \subseteq & I_0^{-1}, & \overline{P} \cap F_1(\overline{P}) & \subseteq & I_1^{-1}, \\
F_0(\overline{P}) \cap F_{01}(\overline{P}) & \subseteq & F_0(I_1^{-1})), & F_1(\overline{P}) \cap F_{10}(\overline{P}) & \subseteq & F_1(I_0^{-1}), \\
F_{01}(\overline{P}) \cap F_{010}(\overline{P}) & \subseteq & F_{01}(I_0^{-1}), & F_{10}(\overline{P}) \cap F_{101}(\overline{P}) & \subseteq & F_{10}(I_1^{-1}), \\
& \vdots & & & \vdots & \\
F_{(01)^{m-1}0}(\overline{P}) \cap \overline{P} & \subseteq & F_{(01)^{m-1}0}(I_1^{-1}), & F_{(10)^{m-1}1}(\overline{P}) \cap \overline{P} & \subseteq & F_{(10)^{m-1}1}(I_0^{-1}).
\end{array}$$

Since the inner angles of P, $F_0(P)$, $F_{01}(P)$, $F_{010}(P), \ldots, F_{(01)^{m-1}0}(P)$ at s_{01} are π/m, these sets are arranged clockwise around s_{01} without intersections and their closures fill out a neighborhood of s_{01}. Similarly, P, $F_1(P)$, $F_{10}(P)$, $F_{101}(P),\ldots,$ $F_{(10)^{m-1}1}(P)$ are arranged clockwise around s_{10} without intersections and their closures fill out a neighborhood of s_{10}. Thus the conditions of Poincaré theorem (see Beardon [2]) are met, so $\mathbb{G}(F)$ is a discrete group with a side at the Euclidean boundary. $\square$

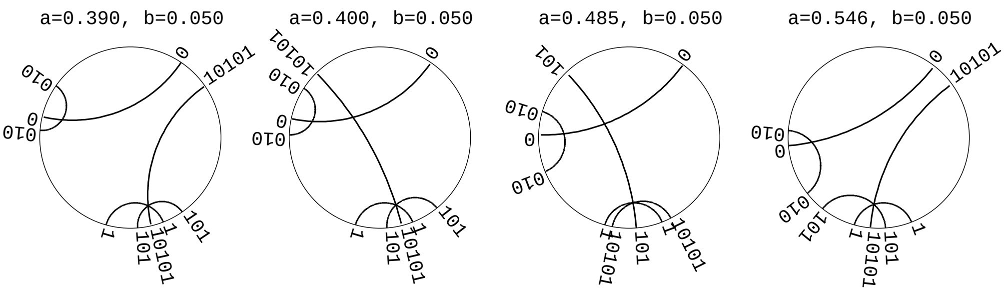

FIGURE 4. Rotation of isometric circles

Proposition 8. *If* $\mathbf{rot}(F_{01}) = 2\pi/m$ *for some* $m \geq 3$ *and* F_{01}, F_{10} *are the only elliptic transformations, then* $\mathbb{G}(F)$ *is a discrete group whose fundamental region has a side at the Euclidean boundary, and* $(\partial\mathbb{D}, F)$ *is not minimal (see Figure 2(3)).*

Proof. Assume $a > b$. The isometric circles $I_0, I_{010}, \ldots, I_{(01)^{m-1}0} = I_1^{-1}$ all pas through the fixed point s_{10} of F_{10}. The angle between $I_{(01)^k 0}$ and $I_{(01)^{k+1}0}$ is $-2\pi/2m = -\pi/m$, so the inner angle of $U(F_{(01)^k 0}) \cap U(F_{(01)^{k+1}0})$ is either π/m or $\pi - \pi/m$. Consider a path in the parameter space with small constant b and increasing a passing from $\mathbf{rot}(F_{01}) = 2\pi(m-1)/m$ to $\mathbf{rot}(F_{01}) = 2\pi/m$ (see Figure 4). The angle of I_{010} with I_0 goes from π/m through $\pi/2$ to $-\pi/m$ and the inner angle of $U(F_0) \cap U(F_{010})$ goes from $\pi - \pi/m$ through $\pi/2$ to π/m. From the symmetry φ we get that the inner angle of $U(F_0^{-1}) \cap U(F_{010}^{-1})$ is π/m as well. We show that $P = U(F_0) \cap U(F_0^{-1}) \cap U(F_{010}) \cap U(F_{010}^{-1})$ is a fundamental region. For $m = 3$ and $a > b$ we have $F_{010}^{-1} = F_{101}$ and we get

