Cellular Automata with an Infinite Number of Subshift Attractors

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We argue that the complexity of a cellular automaton is reflected in the complexity of the lattice of its subshift attractors (i.e., of those attractors which are subshifts). We construct cellular automata with an infinite lattice of subshift attractors of arbitrarily high complexity.

1. Introduction

The concept of an attractor is essential for understanding the dynamics of cellular automata. An *attractor* is defined as a limit set (intersection of forward images) of a nonempty clopen (closed and open) invariant set. Classification of cellular automata based on the system of their attractors has been considered in Hurley [1] or Kůrka [2]. The basic distinction is between cellular automata which possess disjoint attractors and those with attractors that all have nonempty intersections.

Another distinction investigated in Formenti and Kůrka [3] is between attractors which are subshifts and those which are not. For example, each clopen set is an attractor for the identity map, since it is invariant and its limit set is itself. Attractors of this kind, however, do not correspond very well with the intuitive idea of attraction.

Much more interesting are attractors which are subshifts. It turns out that the limit set of a clopen invariant set is a subshift if and only if the set is spreading, that is, if it propagates both to the left and to the right. This is much closer to the spirit of attraction and gives more relevant information on the dynamics of the cellular automaton in question.

The number of attractors may be either finite or countable. Cellular automata with an infinite number of subshift attractors have much higher complexity because they must perform actions at a distance. Their spreading sets may have the same structure but they must have arbitrarily large sizes. Some of these cellular automata perform a kind of self-reproduction when a seed pattern (of arbitrary size) spreads all over the cellular space.

We construct two basic examples of such behavior. In Example 8 there is a stationary particle which is never created and sometimes it is destroyed. The nth spreading set consists of patterns (words) in which the distance between neighboring stationary particles is at least n. A particle whose nearest neighbor is closer is destroyed. In Example 9 there is a stationary particle which is never destroyed and sometimes it is created. A pair of particles at distance n creates other particles at distance n to both their left and right. While both of these cellular automata have an infinite number of attractors, their attractors are ordered differently by the inclusion. In Example 8 we have a decreasing sequence of attractors, in Example 9 we have an increasing sequence of attractors all included in the maximal attractor.

The system of subshift attractors of a cellular automaton forms a lattice with maximum element. The union of two subshift attractors is again a subshift attractor. Their intersection need not be an attractor, but it is nonempty and contains a subshift attractor.

These lattices can have quite complex structures. In the case of linearly ordered lattices, Example 8 yields the lattice of negative integers and Example 9 yields the lattice of positive integers together with infinity. We can combine these two examples to obtain linearly ordered lattices with an arbitrary number of segments isomorphic to $\mathbb{N}^-$, $\mathbb{N}\cup\{\infty\}$, or $\mathbb{Z}\cup\{\infty\}$. There are also many cellular automata whose lattices of subshift attractors are not linearly ordered. The complexity of the lattice of subshift attractors thus may serve as a suitable measure for the complexity of a cellular automaton.

2. Cellular automata and subshifts

For a finite alphabet A, denote by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A. The length of a word $u = u_0 \dots u_{n-1} \in A^n$ is denoted by $|u| := n$. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{Z}}$ the space of *A-configurations*, or doubly-infinite sequences of letters of A equipped with the metric

$$d(x,y) := 2^{-n}, \text{ where } n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}.$$

For any nonzero $u \in A^*$ we have a periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. The *cylinder* $[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l,l+|u|)} = u\}$ of a word $u \in A^*$ located at $l \in \mathbb{Z}$ is a clopen set (closed and open). We abbreviate $[u] := [u]_0$. Any clopen set is a finite union of cylinders.

The *shift map* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i := x_{i+1}$. A *subshift* is a nonempty subset $\Sigma \subseteq A^{\mathbb{Z}}$ which is closed and strongly σ-invariant, that is, $\sigma(\Sigma) = \Sigma$. A subshift is uniquely determined by its *language* $\mathcal{L}(\Sigma) := \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$.

A *cellular automaton* is a continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which commutes with the shift map, that is, $F\sigma = \sigma F$. For any cellular automaton

F there exists a *local rule* $f : A^{2r+1} \to A$ such that $F(x)_i = f(x_{[i-r,i+r]})$ for some *radius* $r \geq 0$. A set $Y \subseteq A^{\mathbb{Z}}$ is F-*invariant* if $F(Y) \subseteq Y$. The *omega-limit* of a closed F-invariant set Y is $\Omega_F(Y) := \bigcap_{n\geq 0} F^n(Y)$. A set $Y \subseteq A^{\mathbb{Z}}$ is an *attractor* of F if there exists a clopen invariant set V such that $\Omega_F(V) = Y$. The maximal attractor $\Omega_F := \Omega_F(A^{\mathbb{Z}})$ is also called the *limit set* of F. We denote by $\mathcal{A}(F)$ the system of attractors of F and $\mathcal{A}_\sigma(F)$ the system of the subshift attractors of F.

