Möbius number systems based on interval covers

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Abstract. Given a finite alphabet A, a system of real orientation-preserving Möbius transformations $(F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{a\in A}$, a subshift $\Sigma \subseteq A^{\mathbb{N}}$, and an interval cover $\mathcal{W} = \{W_a :\ a \in A\}$ of $\overline{\mathbb{R}}$, we consider the expansion subshift $\Sigma_{\mathcal{W}} \subseteq \Sigma$ of all expansions of real numbers with respect to $\mathcal{W}$. If the expansion quotient $\mathbf{Q}(\Sigma, \mathcal{W})$ is greater than 1 then there exists a continuous and surjective symbolic mapping $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ and we say that $(F, \Sigma_{\mathcal{W}})$ is a Möbius number system. We apply our theory to the system of binary continued fractions which is a combination of the binary signed system with the continued fractions, and to the binary square system whose transformations have stable fixed points -1, 0, 1 and ∞.

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1. Introduction

Möbius number systems introduced in Kůrka [5] and [8] are based on iterative systems of Möbius transformations $(F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{a\in A}$ indexed by a finite alphabet A, acting on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The convergence space $\mathbb{X}_F$ consists of all infinite words $u \in A^{\mathbb{N}}$ such that $F_{u_0} \cdots F_{u_n}(z)$ converge to a real number $\Phi(u) \in \overline{\mathbb{R}}$ whenever z is a complex number with positive imaginary part. Since the symbolic representation $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ is usually not continuous, we search for a subshift $\Sigma \subseteq \mathbb{X}_F$ such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective. In this case we say that (F, Σ) is a Möbius number system.

In Kůrka [8] we have developed a theory of Möbius number systems with sofic subshifts. In the present paper we consider expansion subshifts which are obtained when we expand real numbers $x \in \overline{\mathbb{R}}$ into symbolic sequences $u \in A^{\mathbb{N}}$ with $\Phi(u) = x$. The expansion procedure uses an interval cover $\mathcal{W} = \{W_a :\ a \in A\}$ of $\overline{\mathbb{R}}$ and tests conditions $x \in W_{u_0}$, $F_{u_0}^{-1}(x) \in W_{u_1}$, $F_{u_0u_1}^{-1}(x) \in W_{u_2}$ etc. We limit the expansions to a predefined subshift $\Sigma \subseteq A^{\mathbb{N}}$ which may be the full shift or the subshift which forbids identities u with $F_{u_0} \cdots F_{u_{n-1}} = \mathrm{Id}$. The **expansion subshift** $\Sigma_{\mathcal{W}}$ consists of all expansions which belong to Σ and respect $\mathcal{W}$. We define the expansion quotient $\mathbf{Q}(\Sigma, \mathcal{W})$ and show that if $\mathbf{Q}(\Sigma, \mathcal{W}) > 1$, then $(F, \Sigma_{\mathcal{W}})$ is a Möbius number system (Corollary 12). There is a converse (Theorem 13) which uses an additional condition on images of cylinders of finite words.

While the expansion subshifts $\Sigma_{\mathcal{W}}$ are in general not sofic, the arithmetical algorithms which work with them are simpler than in the sofic case. If all transformations F_a have rational entries, and if their expansion intervals W_a overlap and have rational endpoints, then there exist arithmetical algorithms which generalize the arithmetical algorithms for exact real computation described in Gosper [1], Vuillemin [13], or Potts et al. [11]. We present algorithms for expansions of rational or algebraic numbers and for computation of rational functions and fractional bilinear functions such as the sum and product. Using the continued fractions of Gauss (see Wall [14]), algorithms for many transcendental functions can be obtained similarly as in Potts [10].

We apply our theory to several examples. We show that both the binary signed system and the system of continued fractions can be conceived as Möbius number systems. The system of binary continued fractions considered in Kůrka [8] is a combination of these two classical systems and corresponds to the continued logarithms of Gosper [1]. Then we treat polygonal number systems considered in Kůrka [5], in particular the binary square system consisting of transformations with stable fixed points -1, 0, 1 and ∞. Some of the present results have appeared in Kůrka [7] and Kazda [3].

2. Möbius transformations

The **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ can be regarded as a projective space, i.e., the space of one-dimensional subspaces of the two-dimensional vector space. On $\overline{\mathbb{R}}$ we have homogenous coordinates $x = (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with equality $x = y$ iff $x_0y_1 = x_1y_0$. We regard $x \in \overline{\mathbb{R}}$ as a column vector, and write it usually as $x = x_0/x_1$, for example $\infty = 1/0$. The open and closed intervals with endpoints $a, b \in \overline{\mathbb{R}}$ are defined by

$$\begin{aligned}(a,b) &= \{x \in \overline{\mathbb{R}} :\ (a_0x_1 - a_1x_0)(x_0b_1 - x_1b_0)(b_0a_1 - b_1a_0) > 0\},\\ [a,b] &= \{x \in \overline{\mathbb{R}} :\ (a_0x_1 - a_1x_0)(x_0b_1 - x_1b_0)(b_0a_1 - b_1a_0) \geq 0\}.\end{aligned}$$

For distinct $a, b \in \mathbb{R}$, we have $(a,b) = \{x \in \mathbb{R} :\ a < x \text{ or } x < b\} \cup \{\infty\}$ if $a > b$ and $(a,b) = \{x \in \mathbb{R} :\ a < x < b\}$ if $a < b$. Moreover, $(a,a) = \emptyset$ and $[a,a] = \overline{\mathbb{R}}$.

A real **orientation-preserving Möbius transformation** (MT) is a self-map of $\overline{\mathbb{R}}$ of the form $M_{(a,b,c,d)}(x) = (ax+b)/(cx+d) = (ax_0+bx_1)/(cx_0+dx_1)$, where $a,b,c,d \in \mathbb{R}$ and $ad-bc > 0$. An MT acts also on the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and on the **upper half-plane** $\mathbb{U} = \{z \in \mathbb{C} :\ \Im(z) > 0\}$: if $z \in \mathbb{U}$ then $M(z) \in \mathbb{U}$. The map $\mathbf{d}(z) = (iz+1)/(z+i)$ maps $\mathbb{U}$ conformally to the **unit disc** $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$ and $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} :\ |z| = 1\}$. Define the **circle distance** on $\overline{\mathbb{R}}$ by

$$\varrho(x,y) = 2 \arcsin \frac{|x_0y_1 - x_1y_0|}{\sqrt{(x_0^2+x_1^2)(y_0^2+y_1^2)}}$$

which is the length of the shortest arc joining $\mathbf{d}(x)$ and $\mathbf{d}(y)$ in $\partial\mathbb{D}$. The length of a closed interval $B_r(a) = \{x \in \overline{\mathbb{R}} :\ \varrho(x,a) \leq r\}$ is $||B_r(a)|| = \min\{2r, 2\pi\}$. The length $||J||$ of a set $J \subseteq \overline{\mathbb{R}}$ is the length of the shortest interval which contains J. For $x \in \overline{\mathbb{R}}$ and $0 < \varepsilon < 2\pi$ denote by $x \oplus \varepsilon, x \ominus \varepsilon \in \overline{\mathbb{R}}$ the unique points for which $||[x, x\oplus\varepsilon]|| = \varepsilon$ and $||[x\ominus\varepsilon, x]|| = \varepsilon$. On the closed disc $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ we get **disc Möbius transformations** $\widehat{M} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ defined by

$$\widehat{M}_{(a,b,c,d)}(z) = \mathbf{d} \circ M_{(a,b,c,d)} \circ \mathbf{d}^{-1}(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. Define the **norm** of a Möbius transformation $M = M_{(a,b,c,d)}$ by $||M|| := (a^2+b^2+c^2+d^2)/(ad-bc)$. We have

$||M|| \geq 2$, and $||M|| = 2$ iff M is a **rotation**, i.e., if $\widehat{M}(z) = \widehat{R}_\alpha(z) = e^{i\alpha}z$ for some $\alpha \in \mathbb{R}$.

Definition 1 *The* **circle derivation** $M^\bullet : \overline{\mathbb{R}} \to (0, \infty)$ *and the* **expansion interval** $\mathbf{V}(M)$ *of* $M = M_{(a,b,c,d)}$ *are defined by*

$$M^\bullet(x) := |\widehat{M}'(\mathbf{d}(x))| = \frac{(ad-bc)(x_0^2 + x_1^2)}{(ax_0 + bx_1)^2 + (cx_0 + dx_1)^2},$$
$$\mathbf{V}(M) := \{x \in \overline{\mathbb{R}} : \ (M^{-1})^\bullet(x) > 1\}.$$

We have $M^\bullet(x) = \lim_{y \to x} \varrho(M(y), M(x))/\varrho(x, y)$, so the circle derivation is the derivation with respect to ϱ, and $(MN)^\bullet(x) = M^\bullet(N(x)) \cdot N^\bullet(x)$. If M is a rotation, then $\mathbf{V}(M) = \emptyset$, otherwise $\mathbf{V}(M)$ is a nonempty open interval. The expansion interval $\mathbf{V}(M)$ is related to the value $\widehat{M}(0) \in \mathbb{D}$ of the corresponding disc MT. If $\mathbf{V}(M) = (l, r)$, then $\widehat{M}(0)$ is the middle of the line joining $\mathbf{d}(l)$ and $\mathbf{d}(r)$. This follows from Lemma 7 of Kazda [2]. Further usefull relations between $||M||$, $\widehat{M}(0)$ and $M^\bullet(x)$ have been given in Kůrka [8]: $|\widehat{M}(0)|^2 = (||M|| - 2)/(||M|| + 2)$,

$$\min\{M^\bullet(x) : \ x \in \overline{\mathbb{R}}\} = \tfrac{1}{2}(||M|| - \sqrt{||M||^2 - 4}),$$
$$\max\{M^\bullet(x) : \ x \in \overline{\mathbb{R}}\} = \tfrac{1}{2}(||M|| + \sqrt{||M||^2 - 4}).$$

3. Möbius number systems

For a finite alphabet A denote by $A^* := \bigcup_{m \geq 0} A^m$ the set of finite words and by $A^+ := A^* \setminus \{\lambda\}$ the set of finite non-empty words. The length of a word $u = u_0 \dots u_{m-1} \in A^m$ is $|u| := m$. We denote by $A^{\mathbb{N}}$ the Cantor space of infinite words $u = u_0 u_1 \dots$ equipped with metric $d(u, v) := 2^{-k}$, where $k = \min\{i \geq 0 : u_i \neq v_i\}$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$. We say that $v \in A^*$ is a subword of $u \in A^* \cup A^{\mathbb{N}}$ and write $v \sqsubseteq u$, if $v = u_{[i,j)}$ for some $0 \leq i \leq j \leq |u|$, where we put $|u| = \infty$ for $u \in A^{\mathbb{N}}$. Given $u \in A^n$, $v \in A^m$ with $n \geq 0$, $m > 0$, denote by $u.v \in A^{\mathbb{N}}$ the periodic word with preperiod u and period v defined by $(u.v)_i = u_i$ for $i < n$ and $(u.v)_{n+km+i} = v_i$ for $i < m$ and for all $k \geq 0$. Denote by $[u]$ the **cylinder** $\{v \in A^{\mathbb{N}} : \ v_{[0,|u|)} = u\}$ of $u \in A^*$.