$$\begin{array}{rclrcl}
\overline{P} \cap F_0(\overline{P}) & \subseteq & I_0^{-1}, & \overline{P} \cap F_{101}(\overline{P}) & \subseteq & I_{010}, \\
F_0(\overline{P}) \cap F_{0101}(\overline{P}) & \subseteq & F_0(I_{010}), & F_{101}(\overline{P}) \cap F_{1010}(\overline{P}) & \subseteq & F_{101}(I_0^{-1}), \\
F_{0101}(\overline{P}) \cap F_{01010}(\overline{P}) & \subseteq & F_{0101}(I_0^{-1}), & F_{1010}(\overline{P}) \cap F_1(\overline{P}) & \subseteq & F_{1010}(I_{010}), \\
F_{01010}(\overline{P}) \cap F_{01}(\overline{P}) & \subseteq & F_{01010}(I_{010}), & F_1(\overline{P}) \cap F_{10}(\overline{P}) & \subseteq & F_1(I_0^{-1}), \\
F_{01}(\overline{P}) \cap F_{010}(\overline{P}) & \subseteq & F_{01}(I_0^{-1}), & F_{10}(\overline{P}) \cap F_{10101}(\overline{P}) & \subseteq & F_{10}(I_{010}), \\
F_{010}(\overline{P}) \cap \overline{P} & \subseteq & F_{010}(I_{010}), & F_{10101}(\overline{P}) \cap \overline{P} & \subseteq & F_{10101}(I_0^{-1}).
\end{array}$$

For $m \geq 3$ and $a > b$ we have $F_{010}^{-1} = F_{(10)^{m-2}1}$ and we get

$$\begin{array}{rcl}
\overline{P} \cap F_0(\overline{P}) & \subseteq & I_0^{-1}, \\
\overline{P} \cap F_{(10)^{m-2}1}(\overline{P}) & \subseteq & I_{010}, \\
F_0(\overline{P}) \cap F_{(01)^{m-1}}(\overline{P}) & \subseteq & F_0(I_{010}), \\
F_{(10)^{m-2}1}(\overline{P}) \cap F_{(10)^{m-1}}(\overline{P}) & \subseteq & F_{(10)^{m-2}}(I_0^{-1}), \\
F_{(01)^{m-1}}(\overline{P}) \cap F_{(01)^{m-1}0}(\overline{P}) & \subseteq & F_{(01)^{m-2}0}(I_0^{-1}), \\
F_{(10)^{m-1}}(\overline{P}) \cap F_{(10)^{m-3}1}(\overline{P}) & \subseteq & F_{(10)^{m-2}1}(I_{010}), \\
 & \vdots & \\
F_{010}(\overline{P}) \cap \overline{P} & \subseteq & F_{010}(I_{010}), \\
F_{(10)^{m-1}1}(\overline{P}) \cap \overline{P} & \subseteq & F_{(10)^{m-1}1}(I_0^{-1}),
\end{array}$$

Thus $\mathbb{G}(F)$ is a discrete group with a side at the Euclidean boundary. $\square$

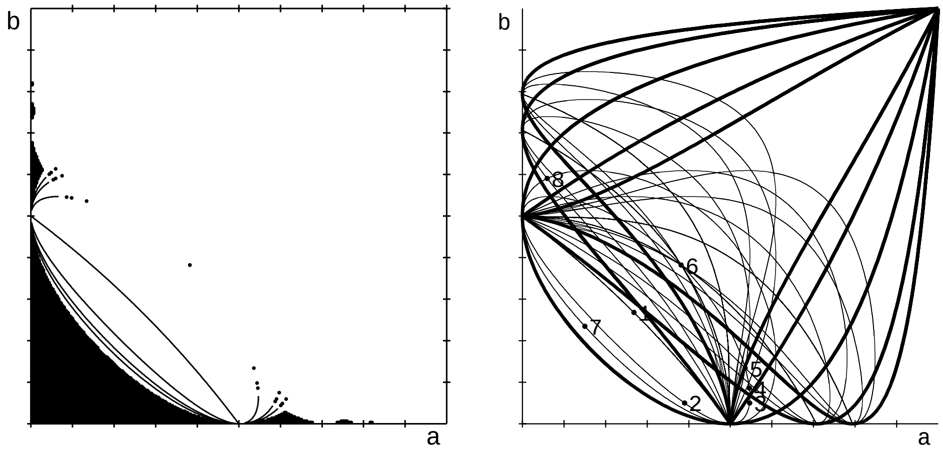

FIGURE 5. Parameter space: Minimal systems are in the white region (left), Curves $|\mathbf{tr}(F_{0^n1})| = 2$ and $|\mathbf{tr}(F_{01^n})| = 2$ for $n = 1, 2, 3$ (thick), curves $\mathbf{rot}(F_u) = \pm 2\pi/m$ (thin), positions of the systems $1, 2, \ldots, 8$ from Figures 2, 3 (right).