An ε-chain is a sequence of configurations $x_0, x_1, \ldots, x_n$ such that $d(F(x_i), x_{i+1}) < \varepsilon$ for $i < n$. The *chain relation* $C \subseteq A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ is defined by $(x, y) \in C$ if and only if for every $\varepsilon > 0$ there exists an ε-chain from x to y. The set of *chain-recurrent* configurations is the diagonal $|C| := \{x \in A^{\mathbb{Z}} : (x, x) \in C\}$. A subset $\mathrm{C} \subseteq |C|$ is a *chain-terminal set*, if $x \in \mathrm{C}$ and $(x, y) \in C$ implies $y \in \mathrm{C}$. The attractors are characterized by their chain-recurrent configurations. If Y is an attractor, then $Y \cap |C|$ is a chain terminal set. Moreover, if $Y_0 \neq Y_1$ are distinct attractors, then $Y_0 \cap |C| \neq Y_1 \cap |C|$ (see Akin [4] or Kůrka [5]). The chain relation is transitive and its symmetrization $C \cap C^{-1}$ restricted to $|C|$ is an equivalence. The equivalence classes are called *chain-components*. A chain-terminal set is a union of chain-components. Chain-terminal sets can be read off from the *graph of chain-components* which is an oriented graph whose vertices are chain-components and there is an edge $C_0 \to C_1$ if and only if there exist $x_0 \in C_0$, $x_1 \in C_1$ such that $(x_0, x_1) \in C$.

3. Subshift attractors and the small quasi-attractor

A *subshift attractor* of a cellular automaton is an attractor which is a subshift. For example, the maximal attractor Ω_F is a subshift attractor. To make the paper self-contained, we reproduce here the basic properties of subshift attractors given in Kůrka [6] or Formenti and Kůrka [3]. First we characterize clopen invariant sets whose limit sets are subshifts.

Definition 1. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. We say that a clopen F-invariant set $U \subseteq A^{\mathbb{Z}}$ is *spreading to the right* if $F^k(U) \subseteq \sigma^{-1}(U)$ for some $k > 0$, and *spreading to the left* if $F^k(U) \subseteq \sigma(U)$ for some $k > 0$. We say that U is *spreading* if it is spreading to both the right and left.

Proposition 1. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and U a clopen F-invariant set. Then $\Omega_F(U)$ is a subshift attractor if and only if U is spreading.

Proof. If $\Omega_F(U)$ is a subshift, then $\Omega_F(U) = \Omega_F(\sigma(U)) \subseteq \sigma(U)$, and by compactness there exists $k > 0$ such that $F^k(U) \subseteq \sigma(U)$. A similar argument applies to $\sigma^{-1}(U)$. Conversely, assume that $F^k(U) \subseteq \sigma^{-1}(U) \cap U \cap \sigma(U)$. If $y \in \Omega_F(U)$ and $n \geq 0$ then $y \in F^{n+k}(U)$, so there exists

$x \in U$ with $y = F^{n+k}(x) = F^n(F^k(x)) \in F^n(\sigma^{-1}(U))$, and $y \in \Omega_F(\sigma^{-1}(U)) = \bigcap_{n\geq 0} \sigma^{-1} F^n(U)$. Thus $\sigma(y) \in \Omega_F(U)$, and $\sigma(\Omega_F(U)) \subseteq \Omega_F(U)$. Similarly we prove $\sigma^{-1}(\Omega_F(U)) \subseteq \Omega_F(U)$, so $\Omega_F(U)$ is a subshift. ■

Proposition 2. If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton and U_0, U_1 spreading sets, then both $U_0 \cup U_1$ and $U_0 \cap U_1$ are nonempty spreading sets, and

$$\Omega_F(U_0 \cup U_1) = \Omega_F(U_0) \cup \Omega_F(U_1)$$
$$\Omega_F(U_0 \cap U_1) \subseteq \Omega_F(U_0) \cap \Omega_F(U_1).$$

Proof. The case of the union is elementary (see Kůrka [5] for details). In the case of intersection, the only nontrivial part is to show that $U_0 \cap U_1$ is nonempty. Since a clopen set is a finite union of cylinders, there exists a configuration $x \in A^{\mathbb{Z}}$ such that the σ-orbit of x visits each U_i with bounded gaps, that is,

$$\exists n > 0, \quad \forall i \leq 1, \quad \forall p \in \mathbb{Z}, \quad \exists j < n, \quad \sigma^{p+j}(x) \in U_i.$$

