# DYNAMICALLY DEFINED RECURRENCE DIMENSION

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**Abstract.** We modify the idea of a previous article [8] and introduce polynomial and exponential *dynamically defined recurrence dimensions*, topological invariants which express how the Poincaré recurrence time of a set grows when the diameter of the set shrinks. We introduce also the concept of polynomial entropy which applies in the case that topological entropy is zero and complexity function is polynomial. We compare recurrence dimensions with topological and polynomial entropies, evaluate recurrence dimensions of Sturmian subshifts and show some examples with Toeplitz subshifts.

## 1. Introduction

In [1] V. Afraimovich proposed to study the scaling properties of recurrence time by defining, through the Carathéodory construction, a dimension called either *recurrence dimension* or *Afraimovich-Pesin* (AP) *dimension* in [8]. This dimension is a variant of the Hausdorff dimension, with the diameter of a set replaced by some gauge function of the smallest return time of a set into itself (Poincaré recurrence time of a set). Afraimovich investigated mainly irrational rotations and used $1/t$ as a gauge function. In [8], a systematic study of the AP-dimension was carried out with the use of another gauge function, $e^{-t}$. We review here the principal results of this analysis:

(1) The AP-dimension is a topological invariant.
(2) The AP-dimension coincides with the topological entropy on a large class of dynamical systems (subshifts of finite type and Axiom-A systems). There exist examples where the AP-dimension is different from the topological entropy.
(3) The AP-dimension is bounded from below by the asymptotic distribution of periodic points $\limsup_{n\to\infty} \frac{\ln \mathrm{Per}(n)}{n}$.
(4) The AP-dimension of a measure has been defined as the infimum of the AP-dimensions of the sets of full measures; it has been proved that it is a metric invariant and that for aperiodic dynamical systems its value is always zero.

All these results have been obtained with the use of general open and closed covers in the Carathéodory construction, in the attempt to conform the theory to the usual construction of the Hausdorff measures. If, on one hand, this approach yields general topological and metric characterizations like those quoted above, on the other hand it turned out to be difficult to compute the AP-dimension in particular examples.

The basic question raised in [8] was whether there exists a system where the AP-dimension is different from the asymptotic distribution of periodic points. In

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particular, does there exist a minimal system with positive AP-dimension? In this case the AP-dimension would really impose as a new topological invariant to classify such systems and describe their recurrence properties.

A first step in this direction has been done by Bruin [5]. He considered subshifts and proved that the AP-dimension does not exceed the topological entropy. Moreover he provided a non trivial example where the bound is strict. A second improvement was given by Kůrka and Maass [7]. Trying to answer the question addressed above and motivated by Bruin's result, these authors studied large classes of minimal systems with positive topological entropy (Toeplitz subshifts and Vershik systems defined by Bratelli diagrams). In all these cases they showed that the AP-dimension is zero. The conclusion was that the class of open covers used in the definition of the AP-dimension was too large to get a distinguishing topological invariant. For subshifts, they modified the definition of AP-dimension by replacing open covers by dynamically defined clopen partitions similarly as these partitions are used in computation of topological entropy and pressure. The resulting concept called recurrence dimension is a topological invariant. Bruin's results can be modified to show that the recurrence dimension does not exceed the topological entropy, but may be different from it and positive.

In the present article we follow the same strategy with a few differences and extensions, using the gauge function $1/t$ instead of $e^{-t}$. Given a zero-dimensional dynamical system we consider clopen partitions and their associated dynamically defined partitions, and perform the Carathéodory construction. The critical exponent of the resulting Borel measure is called *polynomial recurrence dimension.* We show that the polynomial recurrence dimension is concentrated on the nonwandering set similarly as topological entropy (Bowen theorem). Using Kac theorem we show that the polynomial recurrence time is at least 1. For subshifts we obtain an upper bound related to a quantity which we call polynomial entropy. This is the exponent of the algebraic growth of the complexity function, when the topological entropy is zero. We show that the polynomial recurrence dimension of every Sturmian subshift is 1, thus completing the previous work of Afraimovich [1] on irrational rotations. Then we construct examples with Toeplitz subshifts showing that polynomial entropy can attain arbitrary values and that polynomial recurrence dimension may be different from it.

## 2. Polynomial entropy

A dynamical system is a pair $(X, f)$, where $X$ is a compact metric space and $f : X \to X$ is a continuous mapping. An open cover of a compact metric space $X$ is a finite system $\mathcal{V} = \{V_a : a \in A\}$ of nonempty open sets whose union is $X$. Its diameter is $\operatorname{diam}(\mathcal{V}) = \max\{\operatorname{diam}(V) : V \in \mathcal{V}\}$, its size $|\mathcal{V}|$ is the number of elements of its smallest subcover.

For a finite alphabet $A$ denote by $A^* = \bigcup_{n\geq 0} A^n$ the set of all words of $A$. The length of a word $u \in A^n$ is denoted by $|u| = n$. Denote by $A^{\mathbb{N}}$ the space of one-sided sequences of letters of $A$ and $(A^{\mathbb{N}}, \sigma)$ the shift dynamical system. The cylinders are sets $[u] = \{x \in A^{\mathbb{N}} : x_{[0,n)} = u\}$, where $u \in A^n$.