The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty set $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. For a subshift Σ there exists a set $D \subseteq A^+$ of **forbidden words** such that $\Sigma = \mathcal{S}(D) := \{x \in A^{\mathbb{N}} : \ \forall u \sqsubseteq x, u \not\in D\}$. We denote by $\mathcal{S} = A^{\mathbb{N}}$ the full shift with the alphabet A. A subshift is uniquely determined by its **language** $\mathcal{L}(\Sigma) := \{u \in A^* : \ \exists x \in \Sigma, u \sqsubseteq x\}$. We denote by $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n$. A subshift is of **finite type** (SFT), if the set D of forbidden words is finite. If $D \subseteq A^n$, then we say that the **order** of Σ is at most n. A subshift is **sofic**, if its language is regular (Lind and Marcus [9]). An **iterative system** is a continuous map $F : A^* \times X \to X$, or a family of continuous maps $(F_u : X \to X)_{u \in A^*}$ satisfying $F_{uv} = F_u F_v$, and $F_\lambda = \mathrm{Id}$. It is determined by generators $(F_a : X \to X)_{a \in A}$.

Definition 2 *We say that* $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$*, is a* **Möbius iterative system***, if all* $F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *are orientation-preserving Möbius transformations. The* **convergence space** $\mathbb{X}_F \subseteq A^{\mathbb{N}}$ *and the* **symbolic representation** $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ *are defined by*

$$\mathbb{X}_F := \{u \in A^{\mathbb{N}} : \ \lim_{n \to \infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\},$$
$$\Phi(u) := \lim_{n \to \infty} F_{u_{[0,n)}}(i),$$

where $i \in \mathbb{U}$ is the imaginary unit. If $\Sigma \subseteq \mathbb{X}_F$ is a subshift such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective, then we say that (F, Σ) is a **Möbius number system***. We say that a Möbius number system is* **redundant***, if for every continuous map $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $\Phi f = g\Phi$.*

If $u \in \mathbb{X}_F$ then $\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(z)$ for every $z \in \mathbb{U}$ (see Kazda [2]). For $v \in A^+$, $u \in A^{\mathbb{N}}$ we have $vu \in \mathbb{X}_F$ iff $u \in \mathbb{X}_F$, and then $\Phi(vu) = F_v(\Phi(u))$. Since Φ is usually not continuous on $\mathbb{X}_F$, we search for a subshift $\Sigma \subseteq \mathbb{X}_F$ such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective, so that we get a symbolic representation of $\overline{\mathbb{R}}$ by a Cantor space Σ. If the system is redundant, then continuous functions $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ can be lifted to their continuous symbolic extensions $f : \Sigma \to \Sigma$. The continuity of a function $f : \Sigma \to \Sigma$ is necessary for its computability (see e.g. Weihrauch [15]). We show that the system is redundant if the images of its cylinders overlap (Theorem 11). We give some sufficient and some necessary conditions for $\Phi(u) = x$, which will be needed in the proofs.

Lemma 3 (Kůrka [8]) *Let $u \in A^{\mathbb{N}}$ and $x \in \overline{\mathbb{R}}$. Then $\Phi(u) = x$ iff there exists $c > 0$ and a sequence of intervals $I_m \ni x$ such that $\liminf_{n\to\infty} ||F^{-1}_{u_{[0,n)}}(I_m)|| > c$ for each m, and $\lim_{m\to\infty} ||I_m|| = 0$.*

Lemma 4 *Let $u \in A^{\mathbb{N}}$ and $x \in \overline{\mathbb{R}}$.*

(1) If $\lim_{n\to\infty}(F^{-1}_{u_{[0,n)}})^\bullet(x) = \infty$, then $\Phi(u) = x$.

(2) If $\Phi(u) = x$, then for every $r < 1$ there exists n such that $(F^{-1}_{u_{[0,n)}})^\bullet(x) \geq r^n$.

Proof: (1) Denote $z = \mathbf{d}(x)$, $\widehat{M}_n(z) = \widehat{F}_{u_{[0,n)}}(z) = (\alpha_n z + \beta_n)/(\overline{\beta_n} z + \overline{\alpha_n})$, where $|\alpha_n|^2 - |\beta_n|^2 = 1$, so $(M_n^{-1})^\bullet(x) = |(\widehat{M}_n^{-1})'(z)| = |\alpha_n - \overline{\beta_n} z|^{-2}$. If $\lim_{n\to\infty}(M_n^{-1})^\bullet(z) = \infty$ then $\lim_{n\to\infty} |\alpha_n - \overline{\beta_n} z| = 0$. Since $|\alpha_n| > 1$ for n sufficiently large, there exists $0 < r < 1$ such that $|\beta_n| > r$ for all sufficiently large n. It follows

$$\lim_{n\to\infty} \frac{\alpha_n}{\overline{\beta_n}} = z = \frac{1}{\overline{z}} = \lim_{n\to\infty} \frac{\beta_n}{\overline{\alpha_n}} = \lim_{n\to\infty} \widehat{M}_n(0) \;\Rightarrow\; \Phi(u) = x.$$

(2) Assume by contradiction that there exists $r < 1$ such that $(F^{-1}_{u_{[0,n)}})^\bullet(x) < r^n$ for all n and choose $s < 1$ with $r < s$. Since $\ln(F_a^{-1})^\bullet(x)$ is uniformly continuous on $\overline{\mathbb{R}}$, there exists $\delta > 0$ such that if $\varrho(x, y) < \delta$, then $(F_a^{-1})^\bullet(y) < (F_a^{-1})^\bullet(x) \cdot s/r$ for each $a \in A$. Denote by $I = (x \ominus \delta, x \oplus \delta)$. For $y \in I$ denote by $y_n = F^{-1}_{u_{[0,n)}}(y)$, $x_n = F^{-1}_{u_{[0,n)}}(x)$. We prove by induction that $(F^{-1}_{u_{[0,n)}})^\bullet(y) < s^n$ and $\varrho(y_n, x_n) < \delta$. If the condition holds for all $k \leq n$, then

$$\begin{aligned}
(F^{-1}_{u_{[0,n+1)}})^\bullet(y) &= (F^{-1}_{u_0})^\bullet(y_0) \cdots (F^{-1}_{u_n})^\bullet(y_n) \\
&< (F^{-1}_{u_0})^\bullet(x_0) \cdots (F^{-1}_{u_n})^\bullet(x_n) \cdot (s/r)^{n+1} \\
&= (F^{-1}_{u_{[0,n+1)}})^\bullet(x) \cdot (s/r)^{n+1} \leq s^{n+1}.
\end{aligned}$$

Since $F^{-1}_{u_{[0,n+1)}}$ is a contraction on I, we get $\varrho(x_{n+1}, y_{n+1}) < \delta$. Since each $F^{-1}_{u_{[0,n)}}$ is a contraction on I, we cannot have $\Phi(u) = x$ by Lemma 3. □

4. Interval systems

We define the expansion subshift of all expansions of real numbers which are chosen from a predefined subshift Σ and respect an interval cover or almost-cover $\{W_a : a \in A\}$ of $\overline{\mathbb{R}}$. A natural choice for Σ is either the full shift $\mathcal{S} = A^{\mathbb{N}}$ or the **subshift with forbidden identities** $D = \{u \in A^+ : F_u = \mathrm{Id}\}$.

Definition 5 *Let $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and $\Sigma \subseteq A^{\mathbb{N}}$ a subshift.*

(1) We say that $\mathcal{W} = \{W_u : u \in \mathcal{L}(\Sigma)\}$ is an **interval system** *for F and Σ, if each $W_u \subseteq \overline{\mathbb{R}}$ is a finite union of open intervals, $W_u = \overline{\mathbb{R}}$ iff $u = \lambda$, and $W_{uv} = W_u \cap F_u(W_v)$ for each $uv \in \mathcal{L}(\Sigma)$.*

(2) We say that $\mathcal{W}$ is **almost-compatible** *with F and Σ if $\overline{W_u} = \bigcup\{\overline{W_{ua}} : a \in A, ua \in \mathcal{L}(\Sigma)\}$ for each $u \in \mathcal{L}(\Sigma)$.*

(3) We say that $\mathcal{W}$ is **compatible** *with F and Σ if $W_u = \bigcup\{W_{ua} : a \in A, ua \in \mathcal{L}(\Sigma)\}$ for each $u \in \mathcal{L}(\Sigma)$.*

(4) Denote by $\Sigma_{\mathcal{W}} := \{u \in \Sigma : \forall n, W_{u_{[0,n)}} \neq \emptyset\}$ the **expansion subshift** *of Σ and $\mathcal{W}$.*

(5) Denote by $\mathcal{W}^n(\Sigma) := \{W_u : u \in \mathcal{L}^n(\Sigma_{\mathcal{W}})\}$.