By Proposition 1, $F^m(x) \in \widetilde{U} := \bigcap_{n\in\mathbb{Z}} \sigma^n(U_0 \cap U_1)$ for some $m \geq 0$. Thus $\widetilde{U}$ is nonempty and F-invariant. The nonempty set $\Omega_F(\widetilde{U})$ is contained in both $\Omega_F(U_0)$ and $\Omega_F(U_1)$. ■

If F is a surjective cellular automaton, then its only subshift attractor is the full space $A^{\mathbb{Z}}$ (see Kůrka [6] for a proof). It follows from Proposition 2 that the intersection of all subshift attractors is nonempty. In Formenti and Kůrka [3] the following stronger theorem was proved.

Theorem 3. Let F be a cellular automaton. The intersection of all subshift attractors of all $F^q\sigma^p$, where $q > 0$ and $p \in \mathbb{Z}$, is a nonempty F-invariant subshift called the *small quasi-attractor* Q_F. Moreover, $F : Q_F \to Q_F$ is surjective.

Example 4. There exists a cellular automaton whose small quasi-attractor is properly contained in the intersection of all its attractors.

Proof. The elementary cellular automaton $F(x)_i = x_i x_{i+1}$ has a unique subshift attractor $\Omega_F = \{x \in \{0,1\}^{\mathbb{Z}} : \ \forall n > 0, 10^n 1 \not\sqsubseteq x\}$. However, $F^2\sigma^{-1}(x)_i = x_{i-1}x_i x_{i+1}$ has as a minimal attractor $\{0^\infty\}$, so $Q_F = \{0^\infty\}$. ■

4. The lattice of subshift attractors

A lattice is a partially ordered set $(L, \leq)$, such that any two elements $x, y \in L$ have their meet $x \wedge y$ and join $x \vee y$. The meet $x \wedge y$ is the (unique) largest element which is less than both x and y. The join $x \vee y$ is the smallest element which is larger than both x and y. A lattice is therefore a structure $(L, \leq, \wedge, \vee)$ satisfying axioms

$$
\begin{aligned}
x \leq y \ \ \& \ \ y \leq z \ &\Longrightarrow \ x \leq z, \\
x \leq y \ \ \& \ \ y \leq x \ &\Longleftrightarrow \ x = x, \\
z \leq x \ \ \& \ \ z \leq y \ &\Longleftrightarrow \ z \leq x \wedge y, \\
x \leq z \ \ \& \ \ y \leq z \ &\Longleftrightarrow \ x \vee y \leq z.
\end{aligned}
$$

A finite lattice has a unique minimal and a unique maximal element. If Σ_0, Σ_1 are subshift attractors of a cellular automaton F, then their union $\Sigma_0 \cup \Sigma_1$ is a subshift attractor. Their intersection is not necessarily an attractor but it is nonempty and its omega-limit $\Sigma_0 \wedge \Sigma_1 := \Omega_F(\Sigma_0 \cap \Sigma_1)$ is the largest attractor contained in both Σ_0 and Σ_1. Thus the system $\mathcal{A}_\sigma(F)$ of subshift attractors of F ordered by inclusion forms a lattice with join $\cup$ and meet $\wedge$ and with maximal element Ω_F.

Example 5. For each $n \geq 0$ there exists a cellular automaton with a linearly ordered lattice of subshift attractors $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_n$.

Proof. For $n = 0$ take the zero cellular automaton $F(x)_i = 0$ in the binary alphabet. For $n > 0$ take alphabet $A := \{0, 1, \ldots, n\}$ and cellular automaton $F(x)_i = \min\{x_{i-1}, x_i, x_{i+1}\}$. Then $U_k := [0] \cup \cdots \cup [k]$ is a spreading set and $\Sigma_k := \Omega_F(U_k)$ is a subshift attractor. The inclusions $\Sigma_{k-1} \subset \Sigma_k$ are proper since $k^\infty \in \Sigma_k \setminus \Sigma_{k-1}$. ■

Example 6. The intersection of two subshift attractors need not be an attractor.