A subshift is any subset $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and $\sigma$-invariant. We denote by

$$\mathcal{L}(\Sigma) = \{u \in A^* : \exists v \in \Sigma, \exists i \geq 0 : v_{[i,i+|u|)} = u\}$$

the language of words appearing in $\Sigma$ and $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n$. The complexity function of a subshift is defined by $P(n) = |\mathcal{L}^n(\Sigma)|$.

Let $(X, f)$ be a dynamical system and $\mathcal{V} = \{V_a : a \in A\}$ an open cover of $X$. For $u \in A^* \cup A^{\mathbb{N}}$ put

$$V_u = \{x \in X : \forall i < |u|, f^i(x) \in V_{u_i}\}.$$

Then $\mathcal{V}^n = \{V_u : u \in A^n, V_u \neq \emptyset\}$ is also an open cover. We say that $\mathcal{V}$ is generating, if $\mathrm{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$. The entropy $H(X, f, \mathcal{V})$ of an open cover $\mathcal{V}$ and the topological entropy $h(X, f)$ are defined by

$$\begin{aligned} H(X, f, \mathcal{V}) &= \lim_{n\to\infty} \frac{\ln|\mathcal{V}^n|}{n}, \\ h(X, f) &= \lim_{\varepsilon\to 0} \sup\{H(X, f, \mathcal{V}) : \mathrm{diam}(\mathcal{V}) < \varepsilon,\ \mathcal{V} \text{ open cover of } X\}. \end{aligned}$$

When the entropy of $\mathcal{V}$ is zero, the function $|\mathcal{V}^n|$ grows slower than any exponential. Its growth may be then polynomial $|\mathcal{V}^n| \approx n^a$. We call $a \approx \frac{\ln|\mathcal{V}^n|}{\ln n}$ the polynomial entropy. In contrast to the (exponential) entropy, the limit in question need not exist, so we get lower and upper polynomial entropies.

**Definition 2.1.** *Let $(X, f)$ be a dynamical system and $\mathcal{V}$ an open cover of $X$. The lower ($\underline{h}(X, f)$) and upper ($\overline{h}(X, f)$) polynomial entropies are defined by*

$$\underline{H}_p(X, f, \mathcal{V}) = \liminf_{n\to\infty} \frac{\ln|\mathcal{V}^n|}{\ln n}, \quad \overline{H}_p(X, f, \mathcal{V}) = \limsup_{n\to\infty} \frac{\ln|\mathcal{V}^n|}{\ln n},$$

$$\overline{\underline{h}}_p(X, f) = \lim_{\varepsilon\to 0} \sup\{\overline{\underline{H}}_p(X, f, \mathcal{V}) : \mathrm{diam}(\mathcal{V}) < \varepsilon,\ \mathcal{V} \textit{ open cover of } X\}.$$

The last line represents two formulas, one with upper bars $\overline{h}_p$, $\overline{H}_p$ and the other with lower bars. Similarly as for the (exponential) topological entropy we get $\overline{\underline{h}}_p(X, f) = \lim_{n\to\infty} \overline{\underline{H}}_p(X, f, \mathcal{V}_n)$ whenever $(\mathcal{V}_n)_{n\geq 0}$ is a sequence of open covers with $\mathrm{diam}(\mathcal{V}_n) \to 0$ as $n \to \infty$. In particular for a subshift $\Sigma$ we get $h(\Sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{n}$ and

$$\underline{h}_p(\Sigma) = \liminf_{n\to\infty} \frac{\ln P(n)}{\ln n}, \quad \overline{h}_p(\Sigma) = \limsup_{n\to\infty} \frac{\ln P(n)}{\ln n}.$$

## 3. Polynomial recurrence dimension

In this section we consider dynamical systems on zero-dimensional spaces. A compact metric space $X$ is zero-dimensional iff it is homeomorphic to a closed subspace of $A^{\mathbb{N}}$ for some finite alphabet $A$. A clopen partition of a zero-dimensional space $X$ is a system $\mathcal{V} = \{V_a : a \in A\}$ of nonempty clopen sets $V_a \subseteq X$ which are pairwise disjoint and their union is $X$. Denote by $\mathcal{P}(X)$ the set of clopen partitions of $X$. The Poincaré recurrence time of a subset $Y \subseteq X$ is

$$\tau(Y) = \min\{k > 0 : f^k(Y) \cap Y \neq \emptyset\}$$

**Definition 3.1.** *Let $(X, f)$ be a zero-dimensional dynamical system, $\mathcal{V}$ a clopen partition of $X$, $Y \subseteq X$, and $\alpha > 0$. Using upper and lower limits we define the upper and lower polynomial recurrence dimensions $\overline{r}_p(Y, f)$ and $\underline{r}_p(Y, f)$ of $Y$ as*

$$\begin{aligned} \underline{M}_p(Y, f, \mathcal{V}, \alpha) &= \liminf_{n\to\infty} \sum_{V\in\mathcal{V}^n, V\cap Y\neq\emptyset} \tau(V)^{-\alpha}, \\ \overline{M}_p(Y, f, \mathcal{V}, \alpha) &= \limsup_{n\to\infty} \sum_{V\in\mathcal{V}^n, V\cap Y\neq\emptyset} \tau(V)^{-\alpha}, \end{aligned}$$

$$\begin{aligned}
\underline{R}_p(Y,f,\mathcal{V}) &= \sup\{\alpha>0:\ \underline{M}_p(Y,f,\mathcal{V},\alpha)=\infty\},\\
\overline{m}_p(Y,f,\alpha) &= \lim_{\varepsilon\to 0}\sup\{\overline{M}_p(Y,f,\mathcal{V},\alpha):\ \mathcal{V}\in\mathcal{P}(X),\ \mathrm{diam}(\mathcal{V})<\varepsilon\},\\
\overline{r}_p(Y,f) &= \sup\{\alpha>0:\ \overline{m}_p(Y,f,\alpha)=\infty\}.
\end{aligned}$$