(6) Denote by $\ell(\mathcal{W}) := \sup\{l > 0 : (\forall I \subseteq \overline{\mathbb{R}})(||I|| < l \Rightarrow \exists a \in A, I \subseteq W_a)\}$ the **Lebesgue number** *of $\mathcal{W}$.*

We have $W_u = W_{u_0} \cap F_{u_0}(W_{u_1}) \cap F_{u_{[0,2)}}(W_{u_2}) \cap \cdots \cap F_{u_{[0,n)}}(W_{u_n})$ for each $u \in \mathcal{L}^{n+1}(\Sigma)$, so $\mathcal{W}$ is determined by $\mathcal{W}^1 = \{W_a : a \in A\}$. In the examples, W_a are usually open intervals, but their intersections need not be always intervals. A compatible interval system expresses the process of expansion of real numbers: Given $x \in \overline{\mathbb{R}}$, we construct nondeterministically its expansion $u \in \Sigma_{\mathcal{W}}$ as follows: Choose u_0 with $x \in W_{u_0}$, choose u_1 with $F_{u_0}^{-1}(x) \in W_{u_1}$ and $u_0 u_1 \in \mathcal{L}(\Sigma)$, choose u_2 with $F_{u_0 u_1}^{-1}(x) \in W_{u_2}$ and $u_0 u_1 u_2 \in \mathcal{L}(\Sigma)$, etc. Then $x \in W_{u_{[0,n)}}$ for each n. In Theorem 9 we show that if $W_a = \mathbf{V}(F_a)$ are the expansion intervals of F_a (Definition 1) and if u is the expansion of x, then $\Phi(u) = x$.

If $\mathcal{W}$ is almost-compatible with F and Σ then $\mathcal{W}^1(\Sigma)$ is an **almost-cover** of $\overline{\mathbb{R}}$, i.e., $\bigcup_a \overline{W_a} = \overline{\mathbb{R}}$. If $\mathcal{W}$ is compatible with F and Σ then $\mathcal{W}^1(\Sigma)$ is a **cover** of $W_\lambda = \overline{\mathbb{R}}$, i.e., $\bigcup_a W_a = \overline{\mathbb{R}}$. If $\Sigma = \mathcal{S} = A^{\mathbb{N}}$ is the full shift, then we have equivalencies: $\mathcal{W}$ is almost-compatible iff $\mathcal{W}^1(\mathcal{S})$ is an almost-cover, and $\mathcal{W}$ is compatible iff $\mathcal{W}^1(\mathcal{S})$ is a cover. This is a special case of Proposition 6.

Proposition 6 *Let Σ be a SFT of order at most $n+1$ and let $\mathcal{W}$ be an interval system.*

(1) If $\overline{W_u} = \bigcup\{\overline{W_{ua}} : a \in A, ua \in \mathcal{L}(\Sigma)\}$ holds for all $u \in \mathcal{L}(\Sigma)$ with $|u| \le n$, then $\mathcal{W}$ is almost-compatible with F and Σ.

(2) If $W_u = \bigcup\{W_{ua} : a \in A, ua \in \mathcal{L}(\Sigma)\}$ holds for all $u \in \mathcal{L}(\Sigma)$ with $|u| \le n$, then $\mathcal{W}$ is compatible with F and Σ.

Proof: (1) We use the fact that if U, V are finite unions of open intervals, then $U \cap \overline{V} \subseteq \overline{U \cap V}$. Let $v, u \in A^+$ and $|u| = n$. If $x \in \overline{W_{vu}}$, then for some $y \neq x$ either

$(x,y) \subseteq W_{vu}$ or $(y,x) \subseteq W_{vu}$. In the former case we have

$$\begin{aligned}(x,y) \subseteq W_v \cap F_v(W_u) &\subseteq W_v \cap F_v\left(\bigcup\{\overline{W_{ua}}:\ a \in A, ua \in \mathcal{L}(\Sigma)\}\right)\\ &= \bigcup\{W_v \cap F_v(\overline{W_{ua}}):\ a \in A, vua \in \mathcal{L}(\Sigma)\}\\ &\subseteq \bigcup\{\overline{W_v \cap F_v(W_{ua})}:\ a \in A, vua \in \mathcal{L}(\Sigma)\}\\ &= \bigcup\{\overline{W_{vua}}:\ a \in A, vua \in \mathcal{L}(\Sigma)\},\end{aligned}$$

so $x \in \bigcup\{\overline{W_{vua}}:\ a \in A, vua \in \mathcal{L}(\Sigma)\}$. The proof of (2) is analogous. □

Proposition 7 *Let $\mathcal{W}$ be almost-compatible with F and Σ. Then each $\mathcal{W}^n(\Sigma)$ is an almost-cover of $\overline{\mathbb{R}}$. If $\mathcal{W}$ is compatible, then each $\mathcal{W}^n(\Sigma)$ is a cover of $\overline{\mathbb{R}}$.*

Proof: Given $x \in \overline{\mathbb{R}}$ there exists $u_0 \in A$, $y_0 \in \overline{\mathbb{R}}$ such that $(x, y_0) \subseteq W_{u_0}$. There exist $u_1 \in A$, $y_1 \in \overline{\mathbb{R}}$ with $u_{[0,1]} \in \mathcal{L}(\Sigma)$ and $(x, y_1) \subseteq W_{u_{[0,1]}}$, and we continue in this way till we get $(x, y_{n-1}) \subseteq W_{u_{[0,n)}}$, so $x \in \overline{W_{u_{[0,n)}}}$. □

Proposition 8 *Let $\mathcal{W}$ be an interval system for $\mathcal{S} = A^{\mathbb{N}}$ and assume that for each $u \in A^m$, $a \in A$ we have $W_a \cap F_a(W_u) \neq \emptyset \ \Rightarrow\ F_a(W_u) \subseteq W_a$. Then $\mathcal{S}_{\mathcal{W}}$ is a SFT of order at most $m+1$.*

Proof: Recall that a subshift Σ has order at most p, if $u \in A^{\mathbb{N}}$ belongs to Σ whenever all subwords of u of length p belong to $\mathcal{L}(\Sigma)$. Assume that $u \in A^{n+1}$, and for all $v \sqsubseteq u$ with $|v| = m+1$ we have $W_v \neq \emptyset$. Then

$$\begin{aligned}W_u &= W_{u_0} \cap F_{u_0}(W_{u_{[1,m]}}) \cap F_{u_{[0,m]}}(W_{u_{[m+1,n]}})\\ &= F_{u_0}(W_{u_{[1,m]}}) \cap F_{u_{[0,m]}}(W_{u_{[m+1,n]}})\\ &= F_{u_0}(W_{u_{[1,n]}}) = \cdots = F_{u_{[0,n-m)}}(W_{u_{[n-m,n]}})\end{aligned}$$

and $W_{u_{[n-m,n]}} \neq \emptyset$, so $W_u \neq \emptyset$. □

Theorem 9 *Let $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and assume that $\mathcal{V} = \{V_a = \mathbf{V}(F_a):\ a \in A\}$ (see Definition 1) is almost-compatible with $\mathcal{S} = A^{\mathbb{N}}$, i.e., $\bigcup_{a\in A} \overline{V_a} = \overline{\mathbb{R}}$. Then $(F, \mathcal{S}_{\mathcal{V}})$ is a Möbius number system and for each $u \in \mathcal{L}(\mathcal{S}_{\mathcal{V}})$ we have $\Phi([u] \cap \mathcal{S}_{\mathcal{V}}) = \overline{V_u}$.*

Proof: Denote by l_a, r_a the endpoints of $V_a = (l_a, r_a)$ and by $V_a^{\varepsilon} := (l_a \oplus \varepsilon, r_a \ominus \varepsilon)$ provided $2\varepsilon < ||V_a||$. Since F_a are contractions on $F_a^{-1}(V_a)$, there exists an increasing continuous function $\psi : [0, 2\pi) \to [0, 2\pi)$ such that $\psi(0) = 0$, $0 < \psi(t) < t$ for $t > 0$, and $||F_a(I)|| \le \psi(||I||)$ for each $a \in A$ and $I \subseteq F_a^{-1}(V_a)$. Given $u \in \mathcal{S}_{\mathcal{V}}$ and $m \le n$ we have $F^{-1}_{u_{[0,m]}}(V_{u_{[0,n]}}) \subseteq F^{-1}_{u_{[0,m]}} F_{u_{[0,m)}}(V_{u_m}) = F^{-1}_{u_m}(V_{u_m})$, so

$$\begin{aligned}||V_{u_{[0,n]}}|| = ||F_{u_0} F^{-1}_{u_0}(V_{u_{[0,n]}})|| &\le \psi(||F^{-1}_{u_0}(V_{u_{[0,n]}})||)\\ &= \psi(||F_{u_1} F^{-1}_{u_{[0,1]}}(V_{u_{[0,n]}})||) \le \psi^2(||F^{-1}_{u_{[0,1]}}(V_{u_{[0,n]}})||) \le \cdots\\ &\le \psi^n(||F^{-1}_{u_{[0,n]}}(V_{u_{[0,n]}})||) \le \psi^n(2\pi).\end{aligned}$$

Since $\psi(t) < t$ and the only fixed point of ψ is zero, we get $\lim_{n\to\infty} ||V_{u_{[0,n)}}|| = 0$, so there exists a unique point $x \in \bigcap_n \overline{V_{u_{[0,n)}}}$. We show that $\Phi(u) = x$. There exists

$\varepsilon > 0$ such that whenever $F_a^{-1}([l_a, l_a \oplus \varepsilon]) \cap (V_b \setminus V_b^\varepsilon) \neq \emptyset$ then $F_a^{-1}(l_a) \in \{l_b, r_b\}$ and similarly, if $F_a^{-1}([r_a \ominus \varepsilon, r_a]) \cap (V_b \setminus V_b^\varepsilon) \neq \emptyset$ then $F_a^{-1}(r_a) \in \{l_b, r_b\}$. By choosing ε small enough, we can also guarantee that $F_a^{-1}([l_a, l_a \oplus \varepsilon]) \cap V_b$ and $F_a^{-1}([r_a \ominus \varepsilon, r_a]) \cap V_b$ are intervals (as opposed to unions of two nonempty disjoint intervals) for all a, b.

