Recurrence balance

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Abstract

We introduce a new topological invariant which describes the balance of recurrence time between different regions of the state space. When the recurrence time of a set decreases with its size uniformly over the whole space (e.g. in adding machines) the recurrence balance is one. When there are great disproportions between different regions, the recurrence balance is smaller. For the systems with periodic points the recurrence balance is zero. We construct a minimal system with recurrence balance strictly smaller than 1.

A dynamical system is a pair (X, F) where X is a compact metric space and $F : X \to X$ is a continuous map. Let $\mathfrak{U}_\varepsilon(X)$ be the family of open covers of X with diamater less than ε. For $U \subseteq X$ let $\tau(U) = \min\{k > 0 :\ F^k(U) \cap V \neq \emptyset\}$ be its Poincaré recurrence time. For an open cover $\mathcal{U}$ of X let $P(\mathcal{U})$ be the smallest size of its finite subcover and

$$Q(\mathcal{U}) = \min\{\tau(U) :\ U \in \mathcal{U}\}$$

the minimum recurrence time of $\mathcal{U}$. The recurrence balance of (X, F) is

$$\mathbf{q}(X, F) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{Q(\mathcal{U})}{P(\mathcal{U})} :\ \mathcal{U} \in \mathfrak{U}_\varepsilon(X) \right\}.$$

If X is a Cantor space, then open covers can be replaced by clopen partitions without changing the value of $\mathbf{q}(X, F)$. Since $Q(\mathcal{U}) \leq P(\mathcal{U})$, we get $\mathbf{q}(X, F) \leq 1$. If (X, F) has a periodic point then $\mathbf{q}(X, F) = 0$. It is easy to see that all adding machines have recurrence balance $\mathbf{q}(X, F) = 1$.

We construct a minimal system with recurrence balance strictly smaller than 1. We construct the system (X, F) as an inverse limit of graphs G_k using the representation of Gambaudo and Martens [**?**]. The vertices of G_k are words of an alphabet $A_k \subseteq \mathbb{N}^{k+1}$. We have projections $\pi_k : A_k \to A_{k-1}$ defined by $\pi_k(u) = u_{[0,k)}$. Set

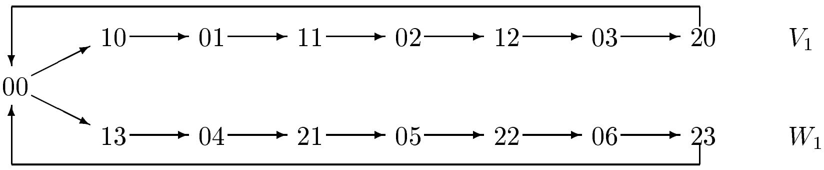
$$X = \lim(A_k, \pi_k) = \{x \in \mathbb{N}^\infty :\ x_{[0,k]} \in A_k, \pi_k(x_{[0,k]}) = x_{[0,k-1]}\}$$

In G_k all vertices have outdegree 1 except 0^{k+1}, which has outdegree 2 and if $0^{k+1} \to u$, $0^{k+1} \to v$, then $u_{[0,k)} = v_{[0,k)}$ have the common prefix of length k. For $x \in X$, define $F(x) \in X$ so that $F(x)_{[0,k)}$ is the common prefix of length k of all vertices v in G_k with $x_{[0,k]} \to v$. Then $F : X \to X$ is continuous, and

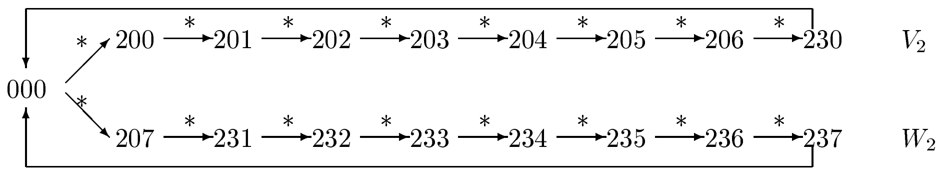
$u \to v$ in G_k iff $F([u]) \cap [v] \neq \emptyset$, where $[u] = \{x \in X : \; x_{[0,k]} = u\}$ is the cylinder of u. We start with $A_0 = \{0,1,2\}$ and graph G_0

$$1 \rightleftarrows 0 \rightleftarrows 2$$

which consists of two loops $V_0 = \{0,1\}$, $W_0 = \{0,2\}$. In the second step set $A_1 = \{00, 10, 11, \ldots, 06, 23\}$ with graph G_1:



consisting of two loops of length 8. $V_1 = \{00, 10, 01, 11, 02, 12, 03, 20\}$ runs thrice through V_0 and once through W_0. $W_1 = \{00, 13, 04, 21, 05, 22, 06, 23\}$ runs once through V_0 and thrice through W_0. In general, given a sequence $(n_k)_{k\geq 1}$ with $n_k > 2$, G_k consists of two loops V_k, W_k of length $2n_1n_2\cdots n_k$, whose only common vertex is 0^{k+1}. V_k runs first $(n_k - 1)$ times through V_{k-1} and once through W_{k-1}, while W_k runs once through V_{k-1} and $(n_k - 1)$ times through W_{k-1}. We assume $n_1 = 4$, $n_2 = 8$, so the graph G_2 is