Clearly, the upper and lower polynomial recurrence dimensions are topological invariants and $\underline{r}_p(Y,f)\le\overline{r}_p(Y,f)$. We now explain how this definition has been chosen :

- The set function $M_p$ is defined similarly as in [8], except that we restrict ourself to a specific family of covers, namely clopen partitions and their powers.
- We then define $m_p$ over all clopen partitions in order to get rid of the arbitrary choice of the initial partition. The choice of the upper limit in $\varepsilon$ yields the nice property (Proposition 3.7) that the dimension can be calculated with the use of any generating clopen partition. This property cannot be obtained with lower limit.
- Finally we define the critical exponent $r_p$ as the supremum over all $\alpha$ such that $m_p(Y,f,\alpha)=\infty$. Taking the infimum over all $\alpha$ such that $m_p(Y,f,\alpha)=0$ gives a less interesting quantity since it would give us $\infty$ for every dynamical systems containing periodic points.

From now on, we write $r_p(Y,f)$ whenever the same arguments or statements apply for both $\overline{r}_p(Y,f)$ and $\underline{r}_p(Y,f)$, and similarly for $m_p$, $M_p$ and $R_p$.

**Proposition 3.2.** *If $(X,f)$ is minimal, then*

$$r_p(Y,f)=\inf\{\alpha>0:\ m_p(Y,f,\alpha)=0\}.$$

The proof of this fact can be easily adapted from Afraimovich [1].

**Proposition 3.3.** *The set function $m_p(.,f,\alpha)$ is a Borel measure.*

*Proof.* These are standard arguments for construction of this type, see e.g., [9]. We check that $m_p$ is an outer measure. If $Y\subset Z$, then $m_p(Y,f,\alpha)\le m_p(Z,f,\alpha)$. Moreover, $m_p(\bigcup_k U_k,f,\alpha)\le\sum_k m_p(U_k,f,\alpha)$. To show that $m_p$ is a Borel measure, we have to verify the following property :

$$d(Y,Z)>0\Rightarrow m_p(Y\cup Z,f,\alpha)=m_p(Y,f,\alpha)+m_p(Z,f,\alpha).$$

This follows immediately from the fact that in the definition of $M_p$, we take partitions $\mathcal{V}$ with diameter going to zero, thus $Y$ and $Z$ are covered by different sets of the same partitions. □

**Proposition 3.4.** *Let $(X,f)$ be a dynamical system and $Y\subseteq X$ a closed invariant subset. Then*

$$m_p(Y,f_{|Y},\alpha)\le m_p(Y,f,\alpha)\le m_p(X,f,\alpha)$$
$$r_p(Y,f_{|Y})\le r_p(Y,f)\le r_p(X,f)$$

*Proof.* Let $\mathcal{V}$ be a clopen partition of $Y$, $\mathrm{diam}(\mathcal{V})<\varepsilon$, $\eta>0$, and $V\in\mathcal{V}$. Since $V$ is open in $Y$, for every $x\in V$ there exists $0<\delta_x<\eta$, such that $B_{\delta_x}(x)\cap Y\subseteq V$, and $B_{2\delta_x}(x)\cap V'=\emptyset$ for every $V\ne V'\in\mathcal{V}$. Let $K\subset V$ be a finite subset such that $\{B_{\delta_x}(x):x\in K\}$ is a finite cover of $V$, and let $W_V=\bigcup_{x\in K}B_{\delta_x}(x)$. Since $X$ is totally disconnected, the balls are clopen, so $W_V$ is clopen too, $V=Y\cap W_V$, and $W_V\cap W_{V'}=\emptyset$, if $V\ne V'$. Let $\mathcal{W}$ be the clopen partition consisting of all $W_V, V\in\mathcal{V}$ and a clopen partition of the complement of $\bigcup_{V\in\mathcal{V}}W_V$. Then $\mathcal{W}$ is a clopen partition of $X$. We get $\mathcal{V}^n = \{Y \cap W : W \in \mathcal{W}^n\}$ and $\tau(Y \cap W) \geq \tau(W)$, so $M_p(Y, f_{|Y}, \mathcal{V}, \alpha) \leq M_p(Y, f, \mathcal{W}, \alpha)$. Since $\mathrm{diam}(\mathcal{W}) \leq \varepsilon + 2\eta$ for arbitrarily small $\eta$, $m_p(Y, f_{|Y}, \alpha) \leq m_p(Y, f, \alpha)$. From Proposition 3.3 we get $m_p(Y, f, \alpha) \leq m_p(X, f, \alpha)$. □

From now on, we leave out $f$ in the formulas for $m, M, R, r$.