Denote by $x_n := F_{u_{[0,n)}}^{-1}(x)$. Since $V_{u_{[0,n]}} \subseteq F_{u_{[0,n)}}(V_{u_n}) \subseteq F_{u_{[0,n)}}(\overline{V_{u_n}})$, we get $x \in \overline{V_{u_{[0,n]}}} \subseteq F_{u_{[0,n)}}(\overline{V_{u_n}})$, so $x_n \in \overline{V_{u_n}}$. We have

$$(F_{u_{[0,n)}}^{-1})^\bullet(x) = (F_{u_0}^{-1})^\bullet(x_0) \cdot (F_{u_1}^{-1})^\bullet(x_1) \cdots (F_{u_{n-1}}^{-1})^\bullet(x_{n-1}),$$

and each factor in this product is at least 1. If $x_n \in V_{u_n}^\varepsilon$ for an infinite number of n, then $\lim_{n\to\infty}(F_{u_{[0,n)}}^{-1})^\bullet(x) = \infty$ and $\Phi(u) = x$ by Lemma 4. Assume therefore that there exists n_0 such that $x_n \in \overline{V_{u_n}} \setminus V_{u_n}^\varepsilon$ for each $n \geq n_0$. Assume that $x_n \in [l_{u_n}, l_{u_n} \oplus \varepsilon]$ and observe that $x_{n+1} = F_{u_n}^{-1}(x_n)$. By the definition of ε, we then have that $F_{u_n}^{-1}(l_{u_n}) \in \{l_{u_{n+1}}, r_{u_{n+1}}\}$.

However, if $F_{u_n}^{-1}(l_{u_n}) = r_{u_{n+1}}$, then $x_n \notin \overline{V_{u_n} \cap F_{u_n}(V_{u_{n+1}})}$ and therefore $x \notin \overline{V_{u_{[0,n+1]}}}$ which is a contradiction. Thus $F_{u_{n+1}}^{-1}(l_{u_n}) = l_{u_{n+1}}$ and $x_n \in [l_{u_n}, l_{u_n} \oplus \varepsilon]$ for all $n > n_0$. If $I \subseteq V_a$ then $||F_a^{-1}(I)|| \geq \psi^{-1}(||I||)$. Given any interval $I_\delta = [x, x \oplus \delta]$, where $\delta > 0$, there exists $n > n_0$ such that $||F_{u_{[0,n)}}^{-1}(I_\delta)|| \geq \max\{||V_a|| : a \in A\} - \varepsilon$, so $\Phi(u) = x$ by Lemma 3. If $x_n \in [r_{u_n} \ominus \varepsilon, r_{u_n}]$ we proceed similarly. Therefore in all cases $\Phi(u) = x$ and $\mathcal{S}_\mathcal{V} \subseteq \mathbb{X}_F$.

We show $\Phi([v] \cap \mathcal{S}_\mathcal{V}) = \overline{V_v}$ for each $v \in \mathcal{L}(\mathcal{S}_\mathcal{V})$. If $u \in \mathcal{S}_\mathcal{V} \cap [v]$, then $\Phi(u) \in \overline{V_v}$ by the preceding proof, so $\Phi([v] \cap \mathcal{S}_\mathcal{V}) \subseteq \overline{V_v}$. If $x \in \overline{V_v}$, then $x_n \in \overline{V_{v_n}}$ for each $n < |v|$. If there are $j < k < |v|$ with $x_j = l_{u_j}$, $x_k = r_{u_k}$, then $x_j \notin \overline{V_{u_j} \cap F_{u_{[j,k)}}(V_{u_k})}$, which is a contradiction. Thus either $x_n \neq r_{v_n}$ for each $n < |v|$ or $x_n \neq l_{v_n}$ for each $n < |v|$. We define $u \in [v] \cap \mathcal{S}_\mathcal{V}$ by $u_n = v_n$ for $n < |v|$ and for $n \geq |v|$ by induction. If $x_k \neq r_{v_k}$ for all $k < n$ then there exists u_n such that $x_n = F_{u_{[0,n)}}^{-1}(x) \in [l_{u_n}, r_{u_n})$. Then $u \in \mathcal{S}_\mathcal{V} \cap [v]$ and $\Phi(u) = x$ so $\overline{V_v} \subseteq \Phi([v] \cap \mathcal{S}_\mathcal{V})$. This works also for $V_\lambda = \overline{\mathbb{R}}$, so $\Phi : \mathcal{S}_\mathcal{V} \to \overline{\mathbb{R}}$ is surjective. Finally, since $\lim_{n\to\infty} ||\Phi([u_{[0,n)}] \cap \mathcal{S}_\mathcal{V})|| = 0$, $\Phi : \mathcal{S}_\mathcal{V} \to \overline{\mathbb{R}}$ is continuous. □

5. Expansion quotients

The condition of Theorem 9 is not necessary for the inclusion of $\Sigma_\mathcal{W}$ in $\mathbb{X}_F$ and much larger interval covers may yield Möbius number systems as well. We obtain a better condition with the concept of the **expansion quotient** $\mathbf{Q}(\Sigma, \mathcal{W})$ of Σ and $\mathcal{W}$.

Definition 10 *If $\mathcal{W}$ is almost-compatible with F and Σ, then we define*

$$\begin{aligned} \mathbf{q}(u) &:= \inf\{(F_u^{-1})^\bullet(x) : x \in W_u\},\ u \in \mathcal{L}(\Sigma_\mathcal{W}) \\ \mathbf{Q}_n(\Sigma, \mathcal{W}) &:= \min\{\mathbf{q}(u) : u \in \mathcal{L}^n(\Sigma_\mathcal{W})\}, \\ \mathbf{Q}(\Sigma, \mathcal{W}) &:= \lim_{n\to\infty} \sqrt[n]{\mathbf{Q}_n(\Sigma, \mathcal{W})}. \end{aligned}$$

For $x \in W_{uv}$ we have $(F_{uv}^{-1})^\bullet(x) = (F_u^{-1})^\bullet(x) \cdot (F_v^{-1})^\bullet(F_u^{-1}(x)) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$, and therefore $\mathbf{q}(uv) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$. It follows $\mathbf{Q}_{n+m}(\Sigma, \mathcal{W}) \geq \mathbf{Q}_n(\Sigma, \mathcal{W}) \cdot \mathbf{Q}_m(\Sigma, \mathcal{W})$, so the limit $\mathbf{Q}(\Sigma, \mathcal{W})$ exists and $\mathbf{Q}(\Sigma, \mathcal{W}) \geq \sqrt[n]{\mathbf{Q}_n(\Sigma, \mathcal{W})}$ for each n.

Theorem 11 *Let $\mathcal{W}$ be almost-compatible with Σ, and assume that for some $n > 0$ we have $\mathbf{Q}_n(\Sigma, \mathcal{W}) \geq 1$ and no F_u with $u \in \mathcal{L}^n(\Sigma)$ is a rotation. Then $(F, \Sigma_\mathcal{W})$ is a Möbius number system and $\Phi([u] \cap \Sigma_\mathcal{W}) = \overline{W_u}$ for each $u \in \mathcal{L}(\Sigma_\mathcal{W})$. If $\mathcal{W}$ is compatible with F and Σ then $(F, \Sigma_\mathcal{W})$ is redundant.*

Proof: For each $u \in \mathcal{L}^n(\Sigma_\mathcal{W})$ we have $W_u \subseteq \mathbf{V}(F_u)$. Consider the alphabet $B = \mathcal{L}^n(\Sigma_\mathcal{W})$ and the Möbius number system $G : B^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by $G_u = F_u$. Then $\mathcal{V} = \{\mathbf{V}(F_u) : u \in B\}$ is an almost-cover, so $(G, (B^\mathbb{N})_\mathcal{V})$ is a Möbius number system by Theorem 9. Given $u \in A^\mathbb{N}$, define $\tilde{u} \in B^\mathbb{N}$ by $\tilde{u}_k = u_{[kn,(k+1)n)}$. If $u \in \Sigma_\mathcal{W}$, then $\tilde{u} \in (B^\mathbb{N})_\mathcal{V}$, so $\lim_{k\to\infty} F_{u_{[0,kn)}}(z) = \Phi_G(\tilde{u})$ for any $z \in \mathbb{U}$. In particular the condition is satisfied for each $z = F_v(i)$, where $v \in A^+$, $|v| < n$. It follows $\lim_{k\to\infty} F_{u_{[0,k)}}(i) = \Phi_G(\tilde{u})$, so $\Phi_F(u) = \Phi_G(\tilde{u})$. Thus $\Sigma_\mathcal{W} \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_\mathcal{W} \to \overline{\mathbb{R}}$ is continuous, since $\lim_{n\to\infty} ||W_{u_{[0,n)}}|| = 0$. Since each $\mathcal{W}^n(\Sigma)$ is an almost-cover, $\Phi : \Sigma_\mathcal{W} \to \overline{\mathbb{R}}$ is surjective. By Theorem 9 we get $\Phi([u] \cap \Sigma_\mathcal{W}) = \overline{W_u}$ for each $u \in \mathcal{L}(\Sigma_\mathcal{W})$.

