Dynamics of cellular automata in non-compact spaces

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1 Glossary

Almost equicontinuous CA: a CA which has at least one equicontinuous configuration.

Attraction basin: the set of configurations whose orbit is eventually attracted by an attractor.

Attractor: a closed invariant set which attracts all orbits in some of its neighbourhood.

Besicovitch pseudometrics: a pseudo-metric that quantifies the upper-density of differences.

Blocking word: a word that interrupts the information flow. A configuration containing an infinite number of blocking words both to the right and to the left gives rise to an equicontinuous configuration.

Equicontinuous CA: a CA in which all configurations are equicontinuous.

Equicontinuous configuration: a configuration for which nearby configurations remain close.

Expansive CA: two distinct configurations, no matter how close, eventually separate during the evolution.

Generic space: the space of configurations for which upper-density and lower-density coincide.

Sensitive CA: in any neighbourhood of any configuration there exists a configuration such that the orbits of the two configurations eventually separate.

Spreading set: a clopen invariant set propagating both to the left and to the right.

Toeplitz space: the space of regular quasi-periodic configurations.

Weyl pseudometrics: a pseudo-metric that quantifies the upper density of differences with respect to all possible cell indices.

2 Definition

In topological dynamics, the assumption of compactness is usually adopted as it has far reaching consequences. Each compact dynamical system has an almost periodic point, contains a minimal subsystem, and each trajectory has a limit point. Nevertheless, there are important examples of non-compact dynamical systems like linear systems on $\mathbb{R}^n$ and the theory should cover these examples as well. The study of dynamics of cellular automata (CA) in the compact Cantor space of symbolic sequences starts with Hedlund [7] and is by now a firmly established discipline (see e.g., Kůrka [15]). The study of dynamics of CA in non-compact spaces like Besicovitch or Weyl spaces is more recent and provides an interesting alternative perspective.

The study of dynamics of cellular automata in non-compact spaces has at least two distinct origins. The first concerns the study of dynamical properties on peculiar countable dense sub-spaces of the Cantor space (the space of finite configuration or the space of spatially periodic configurations, for instance). The idea is that on those spaces, some properties are easier to prove than on the full Cantor space. Once a property is proved on such a sub-space, one can try to lift it to the original Cantor space by using denseness. Another advantage is that the configurations on these spaces are easily representable on computers. Indeed, computer simulations and practical applications of CA usually take place in these subspaces.

The second origin is connected to the question of suitability of the classical Cantor topology for the study of chaotic behavior of CA and of symbolic systems in general. We briefly recall the motivations. Consider sensitivity to initial conditions for a CA in the Cantor topology. The shift map σ, which is a very simple CA, is sensitive to initial conditions since small perturbations far from the central region are eventually brought to the central part. However, from an algorithmic point of view, the shift map is very simple. We are inclined to regard a system as chaotic if its behavior cannot easily be reconstructed. This is not the case of the shift map whose chaoticity is more an artifact of the Cantor metric, rather than an intrinsic property of the system. Therefore, one may want to define another metric in which sensitive CA not only transport information (like the shift map) but also build/destroy new information at each time step.

This basic requirement stimulated the quest for alternative topologies to the classical Cantor space. This lead first to the Besicovitch topology and then to the Weyl topology in Cattaneo et al [4]. used to investigate almost periodic real functions (see Besicovitch [1] or Iwanik [9]). Both these pseudometrics can be defined starting from suitable semi-measures on the set $\mathbb{Z}$ of integers. This way of construction had a *Pandora effect* opening the way to many new interesting topological spaces. Some of them are reported in this paper; others can be found in Cervelle and Formenti [5].

Each topology focuses on some peculiar aspects of the dynamics under study but all of them have a common denominator, namely non-compactness.

3 Introduction

A given CA in alphabet A can be regarded as a dynamical system in several topological spaces: Cantor configuration space $\mathcal{C}_A$, space $\mathcal{M}_A$ of shift-invariant Borel probability measures on $A^{\mathbb{Z}}$, the Weyl space $\mathcal{W}_A$, the Besicovitch space $\mathcal{B}_A$, the generic space $\mathcal{G}_A$, the Toeplitz space $\mathcal{T}_A$ and the periodic space $\mathcal{P}_A$. We refer to various topological properties of these systems by prefixing the name of space in question. Basic results correlate various dynamical properties of CA in these spaces.

The Cantor topology corresponds to the point of view of an observer who can distinguish only a finite central part of a configuration and sites outside this central part of the configuration are not taken into account. The Besicovitch and Weyl topologies, on the other hand, correspond to a god-like position of someone who sees whole configurations and can distinguish the frequency of differences. In the Besicovitch topology, the centers of configurations still play a distinguished role, as the frequencies of differences are computed from the center. In the Weyl topology, on the other hand, no site has a privileged position. Both Besicovitch and Weyl topologies are defined by pseudometrics. Different configurations can have zero distance and the topological space consists of equivalence classes of configurations which have zero distance.

The generic space $\mathcal{G}_A$ is a subspace of the Besicovich space of those configurations, in which each finite word has a well defined frequency. These frequencies define a Borel probability measure on the Cantor space of configurations, so we have a projection from the generic space $\mathcal{G}_A$ to the space $\mathcal{M}_A$ of Borel probability measures equipped with the weak* topology. This is a natural space for investigating the dynamics of CA on random configurations.

The Toeplitz space $\mathcal{T}_A$ consists of regular quasi-periodic configurations. This means that each pattern repeats periodically but different patterns have different periods. The Besicovitch and Weyl pseudometrics are actually metrics on the Toeplitz space and moreover they coincide on $\mathcal{T}_A$.

4 Dynamical systems

A **dynamical system** is a continuous map $F : X \to X$ of a nonempty metric space X to itself. The n-th iteration $F^n : X \to X$ of F is defined by $F^0(x) = x$, $F^{n+1}(x) = F(F^n(x))$. A point $x \in X$ is **fixed**, if $F(x) = x$. It is **periodic**, if $F^n(x) = x$ for some $n > 0$. The least positive n with this property is called the **period** of x. The **orbit** of x is the set $\mathcal{O}(x) = \{F^n(x) : \; n > 0\}$. A set $Y \subseteq X$ is **positively invariant**, if $F(Y) \subseteq Y$ and **strongly invariant** if $F(Y) = Y$. A point $x \in X$ is **equicontinuous** ($x \in \mathcal{E}_F$) if the family of maps F^n is equicontinuous at X, i.e. $x \in \mathcal{E}_F$ iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B_\delta(x))(\forall n > 0)(d(F^n(y), F^n(x)) < \varepsilon).$$

The system (X, F) is **almost equicontinuous** if $\mathcal{E}_F \neq \emptyset$ and **equicontinuous**, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)(\forall y \in B_\delta(x))(\forall n > 0)(d(F^n(y), F^n(x)) < \varepsilon).$$

For an equicontinuous system $\mathcal{E}_F = X$. Conversely, if $\mathcal{E}_F = X$ and if X is compact, then F is equicontinuous; this needs not be true in the non-compact case. A system (X, F) is **sensitive** (to initial conditions), if

$$(\exists \varepsilon > 0)(\forall x \in X)(\forall \delta > 0)(\exists y \in B_\delta(x)) \; (\exists n > 0)(d(f^n(y), f^n(x)) \geq \varepsilon).$$