**Proposition 3.5.** *Let $(X, f)$ be a zero-dimensional dynamical system and set $\mathrm{Per}(n) = \mathrm{card}\{x \in X : f^n(x) = x\}$. Then*

$$\overline{r}_p(X) \geq \limsup_{n\to\infty} \frac{\ln \mathrm{Per}(n)}{\ln n}, \quad \underline{r}_p(X) \geq \liminf_{n\to\infty} \frac{\ln \mathrm{Per}(n)}{\ln n}.$$

*Proof.* Suppose that for every $n$, $\mathrm{Per}(n)$ is finite. For every $n$ there exists $\varepsilon > 0$, such that if $f^n(x) = x$, $f^n(y) = y$, then $d(x, y) > \varepsilon$. If $\mathrm{diam}(\mathcal{V}) < \varepsilon$, then for every $k$

$$\sum_{V\in\mathcal{V}^k} \tau(V)^{-\alpha} \geq \mathrm{Per}(n) \cdot n^{-\alpha}.$$

If $\alpha < \beta < \limsup_{n\to\infty} \frac{\ln \mathrm{Per}(n)}{\ln n}$, then for an infinite number of $n$, $\mathrm{Per}(n) \cdot n^{-\alpha} \geq n^{\beta-\alpha}$, so $\overline{M}_p(X, \mathcal{V}, \alpha) = \infty$, $\overline{m}_p(X, \alpha) = \infty$ and $\alpha < \overline{r}_p(X)$. Similarly for lower limits. □

For next proposition we prove the following lemma :

**Lemma 3.6.** *Let $\mathcal{U}$ and $\mathcal{V}$ two clopen* generating *partitions of $X$. Then one can find a constant $0 < C < \infty$, such that for every $\alpha > 0$,*

$$\frac{1}{C} \cdot M_p(X, \mathcal{V}, \alpha) \leq M_p(X, \mathcal{U}, \alpha) \leq C \cdot M_p(X, \mathcal{V}, \alpha).$$

*Proof.* We have $\mathrm{diam}(\mathcal{U}^n) \to 0$, $\mathrm{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$. Thus there exist positive integers $m, n$ such that $\mathcal{U}^{m+1} \succeq \mathcal{V}$ and $\mathcal{V}^{n+1} \succeq \mathcal{U}$, so for every $k > n$, $\mathcal{U}^{k+m} \succeq \mathcal{V}^k \succeq \mathcal{U}^{k-n}$. Every $V \in \mathcal{V}^k$ is a subset of some set $U \in \mathcal{U}^{k-n}$ and a union $V = U_1 \cup \cdots \cup U_p$, of at most $p \leq |\mathcal{U}|^{n+m}$ sets $U_i \in \mathcal{U}^{k+m}$. We get $\tau(V) \leq \tau(U_i)$, so

$$p \cdot \tau(V)^{-\alpha} \geq \tau(U_1)^{-\alpha} + \cdots + \tau(U_p)^{-\alpha}$$

$$\sum_{V\in\mathcal{V}^k} \tau(V)^{-\alpha} \geq |\mathcal{U}|^{-n-m} \sum_{U\in\mathcal{U}^{k+m}} \tau(U)^{-\alpha}.$$

Thus $M_p(X, \mathcal{V}, \alpha) \geq |\mathcal{U}|^{-m-n} \cdot M_p(X, \mathcal{U}, \alpha)$. Interchanging $\mathcal{U}$ and $\mathcal{V}$, we obtain the result. □

**Proposition 3.7.** *Let $\mathcal{V}$ be a generating clopen partition. Then $r_p(X) = R_p(X, \mathcal{V})$.*

*Proof.* If there exists a generating clopen partition, then there exists $\varepsilon > 0$, such that any clopen partition of diameter less than $\varepsilon$ is also generating. For a given $\alpha$, let us choose a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of clopen partitions with diameter going to zero such that $M_p(X, \mathcal{V}_n, \alpha)$ decreases and converges to $m_p(X, \alpha)$. Thus, for $n$ large enough, $\mathcal{V}_n$ is generating, and we can apply Lemma 3.6 with $\mathcal{V}$ and $\mathcal{V}_n$ for each $n$. If $\alpha < R_p(X, \mathcal{V})$, then by Lemma 3.6, $C \cdot M_p(X, \mathcal{V}_n, \alpha) \geq M_p(X, \mathcal{V}, \alpha) = \infty$. Since this applies for any $n$ large enough and any $\alpha < R_p(X, \mathcal{V})$, this shows $m_p(X, \alpha) = \infty$ and $r_p(X) \geq R_p(X, \mathcal{V})$. If $\alpha > R_p(X, \mathcal{V})$, then for some $k$ large enough, $M_p(X, \mathcal{V}_k, \alpha) \leq C \cdot M_p(X, \mathcal{V}, \alpha) < \infty$. Since the sequence $M_p(X, \mathcal{V}_n, \alpha)$ is decreasing, this shows that $m_p(X, \alpha) < \infty$. This is true for any $\alpha > R_p(X, \mathcal{V})$, thus $r_p(X) \leq R_p(X, \mathcal{V})$. □

Recall that a point $x \in X$ is non-wandering, if every its neighborhood has finite recurrence time. By Bowen theorem [4], the topological entropy of a dynamical system is equal to the topological entropy of the system restricted to the set of non-wandering points.