We show that if $\mathcal{W}$ is compatible with F and Σ, then $(F, \Sigma_\mathcal{W})$ is redundant. If $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is continuous, then $g\Phi : \Sigma_\mathcal{W} \to \overline{\mathbb{R}}$ is uniformly continuous. Given $u \in \Sigma_\mathcal{W}$, we construct $v = f(u) \in \Sigma_\mathcal{W}$ by induction so that for each n there exists k_n such that $g\Phi([u_{[0,k_n)}]) \subseteq W_{v_{[0,n)}}$. If the condition holds for n, then there exists $k_{n+1} > k_n$ such that $g\Phi([u_{[0,k_{n+1})}])$ is a subset of $W_{v_{[0,n+1)}}$ for some v_n with $v_{[0,n+1)} \in \mathcal{L}(\Sigma_\mathcal{W})$. Thus $f : \Sigma_\mathcal{W} \to \Sigma_\mathcal{W}$ is continuous and $\Phi f = g\Phi$. □

Corollary 12 *If $\mathcal{W}$ is almost-compatible with F and Σ, and if $\mathbf{Q}(\Sigma, \mathcal{W}) > 1$, then $(F, \Sigma_\mathcal{W})$ is a Möbius number system and $\Phi([u] \cap \Sigma_\mathcal{W}) = \overline{W_u}$ for each $u \in \mathcal{L}(\Sigma_\mathcal{W})$.*

Proof: We get $\mathbf{Q}_n(\Sigma, \mathcal{W}) > 1$ for some n and it follows that no F_u with $u \in \mathcal{L}^n(\Sigma_\mathcal{W})$ is a rotation. □

Theorem 13 *Assume that $\mathcal{W}$ is almost-compatible with F and Σ, $(F, \Sigma_\mathcal{W})$ is a Möbius number system, and $\Phi([u] \cap \Sigma_\mathcal{W}) = \overline{W_u}$ for each $u \in \mathcal{L}(\Sigma_\mathcal{W})$. Then $\mathbf{Q}(\Sigma, \mathcal{W}) \geq 1$.*

Proof: Assume the contrary and choose r with $\mathbf{Q}(\Sigma, \mathcal{W}) < r < 1$. Then $\mathbf{Q}_n(\Sigma, \mathcal{W}) < r^n$ for n larger than some n_0. For $v \in \mathcal{L}(\Sigma_\mathcal{W})$ denote by $\alpha(v) = \mathbf{q}(v)/r^{|v|}$, and $a_n = \min\{\alpha(v) : v \in \mathcal{L}(\Sigma_\mathcal{W}), |v| \leq n\}$, so $a_n \leq \mathbf{Q}_n(\Sigma, \mathcal{W})/r^n \leq (\mathbf{Q}(\Sigma, \mathcal{W})/r)^n$. Denote by $v^{(n)} \in \mathcal{L}(\Sigma_\mathcal{W})$ the word of minimal length such that $\alpha(v^{(n)}) = a_n$ and by $x_n \in \overline{\mathbb{R}}$ the corresponding number for which the minimum is attained. We show that for each prefix u of $v^{(n)}$ we have $(F_u^{-1})^\bullet(x_n) \leq r^{|u|}$. Assume the contrary with $v^{(n)} = uw$. Then

$$\alpha(uw) = \frac{(F_{uw}^{-1})^\bullet(x_n)}{r^{|uw|}} = \frac{(F_w^{-1})^\bullet(F_u^{-1}(x_n))}{r^{|w|}} \cdot \frac{(F_u^{-1})^\bullet(x_n)}{r^{|u|}}$$

$$> \frac{(F_w^{-1})^\bullet(F_u^{-1}(x_n))}{r^{|w|}} \geq \alpha(w),$$

which is a contradiction since w is shorter than uw and therefore $\alpha(uw) \leq \alpha(w)$. Since $\Sigma_\mathcal{W}$ is compact, there exist $v \in \Sigma_\mathcal{W}$ such that $(F_{v_{[0,n)}}^{-1})^\bullet(x_m) \leq r^n$ for each $n \leq m$, and $x_m \in \overline{W_{v_{[0,m)}}}$. By compactness of $\overline{\mathbb{R}}$ there exists $x \in \overline{\mathbb{R}}$ such that $(F_{v_{[0,n)}}^{-1})^\bullet(x) \leq r^n$ for each $n \leq m$, so $\Phi(v) \neq x$ by Lemma 4. On the other hand $x \in \bigcap_n \overline{W_{v_{[0,n)}}} = \Phi([v_{[0,n)}] \cap \Sigma_\mathcal{W})$, so $x = \Phi(v)$ by the continuity of Φ, and this is a contradiction. □

If (F, Σ) is a Möbius number system and $W_a = \overline{\mathbb{R}}$ is the trivial interval system then $\Sigma_\mathcal{W} = \Sigma$ so $(F, \Sigma_\mathcal{W})$ is a Möbius number system but $\mathbf{Q}(\Sigma, \mathcal{W}) < 1$. While the trivial interval cover is excluded in Definition 5, the example shows that some condition like $\Phi([u]\cap\Sigma_{\mathcal{W}}) = \overline{W_u}$ is necessary in Theorem 13.

There are some constraints on expansion subshifts of Möbius number systems. For example, no full shift is an expansion subshift of any Möbius number systems (see Kazda [3]).

6. Arithmetical algorithms

In arithmetical algorithms we work with the extended rational numbers $\overline{\mathbb{Q}} = \mathbb{Q}\cup\{\infty\}$ with homogenous integer coordinates $x = x_0/x_1 \in \mathbb{Z}^2 \setminus \{0/0\}$. Denote by $\mathcal{I}$ the set of closed intervals $I = [I_0, I_1]$ with endpoints $I_0 \neq I_1$ in $\overline{\mathbb{Q}}$. Denote by $\mathcal{M}_1$ the set of MT $M = M_{(a,b,c,d)}$ whose coefficients $a, b, c, d \in \mathbb{Z}$ are integers with $ad - bc > 0$.

We assume that Σ is a sofic subshift, whose language is accepted by a **deterministic finite automaton** (DFA). This is a triple $\mathcal{A} = (Y, \delta, \mathbf{i})$, where Y is a finite set of states, $\delta : Y \times A \to Y$ is a partial **transition function**, and $\mathbf{i} \in Y$ is the initial state. The function δ is extended to a partial function $\delta : Y \times A^* \to Y$ by $\delta(p, \lambda) = p$, $\delta(p, ua) = \delta(\delta(p, u), a)$, where the left-hand-side is defined iff the right-hand-side is defined. The language $\mathcal{L}(\mathcal{A})$ of $\mathcal{A}$ consists of words $u \in A^*$ which are **accepted**, i.e., for which $\delta(\mathbf{i}, u)$ is defined.

We assume that $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Möbius iterative system, $\Sigma \subseteq A^{\mathbb{N}}$ is a sofic subshift whose language is accepted by a DFA $\mathcal{A} = (Y, \delta, \mathbf{i})$, and $\mathcal{W} = \{W_u : u \in \mathcal{L}(\Sigma)\}$ is an interval system compatible with F and Σ, such that $F_a \in \mathcal{M}_1$ and $\overline{W_a} \in \mathcal{I}$ for each $a \in A$. We also assume that for some $n > 0$, $\mathbf{Q}_n(\Sigma, \mathcal{W}) \geq 1$ and no F_u with $u \in \mathcal{L}^n(\Sigma_{\mathcal{W}})$ is a rotation. Finally we assume that $\ell(\mathcal{W}) > 0$, so $(F, \Sigma_{\mathcal{W}})$ is a redundant Möbius number system. We generalize arithmetical algorithms of Gosper [1] (see also Vuillemin [13], Kornerup and Matula [4] or Potts et al. [11]) using infinite labelled graphs. This is a structure $G = (V, E, s, t, l)$ where V is the set of vertices, E is the set of edges, $s, t : E \to V$ is the source and target map and $l : E \to A \cup \{\lambda\}$ is the labelling function. An infinite path with source $s(u_0)$ is any sequence of edges $u \in E^{\mathbb{N}}$ such that $t(u_i) = s(u_{i+1})$. Its label is $l(u_0)l(u_1)\ldots \in A^{\mathbb{N}} \cup A^*$.

Definition 14 *The vertices of the* **rational expansion graph** *are pairs* $(x, q) \in \overline{\mathbb{Q}}\times Y$. *Its labelled edges are* $(x, q) \xrightarrow{a} (F_a^{-1}(x), \delta(q, a))$ *such that* $\exists\delta(q, a)$ *and* $x \in W_a$.

Proposition 15 *For each* $x \in \overline{\mathbb{Q}}$ *there exists an infinite path with source* $(x, \mathbf{i})$. *If* $u \in A^{\mathbb{N}}$ *is its label, then* $u \in \Sigma_{\mathcal{W}}$ *and* $\Phi(u) = x$.

Proof: Set $x_0 = x$, $q_0 = \mathbf{i}$. At step n choose u_n such that $x_n \in W_{u_n}$ and $\exists\delta(q_n, u_n)$. Set $x_{n+1} = F_{u_n}^{-1}(x_n)$, $q_{n+1} = \delta(q_n, u_n)$ and continue. Then we have $x \in W_{u_{[0,n)}}$, and $u_{[0,n)} \in \mathcal{L}(\Sigma_{\mathcal{W}})$, so $u \in \Sigma_{\mathcal{W}}$ and $\Phi(u) = x$. □

The nondeterministic expansion algorithm based on the rational expansion graph of Definition 14 can be realized by a machine whose inner states are pairs $(x, q) \in \overline{\mathbb{Q}}\times Y$. The machine is attached to an output tape where the expansion is written. The algorithm can be made deterministic by further specifications, like in the arithmetic expansion algorithm of Definition 21.