A sensitive system has no equicontinuous point. However, there exist systems with no equicontinuity points which are not sensitive. A system (X, F) is **positively expansive**, if

$$(\exists \varepsilon > 0)(\forall x \neq y \in X)(\exists n \geq 0)(d(f^n(x), f^n(y)) \geq \varepsilon).$$

A positively expansive system on a perfect space is sensitive. A system (X, F) is (topologically) **transitive**, if for any nonempty open sets $U, V \subseteq X$ there exists $n \geq 0$ such that $F^{-n}(U) \cap V \neq \emptyset$. If X is perfect and if the system has a dense orbit, then it is transitive. Conversely, if (X, F) is topologically transitive and if X is compact, then (X, F) has a dense orbit. A system (X, F) is **mixing**, if for any nonempty open sets $U, V \subseteq X$ there exists $k > 0$ such that for every $n \geq k$ we have $F^{-n}(U) \cap V \neq \emptyset$. An ε-chain (from x_0 to x_n) is a sequence of points $x_0, \ldots, x_n \in X$ such that $d(F(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i < n$. A system (X, F) is **chain-transitive**, if for any $\varepsilon > 0$ and any $x, y \in X$ there exists an ε-chain from x to y.

A strongly invariant closed set $Y \subseteq X$ is **stable**, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X, (d(x, Y) < \delta \implies \forall n > 0, d(F(x), Y) < \varepsilon).$$

A strongly invariant closed stable set $Y \subseteq X$ is an **attractor**, if

$$\exists \delta > 0, \forall x \in X, (d(x, Y) < \delta \implies \lim_{n \to \infty} d(F(x), Y) < \varepsilon).$$

A set $W \subseteq X$ is **inward**, if $F(\overline{W}) \subseteq W^\circ$. In compact spaces, attractors are exactly Ω-limits $\Omega_F(W) = \bigcap_{n>0} F(W)$ of inward sets.

Theorem 1 (Knudsen [11]) *Let (X, F) be a DS and $Y \subseteq X$ a dense, F-invariant subset.*

(1) (X, F) is sensitive iff (Y, F) is sensitive.

(2) (X, F) is transitive iff (Y, F) is transitive.

Recall that a space X is **separable**, if it has a countable dense set.

Theorem 2 (Blanchard, Formenti, and Kůrka [3]) *Let (X, F) be a dynamical system on a non-separable space. If (X, F) is transitive, then it is sensitive.*

5 Cellular automata

For a finite alphabet A, denote by $|A|$ the number of its elements, by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A, and by $A^+ := \bigcup_{n>0} A^n = A^* \setminus \{\lambda\}$ the set of nonempty words. The length of a word $u \in A^n$ is denoted by $|u| := n$. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{Z}}$ the set of **A-configurations**, or doubly-infinite sequences of letters of A. For any $u \in A^+$ we have a periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. The **cylinder** of a word $u \in A$ located at $l \in \mathbb{Z}$ is the set $[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l,l+|u|)} = u\}$. The cylinder set of a set of words $U \subseteq A^+$ located at $l \in \mathbb{Z}$ is the set $[U]_l = \bigcup_{u\in U} [u]_l$.

A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{Z}}$ such that there exists a set $D \subseteq A^+$ of **forbidden words** and $\Sigma = \Sigma_D := \{x \in A^{\mathbb{Z}} :\ \forall u \sqsubseteq x, u \notin D\}$. A subshift Σ_D is of **finite type** (SFT), if D is finite. A subshift is uniquely determined by its **language**

$$\mathcal{L}(\Sigma) := \bigcup_{n\geq 0} \mathcal{L}^n(\Sigma), \quad \text{where} \quad \mathcal{L}^n(\Sigma) := \{u \in A^n :\ \exists x \in \Sigma, u \sqsubseteq x\}.$$

A **cellular automaton** is a map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $F(x)_i = f(x_{[i-r,i+r]})$, where $r \geq 0$ is a radius and $f : A^{2r+1} \to A$ is a local rule. In particular the **shift map** $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i := x_{i+1}$. A local rule extends to the map $f : A^* \to A^*$ by $f(u)_i = f(u_{[i,i+2r]})$ so that $|f(u)| = \max\{|u| - 2r, 0\}$.

Definition 3 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA.*

(1) A word $u \in A$ is m-blocking, if $|u| \geq m$ and there exists offset $d \leq |u| - m$ such that $\forall x, y \in [u]_0, \forall n > 0, F^n(x)_{[d,d+m)} = F^n(y)_{[d,d+m)}$.

(2) A set $U \subseteq A^+$ is spreading, if $[U]$ is F-invariant and there exists $n > 0$ such that $F^n([U]) \subseteq \sigma^{-1}([U]) \cap \sigma([U])$.

The following results will be useful in the sequel.

Proposition 4 (Formenti, Kůrka [6]) *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA and let $U \subseteq A^+$ be an invariant set. Then $\Omega_F([U])$ is a subshift iff U is spreading.*

Theorem 5 (Hedlund [7]) *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA with local rule $f : A^{2r+1} \to A$. Then F is surjective iff $f : A^* \to A^*$ is surjective iff $|f^{-1}(u)| = |A|^{2r}$ for each $u \in A^+$.*

6 Submeasures

A pseudometric on a set X is a map $d : X \times X \to [0, \infty)$ which satisfies the following conditions:
1. $d(x, y) = d(y, x)$,
2. $d(x, z) \leq d(x, y) + d(y, z)$.
If moreover $d(x, y) > 0$ for $x \neq y$, then we say that d is a metric. There is a standard method to create pseudometrics from submeasures. A bounded submeasure (with bound $M \in \mathbb{R}^+$) is a map $\varphi : \mathcal{P}(\mathbb{Z}) \to [0, M]$ which satisfies the following conditions:
1. $\varphi(\emptyset) = 0$,
2. $\varphi(U) \leq \varphi(U \cup V) \leq \varphi(U) + \varphi(V)$ for $U, V \subseteq \mathbb{Z}$.
A bounded submeasure φ on $\mathbb{Z}$ defines a pseudometric $d_\varphi : A^{\mathbb{Z}} \times A^{\mathbb{Z}} \to [0, \infty)$

by $d_\varphi(x,y) := \varphi(\{i \in \mathbb{Z} : x_i \neq y_i\})$. The Cantor, Besicovich and Weyl pseudometrics on $A^{\mathbb{Z}}$ are defined by the following submeasures:

$$\begin{aligned}
\varphi_{\mathcal{C}}(U) &:= 2^{-\min\{|i|:\, i\in U\}} \\
\varphi_{\mathcal{B}}(U) &:= \limsup_{l\to\infty} \frac{|U\cap[-l,l)|}{2l} \\
\varphi_{\mathcal{W}}(U) &:= \limsup_{l\to\infty} \sup_{k\in\mathbb{Z}} \frac{|U\cap[k,k+l)|}{l}
\end{aligned}$$

7 The Cantor space

The Cantor metric on $A^{\mathbb{Z}}$ is defined by

$$d_{\mathcal{C}}(x,y) = 2^{-k} \quad \text{where} \quad k = \min\{|i| :\ x_i \neq y_i\}$$

so $d_{\mathcal{C}}(x,y) < 2^{-k}$ iff $x_{[-k,k]} = y_{[-k,k]}$. We denote by $\mathcal{C}_A = (A^{\mathbb{Z}}, d_{\mathcal{C}})$ the metric space of two-sided configurations with metric $d_{\mathcal{C}}$. The cylinders are clopen sets in $\mathcal{C}_A$. All Cantor spaces (with different alphabets) are homeomorphic. The Cantor space is compact, totally disconnected and perfect, and conversely, every space with these properties is homeomorphic to a Cantor space. Literature about CA dynamics in Cantor spaces is really huge. In this section, we just recall some results and definitions which will be used later.