Proof. Take the alphabet $A = \{0, 1, 2, 3\}$ and local rule $f : A^3 \to A$ given by

$$x1x{:}0, \quad z2x{:}1, \quad x2z{:}1, \quad y3x{:}1, \quad x3y{:}1,$$

where $x \in A$, $y \in \{0, 1, 2\}$, $z \in \{0, 1, 3\}$ and the first applicable production is used, otherwise the letter is left unchanged (see the simulation depicted in Figure 1). Then $U_2 := [0] \cup [1] \cup [2]$, $U_3 := [0] \cup [1] \cup [3]$

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000222222111113333322222323232333333000
000122221000001333112221111111133331000
000012210000000131001210000000013310000
000001100000000010000100000000001100000
000000000000000000000000000000000000000
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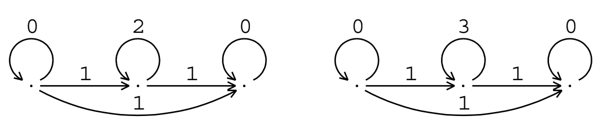
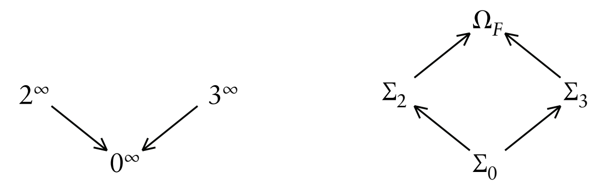


$_F(U_2)$ $_F(U_3)$

Figure 1. Intersection of attractors.



(a) (b)

Figure 2. Chain-components and attractors for Example 6.

are spreading sets whose attractors $\Sigma_2 := \Omega_F(U_2)$, $\Sigma_3 := \Omega_F(U_3)$ are sofic subshifts with labeled graphs as shown in Figure 1. The subshifts consist of all labels of all (doubly infinite) paths in their corresponding graphs. Then $\Sigma_0 := \Sigma_2 \wedge \Sigma_3 = \{0^\infty\}$ and $0^\infty 10^\infty \in (\Sigma_2 \cap \Sigma_3) \setminus \Sigma_0$. There is one more subshift attractor: $\Omega_F = \Sigma_2 \cup \Sigma_3$. ■

The lattice of subshift attractors of the cellular automaton in Example 6 can be analyzed using the chain relation. The only chain-recurrent configurations of F are $|C| = \{0^\infty, 2^\infty, 3^\infty\}$ and the chain-components are singletons. The graph of chain-components is shown in Figure 2(a). We see that there are exactly four chain-terminal sets, so there are no more than four attractors. In general, not every chain-terminal set corresponds to an attractor (see Kůrka [5]). In the present case, however, there are exactly four attractors and all of them are subshifts:

$$\Sigma_0 \cap |C| = \{0^\infty\}, \qquad \Sigma_2 \cap |C| = \{0^\infty, 2^\infty\},$$
$$\Sigma_3 \cap |C| = \{0^\infty, 3^\infty\}, \qquad \Omega_F \cap |C| = \{0^\infty, 2^\infty, 3^\infty\}.$$

The lattice of subshift attractors can be seen in Figure 2(b), where the arrows mean inclusions.

Lattices of higher complexity may be obtained by taking the cartesian product of cellular automata. If U, V are spreading sets for a cellular automaton F, then $U \times V$ is a spreading set for $F \times F$. Further spreading sets can be obtaind by taking the unions. Take, for example, the elementary cellular automaton $F(x)_i = x_{i-1}x_ix_{i+1}$ in the binary alphabet. It has two attractors $\Omega_F([0]_0) = \{0^\infty\}$, $\Omega_F = \{x \in \{0,1\}^{\mathbb{Z}} : \ \forall n > 0, 10^n1 \not\sqsubseteq x\}$. The product $F \times F$ is a cellular automaton in alphabet $A = \{0,1,2,3\}$, where $0 = (0,0)$, $1 = (1,0)$, $2 = (0,1)$, $3 = (1,1)$. The only chain-recurrent configurations are $|C| = \{0^\infty, 1^\infty, 2^\infty, 3^\infty\}$ and the chain-components are singletons. The graph of chain-components is shown in Figure 3(a). There are exactly five chain-terminal sets and each of them corresponds to an attractor. The spreading sets of these attractors are $U_0 := [0]$, $U_1 := [0] \cup [1]$, $U_2 := [0] \cup [2]$, $U_{1,2} := U_1 \cup U_2$, and $U_3 = A^{\mathbb{Z}}$. The lattice of subshift attractors is shown in Figure 3(b).