**Proposition 3.8.** *Let $(X, f)$ a zero-dimensional dynamical system, and $\Omega$ the set of non-wandering points. Then $r_p(X) = r_p(\Omega)$.*

*Proof.* If we denote $W = \Omega^c$ the complement of $\Omega$, then for any point $x \in W$, one can find an open set $U(x)$ such that $\tau(U(x)) = \infty$. Since $W$ is separable, the cover $\{U(x) : x \in W\}$ of $W$ has a countable sub-cover $\mathcal{F}$. By $\sigma$-additivity, we have $m(W, \alpha) \leq \sum_{F \in \mathcal{F}} m_p(F, \alpha) = 0$, $m_p(\Omega, \alpha) = m_p(X, \alpha) - m_p(W, \alpha) = m_p(X, \alpha)$, and $r_p(X) = r_p(\Omega)$. □

**Proposition 3.9.** *For every zero-dimensional, minimal dynamical system $(X, f)$, $\underline{r}_p(X, f) \geq 1$.*

*Proof.* There exists an ergodic measure $\mu$, which is positive on open sets. By the Kac's theorem (see e.g., [6], page 133),

$$\tau(U) \leq \int_U \tau_U(x) \frac{d\,\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}.$$

Here $\tau_U(x) = \min\{k > 0 : \; f^k(x) \in U\}$. For every clopen partition

$$\sum_{U \in \mathcal{U}} \tau(U)^{-1} \geq \sum_{U \in \mathcal{U}} \mu(U) \geq 1$$

so $\underline{m}_p(X, f, \mathcal{U}, 1) > 0$. Since $(X, f)$ is minimal, $\underline{r}_p(X, f, \mathcal{U}) \geq 1$ by Proposition 3.2. □

**Proposition 3.10.** *For every subshift $\Sigma \subseteq A^{\mathbb{N}}$, $\overline{r}_p(\Sigma) \leq \overline{h}_p(\Sigma) + 1$*

*Proof.* If $\overline{h}_p(\Sigma) + 1 < \alpha$, pick some $\beta$ with $\overline{h}_p(\Sigma) < \beta < \alpha - 1$. There exists $k_0$, such that for all $k \geq k_0$, $P(k) \leq k^\beta$. If $u \in \mathcal{L}^n(\Sigma)$ and $\tau([u]) = k < n$, then $u_{k+i} = u_i$ for all $i \in [0, |u|)$, so the number of cylinders of length $n$ with the return time $k$ is at most the number of cylinders of length $k$. For $\tau([u]) = k \geq n$ we use $e^{-\tau[u]} \leq e^{-n}$ to obtain

$$\begin{aligned} \sum_{u \in \mathcal{L}^n(\Sigma)} \tau([u])^{-\alpha} &\leq \sum_{k=1}^{n} P(k) \cdot k^{-\alpha} \leq \sum_{k=1}^{n} k^{\beta - \alpha} \\ \overline{m}_p(\Sigma, \alpha) &\leq \sum_{k=1}^{\infty} k^{\beta - \alpha} < \infty \end{aligned}$$

so $\overline{m}_p(\Sigma, \alpha) < \infty$, and $\overline{r}_p(\Sigma) \leq \alpha$. □

**Proposition 3.11.** *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift and $a, b > 0$. Then*

$$\begin{aligned} \forall n_0, \exists n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), a|u| \leq \tau[u] &\Rightarrow \underline{r}_p(\Sigma) \leq \overline{h}_p(\Sigma) \\ \exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), a|u| \leq \tau[u] &\Rightarrow \underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma) \\ \exists n_0, \forall n \geq n_0, \forall u \in \mathcal{L}^n(\Sigma), \tau[u] \leq b|u| &\Rightarrow \underline{h}_p(\Sigma) \leq \overline{r}_p(\Sigma) \end{aligned}$$

*Proof.* 1. Let $\overline{h}_p(\Sigma) < \beta < \alpha$ so for every $n \geq n_0$, $P(n) < n^\beta$. There exists an infinite number of $n$ such that $a|u| \leq \tau[u]$ whenever $|u| = n$. We get

$$\sum_{u \in \mathcal{L}^n(\Sigma)} \tau[u]^{-\alpha} \leq P(n)(an)^{-\alpha} \leq a^{-\alpha} \cdot n^{\beta - \alpha} \to 0$$

Thus $\underline{m}_p(\Sigma,\mathcal{A},\alpha)=0$, so $\underline{r}_p(\Sigma)\leq\alpha$ and $\underline{r}_p(\Sigma)\leq\overline{h}_p(\Sigma)$.
2. Let $\underline{h}_p(\Sigma)<\beta<\alpha$ so there exists an infinite number of $n$ for which $P(n)<n^\beta$, and $a|u|\leq\tau[u]$ for all $u$ with $|u|=n$. We get again $\underline{m}_p(\Sigma,\mathcal{A},\alpha)=0$, so $\underline{r}_p(\Sigma)\leq\alpha$ and $\underline{r}_p(\Sigma)\leq\underline{h}_p(\Sigma)$.
3. Let $\underline{h}_p(\Sigma)>\beta>\alpha$ so for every $n\geq n_0$, $P(n)>n^\beta$ and

$$\sum_{u\in\mathcal{L}^n(\Sigma)}\tau[u]^{-\alpha}\geq P(n)(bn)^{-\alpha}\geq b^{-\alpha}\cdot n^{\beta-\alpha}\to\infty.$$

Thus $\overline{m}_p(\Sigma,\mathcal{A},\alpha)=\infty$, so $\overline{r}_p(\Sigma)\geq\alpha$ and $\overline{r}_p(\Sigma)\geq\underline{h}_p(\Sigma)$. □