Theorem 6 (Kůrka [12]) *Let $(\mathcal{C}_A, F)$ be a CA with radius r.*

(1) $(\mathcal{C}_A, F)$ is almost equicontinuous iff there exists a r-blocking word for F

(2) $(\mathcal{C}_A, F)$ is equicontinuous iff all sufficiently long words are r-blocking.

Denote by $\mathcal{E}_F$ the set of equicontinuous points of F. The sets of **equicontinuous directions** and **almost equicontinuous directions** of a CA $(\mathcal{C}_A, F)$ (see Sablik [18]) are defined by

$$\begin{aligned}
\mathfrak{E}(F) &= \left\{\frac{p}{q} :\ p\in\mathbb{Z}, q\in\mathbb{N}^+, \mathcal{E}_{F^q\sigma^p} = A^{\mathbb{Z}}\right\}, \\
\mathfrak{A}(F) &= \left\{\frac{p}{q} :\ p\in\mathbb{Z}, q\in\mathbb{N}^+, \mathcal{E}_{F^q\sigma^p} \neq \emptyset\right\}.
\end{aligned}$$

8 The periodic space

Definition 7 *The* **periodic space** *$\mathcal{P}_A = \{x \in A^{\mathbb{Z}} :\ \exists n > 0, \sigma^n(x) = x\}$ over an alphabet A consists of shift periodic configurations with Cantor metric $d_{\mathcal{C}}$.*

All periodic spaces (with different alphabets) are homeomorphic. The periodic space is not compact, but it is totally disconnected and perfect. It is dense in $\mathcal{C}_A$. If $(\mathcal{C}_A, F)$ is a CA, Then $F(\mathcal{P}_A) \subseteq \mathcal{P}_A$. We denote by $F_{\mathcal{P}} : \mathcal{P}_A \to \mathcal{P}_A$ the restriction of F to $\mathcal{P}_A$, so $(\mathcal{P}_A, F_{\mathcal{P}})$ is a (non-compact) dynamical system. Every $F_{\mathcal{P}}$-orbit is finite, so every point $x \in \mathcal{P}_A$ is $F_{\mathcal{P}}$-eventually periodic.

Theorem 8 *Let F be a CA over alphabet A.*

(1) $(\mathcal{C}_A, F)$ is surjective iff $(\mathcal{P}_A, F_{\mathcal{P}})$ is surjective.

(2) $(\mathcal{C}_A, F)$ is equicontinuous iff $(\mathcal{P}_A, F_{\mathcal{P}})$ is equicontinuous.

(3) $(\mathcal{C}_A, F)$ is almost equicontinuous iff $(\mathcal{P}_A, F_{\mathcal{P}})$ is almost equicontinuous.

(4) $(\mathcal{C}_A, F)$ is sensitive iff $(\mathcal{P}_A, F_{\mathcal{P}})$ is sensitive.

(5) $(\mathcal{C}_A, F)$ is transitive iff $(\mathcal{P}_A, F_{\mathcal{P}})$ is transitive.

Proof: (1a) Let F be surjective, let $y \in \mathcal{P}_A$ and $\sigma^n(y) = y$. There exists $z \in F^{-1}(y)$ and integers $i < j$ such that $z_{[inr,inr+r)} = z_{[jnr,jnr+r)}$. Then $x = (z_{[inr,jnr)})^\infty \in \mathcal{P}_A$ and $F_\mathcal{P}(x) = y$, so $F_\mathcal{P}$ is surjective.
(1b) Let $F_\mathcal{P}$ be surjective, and $u \in A^+$. Then u^∞ has $F_\mathcal{P}$-preimage and therefore u has preimage under the local rule. By Hedlund Theorem, $(\mathcal{C}_A, F)$ is surjective.
(2a) Since $\mathcal{P}_A \subset \mathcal{C}_A$, the equicontinuity of F implies trivially the equicontinuity of $F_\mathcal{P}$.
(2b) Let $F_\mathcal{P}$ be equicontinuous. There exist $m > r$ such that if $x, y \in \mathcal{P}_A$ and $x_{[-m,m]} = y_{[-m,m]}$, then $F^n(x)_{[-r,r]} = F^n(y)_{[-r,r]}$ for all $n \geq 0$. We claim that all words of length $2m+1$ are $(2r+1)$-blocking with offset $m-r$. If not, then for some $x, y \in A^\mathbb{Z}$ with $x_{[-m,m]} = y_{[-m,m]}$, there exists $n > 0$ such that $F^n(x)_{[-r,r]} \neq F^n(y)_{[-r,r]}$. For periodic configurations $x' = (x_{[-m-nr,m+nr]})^\infty$, $y' = (y_{[-m-nr,m+nr]})^\infty$ we get $F^n(x')_{[-r,r]} \neq F^n(y')_{[-r,r]}$ contradicting the assumption. By Theorem 6, F is $\mathcal{C}$-equicontinuous.
(3a) If $(\mathcal{C}_A, F)$ is almost equicontinuous, then there exists a r-blocking word u and $u^\infty \in \mathcal{P}_A$ is an equicontinuous configuration for $(\mathcal{P}_a, F_\mathcal{P})$.
(3b) The proof is analogous as (2b).
(4) and (5) follow from the Theorem 1 of Knudsen. □

9 The Toeplitz space

Definition 9 *Let A be an alphabet*

(1) The **Besicovitch pseudometric** *on $A^\mathbb{Z}$ is defined by*

$$d_\mathcal{B}(x, y) = \limsup_{l\to\infty} \frac{|\{j \in [-l, l) :\ x_j \neq y_j\}|}{2l}$$

(2) The **Weyl pseudometric** *on $A^\mathbb{Z}$ is defined by*

$$d_\mathcal{W}(x, y) = \limsup_{l\to\infty} \max_{k\in\mathbb{Z}} \frac{|\{j \in [k, k+l) :\ x_j \neq y_j\}|}{l}$$

Clearly $d_\mathcal{B}(x, y) \leq d_\mathcal{W}(x, y)$ and

$$d_\mathcal{B}(x, y) < \varepsilon \iff \exists l_0 \in \mathbb{N}, \forall l \geq l_0, |\{j \in [-l, l] :\ x_j \neq y_j\}| < (2l+1)\varepsilon.$$
$$d_\mathcal{W}(x, y) < \varepsilon \iff \exists l_0 \in \mathbb{N}, \forall l \geq l_0, \forall k \in \mathbb{Z}, |\{j \in [k, k+l) :\ x_j \neq y_j\}| < l\varepsilon$$

Both $d_\mathcal{B}$ and $d_\mathcal{W}$ are symmetric and satisfy the triangle inequality, but they are not metrics. Distinct configurations $x, y \in A^\mathbb{Z}$ can have zero distance. We construct a set of **regular quasi-periodic** configurations, on which $d_\mathcal{B}$ and $d_\mathcal{W}$ coincide and are metrics.