We now describe a binary operation on cellular automata which yields the concatenation of their lattices. Given lattices $(L_0, \le_0)$ and $(L_1, \le_1)$, define their *concatenation* L_0L_1 as the lattice on the disjoint

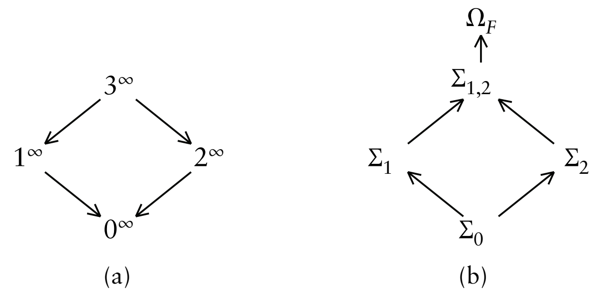


Figure 3. Chain-components and attractors for the product.

union $L_0 \cup L_1$, such that every element of L_0 is less than or equal to every element of L_1:

$$x \le y \iff (x \le_0 y) \quad \text{or} \quad (x \le_1 y) \quad \text{or} \quad (x \in L_0 \ \& \ y \in L_1).$$

Proposition 7. Let F, G be cellular automata with disjoint alphabets, radius $r \ge 1$, and local rules $f : A^{2r+1} \to A$, $g : B^{2r+1} \to B$. Let $h : A \cup B \to A$ be a map which is identity on A. Define a cellular automaton H on $C = A \cup B$ by

$$H(x)_i := \begin{cases} g(x_{[i-r,i+r]}) & \text{if} \quad x_{[i-r,i+r]} \in B^{2r+1}, \\ fh(x_{[i-r,i+r]}) & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}_\sigma(H) = \mathcal{A}_\sigma(F)\mathcal{A}_\sigma(G)$.

Proof. If U is a spreading set for F, it is also spreading for H. Let V be a spreading set for G which is a union of cylinders of words of length m. Take a word $v \in B^m$ and replace some of its prefix (length $i \le m$) by a word $u \in A^i$ and some of its suffix (length $j \le m - i$) by a word $w \in A^j$. We get a clopen set

$$V_H := \bigcup \{[uv_{[|u|,|v|-|w|)}w : \ [v] \subseteq V, u, w \in A^*, |u| + |w| \le |u|\},$$

which contains $A^{\mathbb{Z}}$ and is spreading for H. Thus, $\Omega_G(V) \subseteq \Omega_H(V_H)$. Moreover, for spreading sets V, V' of G we have $\Omega_G(V) \subseteq \Omega_G(V')$ if and only if $\Omega_H(V_H) \subseteq \Omega_H(V'_H)$. ■

5. Infinite number of subshift attractors

Example 8. There exists a cellular automaton with an infinite sequence of subshift attractors $\Sigma_{n+1} \subset \Sigma_n$ indexed by positive integers.

Proof. We construct first a cellular automaton with an infinite number of clopen invariant sets $U_{n+1} \subset U_n$ which spread to the right. We have

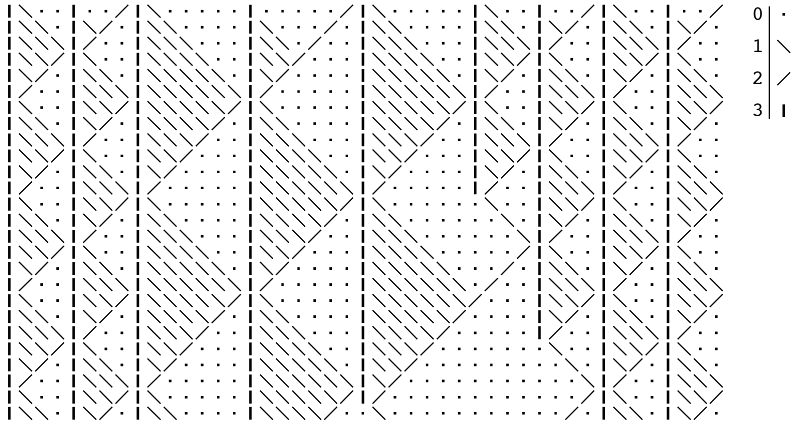


Figure 4. Decreasing sequence of subshift attractors.

the alphabet $A = \{0, 1, 2, 3\}$ and a cellular automaton $F(x)_i = f(x_{[i-1,i+1]})$, where $f : A^3 \to A$ is the local rule given by

$$\begin{array}{lllll} x33{:}0, & 132{:}3, & x32{:}0, & xx2{:}2, & x13{:}2, \\ 3xx{:}1, & x2x{:}0, & 10x{:}1, & 11x{:}1, & x1x{:}0. \end{array}$$

Here $x \in A$ and the first applicable production is used, otherwise the letter is left unchanged (see the simulation depicted in Figure 4). Letter 3 is a stationary particle which generates right-going particles 1 *via* the production *3xx:1*. When a 1 particle reaches another 3, it changes to a left-going particle 2 *via* the production *x13:2* and erases all 1 particles that it encounters. When this 2 particle reaches a 3 particle simultaneously with a 1 particle, the 3 particle is preserved by the production *132:3*. Otherwise 3 is destroyed by the production *x32:0*. Since the 3 particles are never created, they successively disappear unless they are distributed periodically. Set

$$U_n = \{x \in A^{\mathbb{Z}} : \ (0 \le i < j < 2n-2) \& (x_i = x_j = 3) \implies j - i \ge n\}.$$