Definition 16 *The vertices of the* **linear graph** *are triples* (M, q, u) *where* $M \in \mathcal{M}_1$, $q \in Y$ *and* $u \in \Sigma_{\mathcal{W}}$. *Its edges are*

$$(M, q, u) \xrightarrow{a} (F_a^{-1}M, \delta(q, a), u) \quad \textit{if} \quad \exists\delta(q, a)\ \&\ M(W_{u_0}) \subseteq W_a,$$

$$(M, q, u) \xrightarrow{\lambda} (MF_{u_0}, q, \sigma(u)).$$

Proposition 17 *If $u \in \Sigma_{\mathcal{W}}$ and $M \in \mathcal{M}_1$ then there exists an infinite path with source $(M, \mathbf{i}, u)$ whose label w belongs to $A^{\mathbb{N}}$. For any such w we have $w \in \Sigma_{\mathcal{W}}$ and $\Phi(w) = M(\Phi(u))$.*

Proof: We show by induction that if $w \in A^*$ is the label of a path with source $(M, \mathbf{i}, u)$, then its target is $(F_w^{-1} M F_{u_{[0,k)}}, \delta(\mathbf{i}, w), u_{[k,\infty)})$ for some $k \geq 0$, and $M(W_{u_{[0,k]}}) \subseteq W_w$. The statement holds for $w = \lambda$ and $k = 0$. If the statement holds for w and some $k \geq 0$ then we have two possible continuations:

$$(F_w^{-1} M F_{u_{[0,k)}}, \delta(\mathbf{i}, w), u_{[k,\infty)}) \xrightarrow{\lambda} (F_w^{-1} M F_{u_{[0,k]}}, \delta(\mathbf{i}, w), u_{[k+1,\infty)}),$$
$$(F_w^{-1} M F_{u_{[0,k)}}, \delta(\mathbf{i}, w), u_{[k,\infty)}) \xrightarrow{a} (F_{wa}^{-1} M F_{u_{[0,k)}}, \delta(\mathbf{i}, wa), u_{[k,\infty)}).$$

In the former case we get $M(W_{u_{[0,k+1]}}) \subseteq M(W_{u_{[0,k]}}) \subseteq W_w$. In the latter case we use the assumption $F_w^{-1} M F_{u_{[0,k)}}(W_{u_k}) \subseteq W_a$, to get $M(W_{u_{[0,k]}}) \subseteq M(F_{u_{[0,k)}}(W_{u_k})) \subseteq F_w(W_a)$, and $M(W_{u_{[0,k]}}) \subseteq W_w \cap F_w(W_a) = W_{wa}$. Since $u \in \Sigma_{\mathcal{W}}$, we have $\lim_{k\to\infty} ||W_{u_{[0,k]}}|| = 0$, so for each j there exists k and w_j such that $M(W_{u_{[0,k)}}) \subseteq W_{w_{[0,j)}}$. Thus there exists a path with label $w \in \Sigma_{\mathcal{W}}$ and $M(\Phi(u)) = \Phi(w)$. □

The algorithm based on the linear graph of Definition 16 can be realized by a machine whose inner states are $(M, q) \in \mathcal{M}_1 \times Y$. The machine is attached to an input and output tapes and can see the first letter of the input tape.

7. Fractional bilinear functions

To obtain algorithms for functions of two variables like the sum or product, we consider **orientation-reversing** MT $M_{(a,b,c,d)}$ with $ad-bc < 0$, **singular** MT with $ad-bc = 0$ and $|a|+|b|+|c|+|d| > 0$, and the **zero** MT $M_{(0,0,0,0)} = 0$. Each MT defines a closed relation

$$\widetilde{M} = \{(x, y) \in \overline{\mathbb{R}}^2 : \ (ax_0 + bx_1)y_1 = (cx_0 + dx_1)y_0\}.$$

We define the **stable point** $s_M \in \overline{\mathbb{R}}$ of a singular MT by $s_M \in \{\frac{a}{c}, \frac{b}{d}\} \cap \overline{\mathbb{R}}$. Note that $\frac{a}{c}$ or $\frac{b}{d}$ can be $\frac{0}{0} \notin \overline{\mathbb{R}}$, but not both. Similarly the **unstable point** $u_M \in \overline{\mathbb{R}}$ of a singular MT is defined by $u_M \in \{-\frac{b}{a}, -\frac{d}{c}\} \cap \overline{\mathbb{R}}$. A singular MT yields the relation $\widetilde{M} = (\overline{\mathbb{R}} \times \{s_M\}) \cup (\{u_M\} \times \overline{\mathbb{R}})$, and the zero MT yields the full relation $\widetilde{M} = \overline{\mathbb{R}}^2$. The value $M(I) = \{y \in \overline{\mathbb{R}} : \ \exists x \in I, (x, y) \in \widetilde{M}\}$ of M on a closed interval $I = [I_0, I_1]$ is

$$M(I) = \begin{cases} [M(I_0), M(I_1)] & \text{if} \quad ad - bc > 0, \\ [M(I_1), M(I_0)] & \text{if} \quad ad - bc < 0, \\ \{s_M\} & \text{if} \quad ad - bc = 0 \ \& \ M \neq 0 \ \& \ u_M \notin I, \\ \overline{\mathbb{R}} & \text{if} \quad ad - bc = 0 \ \& \ M \neq 0 \ \& \ u_M \in I, \\ \overline{\mathbb{R}} & \text{if} \quad M = 0. \end{cases}$$

Denote by $\mathcal{M}_{(1,1)}$ the set of **fractional bilinear functions** with integer coefficients. These are functions $P : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}} \cup \{\frac{0}{0}\}$ of two variables $x, y \in \overline{\mathbb{R}}$ of the form

$$P(x, y) = \frac{ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1}{ex_0y_0 + fx_0y_1 + gx_1y_0 + hx_1y_1}$$

which represent closed relations

$$\widetilde{P} = \{(x, y, z) \in \overline{\mathbb{R}}^3 : (ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1)z_1 = (ex_0y_0 + fx_0y_1 + gx_1y_0 + hx_1y_1)z_0\}.$$

For each $x, y \in \overline{\mathbb{R}}$, $P(x,-)$, $P(-,y)$ are MT with matrices

$$P(x,-) = \begin{bmatrix} ax_0 + cx_1 & bx_0 + dx_1 \\ ex_0 + gx_1 & fx_0 + hx_1 \end{bmatrix}, \; P(-,y) = \begin{bmatrix} ay_0 + by_1 & cy_0 + dy_1 \\ ey_0 + fy_1 & gy_0 + hy_1 \end{bmatrix},$$

so if $x, y \in \overline{\mathbb{R}}$ and I, J are closed intervals, then $P(x, J)$, $P(I, y)$ are well defined intervals. Since $P(x,-)$ and $P(-,y)$ are monotone, the image $P(I, J)$ of closed intervals $I = [I_0, I_1]$, $J = [J_0, J_1]$ coincides with the image of the boundary of $I \times J$:

$$\begin{aligned} P(I,J) &:= \{z \in \overline{\mathbb{R}} : \; \exists x \in I, \exists y \in J, (x,y,z) \in \widetilde{P}\} \\ &= P(I_0, J) \cup P(I, J_1) \cup P(I_1, J) \cup P(I, J_0). \end{aligned}$$

Given a (2×2)-matrix $M = M_{(a,b,c,d)}$, define the (4×4)-matrices M^x and M^y by

$$M^x = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}, \; M^y = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

Then both $P(Mx, y) = PM^x(x, y)$ and $P(x, My) = PM^y(x, y)$, as well as MP are fractional bilinear functions. In these formulas, P is regarded as a (2×4)-matrix.

Definition 18 *The* **bilinear graph** *has vertices* (P, q, u, v)*, where* $P \in \mathcal{M}_{(1,1)}$, $q \in Y$ *and* $u, v \in \Sigma_{\mathcal{W}}$*. The edges are*

$$\begin{aligned} &(P,q,u,v) \xrightarrow{a} (F_a^{-1}P, \delta(q,a), u, v) \quad \textit{if} \quad \exists \delta(q,a) \;\&\; P(\overline{W_{u_0}}, \overline{W_{v_0}}) \subseteq W_a, \\ &(P,q,u,v) \xrightarrow{\lambda} (PF^x_{u_0}, q, \sigma(u), v), \\ &(P,q,u,v) \xrightarrow{\lambda} (PF^y_{v_0}, q, u, \sigma(v)). \end{aligned}$$

Proposition 19 *If* $u, v \in \Sigma_{\mathcal{W}}$ *and* $w \in A^{\mathbb{N}}$ *is a label of a path with the source* $(P, \mathbf{i}, u, v)$*, then* $w \in \Sigma_{\mathcal{W}}$ *and* $\Phi(w) = P(\Phi(u), \Phi(v))$*.*

The proof is similar as in Proposition 17. However, for $u, v \in \Sigma_{\mathcal{W}}$ there need not exist any path with source $(P, \mathbf{i}, u, v)$ whose label belongs to $A^{\mathbb{N}}$. The algorithm may give only a finite output, for example when we add words representing $\infty + \infty$.

8. Rational functions

Denote by $\mathcal{M}_n$ the set of rational functions

$$R(x) = \frac{a_0 x_1^n + a_1 x_0 x_1^{n-1} + \cdots + a_n x_0^n}{b_0 x_1^n + b_1 x_0 x_1^{n-1} + \cdots + b_n x_0^n}$$

of degree n with integer coefficients. If $M \in \mathcal{M}_1$, then both composed functions RM and MR belong to $\mathcal{M}_n$. Denote by $\mathbf{t}(x) := \arg \mathbf{d}(x) = 2 \arctan x$ the isomorphism of $\mathbb{R}$ with $(-\pi, \pi)$. The **circle derivation** $R^\bullet : \overline{\mathbb{R}} \to \mathbb{R}$ of R is defined by

$$R^\bullet(x) := (\mathbf{t}R\mathbf{t}^{-1})'(\mathbf{t}(x)) = \frac{R'(x)(1 + x^2)}{1 + R^2(x)}.$$

We have $|R^\bullet(x)| = \lim_{y \to x} \varrho(R(y), R(x))/\varrho(y, x)$. A **monotone element** is a pair (R, I) such that I is a closed interval, $R \in \mathcal{M}_n$ is monotone on I, i.e., $R^\bullet(x)$ does not change sign on I, and R has no poles in I, i.e., $R(x) \neq \infty$ for $x \in I$. We say that a monotone element (R, I) is **sign-changing**, if either $R(I_0) < 0 < R(I_1)$ and $R^\bullet(x) > 0$ for $x \in I$, or $R(I_0) > 0 > R(I_1)$ and $R^\bullet(x) < 0$ for $x \in I$. A sign-changing monotone element (R, I) has a unique **root** $x \in I$ with $R(x) = 0$.

Proposition 20

(1) There exists an algorithm which for $R \in \mathcal{M}_n$ gives a list of sign-changing elements $(R_1, I_1), \ldots, (R_k, I_k)$, such that each root of R is a root of some (R_j, I_j).

(2) It is decidable, whether (R, I) is a monotone element.

(3) Given closed intervals I, J, it is decidable whether $R(I) \subseteq J$.

These algorithms can be obtained from the Sturm theorem (see e.g., van der Waerden [12]), which counts the number of roots of a polynomial P in an interval using the Euclidean algorithm applied to P and its derivative P'. The condition (3) can be decided without evaluating the interval $R(I)$, which may have irrational endpoints: We use the the Sturm theorem to test whether $R(x) - J_0$ and $R(x) - J_1$ have roots in I.