Definition 10

(1) The **period** *of $k \in \mathbb{Z}$ in $x \in A^\mathbb{Z}$ is $r_k(x) := \inf\{p > 0 :\ \forall n \in \mathbb{Z}, x_{k+np} = x_k\}$. We set $r_k(x) = \infty$ if the defining set is empty.*

(2) $x \in A^\mathbb{Z}$ is **quasi-periodic**, *if $r_k(x) < \infty$ for all $k \in \mathbb{Z}$.*

(3) A **periodic structure** *for a quasi-periodic configuration x is a sequence of positive integers $\mathbf{p} = (p_i)_{i<|\mathbf{p}|\leq\infty}$, such that $p_i|p_{i+1}$ (p_i divides p_{i+1}), and for every $k \in \mathbb{Z}$, $r_k(x)|p_i$ for some i.*

(3) A quasi-periodic configuration $x \in A^\mathbb{Z}$ is **regular**, *if for some periodic structure $\mathbf{p}$ of x we have $\lim_{i\to\infty} q_i(x)/p_i = 0$, where $q_i(x) := |\{k \in [0, p_i)_i :\ r_k(x) \not| p_i\}|$ ($r_k(x)$ does not divide p_i).*

Clearly every σ-periodic configuration is quasi-periodic and has a finite periodic structure.

Proposition 11

(1) If x, y are regular quasi-periodic configurations, then $d_{\mathcal{W}}(x,y) = d_{\mathcal{B}}(x,y)$.

(2) If $x \neq y$ are quasi-periodic configurations, then $d_{\mathcal{W}}(x,y) \geq d_{\mathcal{B}}(x,y) > 0$.

Proof: (1) We must show $d_{\mathcal{W}}(x,y) \leq d_{\mathcal{B}}(x,y)$. Let $\mathbf{p}^x$, $\mathbf{p}^y$ be the periodic structures for x and y and let $p_i = k_i^x p_i^x = k_i^y p_i^y$ be the lowest common multiple of p_i^x and p_i^y. Then $\mathbf{p} = (p_i)_i$ is a periodic structure for both x and y. For each $i > 0$ and for each $k \in \mathbb{Z}$ we have

$$|\{j \in [k - p_i, k + p_i) : \ x_j \neq y_j\}| \leq 2k_i^x q_i^x + 2k_i^y q_i^y + |\{j \in [-p_i, p_i) : \ x_j \neq y_j\}|$$

$$\begin{aligned} d_{\mathcal{W}}(x,y) &\leq \lim_{i\to\infty} \max_{k\in\mathbb{Z}} |\{j \in [k - p_i, k + p_i) : \ x_j \neq y_j\}| \\ &\leq \lim_{i\to\infty} \left(\frac{2k_i^x q_i^x}{2k_i^x p_i^x} + \frac{2k_i^y q_i^y}{2k_i^y p_i^y} + \frac{|\{j \in [-p_i, p_i) : \ x_j \neq y_j\}|}{2p_i} \right) \\ &= d_{\mathcal{B}}(x,y) \end{aligned}$$

(2) Since $x \neq y$, there exists i such that for some $k \in [0, p_i)$ and for all $n \in \mathbb{Z}$ we have $x_{k+np_i} = x_k \neq y_k = y_{k+np_i}$. It follows $d_{\mathcal{B}}(x,y) \geq 1/p_i$. □

Definition 12 *The* **Toeplitz space** $\mathcal{T}_A$ *over* A *consists of all regular quasi-periodic configurations with metric* $d_{\mathcal{B}} = d_{\mathcal{W}}$.

Toeplitz sequences are constructed by filling in periodic parts successively. For an alphabet A put $\widetilde{A} = A \cup \{*\}$.

Definition 13

(1) The p**-skeleton** $S_p(x) \in \widetilde{A}^{\mathbb{Z}}$ *of* $x \in A^{\mathbb{Z}}$ *is defined by*

$$S_p(x)_k = \begin{cases} x_k & \text{if} \ \ \forall n \in \mathbb{Z}, x_{k+np} = x_k \\ * & \text{otherwise} \end{cases}$$

(2) The **sequence of gaps** *of* $x \in \widetilde{A}^{\mathbb{Z}}$ *is the unique increasing integer sequence* $(t_i)_{a<i<b}$ *such that* $x_{t_i} = *$, $x_k \neq *$ *for* $t_i < k < t_{i+1}$ *and* $t_{-1} < 0 \leq t_0$.

(3) Let $x, y \in \widetilde{A}^{\mathbb{Z}}$ *and let* (t_i) *be the sequence of gaps of* x. *The* **amalgamation** $T(x,y) \in \widetilde{A}^{\mathbb{Z}}$ *of* x, y *is*

$$T(x,y)_i = \begin{cases} x_i & \text{if} \ \ x_i \neq * \\ y_j & \text{if} \ \ x_i = * \ \& \ i = t_j \end{cases}$$

The p-skeleton is p-periodic. If p is its smallest period, we say that p is an **essential** period of x. The sequence of gaps may be two-way infinite (then $a = -\infty$, $b = \infty$), one-way infinite ($a = -\infty$, $b < \infty$ or $-\infty < a$, $b < \infty$), finite ($-\infty < a < b < \infty$) or even empty when $x \in A^{\mathbb{Z}}$. If it is nonempty then it must be defined at least on -1 or 0.

Proposition 14 *Let* $\mathbf{2} := \{0, 1\}$ *be the binary alphabet and* $[0,1]$ *the real unit interval (with standard metric). There exists an isometry* $f : [0,1] \to \mathcal{T}_{\mathbf{2}}$ *such that* $f(0) = 0^\infty$ *and* $f(1) = 1^\infty$.

Proof: Consider a map $h : \mathbf{2}^* \to \widetilde{\mathbf{2}}^{\mathbb{Z}}$ defined by $h(\lambda) = *^\infty$, $h(0) = (0*)^\infty$, $h(1) = (*1)^\infty$, $h(x_0 \ldots x_{n-1} x_n) = T(h(x_0 \ldots x_{n-1}), h(x_n))$. Thus

```
 h(0)    =   0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*...
 h(1)    =   *1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1...
 h(01)   =   0*010*010*010*010*010*010*010*01...
 h(10)   =   01*101*101*101*101*101*101*101*1...
 h(011)  =   0*0101010*0101010*0101010*010101...
 h(100)  =   010101*1010101*1010101*1010101*1...
 h(0111) =   0*010101010101010*01010101010101...
 h(1000) =   01010101010101*101010101010101*1...
```

For $x \in \mathbf{2}^{\mathbb{N}}$, the limit (in the Cantor topology) $h(x) = \lim_{n\to\infty} h(x_0, \dots x_n)$ exists. If no suffix of x is 1^∞, then $h(x) \in \mathbf{2}^{\mathbb{Z}}$, otherwise $h(x)$ contains exactly one star and we replace it by 1. Thus for each $x \in \mathbf{2}^{\mathbb{N}}$, $h(x) \in \mathbf{2}^{\mathbb{Z}}$ is a regular quasi-periodic sequence. It can be verified directly that $h(0111\dots) = (01)^\infty = h(1000\dots)$. Using an easily verifiable formula $T(x, T(y,z)) = T(T(x,y),z)$, we get

$$\begin{aligned} h(x_0 \dots x_n 01^\infty) &= T(h(x_0 \dots x_n), h(01^\infty) = T(h(x_0 \dots x_n), h(10^\infty)) \\ &= h(x_0 \dots x_n 10^\infty) \end{aligned}$$