## 4. Sturmian subshifts

A Sturmian subshift is a coding of a one-dimensional irrational rotation. Denote by $\mathbb{T}$ the one-dimensional torus (circle) parametrized by the unit interval $[0,1)$. For $\theta\in(0,1)$ irrational denote by $(\mathbb{T},R_\theta)$ the rotation given by $R_\theta(x)=x+\theta \mod 1$. Let $\mathcal{V}=\{[0,1-\theta),[1-\theta,1)\}=\{V_0,V_1\}$ be the cannonical semiopen cover of $\mathbb{T}$. For every binary word $u\in\{0,1\}^*\cup\{0,1\}^{\mathbb{N}}$, $V_u=\{x\in\mathbb{T}:\ \forall i<|u|, R_\theta^i(x)\in V_{u_i}\}$ is a semiopen interval (possibly empty). The Sturmian subshift $\Omega_\theta$ is defined via its language by $\mathcal{L}(\Omega_\theta)=\{u\in\{0,1\}^*:\ V_u\neq\emptyset\}$. Since $\mathcal{V}$ is generating, there exists a factor map $\varphi:(\Omega_\theta,\sigma)\to(\mathbb{T},R_\theta)$. If $u\in\mathcal{L}^n(\Omega_\theta)$ then $\varphi([u])=\overline{V_u}$ and $\tau([u])=\tau(V_u)$.

The return times of cylinders and their corresponding intervals are obtained from continued fraction expansion $\theta=[a_0,a_1,\ldots]$. We use the notation from Schmidt [10] or Alessandri and Berthé [3]. The partial convergents $\frac{p_n}{q_n}$ of $\theta$ satisfy $p_0=a_0=0$, $q_0=1$, $p_1=a_0a_1+1$, $q_1=a_1$ and $p_k=a_kp_{k-1}+p_{k-2}$, $q_k=a_kq_{k-1}+q_{k-2}$ for $k\geq 2$. Then $\eta_k=(-1)^k(q_k\theta-p_k)$ are positive and satisfy $\eta_k=-a_k\eta_{k-1}+\eta_{k-2}$ for $k\geq 2$.

**Proposition 4.1.** *For any Sturmian subshift, the upper and lower polynomial recurrence dimensions are equal to 1,* $\underline{r}_p(\Omega_\theta,\sigma)=\overline{r}_p(\Omega_\theta,\sigma)=1$.

*Proof.* We use the Three distance theorem and the Three gap theorem (see e.g. [3]). Given $n>0$, there exist unique integers $k,m,r$ satisfying

$$n=mq_k+q_{k-1}+r,\quad 1\leq m\leq a_{k+1},\quad 0\leq r<q_k.$$

The Three distance theorem states that the number and the length of intervals in the dynamic partition $\mathcal{V}^n$ are

| | | |
|---|---|---|
| $n+1-q_k$ | intervals of length | $l_1=\eta_k$, |
| $r+1$ | intervals of length | $l_2=\eta_{k-1}-m\eta_k$, |
| $q_k-(r+1)$ | intervals of length | $l_3=\eta_{k-1}-(m-1)\eta_k$. |

The Three gap theorem tells us that the three possible return times of points of an interval of length $\beta$ into itself are

$$t_1=q_{k'},\quad t_2=q_{k'+1}-m'q_{k'},\quad t_3=q_{k'+1}-(m'-1)q_{k'},$$

where $k'$, $m'$ (and $\phi$) are the unique solutions of

$$\beta=m'\eta_{k'}+\eta_{k'+1}+\phi,\quad 0<\phi\leq\eta_{k'},\quad 1\leq m\leq a_{k'+1}.$$

The Poincaré recurrence time of an interval is the minimum of these return times. The third return time $t_3$ is always the largest, so it does not interest us here. For the first two return times we have $t_1<t_2$ except when $m'=a_{k'+1}$. In this case $t_2=q_{k'-1}<t_1$ by the equality $q_{k'+1}=a_{k'+1}q_k+q_{k'-1}$.

We will always choose $\phi=\eta_{k'}$, i.e. the greatest value it can take. If we take

- $\beta = l_1 = \eta_k$ : there are two cases
  - $a_{k+2} = 1$ : we set $k' = k+2, m' = a_{k'+1}$, thus the return time of the interval of length $l_1$ will be $t_2 = q_{k+1}$.
  - $a_{k+2} > 1$ : we set $k' = k+1, m' = a_{k'+1} - 1$, thus the return time will be $t_1 = q_{k+1}$ again.
- $\beta = l_2 = \eta_{k-1} - m\eta_k$ : there are three cases
  - $m < a_{k+1} - 1$ : we set $k' = k, m' = a_{k'+1} - m - 1$, thus the return time of the interval of length $l_2$ will be $t_1 = q_k$.
  - $m = a_{k+1} - 1$ : we set $k' = k+1, m' = a_{k'+1}$, thus the return time will be $t_2 = q_k$.
  - $m = a_{k+1}$ : we set $k' = k+2, m' = a_{k'+1} - 1$, thus the return time will be $t_1 = q_{k+2}$.
- $\beta = l_3 = \eta_{k-1} - (m-1)\eta_k$ : there are two cases
  - $m < a_{k+1}$ : we set $k' = k, m' = a_{k'+1} - m$, thus the return time of the interval of length $l_3$ will be $t_1 = q_k$.
  - $m = a_{k+1}$ : we set $k' = k+1, m' = a_{k'+1}$, thus the return time will be $t_2 = q_k$.