Definition 21 *The* **algebraic expansion graph** *is a graph whose vertices are (R, q, I), where (R, I) is a sign-changing element, and $q \in Y$. Its edges are*

$$(R, q, I) \xrightarrow{a} (RF_a, \delta(q, a), F_a^{-1}(I \cap W_a)),$$

such that $\exists \delta(q, a)$ and $(R, I \cap W_a)$ is a sign-changing element.

Proposition 22 *For each sign-changing element (R, I) there exists an infinite path with source $(R, \mathbf{i}, I)$. If w is its label, then $w \in \Sigma_{\mathcal{W}}$, $\Phi(w) \in I$, and $R\Phi(w) = 0$.*

Proof: We show by induction that if u is a label of a finite path with source $(R, \mathbf{i}, I)$, then its target is $(RF_u, \delta(\mathbf{i}, u), J)$, where $F_u(J) = I \cap W_u$, (RF_u, J) is a sign-changing element, and there exists a path with source $(R, \mathbf{i}, I)$ and label ua for some $a \in A$. If this holds for $u \in A^*$, then $u \in \mathcal{L}(\Sigma_{\mathcal{W}})$, and $(R, F_u(J)) = (R, I \cap W_u)$ is a sign-changing element, so there exists $a \in A$ such that $ua \in \mathcal{L}(\Sigma_{\mathcal{W}})$, and $(R, I \cap W_{ua}) = (R, I \cap W_u \cap F_u(W_a)) = (R, F_u(J \cap W_a))$ is a sign-changing element. Thus $(RF_u, J \cap W_a)$ is a sign-changing element, so there exists a path with label ua and target $(RF_{ua}, \delta(\mathbf{i}, ua), J')$, where $J' = F_a^{-1}(J \cap W_a)$. It follows that (RF_{ua}, J') is a sign-changing element and $F_{ua}(J') = F_u(J) \cap F_u(W_a) = I \cap W_{ua}$. □

The graph for the computation of a rational function $R \in \mathcal{M}_n$ at $\Phi(u)$ has the same structure as the linear graph from Definition 16. To test the condition $R(W_{u_0}) \subseteq W_a$, we use the Sturm theorem rather than the simple comparison of the endpoints of intervals. However, if R is monotone in a neighbourhood I of $\Phi(u)$, the test simplifies. Finally we have a conversion algorithm between two redundant Möbius number systems:

Definition 23 *The $(G, \Theta_{\mathcal{V}})$ to $(F, \Sigma_{\mathcal{W}})$ conversion graph has vertices (M, q, u), where $M \in \mathcal{M}_1$, $q \in Y$ is the state of a DFA for $\Sigma_{\mathcal{W}}$ and $u \in \Theta_{\mathcal{V}}$. The labelled edges are*

$$(M, q, u) \xrightarrow{a} (F_a^{-1}M, \delta(q, a), u) \quad \textit{if} \quad \exists \delta(q, a) \,\&\, V_{u_0} \subseteq W_a,$$
$$(M, q, u) \xrightarrow{\lambda} (MG_{u_0}, q, \sigma(u)).$$

Proposition 24 *Assume that* $(G, \Theta_{\mathcal{V}})$ *and* $(F, \Sigma_{\mathcal{W}})$ *are Möbius number systems and* $(F, \Sigma_{\mathcal{W}})$ *is redundant. If* $u \in \Theta_{\mathcal{V}}$ *then there exists an infinite path with source* $(\mathrm{Id}, \mathbf{i}, u)$ *whose label* v *belongs to* $\Sigma_{\mathcal{W}}$*, and* $\Phi_F(v) = \Phi_G(u)$.

Using the continued fractions of Gauss (see Wall [14]), algorithms for many transcendental functions can be obtained similarly as in Potts [10] or Potts et al. [11].

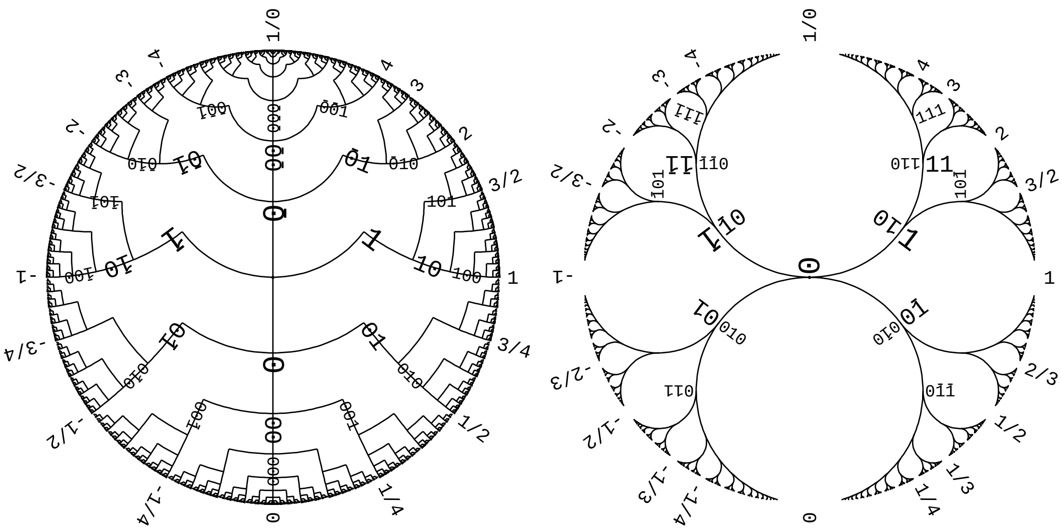


Figure 1. The binary signed system (left) and the regular continued fractions (right)

9. The binary signed system and the system of continued fractions

We give some examples of Möbius number systems based on positional systems and continued fractions. We use the alphabet $A = \{\overline{1}, 0, 1, \overline{0}\}$ whose letters stand for numbers $-1, 0, 1, \infty$.

Example 1 *The Möbius* **binary signed system** *(BSS, Figure 1 left) consists of the alphabet* $A = \{\overline{1}, 0, 1, \overline{0}\}$*, transformations*

$$F_{\overline{1}}(x) = -1 + x,\ F_0(x) = x/2,\ F_1(x) = 1 + x,\ F_{\overline{0}}(x) = 2x,$$

and the interval system $\mathcal{W}^1 = ((-2, -1), (-1, 1), (1, 2), (2, -2))$.

The subshift $\mathcal{S}_{\mathcal{W}}$ is sofic with forbidden words $D = \{\overline{1}1, 0\overline{0}, 1\overline{1}, \overline{0}0, \overline{1}\overline{0}, 1\overline{0}, \overline{1}0^*1, 10^*\overline{1}\}$ and each $u \in \mathcal{L}(\mathcal{S}_{\mathcal{W}})$ can be written as $u = \overline{0}^n v$, where $v \in \{\overline{1}, 0\}^* \cup \{0, 1\}^*$. Then $F_u(x) = 2^n(s_0 + 2^{-1} \cdot s_1 + \cdots + 2^{1-k} \cdot s_{k-1} + 2^{-k} \cdot x)$ for some s_i which all belong either to $\{-1, 0\}$ or to $\{0, 1\}$. The system is not redundant and $\mathbf{Q}(\mathcal{S}, \mathcal{W}) > 1$. A redundant system can be obtained as $\mathcal{S}_{\mathcal{W}_1}$ where $\mathcal{W}_1^1 = ((-3, -\frac{1}{2}), (-1, 1), (\frac{1}{2}, 3), (2, -2))$, or as $\Sigma_{\mathcal{W}_2}$, where $\Sigma = \mathcal{S}(\{\overline{1}1, 0\overline{0}, 1\overline{1}, \overline{0}0\})$ forbids some identities, and $\mathcal{W}_2^1 = ((\infty, 0), (-1, 1), (0, \infty), (1, -1))$. We have $\mathbf{Q}_n(\Sigma, \mathcal{W}_2) = 1$ for each $n \geq 5$.

The means $\widehat{F}_u(0)$ of words $u \in \mathcal{L}(\mathcal{S}_{\mathcal{W}})$ can be seen in Figure 1 left. The curves between these means are constructed as follows. For each MT M there exists a family of MT $(M^t)_{t \in \mathbb{R}}$ such that $M^0 = \mathrm{Id}$, $M^1 = M$, and $M^{t+s} = M^t M^s$. In Figures 1 and 2, each mean $\widehat{F}_u(0)$ is joined to $\widehat{F}_{ua}(0)$ by the curve $(\widehat{F}_u\widehat{F}_a^t(0))_{0\le t\le 1}$. The labels $u \in A^+$ at $\widehat{F}_u(0)$ are written in the direction of the tangent vectors $\widehat{F}'_u(0)$.

Example 2 *The Möbius system of* **regular continued fraction** *(RCF, Figure 1 right) consists of the alphabet* $A = \{\overline{1}, 0, 1\}$*, transformations*

$$F_{\overline{1}}(x) = -1 + x,\ F_0(x) = -1/x,\ F_1(x) = 1 + x,$$

and the interval system $\mathcal{W}^1 = ((\infty, -1), (-1, 1), (1, \infty))$.

By Proposition 7, the subshift $\mathcal{S}_\mathcal{W}$ is of finite type with forbidden words $D = \{00, \overline{1}1, 1\overline{1}, \overline{1}0\overline{1}, 101\}$ and $\mathbf{Q}_n(\mathcal{S}, \mathcal{W}) = 1$ for each n. For each $u \in \mathcal{L}(\mathcal{S}_\mathcal{W})$, the transformation F_u can be written as $F_u(x) = F_1^{a_0}F_0F_1^{a_1}\cdots F_0F_1^{a_n}(x)$ where $a_i \in \mathbb{Z}$, $a_ia_{i+1} \le 0$ and $a_i \ne 0$ for $i > 0$. Thus we obtain a continued fraction whose partial quotients $(-1)^ia_i$ are either all positive or all negative. There is no reasonable way to get redundant continued fractions. We obtain semi-regular CF with the interval system $\mathcal{W}_1^1 = ((\infty, -\frac{1}{2}), (-1, 1), (\frac{1}{2}, \infty))$, but $W_{\overline{1}}$ and W_1 cannot be extended beyond ∞.