For a real number $x \in [0,1]$ with binary expansion $x = \sum_{i=0}^\infty x_i 2^{-i-1}$ put $f(x) = h(x_0x_1x_2x_3\dots)$. If $2^n x$ is an integer for some n, then x has two binary expansions, and $f(x)$ is the same for both expansions. If $|x-y| \le 2^{-m}$, then $x_{[0,m)} = y_{[0,m)}$; therefore $d_{\mathcal{W}}(f(x), f(y)) \le 2^{-m}$ and $f : [0,1] \to \mathcal{T}_A$ is continuous. For dyadic numbers x, y of the form $k/2^m$ we verify $d_{\mathcal{B}}(f(x), f(y)) = |x-y|$, so f is an isometry. □

Proposition 15 *The Toeplitz space $\mathcal{T}_A$ of regular quasi-periodic sequences is pathwise connected and infinite-dimensional.*

Proof: Assume that the alphabet A contains letters $0, 1$ and consider the map $f : [0,1] \to A^{\mathbb{Z}}$ from Proposition 14. For $a \in A$ set $a \cdot 0 = 0$, $a \cdot 1 = a$. Given $u \in \mathbf{2}^{\mathbb{Z}}$ the map $g_u : [0,1] \to \mathbf{2}^{\mathbb{Z}}$ defined by $g_u(x)_i = u_i f(x)_i$ is continuous, $g_u(0) = 0$ and $g_u(1) = u$. Thus $\mathcal{T}_A$ is pathwise connected. To show that $\mathcal{T}_A$ is infinite-dimensional, construct for any n an embedding $f_n : [0,1]^n \to X_W$ of an n-dimensional cube by

$$f_n(x_1, \dots, x_n) = f(x_1)_0 \dots f(x_n)_0 f(x_1)_1 \dots f(x_n)_1 \dots$$

Thus $\mathcal{T}_A$ is at least n-dimensional and therefore infinite-dimensional. □

Proposition 16 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA with radius r.*

(1) If $x \in A^{\mathbb{Z}}$ is a quasi-periodic with periodic structure $\mathbf{p}$, then $F(x)$ is quasi-periodic with periodic structure $\mathbf{p}$.

(2) If x is regular quasi-periodic, then $F(x)$ is regular.

Proof: (1) For $k \in \mathbb{Z}$ denote by $m := \min\{i : \ \forall j \in [k-r, k+r], r_j(x) | p_i\}$. Then p_m is a period of $F(x)_k$.
(2) We have

$$\begin{aligned} q_i(F(x)) &:= |\{k < p_i : \ r_k(F(x)) \not| p_i\}| \\ &\le (2r+1) \cdot |\{k < p_i : \ r_k(x) \not| p_i\}| \\ &= (2r+1) \cdot q_i(x), \end{aligned}$$

so

$$\lim_{i\to\infty} \frac{q_i(F(x))}{p_i} \le (2r+1) \cdot \lim_{i\to\infty} \frac{q_i(x)}{p_i} = 0$$

□

For a CA F we denote by $F_{\mathcal{T}}$ the restriction of F to $\mathcal{T}_A$.

Theorem 17 *Let F be a CA.*

(1) $(\mathcal{C}_A, F)$ is surjective iff $(\mathcal{T}_A, F_\mathcal{T})$ is surjective.

(2) If $\mathfrak{A}(F) \neq \emptyset$, then $(\mathcal{T}_A, F_\mathcal{T})$ is almost equicontinuous.

(3) if $\mathfrak{E}(F) \neq \emptyset$, then $(\mathcal{T}_A, F_\mathcal{T})$ is equicontinuous.

(4) If $(\mathcal{C}_A, F)$ is chain-transitive, then $(\mathcal{T}_A, F_\mathcal{T})$ is chain-transitive.

(5) $(\mathcal{T}_A, F_\mathcal{T})$ is injective iff it is surjective.

Proof: (1) The proof is the same as in Theorem 8(1)
(2) Assume first that F is almost equicontinuous, so there exists $m > r$ and $u \in A^{2m+1}$ such that for any $x, y \in [u]_{-m}$, $F^n(x)_{[-r,r]} = F^n(y)_{[-r,r]}$ for all $n > 0$. We show that u^∞ is $\mathcal{T}$-equicontinuous. For a given $\varepsilon > 0$ set $\delta = \varepsilon/(4m-2r+1)$. If $d_\mathcal{T}(y, x) < \delta$, then there exists l_0 such that for all $l \geq l_0$, $|\{i \in [-l, l] : \; x_i \neq y_i\}| < (2l+1)\delta$. For $k(2m+1) \leq j < (k+1)(2m+1)$, $F^n(y)_j$ can differ from $F^n(x)_j$ only if y differs from x in some $i \in [k(2m+1)-(m-r), (k+1)m+(m-r))$ Thus a change $x_i \neq y_i$ can cause at most $2m+1+2(m-r) = 4m-2r+1$ changes $F^n(y)_j \neq F^n(x)_j$. We get

$$|\{i \in [-l, l) : \; F^n(x)_i \neq F^n(y)_i\}| \leq 2l\delta(4m-2r+1) \leq 2l\varepsilon$$

This shows that $F_\mathcal{T}$ is almost equicontinuous. In the general case that $\mathfrak{A}(F) \neq \emptyset$, we get that $F_\mathcal{T}^q \sigma^p$ is almost equicontinuous for some $p \in \mathbb{Z}$, $q \in \mathbb{N}^+$. Since σ is $\mathcal{T}$-equicontinuous, $F_\mathcal{T}^q$ is almost equicontinuous and therefore $(\mathcal{T}_A, F_\mathcal{T})$ is almost equicontinuous.
(3) The proof is the same as in (2) with the only modification that all $u \in A^m$ are $(2r+1)$-blocking.
(4) The proof of Proposition 8 from [3] works in this case too.
(5) The proof of Proposition 12 of [2] works in this case also. □

10 The Besicovitch space

On $A^\mathbb{Z}$ we have an equivalence $x \approx_\mathcal{B} y$ iff $d_\mathcal{B}(x, y) = 0$. Denote by $\mathcal{B}_A$ the set of equivalence classes of $\approx_B$ and by $\pi_B : A^\mathbb{Z} \to \mathcal{B}_A$ the projection. The factor of $d_\mathcal{B}$ is a metric on $\mathcal{B}_A$. This is the **Besicovitch space** on alphabet A. Using prefix codes, it can be shown that every two Besicovitch spaces (with different alphabets) are homeomorphic. By Proposition 11 each equivalence class contains at most one quasi-periodic sequence.

Proposition 18 *$\mathcal{T}_A$ is dense in $\mathcal{B}_A$*

The proof of Proposition 9 of [2] works also for regular quasi-periodic sequences.

Theorem 19 (Blanchard, Formenti and Kůrka [3])
The Besicovitch space is pathwise connected, infinite-dimensional, homogenous and complete. It is neither separable nor locally compact.