For a fixed $n$, we distinguish two cases. If $m < a_{k+1}$, then

$$\begin{aligned} M_p(\Omega_\theta, \mathcal{V}^n, 1) &= \frac{n+1-q_k}{q_{k+1}} + \frac{q_k}{q_k} = \frac{(m-1)q_k + q_{k-1} + (r+1)}{q_{k+1}} + 1 \\ &\leq \frac{a_{k+1}q_k + q_{k-1} + (r+1-q_k)}{q_{k+1}} + 1 \leq 2 \end{aligned}$$

If $m = a_{k+1}$, then

$$\begin{aligned} M_p(\Omega_\theta, \mathcal{V}^n, 1) &= \frac{(a_{k+1}-1)q_k + q_{k-1} + (r+1)}{q_{k+1}} + \frac{r+1}{q_{k+2}} + \frac{q_k - (r+1)}{q_k} \\ &\leq \frac{q_{k+1}}{q_{k+1}} + \frac{r+1}{q_k} + \frac{q_k - (r+1)}{q_k} = 2 \end{aligned}$$

Since $\mathcal{V}$ is generating, $m_p(\Omega_\theta, \mathcal{V}^n, 1) \leq 2$, so $\overline{r}_p(\Omega_\theta, \sigma) \leq 1$. Using the global lower bound for minimal dynamical systems (Proposition 3.9), we get the result. □

## 5. Toeplitz subshifts

We construct some examples of Toeplitz subshifts. For a point $x \in A^{\mathbb{N}}$, $p > 0$ put

$$\mathrm{Per}_p(x) = \{k \in \mathbb{N} : \forall n \in \mathbb{N}, x_{k+np} = x_k\}$$

The $p$-skeleton $y = S_p(x) \in (A \cup \{*\})^{\mathbb{N}}$ of $x$ is obtained from $x$ by replacing $x_i$ by $*$ for every $i \notin \mathrm{Per}_p(x)$, so $y_i = x_i$ if $i \in \mathrm{Per}_p(x)$, and $y_i = *$ otherwise.

A sequence $x \in A^{\mathbb{N}}$ is a Toeplitz sequence, if the union of all $\mathrm{Per}_p(x)$ is $\mathbb{N}$. A Toeplitz subshift is the orbit closure of a Toeplitz sequence $\Sigma_x = \overline{\mathbf{o}(x)}$. If $x$ is a Toeplitz sequence, then $\sigma^p(S_p(x)) = S_p(x)$. An integer $p > 1$ is an essential period of $x$, if $S_p(x)$ is not periodic with any smaller period than $p$. A periodic structure for an aperiodic Toeplitz sequence $x \in A^{\mathbb{N}}$ is a sequence $p = (p_i)_{i \geq 0}$ such that every $p_i$ is an essential period of $x$, $p_i$ divides $p_{i+1}$ and the union of all $\mathrm{Per}_{p_i}(x)$ is $\mathbb{N}$.

**Example 5.1.** *For every $\alpha \geq 1$ there exists a Toeplitz subshift such that*

$$\underline{h}_p(\Sigma) = \underline{r}_p(\Sigma) = \overline{r}_p(\Sigma) = \overline{h}_p(\Sigma) = \alpha.$$

*Proof.* Let $(n_k)_{k\geq 1}$ be a sequence satisfying $3 \leq n_k \leq n_{k+1} \leq (n_k-1)^2$. Put $A_k = \{0,1,\ldots,n_k-1\}$ and construct injective substitutions $\nu_k : A_{k+1} \to A_k^5$, so that for every $a \in A_{k+1}$, $\nu_k(a) = 00b0c$ for some $b,c \in A_k \setminus \{0\}$ and for every $b \in A_k \setminus \{0\}$, there exists $a \in A_{k+1}$ such that $\nu_k(a) = 00b0b$. For example, if $n_k = k+2$, we have $\nu_1 : \{0,1,2,3\} \mapsto \{00101, 00202, 00102, 00201\}$. Set

$$\Sigma = \bigcap_{k=1}^{\infty} \{\sigma^i \nu_1 \ldots \nu_{k-1}(u) : \ u \in A_k, \ i \geq 0\} \subseteq A_1^{\mathbb{N}}.$$

If $u \in \mathcal{L}^6(\Sigma)$, then there exist $a,b \in A_2$ and a phase $0 \leq m < 5$ such that $u_i = \nu_1(ab)_{m+i}$ for all $i < |u|$. For different phases $m$, $u$ contains zeros in different positions, so $\tau[u] \geq 5$. Moreover, since either $a=0$ or $b=0$, and $0a0a0 \in \mathcal{L}(\Sigma)$ for all $a \in A_1$, we have $\tau[u] \leq 10$. If $u \in \mathcal{L}(\Sigma)$ and $|u| = 5^k$ for some $k > 1$, then there exists $v \in A_k^6$ and a phase $0 \leq m < 5^k$ such that $u_i = \nu_1 \ldots \nu_{k-1}(v)_{m+i}$. Since $5 \leq \tau[v] \leq 10$, we get $5^k \leq \tau[u] \leq 2 \cdot 5^k$. In general, if $5^{k-1} < |u| \leq 5^k$, then