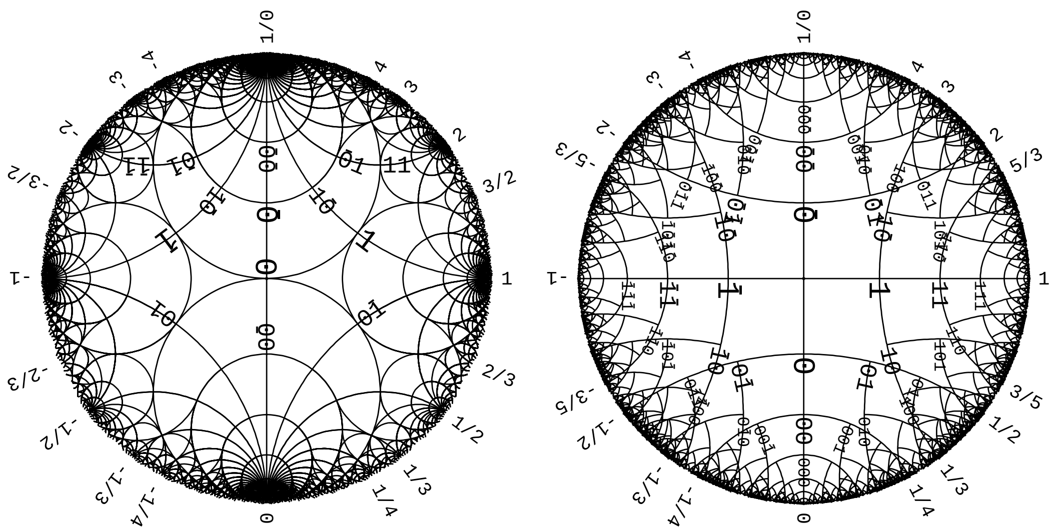


Figure 2. Binary continued fractions (left) and the binary square system (right)

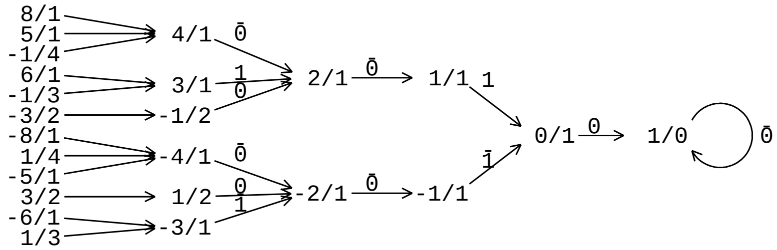
10. Binary continued fractions

Example 3 *The Möbius system of* **binary continued fractions** *(BCF, Figure 2 left) consists of the alphabet* $A = \{\overline{1}, 0, 1, \overline{0}\}$*, transformations*

$$F_{\overline{1}}(x) = -1 + x,\ F_0(x) = -1/x,\ F_1(x) = 1 + x,\ F_{\overline{0}}(x) = 2x,$$

and the interval system $\mathcal{W}^1 = ((\infty, -\frac{1}{2}), (-1, 1), (\frac{1}{2}, \infty), (-2, 2))$.

We have $\ell(\mathcal{W}) > 0$ and $\mathbf{Q}_n(\mathcal{W}) = 1$ for each $n \ge 1$, so $(F, \mathcal{S}_\mathcal{W})$ is redundant and $\mathcal{S}_\mathcal{W}$ is sofic. In BCF, each rational number has an eventually periodic expansion with period length 1 of the form $u.\overline{0}$. The proof of this fact for a variant of the BCF system in Kůrka [8] can be easily adapted to the present situation. To obtain shorter expansions, we can test the parities of the numerator and denominator:



0/1	$0.\overline{0}$	1/4	$0\overline{0}\overline{0}\overline{1}0.\overline{0}$	1/3	$0\overline{1}\overline{0}\overline{1}0.\overline{0}$	2/5	$0\overline{0}\overline{1}0\overline{0}\overline{0}10.\overline{0}$
1/2	$0\overline{0}\overline{1}0.\overline{0}$	3/5	$0\overline{1}010\overline{0}\overline{1}0.\overline{0}$	2/3	$0\overline{1}0\overline{0}10.\overline{0}$	3/4	$10\overline{0}\overline{0}10.\overline{0}$
1/1	$10.\overline{0}$	4/3	$10\overline{1}\overline{0}\overline{1}0.\overline{0}$	3/2	$10\overline{0}\overline{1}0.\overline{0}$	5/3	$10\overline{1}0\overline{0}10.\overline{0}$
2/1	$\overline{0}10.\overline{0}$	5/2	$\overline{0}10\overline{0}\overline{0}\overline{1}0.\overline{0}$	3/1	$1\overline{0}10.\overline{0}$	4/1	$\overline{0}\overline{0}10.\overline{0}$

Figure 3. Arithmetical expansions of rational numbers in the system of binary continued fractions.

Definition 25 *The* **arithmetical expansion graph** *(Figure 3 top) for the BCF system has vertices $x = (x_0, x_1) \in \overline{\mathbb{Q}}$, with $x_1 \geq 0$ and $\gcd(x_0, x_1) = 1$, and labelled edges*

$$\begin{array}{lll}
x \stackrel{\overline{0}}{\to} (x_0/2, x_1) & \textit{if} & |x_0| \geq 2|x_1| \ \& \ 2|x_0, \\
x \stackrel{\overline{0}}{\to} (x_0, 2x_1) & \textit{if} & |x_0| \geq 2|x_1| \ \& \ 2|x_1, \\
x \stackrel{\overline{1}}{\to} (x_0 + x_1, x_1) & \textit{if} & x_0 \leq -x_1 \ \vee \ (2x_0 < -x_1 \ \& \ 2|x_1), \\
x \stackrel{1}{\to} (x_0 - x_1, x_1) & \textit{if} & x_0 \geq x_1 \ \vee \ (2x_0 > x_1 \ \& \ 2|x_1), \\
x \stackrel{0}{\to} (-x_1 \cdot \mathrm{sgn}(x_0), |x_0|) & \textit{otherwise.} &
\end{array}$$

In the expansion procedure of Definition 25, the first applicable rule is used, so each vertex has outdegree 1 and we get a deterministic expansion function $\mathcal{E} : \overline{\mathbb{Q}} \to \Sigma_{\mathcal{W}}$, such that $\mathcal{E}(x)$ is the label of the unique infinite path with source x and $\Phi(\mathcal{E}(x)) = x$ (see Figure 3 top). Each rational number has expansion of the form $u.\overline{0}$ and integers have the same expansions as in the binary system. An integer can be written as $x = x_0 + 2x_1 + \cdots + 2^k x_k$, where $x_i \in \{-1, 0, 1\}$ are either all non-negative, or all non-positive. Then $\mathcal{E}(x) = s_0\overline{0}s_1\overline{0}\ldots\overline{0}s_{k-1}\overline{0}s_k0.\overline{0}$, where s_i is empty if $x_i = 0$, $s_i = \overline{1}$ if $x_i = -1$, and $s_i = 1$ if $x_i = 1$ (see Figure 3 bottom).

11. Hyperbolic polygonal systems

An n-ary hyperbolic polygonal system with quotient $q > 1$ considered in Kůrka [5] has alphabet $A = \{0, 1, \ldots, n-1\}$ and transformations $F_a = R_{2\pi a/n} C_q R_{-2\pi a/n}$, where $C_q(x) = x/q$ and R_α is the rotation by α. In particular for $n = 4$ and $q = 2$ we get

Definition 26 *The Möbius* **binary square system** *(Figure 2 right) with quotient $q = 2$ consists of the alphabet $A = \{\overline{1}, 0, 1, \overline{0}\}$, transformations*

$$F_{\overline{1}}(x) = \frac{3x - 1}{-x + 3}, \ F_0(x) = \frac{x}{2}, \ F_1(x) = \frac{3x + 1}{x + 3}, \ F_{\overline{0}}(x) = 2x,$$

and the interval system $\mathcal{W}^1 = ((-3, -\frac{1}{3}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, 3), (2, -2))$.

Note that all W_a have the same length. Moreover we have $F_0^{-1}(W_0) = (-1, 1)$ and $F_1^{-1}(W_1) = (0, \infty)$. It follows that $\mathcal{S}_\mathcal{W}$ is the sofic subshift

$$\mathcal{S}_\mathcal{W} = \{\overline{1}, 0\}^\mathbb{N} \cup \{0, 1\}^\mathbb{N} \cup \{1, \overline{0}\}^\mathbb{N} \cup \{\overline{0}, \overline{1}\}^\mathbb{N}$$

which is a union of four full shifts, one for each quadrant: $\Phi(\{\overline{1}, 0\}^\mathbb{N}) = [-1, 0]$, $\Phi(\{0, 1\}^\mathbb{N}) = [0, 1]$, $\Phi(\{1, \overline{0}\}^\mathbb{N}) = [1, \infty]$ and $\Phi(\{\overline{0}, \overline{1}\}^\mathbb{N}) = [\infty, -1]$. A disadvantage of this system is, however, that not all rational numbers have periodic expansions. This is shown in Kůrka [6], where some conditions for the periodicity of expansions of rational numbers are given.

12. Conclusions

Arithmetical algorithms simplify if $\Sigma = \mathcal{S}$ is the full shift, since no DFA is necessary in this case. However, if we forbid some identities, we can have larger expansion intervals W_a and therefore a more redundant system with larger Lebesgue number, whose arithmetical algorithms may be faster. If Σ is a SFT, its compatibility can be verified by Proposition 6. In particular cases, the arithmetical algorithms can be further simplified, which is the case of the Binary signed system. While Corollary 12 and Theorem 13 formulate a tight condition on Σ and $\mathcal{W}$ to form a Möbius number system, the case $\mathbf{Q}(\Sigma, \mathcal{W}) = 1$ has not yet been completely analysed.

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