Here each arrow with a star represents 7 or 8 arrows, e.g., $000 \stackrel{*}{\to} 200$ represents $000 \to 100 \to 010 \to 110 \to 020 \to 120 \to 030 \to 200$ and $200 \stackrel{*}{\to} 201$ represents $200 \to 001 \to 101 \to 011 \to 111 \to 021 \to 121 \to 031 \to 201$. Given the sequence $(n_k)_{k\geq 1}$ consider matrices

$$N_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad N_k = \begin{bmatrix} n_k - 1 & 1 \\ 1 & n_k - 1 \end{bmatrix}, \quad M_k = N_k \cdots N_1 = \begin{bmatrix} p_q & q_k \\ q_k & p_k \end{bmatrix}$$

so $p_1 = 3$, $q_1 = 1$, $p_2 = 22$, $q_2 = 10$. Then p_k/q_k is a decreasing sequence and if $(n_k)_{k\geq 1}$ grows fast enough, $\lim_{k\to\infty} p_k/q_k = a > 1$.

Denote by $|V_k|_j$ the number of sets in V_k which belong to cylinder $[j]$, and similarly for $|W_k|_j$, so

$$\begin{array}{lll} |V_1|_0 = 4, & |V_1|_1 = 3, & |V_1|_2 = 1, \\ |W_1|_0 = 4, & |W_1|_1 = 1, & |W_1|_2 = 3 \end{array}$$

and in general

$$\begin{bmatrix} |V_k|_0 & |V_k|_1 & |V_k|_2 \\ |W_k|_0 & |W_k|_1 & |W_k|_2 \end{bmatrix} = M_k \cdot N_0 = \begin{bmatrix} p_k + q_k & p_k & q_k, \\ p_k + q_k & q_k & p_k \end{bmatrix}$$

For the canonical clopen partitions $\mathcal{A}_k = \{[u] :\ u \in A_k\}$ we have

$$Q(\mathcal{A}_k) = 2(p_k + q_k), \quad P(\mathcal{A}_k) = 4(p_k + q_k) - 1$$

so

$$\mathbf{q}(X, F) \geq \lim_{k\to\infty} \frac{Q(\mathcal{A}_k)}{P(\mathcal{A}_k)} = \frac{1}{2}$$

Assume that $\mathcal{U}$ is a clopen partition with $\mathrm{diam}(\mathcal{U}) < 1$. Then $\mathcal{U} \succeq \mathcal{A}_1$ and there exists k such that $\mathcal{A}_k \succeq \mathcal{U}$, so for every $v \in A_k$ there exists a unique $U_v \in \mathcal{U}$ with $[v] \subseteq U_v$. Moreover $Q(\mathcal{U}) \leq 2(p_k + q_k)$ is an even number $Q(\mathcal{U}) = 2l$.

Let V be an initial part of V_k of length $2l$ and let W be the final part of W_k of length $2l$. For $i = 0, 1, 2$ set

$$\begin{aligned} \mathcal{U}_V^i &= \{U_v :\ v \in V, v_0 = i\} \\ \mathcal{U}_W^i &= \{U_w :\ w \in W, v_0 = i\} \end{aligned}$$

If $j \leq k$ and $2l \leq 2(p_j + q_j)$, then $|\mathcal{U}_V^1| \geq lp_j/(p_j + q_j)$ and $|\mathcal{U}_W^2| \geq lp_j/(p_j + q_j)$. This can be proved by induction. Since $Q(\mathcal{U}) = 2l$, any $\mathcal{U}_V^i$ or $\mathcal{U}_W^i$ consists of pairwise disjoint sets. It follows

$$\begin{aligned} P(\mathcal{U}) &\geq \sum_{i=0}^{2} |\mathcal{U}_V^i \cup \mathcal{U}_W^i| \geq |\mathcal{U}_V^0| + |\mathcal{U}_V^1| + |\mathcal{U}_W^2| \geq l + \frac{2lp_k}{p_k + q_k} \\ \frac{Q(\mathcal{U})}{P(\mathcal{U})} &\leq \frac{2p_k + 2q_k}{3p_k + q_k} \leq \frac{2a + 2}{3a + 1} = b < 1 \end{aligned}$$

It follows $\frac{1}{2} \leq \mathbf{q}(X, F) \leq b < 1$.

References

[1] J.M.Gambaudo and M.Martens: Algebraic topology for minimal Cantor sets, manuscript.