Each U_n is clopen and invariant, since the letter 3 is never created. We show that $F^{4n-7}(U_n) \subseteq \sigma^{-1}(U_n)$. Because of the production *x33:0* we get $F(A^{\mathbb{Z}}) \subseteq U_2$, so the formula holds for $n = 2$. Let $n \ge 3$ and assume by contradiction that there exists $x \in U_n$ such that $F^i(x) \notin \sigma^{-1}(U_n)$ for $i \le 4n - 7$. Then $F^i(x)_{n-1} = F^i(x)_{2n-2} = 3$ for all $i \le 4n - 7$ and $x_{[0,n-2]} \in \{0, 1, 2\}^*$. There exists $j \le 2n - 4$ such that $F^j(x)_{[n-2,n]} = 132$. Since $x_{[0,n-2]} \in \{0, 1, 2\}^*$, we get $F^{j+2n-4}(x)_{[n-2,n]} = 032$, so $F^{j+2n-3}(x)_{n-1} = 2$, which is a contradiction. Thus U_n is spreading to the right. Define the mirror image cellular automaton G of F with local rule

$$\begin{array}{lllll} 33x{:}0, & 231{:}3, & 23x{:}0, & 2xx{:}2, & 31x{:}2, \\ xx3{:}1, & x2x{:}0, & x01{:}1, & x11{:}1, & x1x{:}0, \end{array}$$

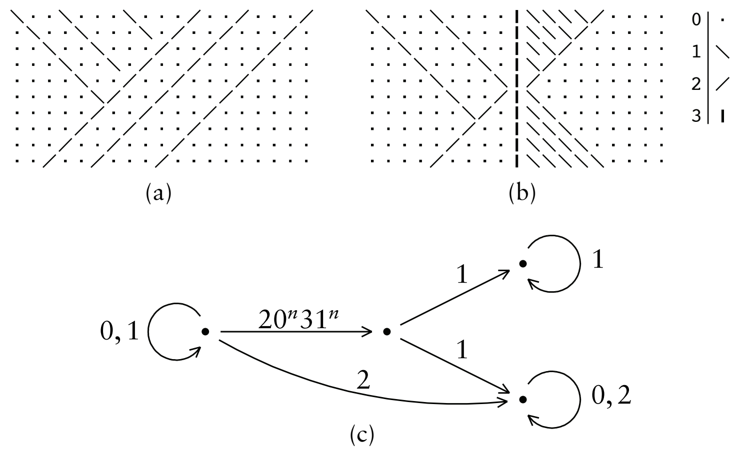


Figure 5. The small quasi-attractor.

so that U_n is spreading to the left for G. Consider now the alphabet $B = \{(a, b) \in A^2 : \ a = 3 \iff b = 3\}$ with 10 letters, and a cellular automaton on B defined by

$$H(x, y)_i = \begin{cases} (0, 0) & \text{if} \quad x_i = y_i = 3 \quad \text{and} \\ & \quad F(x)_i \neq 3 \quad \text{or} \quad G(y)_i \neq 3, \\ (F(x)_i, G(y)_i) & \text{otherwise.} \end{cases}$$

Thus the components F and G of H share the same particles 3 and both F and G can destroy them. Then $V_n := B^{\mathbb{Z}} \cap (U_n \times U_n)$ is a spreading set for H and $\Sigma_n := \Omega_H(V_n)$ is a subshift attractor. Since $V_{n+1} \subset V_n$ we have $\Sigma_{n+1} \subseteq \Sigma_n$. For $n > 2$ there exists a σ-periodic configuration $x = (31^{n-1}320^{n-2})^\infty \in U_n \setminus U_{n+1}$ which is periodic for F, so $x \in \Omega_F(U_n) \setminus \Omega_F(U_{n+1})$. The mirror image configuration $y = (30^{n-2}231^{n-1})^\infty$ is periodic for G, so $(x, y) \in \Sigma_n \setminus \Sigma_{n+1}$. For each configuration $(x, y) \in \Sigma_n \setminus \Sigma_{n+1}$ we have $H^{2n}(x, y) = (x, y)$. ■