Proposition 9. *If $\mathbf{rot}(F_{01}) = 2\pi/m$ for some $m \geq 3$, and $\mathbf{rot}(F_{001}) = 2\pi(n-1)/n$ for some $n \geq 3$, then $\mathbb{G}(F)$ is a discrete group with a side at the Euclidean boundary and $(\partial\mathbb{D}, F)$ is not minimal (see Figure 2(4) and Figure 3(5,8)).*

Proof. Assume $a > b$. The inner angles of both $U(F_0) \cap U(F_{010})$ and $U(F_{010}^{-1}) \cap U(F_0^{-1})$ are again π/m, but F_{010} is now elliptic with $\mathbf{rot}(F_{010}) = 2\pi(n-1)/n$ and $F_{010}^{-1} = F_{(010)^{n-1}} = F_{(10)^{m-2}1}$. The fundamental polygon $P = U(F_0) \cap U(F_0^{-1}) \cap U(F_{010}) \cap U(F_{010}^{-1})$ has one more vertex $s_{010} \in U(F_{010}) \cap U(F_{010}^{-1})$ which forms a cycle of order n. The angle between $I_{(010)^k}$ and $I_{(010)^{k+1}}$ is π/n, so the angle between I_{010} and $I_{(010)^{n-1}}$ is $\pi(n-2)/n$. The inner angle of $U(F_{010}) \cap U(F_{(010)^{n-1}})$ is $2\pi/n$. We have

$$\begin{array}{rcl} \overline{P} \cap F_{010}(\overline{P}) & \subseteq & I_{010}^{-1}, \\ F_{010}(\overline{P}) \cap F_{(010)^2}(\overline{P}) & \subseteq & F_{010}(I_{010}^{-1}), \\ & \vdots & \\ F_{(010)^{n-1}}(\overline{P}) \cap \overline{P} & \subseteq & F_{(010)^{n-1}}(I_{010}^{-1}) \end{array}$$

Thus P, $F_{010}(P), \ldots, F_{010}^{n-1}(P)$ are arranged clockwise around s_{010}, so P is a fundamental domain with a side at the Euclidean boundary. $\square$

Proposition 10. *If $\mathbf{rot}(F_{01}) = 2\pi/3$ and $a = b$, then $\mathbb{G}(F)$ is a discrete group whose fundamental polygon has a side at the Euclidean boundary and $(\partial\mathbb{D}, F)$ is not minimal (see Figure 3(6)).*

Proof. The condition $\mathbf{tr}(F_{01}) = (2a-(1-a)^2)/2a = \cos(\pi/3)$ yields $a^2-3a+1=0$ with the solution $a = (3-\sqrt{5})/2$. Moreover we get

$$\begin{aligned}\mathbf{rot}(F_{001}) = \mathbf{rot}(F_{011}) &= 2\arccos\frac{a^2+a-(1-a^2)(1-a)}{2a\sqrt{a}} = \pi \\ \mathbf{rot}(F_{0001}) = \mathbf{rot}(F_{0111}) &= 2\arccos\frac{a^3+a-(1-a^3)(1-a)}{2a^2} = 4\pi/3\end{aligned}$$

Since $s_{010} = s_{101} = 0$, the Ford fundamental region cannot be constructed. Nevertheless we get a fundamental region $P = U(F_0) \cap U(F_0^{-1}) \cap Q$, where $Q = \{a+bi : a+b > 0\}$ is one of the half-planes of the line joining s_{01} with s_{10}. The vertices of $\overline{P}$ are s_{01}, s_{10}, s_{010}. The point s_{010} forms a cycle of order 2 and $\overline{P}$ with $F_{010}(\overline{P})$ fill a neighbourhood of s_{010} without overlapping. Similarly, the closures of P, $F_{10101}(P)$, $F_{10}(P)$, $F_1(P)$, $F_{1010}(P)$, $F_{101}(P)$ fill the neighbourhood of s_{10} in counter-clockwise order and the closures of P, $F_{010}(P)$, $F_{01}(P)$, $F_{01010}(P)$, $F_{0101}(P)$, $F_0(P)$ fill the neighbourhood of s_{01} in counter-clockwise order, so $\mathbb{G}(F)$ is a discrete group with a side at the Euclidean boundary. □

For other rational angles, minimality depends on parameters in a more complicated way. For example if $\mathbf{rot}(F_{01}) = 2\pi \cdot 4/7$ and $0.17 < a < 0.3$, then $V(F_{0101})$, $V(F_{01010})$, $V(F_{1010})$ and $V(F_{10101})$ cover $\partial\mathbb{D}$, so $\Lambda_F = \partial\mathbb{D}$ and $(\partial\mathbb{D}, F)$ is minimal.

Acknowledgments. The research was supported by the Research Program CTS MSM 0021620845 and by the Czech Science Foundation research project GAČR 201/09/0854. A part of the paper has been written during my stay at the Laboratoire d'Informatique Fondamentale de Marseille (LIF).

REFERENCES

[1] M. F. Barnsley. *Fractals everywhere*. Morgan Kaufmann Pub., 1993.
[2] A. F. Beardon. *The geometry of discrete groups*. Springer-Verlag, Berlin, 1995.
[3] G. A. Edgar. *Measure, Topology and Fractal Geometry*. Springer-verlag, Berlin, 1990.
[4] S. Katok. *Fuchsian Groups*. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago, 1992.
[5] P. Kůrka. Iterative systems of real Möbius transformations. *Discrete and Continuous Dynamical Systems*, 25(2):567–574, 2009.
[6] P. Kůrka and A.Kazda. Möbius number systems based on interval covers. *Nonlinearity*, 23:1031–1046, 2010.

E-mail address: kurka@cts.cuni.cz