The properties of path-connectedness and infinite dimensionality is proved analogously as in Proposition 15. To prove that $\mathcal{B}_A$ is neither separable nor locally compact, Sturmian configurations have been used in [3]. The completeness of $\mathcal{B}_A$ has been proved by Marcinkiewicz [16]. Every cellular automaton $F : A^\mathbb{Z} \to A^\mathbb{Z}$ is uniformly continuous with respect to $d_\mathcal{B}$, so it preserves the equivalence $\approx_B$. If $d_\mathcal{B}(x, y) = 0$, then $d_\mathcal{B}(F(x), F(y)) = 0$. Thus a cellular automaton F defines a uniformly continuous map $F_\mathcal{B} : \mathcal{B}_A \to \mathcal{B}_A$.

Theorem 20 (Blanchard, Formenti and Kůrka [3]) *Let F be a CA on A.*

(1) $(\mathcal{C}_A, F)$ *is surjective iff* $(\mathcal{B}_A, F_{\mathcal{B}})$ *is surjective.*

(2) If $\mathfrak{A}(F) \neq \emptyset$ *then* $(\mathcal{B}_A, F_{\mathcal{B}})$ *is almost equicontinuous.*

(3) if $\mathfrak{E}(F) \neq \emptyset$*, then* $(\mathcal{B}_A, F_{\mathcal{B}})$ *is equicontinuous.*

(4) If $(\mathcal{B}_A, F_{\mathcal{B}})$ *is sensitive, then* $(\mathcal{C}_A, F)$ *is sensitive.*

(5) No cellular automaton $(\mathcal{B}_A, F_{\mathcal{B}})$ *is positively expansive.*

(6) If $(\mathcal{C}_A, F)$ *is chain-transitive, then* $(\mathcal{B}_A, F_{\mathcal{B}})$ *is chain-transitive.*

Theorem 21 (Blanchard, Cervelle and Formenti [2])

(1) No CA $(\mathcal{B}_A, F_{\mathcal{B}})$ *is transitive.*

(2) A CA $(\mathcal{B}_A, F_{\mathcal{B}})$ *has either a unique fixed point and no other periodic point, or it has uncountably many periodic points.*

(3) If a surjective CA has a blocking word, then the set of its $F_{\mathcal{B}}$*-periodic points is dense in* $\mathcal{B}_A$*.*

11 The generic space

For a configuration $x \in A^{\mathbb{Z}}$ and word $v \in A^+$ set

$$\begin{aligned}\underline{\Phi}_v(x) &= \liminf_{n\to\infty} |\{i \in [-n,n) :\ x_{[i,i+|v|)} = v\}|/2n,\\ \overline{\Phi}_v(x) &= \limsup_{n\to\infty} |\{i \in [-n,n) :\ x_{[i,i+|v|)} = v\}|/2n.\end{aligned}$$

For every $v \in A^*$, $\underline{\Phi}_v, \overline{\Phi}_v : A^{\mathbb{Z}} \to [0,1]$ are continuous in the Besicovitch topology. In fact we have

$$\begin{aligned}|\overline{\Phi}_v(x) - \overline{\Phi}_v(y)| &\leq d_{\mathcal{B}}(x,y)\cdot|v|,\\ |\underline{\Phi}_v(x) - \underline{\Phi}_v(y)| &\leq d_{\mathcal{B}}(x,y)\cdot|v|\end{aligned}$$

Define the generic space (over the alphabet A) as

$$\mathcal{G}_A = \{x \in A^{\mathbb{Z}} :\ \forall v \in A^*, \underline{\Phi}_v(x) = \overline{\Phi}_v(x)\}$$

It is a closed subspace of $\mathcal{B}_A$. For $v \in A^*$ denote by $\Phi_v : \mathcal{G}_A \to [0,1]$ the common value of $\underline{\Phi}$ and $\overline{\Phi}$.

Using prefix codes, one can show that all generic spaces (with different alphabets) are homeomorhic. The generic space contains all uniquely ergodic subshifts, in particular all Sturmian sequences and all regular Toeplitz sequences. Thus the proofs in Blanchard Formenti and Kůrka [3] can be applied to the generic space too. In particular the generic space is homogenous. If we regard the alphabet $A = \{0,\ldots,m-1\}$ as the group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, then for every $x \in \mathcal{G}_A$ there is an isometry $H_x : \mathcal{G}_A \to \mathcal{G}_A$ defined by $H_x(y) = x + y$. Moreover, $\mathcal{G}_A$ is pahtwise connected, infinite-dimensional and complete (as a closed subspace the full Besicovitch space). It is neither separable nor locally compact. If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, then $F(\mathcal{G}_A) \subseteq \mathcal{G}_A$. Thus, the restriction of $F_{\mathcal{B}}$ to $\mathcal{G}_A$ defines a dynamical system $(\mathcal{G}_A, F_{\mathcal{G}})$.

Theorem 22 *Let* $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *be a CA.*

(1) $(\mathcal{C}_A, F)$ *is surjective iff* $(\mathcal{G}_A, F_{\mathcal{G}})$ *is surjective.*

(2) If $\mathfrak{A}(F) \neq \emptyset$*, then* $(\mathcal{G}_A, F_{\mathcal{G}})$ *is almost equicontinuous.*

(3) if $\mathcal{E}(F) \neq \emptyset$*, then* $(\mathcal{G}_A, F_{\mathcal{G}})$ *is equicontinuous.*

(4) If $(\mathcal{G}_A, F_{\mathcal{G}})$ *is sensitive, then* $(\mathcal{C}_A, F)$ *is sensitive.*

(5) If F *is* $\mathcal{C}$*-chain transitive, then* F *is* $\mathcal{G}$*-chain transitive.*

The proofs are the same as the proofs of corresponding properties in [3].

12 The space of measures

By a **measure** we mean a **Borel shift-invariant probability measure** on the Cantor space $A^{\mathbb{Z}}$ (see Pivato [17]). This is a countably additive function μ on the Borel sets of $A^{\mathbb{Z}}$ which assigns 1 to the full space and satisfies $\mu(U) = \mu(\sigma^{-1}(U))$. A measure on $A^{\mathbb{Z}}$ is determined by its values on cylinders $\mu(u) := \mu([u]_n)$ which does not depend on $n \in \mathbb{Z}$. Thus a measure can be identified with a map $\mu : A^* \to [0,1]$ subject to **bilateral Kolmogorov compatibility conditions**

$$\sum_{a \in A} \mu(ua) = \sum_{a \in A} \mu(au) = \mu(u), \quad \mu(\lambda) = 1$$

Define the distance of two measures

$$d_{\mathcal{M}}(\mu, \nu) := \sum_{u \in A^+} |\mu(u) - \nu(u)| \cdot |A|^{-2|u|}$$

This is a metric which yields the topology of weak* convergence on the compact space $\mathcal{M}_A := \mathfrak{M}_\sigma(A^{\mathbb{Z}})$ of shift-invariant Borel probability measures. A CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with local rule f determines a continuous and affine map $F_{\mathcal{M}} : \mathcal{M}_A \to \mathcal{M}_A$ by

$$(F_{\mathcal{M}}(\mu))(u) = \sum_{v \in f^{-1}(u)} \mu(v)$$

Moreover F and $F\sigma$ determine the same dynamical system on $\mathcal{M}_A$: $F_{\mathcal{M}} = (F\sigma)_{\mathcal{M}}$.