The small quasi-attractor of the cellular automaton in Example 8 consists of the configurations which contain at most one stationary particle 3 and have an infinite number of preimages. In Figure 5(a) we have a configuration in the first component of Q_H with no 3 particle, and exactly one 3 particle in Figure 5(b). A schematic graph for the first component of Q_H is shown in Figure 5(c). Its language consists of all subwords of the set of words

$$\begin{aligned} L = {} & \{uv : \ u \in \{0, 1\}^*, v \in \{0, 2\}^*\} \cup \\ & \{u20^n31^{n+1}v : \ u \in \{0, 1\}^*, v \in \{0, 2\}^*, n \geq 0\} \cup \\ & \{u20^n31^m : \ u \in \{0, 1\}^*, 0 \leq n < m\}. \end{aligned}$$

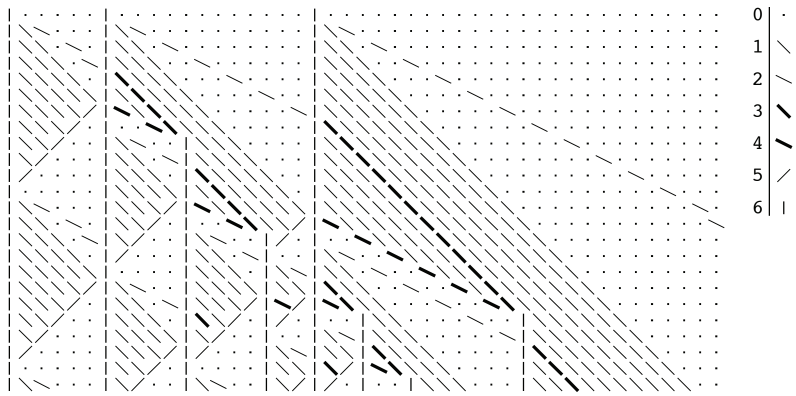


Figure 6. Increasing sequence of subshift attractors.

Example 9. There exists a cellular automaton with an infinite number of subshift attractors $\Sigma_n \subset \Sigma_{n+2} \subset \Omega_H$ indexed by odd positive integers.

Proof. We construct first a cellular automaton with an infinite number of clopen invariant sets $U_n \subset U_{n+1}$ which spread to the right. The alphabet is $A = \{0, 1, 2, 3, 4, 5, 6\}$ and $F(x)_i = f(x_{[i-2,i+1]})$, where $f : A^4 \to A$ is the local rule given by

xx6x:*6*,	*66xx*:*6*,	*x6x6*:*6*,	*43xx*:*6*,	*xx16*:*5*,	*xxx5*:*5*,	*xx5x*:*0*,
x60x:*1*,	*60xx*:*2*,	*x2x6*:*2*,	*26xx*:*3*,	*x3xx*:*3*,	*16xx*:*4*,	*411x*:*4*,
413x:*4*,	*x4xx*:*0*,	*xx3x*:*1*,	*200x*:*2*,	*xx1x*:*1*,	*x1xx*:*1*,	*xxxx*:*0*.

Here $x \in A$ and the first applicable production is used (see the simulation depicted in Figure 6). The letter 6 is a stationary particle which sends to the right a particle 1 with speed 1 and a particle 2 with speed 2 *via* productions *x60x*:*1* and *60xx*:*2*. When these two particles reach another particle 6, then 1 changes to a 3 particle with speed 2 *via* *26xx*:*3* (when the distance between neighboring 6 particles is even, an auxiliary production *x26*:*x* is used). Simultaneously, 2 changes to a 4 particle with speed 1 *via* *16xx*:*4*. When particles 3 and 4 meet, they generate another 6 particle *via* *43xx*:*6*. The clopen set

$$U_n = \{x \in A^{\mathbb{Z}} : \exists i, j \in \mathbb{Z}, 0 \le i < j < 2n \ \& \ x_i = x_j = 6 \ \& \ j - i \le n\}$$

is clearly invariant as no letter 6 is ever destroyed because of the first production *xx6x*:*6*. Because of the second and third productions, U_1 and U_2 are spreading to the right. For $n \ge 3$ we show that $F^{m_n}(U_n) \subseteq \sigma^{-1}(U_n)$, where $m_n := 2n^2 + 2n - 2 + \lfloor (n-1)/2 \rfloor$. Assume by contradiction that there exists $x \in U_n$ such that $F^i(x) \notin \sigma^{-1}(U_n)$ for $i \le m_n$. Then

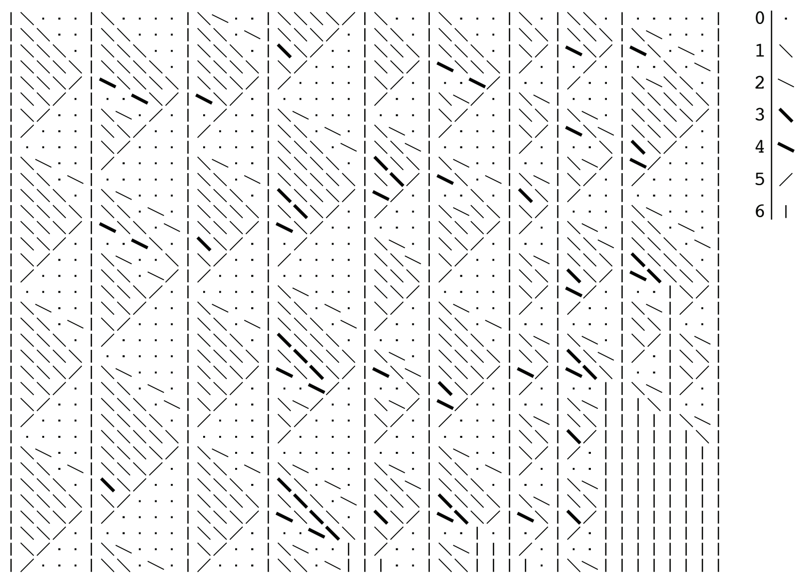


Figure 7. Spreading to the right in quadratic time.

$x_0 = x_n = 6$ and $F^i(x)_j \neq 6$ for $i \leq m_n$ and $j \in (0, 2n] \setminus \{n\}$. There exists $i \leq 2n - 1$ such that for every $k \leq n$, $F^{i+k(2n-1)}(x)_{[0,2]} = 612$ and

$$F^{i+k(2n-1)+\lfloor n/2 \rfloor -1}(x)_{[n-1,n]} = 26, \quad F^{i+k(2n-1)+n-2}(x)_{[n-1,n]} = 16.$$

This produces right-propagating letters 3 and 4 which meet to produce a new 6 letter at position $2n$ or $2n-1$, unless they are prevented to do so by a letter 2 which passes from $2n$ to n. Set $p_n := 2n$ for n odd and $p_n := 2n-1$ for n even. There exists $k \leq n$ such that

$$F^{i+k(2n-1)+n-2+\lfloor (n-1)/2 \rfloor}(x)_{[p_n-2, p_n-1]} = 43,$$
$$F^{i+k(2n-1)+n-1+\lfloor (n-1)/2 \rfloor}(x)_{p_n} = 6.$$

We get $i + k(2n-1) + n - 1 + \lfloor (n-1)/2 \rfloor \leq (n+1)(2n-1) + n - 1 + \lfloor (n-1)/2 \rfloor = m_n$, which is a contradiction. The worst cases for $n = 5, 4, 3$ are displayed in Figure 7 with $m_5 = 60$ and $m_3 = 23$ attained. By a technique similar to that in Example 8 we construct the mirror image G of F and its symmetrization

$$H(x,y)_i = \begin{cases} (6,6) & \text{if} \quad F(x)_i = 6 \quad \text{or} \quad G(y)_i = 6, \\ (F(x)_i, G(y)_i) & \text{otherwise} \end{cases}$$

on the alphabet $B = \{(a,b) \in A^2 : \ a = 6 \iff b = 6\}$ with 37 elements. Thus the componets F and G share the stationary particles 6 and both F and G can create them. Then $V_n = B^{\mathbb{Z}} \times (U_n \times U_n)$ is a spreading set and $\Sigma_n := \Omega_H(V_n)$ is an attractor. Since $V_n \subseteq V_{n+1}$, we get $\Sigma_n \subseteq \Sigma_{n+1}$. However, $\Sigma_{2n-1} = \Sigma_{2n}$ for $n \geq 1$, since V_{2n} generates V_{2n-1} in its neighborhood. For n odd, the inclusions $\Sigma_n \subset \Sigma_{n+2}$ are proper since we have periodic configurations $((60^{n+1})^\infty, (60^{n+1})^\infty)) \in \Sigma_{n+2} \setminus \Sigma_n$. ■

Example 10. There exists a cellular automaton with subshift attractors

$$\cdots \subset \Sigma_{-2} \subset \Sigma_{-1} \subset \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Omega_F.$$

Proof. We use Proposition 7 to the cellular automata in Examples 8 and 9. ■

Using Proposition 7 iteratively we obtain more complex linearly ordered lattices with an arbitrary (but finite) number of segments isomorphic to either $\mathbb{N}^- := \{n \in \mathbb{Z} : \ n \leq 0\}$, $\mathbb{N} \cup \{\infty\}$, or $\mathbb{Z} \cup \{\infty\}$. Taking the cartesian products of these cellular automata, infinite nonlinear lattices of arbitrary complexity may be constructed.

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