For $x \in \mathcal{G}_A$ denote by $\Phi^x : A^* \to [0,1]$ the function $\Phi^x(v) = \Phi_v(x)$. For every $x \in \mathcal{G}_A$, Φ^x is a shift-invariant Borel probability measure. The map $\Phi : \mathcal{G}_A \to \mathcal{M}_A$ is continuous with respect to the Besicovich and weak* topologies. In fact we have

$$\begin{aligned} d_{\mathcal{M}}(\Phi^x, \Phi^y) &\leq d_{\mathcal{B}}(x,y) \sum_{u \in A^+} |u| \cdot |A|^{-2|u|} = d_{\mathcal{B}}(x,y) \sum_{n>0} n \cdot |A|^{-n} \\ &= d_{\mathcal{B}}(x,y) \cdot |A|/(|A|-1)^2 \end{aligned}$$

By a theorem of Kamae [10], Φ is surjective. Every shift-invariant Borel probability measure has a generic point. It follows from the Ergodic Theorem that if μ is a σ-invariant measure, then $\mu(\mathcal{G}_A) = 1$ and for every $v \in A^*$, the measure of v is the integral of its density Φ_v,

$$\mu(v) = \int \Phi_v(x) \, d\mu.$$

If F is a CA, we have a commutative diagram $\Phi F_{\mathcal{G}} = F_{\mathcal{M}} \Phi$.

$$\begin{array}{ccc} \mathcal{G}_A & \xrightarrow{F_{\mathcal{G}}} & \mathcal{G}_A \\ \Phi \downarrow & & \downarrow \Phi \\ \mathcal{M}_A & \xrightarrow{F_{\mathcal{M}}} & \mathcal{M}_A \end{array}$$

Theorem 23 *Let F be a CA over A.*

(1) $(\mathcal{C}_A, F)$ is surjective iff $(\mathcal{M}_A, F_{\mathcal{M}})$ is surjective.

(2) If $(\mathcal{G}_A, F_{\mathcal{G}})$ has dense set of periodic points, then $(\mathcal{M}_A, F_{\mathcal{M}})$ has dense set of periodic points.

(3) If $\mathcal{A}(F) \neq \emptyset$, then $(\mathcal{M}_A, F_{\mathcal{M}})$ is almost equicontinuous.

(4) If $\mathfrak{E}(F) \neq \emptyset$, then $(\mathcal{M}_A, F_{\mathcal{M}})$ is equicontinuous.

Proof: (1) See Kůrka [14] for a proof.
(2) This holds since $(\mathcal{M}_A, F_\mathcal{M})$ is a factor of $(\mathcal{G}_A, F_\mathcal{G})$.
(3) It suffices to prove the claim for the case that F is almost equicontinuous. In this case there exists a blocking word $u \in A^+$ and the Dirac measure δ_u defined by

$$\delta_u(v) = \begin{cases} 1/|u| & \text{if} \quad v \sqsubseteq u \\ 0 & \text{if} \quad v \not\sqsubseteq u \end{cases}$$

is equicontinuous for $(\mathcal{M}_A, F_\mathcal{M})$.
(4) If $(\mathcal{C}_A, F)$ is equicontinuous, then all sufficiently long words are blocking and there exists $d > 0$ such that for all $n > 0$, and for all $x, y \in A^\mathbb{Z}$ such that $x_{[-n-d,n+d]} = y_{[-n-d,n+d]}$ we have $F^k(x)_{[-n,n]} = F^k(y)_{[-n,n]}$ for all $k > 0$. Thus there are maps $g_k : A^* \to A^*$ such that $|g_k(u)| = \max\{|u| - 2d, 0\}$ and for every $x \in A^\mathbb{Z}$ we have $F^k(x)_{[-n,n]} = F_k(x_{[-n-kd,n+kd]}) = g_k(x_{[-n-d,n+d]})$, where f is the local rule for F. We get

$$\begin{aligned} d_\mathcal{M}(F^k_\mathcal{M}(\mu), F^k_\mathcal{M}(\nu)) &= \sum_{n=1}^{\infty} \sum_{u \in A^n} \left| \sum_{v \in f^{-k}(u)} (\mu(v) - \nu(v)) \right| \cdot |A|^{-2n} \\ &= \sum_{n=1}^{\infty} \sum_{u \in A^n} \left| \sum_{v \in g_k^{-1}(u)} (\mu(v) - \nu(v)) \right| \cdot |A|^{-2n} \\ &\le \sum_{n=1}^{\infty} \sum_{v \in A^{n+2d}} |\mu(v) - \nu(v)| \cdot |A|^{-2n} \\ &\le |A|^{4d} \cdot d_\mathcal{M}(\mu, \nu) \end{aligned}$$

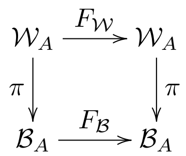
□

13 The Weyl space

Define the following equivalence relation on $A^\mathbb{Z}$: $x \approx_\mathcal{W} y$ iff $d_\mathcal{W}(x, y) = 0$. Denote by $\mathcal{W}_A$ the set of equivalence classes of $\approx_W$ and by $\pi_W : A^\mathbb{Z} \to \mathcal{W}_A$ the projection. The factor of $d_\mathcal{W}$ is a metric on $\mathcal{W}_A$. This is the Weyl space on alphabet A. Using prefix codes, it can be shown that every two Weyl spaces (with different alphabets) are homeomorphic. The Toeplitz space is not dense in the Weyl space (see Blanchard, Cervelle and Formenti [2]).

Theorem 24 (Blanchard, Formenti and Kůrka [3])
The Weyl space is pathwise connected, infinite-dimensional and homogenous. It is neither separable nor locally compact. It is not complete.

Every cellular automaton $F : A^\mathbb{Z} \to A^\mathbb{Z}$ is continuous with respect to $d_\mathcal{W}$, so it preserves the equivalence $\approx_W$. If $d_\mathcal{W}(x, y) = 0$, then $d_\mathcal{W}(F(x), F(y)) = 0$. Thus a cellular automaton F defines a continuous map $F_\mathcal{W} : \mathcal{W}_A \to \mathcal{W}_A$. The shift map $\sigma : \mathcal{W}_A \to \mathcal{W}_A$ is again an isometry, so in $\mathcal{W}_A$ many topological properies are preserved if F is composed with a power of the shift. This is true for example for equicontinuity, almost continuity and sensitivity. If $\pi : \mathcal{W}_A \to \mathcal{B}_A$ is the (continuous) projection and F a CA, then the following diagram commutes.



Theorem 25 (Blanchard, Formenti and Kůrka [3])
Let F be a CA on A.

(1) $(\mathcal{C}_A, F)$ is surjective iff $(\mathcal{W}_A, F_\mathcal{W})$ is surjective.

(2) If $\mathfrak{A}(F) \neq \emptyset$, then $(\mathcal{W}_A, F_\mathcal{W})$ is almost equicontinuous.

(3) if $\mathfrak{E}(F) \neq \emptyset$, then $(\mathcal{W}_A, F_\mathcal{W})$ is equicontinuous.

(4) If $(\mathcal{C}_A, F)$ is chain-transitive, then $(\mathcal{W}_A, F_\mathcal{W})$ is chain-transitive.

Theorem 26 (Blanchard, Cervelle and Formenti [2])
No CA is $(\mathcal{W}_A, F_\mathcal{W})$ is transitive.

Theorem 27 *Let Σ be a subshift attractor of finite type for F (in the Cantor space). Then there exists $\delta > 0$ such that for every $x \in \mathcal{W}_A$ satisfying $d_\mathcal{W}(x, \Sigma) < \delta$, $F^n(x) \in \Sigma$ for some $n > 0$.*

Thus a subshift attractor of finite type is a $\mathcal{W}$-attractor. Example 2 shows that it need not be $\mathcal{B}$-attractor. Example 3 shows that the assertion need not hold if Σ is not of finite type.
Proof: Let $U \subseteq A^\mathbb{Z}$ be a $\mathcal{C}$-clopen set such that $\Sigma = \Omega_F(U)$. Let U be a union of cylinders of words of length q. Set $\widetilde{\Omega}_\sigma(U) = \bigcap_{n\in\mathbb{Z}} \sigma^n(U)$. By a generalization of a theorem of Hurd [8] (see Kůrka [15]), there exists $m > 0$ such that $\Sigma = F^m(\widetilde{\Omega}_\sigma)$. If $d_\mathcal{W}(x, \Sigma) < 1/q$ then there exists $l > 0$ such that for every $k \in \mathbb{Z}$ there exists a nonnegative $j < l$ such that $\sigma^{k+j}(x) \in U$. It follows that there exists $n > 0$ such that $F^n(x) \in \widetilde{\Omega}_\sigma(U)$ and therefore $F^{n+m}(x) \in \Sigma$.
□

14 Examples

Example 1 *The Identity rule* $\mathrm{Id}(x) = x$.

$(\mathcal{B}_A, \mathrm{Id}_\mathcal{B})$ and $(\mathcal{W}_A, \mathrm{Id}_\mathcal{W})$ are chain-transitive (since both $\mathcal{B}_A$ and $\mathcal{W}_A$ are connected). However, $(\mathcal{C}_A, \mathrm{Id})$ is not chain-transitive. Thus the converse of Theorem 20(6) and of Theorem 25(4) does not hold.

Figure 1: The product ECA184

Example 2 *The product rule ECA128 $F(x)_i = x_{i-1} \cdot x_i \cdot x_{i+1}$.*

$(\mathcal{C}_A, F)$, $(\mathcal{B}_A, F_\mathcal{B})$ and $(\mathcal{W}_A, F_\mathcal{W})$ are almost equicontinous and the configuration 0^∞ is equicontinuous in all these versions. By Theorem 27, $\{0^\infty\}$ is a $\mathcal{W}$-attractor. However, contrary to a mistaken Proposition 9 in [3], $\{0^\infty\}$ is not $\mathcal{B}$-attractor. For a given $0 < \varepsilon < 1$ define $x \in A^\mathbb{Z}$ by $x_i = 1$ iff $3^n(1-\varepsilon) < |i| \leq 3^n$ for some $n \geq 0$. Then $d_\mathcal{B}(x, 0^\infty) = \varepsilon$ but x is a fixed point, since $d_\mathcal{B}(F(x), x) = \lim_{n\to\infty} 2n/3^n = 0$ (see Figure 1).

Example 3 *The trafic ECA184 $F(x)_i = 1$ iff $x_{[i-1,i]} = 10$ or $x_{[i,i+1]} = 11$.*

No $F^q\sigma^p$ is $\mathcal{C}$-almost equicontinuous, so $\mathfrak{A}(F) = \emptyset$. However, if $d_\mathcal{W}(x, 0^\infty) < \delta$, then $d_\mathcal{W}(F^n(x), 0^\infty) < \delta$ for every $n > 0$, since F conserves the number of letters 1 in a configuration. Thus 0^∞ is a point of equicontinuity in $(\mathcal{T}_A, F_\mathcal{T})$,

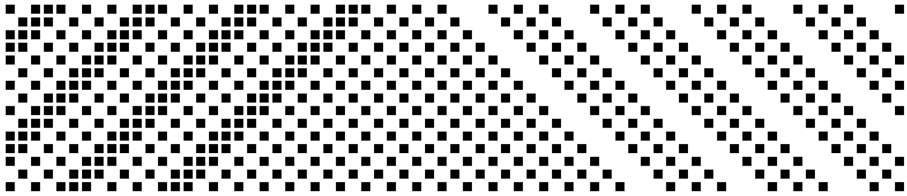


Figure 2: The traffic ECA184

$(\mathcal{B}_A, F_\mathcal{B})$, and $(\mathcal{W}_A, F_\mathcal{W})$. This shows that item (2) of Theorems 17, 20 and 25 cannot be converted. The maximal $\mathcal{C}$-attractor $\Omega_F = \{x \in A^\mathbb{Z} : \forall n > 0, 1(10)^n 0 \not\sqsubseteq x\}$ is not SFT. We show that it does not $\mathcal{W}$-attracts points from any of its neighbourhood. For a given even integer $q > 2$ define $x \in A^\mathbb{Z}$ by

$$x_i = \begin{cases} 0 & \text{if } \exists n \geq 0, i = qn+1 \\ 1 & \text{if } \exists n < 0, i = qn \\ ((01)^\infty)_i & \text{otherwise} \end{cases}$$

Then $d_W(F^k(x), \Omega_F) = 1/q$ for all $k > 0$ (see Figure 2, where $q = 8$).

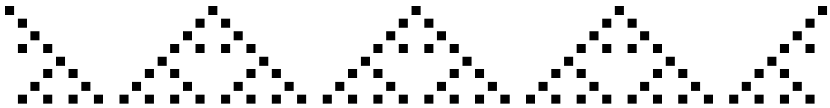


Figure 3: The sum ECA90

Example 4 *The sum ECA90* $F(x)_i = (x_{i-1} + x_{i+1}) \bmod 2$.

Both $(\mathcal{B}_A, F_\mathcal{B})$ and $(\mathcal{W}_A, F_\mathcal{W})$ are sensitive (Cattaneo et al. [4]). For a given $n > 0$ define a configuration z by $z_i = 1$ iff $i = k2^n$ for some $k \in \mathbb{Z}$. Then $F^{2^{n-1}}(z) = (01)^\infty$. For any $x \in A^\mathbb{Z}$, we have $d_\mathcal{W}(x, x+z) = 2^{-n}$ but $d_\mathcal{W}(F^{2^{n-1}}(x), F^{2^{n-1}}(x+z)) = 1/2$. The same argument works for $(\mathcal{B}_A, F_\mathcal{B})$.

Example 5 *The shift ECA170* $F(x)_i = x_{i+1}$.

Since the system has fixed points 0^∞ and 1^∞, it has uncountable number of periodic points. However, the periodic points are not dense in $\mathcal{B}_A$ ([2]).

15 Future directions

One of the promising research directions is the connection between the generic space and the space of Borel probability measures which is based on the factor map Φ. In particular Lyapunov functions based on particle weight functions (see Kůrka [13]) work both for the measure space $\mathcal{M}_A$ and the generic space $\mathcal{G}_A$. The potential of Lyapunov functions for the classification of attractors has not yet been fully explored. This holds also for the connections between attractors in different topologies. While the theory of attractors is well established in compact spaces, in noncompact spaces there are several possible approaches. Finally, the comparison of entropy properties of CA in different topologies may be revealing for classification of CA.

There is even a more general approach to different topologies for CA based on the concept of submeasure on $\mathbb{Z}$. Since each submeasure defines a pseudometric, it would be interesting to know, whether CA are continuous with respect to any of these pseudometrics, and whether some dynamical properties of CA can be derived from the properties of defining submeasures.

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