$$\frac{|u|}{5} \leq 5^{k-1} \leq \tau[u] \leq 2 \cdot 5^k \leq 10|u|.$$

We estimate now the complexity of $\Sigma$. If $|u| = 5^{k-1}$, then $u$ is a sub-word of $\nu_1 \ldots \nu_{k-1}(ab)$ for some $a,b \in A_k$ and either $a=0$ or $b=0$. Thus we have at most $2n_k$ possibilities for $ab$ and $5^{k-1}$ possibilities for the phase. On the other hand there are at least $n_k$ different words of the form $\nu_1 \ldots n_{k-1}(a)$, and at least $2 \cdot 5^{k-2}$ different phases for them, in which the distinguishing part $b0c$ of $\nu_1(a) = 00b0c$ occurs. Thus we get $2 \cdot 5^{k-2} \cdot n_k \leq P(5^{k-1}) \leq 2 \cdot 5^{k-1} \cdot n_k$. In general, if $5^{k-1} \leq m < 5^k$, then

$$2 \cdot 5^{k-2} \cdot n_k \leq P(m) \leq 2 \cdot 5^k \cdot n_{k+1}.$$

To construct a subshift with polynomial entropies $\alpha = 1$, put $n_k = 3$ for all $k$ and get

$$\underline{h}_p(\Sigma) = r_p(\Sigma) = \overline{h}_p(\Sigma) = 1.$$

Let $\alpha > 1$, and let $k_0$ be the first integer with $5^{k_0(\alpha-1)} \geq 3$. For $k \geq k_0$ define $n_k$ by $5^{k(\alpha-1)} \leq n_k < 5^{k(\alpha-1)} + 1$. Using the sequence $(n_k)_{k \geq k_0}$ we construct a subshift in alphabet $A = A_{k_0} = \{0,\ldots,n_{k_0}-1\}$. We verify easily $n_k \leq n_{k+1} \leq (n_k-1)^2$. If $5^{k-1} < m \leq 5^k$, then

$$\frac{\ln(2 \cdot 5^{k-2+k(\alpha-1)}+1)}{k \ln 5} \leq \frac{\ln P(m)}{\ln m} \leq \frac{\ln(2 \cdot 5^{k+(k+1)(\alpha-1)}+1)}{(k-1)\ln 5}.$$

Both sides of this inequality tend to $\alpha$ as $k \to \infty$, so $\underline{h}_p(\Sigma) = \overline{h}_p(\Sigma) = \alpha$. By Proposition 3.11, $\overline{r}_p(\Sigma) = \underline{r}_p(\Sigma) = \alpha$. □

**Example 5.2.** *For every $\alpha > 2$ there exists a Toeplitz subshift with*

$$\underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma) = \alpha, \ \overline{h}_p(\Sigma) = \alpha + 1.$$

*Proof.* We use the construction of the preceding example, but we do not require $n_k \leq n_{k+1}$, so that not all words $00a0a$ may be present. We loose the upper estimate for $\tau[u]$, but we still have $\tau[u] \geq |u|$. Thus we put (for large enough $k$)

$$n_{2k} = \lceil 5^{2k(\alpha-1)} \rceil, \ n_{2k+1} = \lceil 5^{(2k+1)\alpha} \rceil$$

and we get $\underline{h}_p(\Sigma) = \alpha$, $\overline{h}_p(\Sigma) = \alpha + 1$, and $\underline{r}_p(\Sigma) \leq \underline{h}_p(\Sigma)$. □

## 6. Conclusions

If we replace in Definition 3.1 the gauge function $1/t$ by $e^{-t}$ we get (exponential) lower and upper recurrence dimensions $\underline{r}(Y,f)$ and $\overline{r}(Y,f)$ of a subset $Y \subseteq X$ of a zero-dimensional dynamical system $(X,f)$. For these (exponential) recurrence dimensions, Theorems 3.2, 3.3, 3.4, 3.7 and 3.8 remain valid. For subshifts, we get

$$\liminf_{n\to\infty} \frac{\ln \operatorname{Per}(n)}{n} \leq \underline{r}(X,f) \leq h(X,f)$$

$$\limsup_{n\to\infty} \frac{\ln \operatorname{Per}(n)}{n} \leq \overline{r}(X,f) \leq h(X,f)$$

Examples in [5] or [7] show that there exists a subshift with $0 < \underline{r}(\Sigma,\sigma)$ and also a subshift with $\overline{r}(\Sigma,\sigma) = 0$ and positive topological entropy.

The Carathéodory structures used in this paper are close to the partition functions employed in the thermodynamic formalism. Recently Afraimovich and coworkers [2] constructed partition functions where, besides the exponential of negative Poincaré recurrence time of a cylinder, there is some power of the measure of the cylinder. This would allow to get some sort of multifractal analysis of the systems through the definition of a whole spectrum of recurrence dimensions. The meaning of these "generalized recurrence dimensions" is not yet clear and their computation and interpretation in the context of the Toeplitz flows analysed in this paper is surely an interesting challenge.

On can generalize the definition of recurrence dimension to higher-dimensional homogenous spaces. These are spaces whose nonempty open sets have the same topological dimension. A topological partition of a homogenous space is a finite collection of its pairwise disjoint open sets whose closures cover the space. In zero-dimensinal spaces, topological partitions are just clopen covers. In a real interval, topological partitions consist of finite union of open intervals.

We have briefly spoken, in the Introduction, of the recurrence dimension of a measure; its value was always zero with the former definition of the AP-dimension. It would be interesting to reconsider this dimension in the framework developed in this paper. There is no reason that it should be again zero. On the contrary it could classify the invariant measures or, in the case of a uniquely ergodic systems, it could give a new metric invariant.

As a final remark, we would like to point out that the main subject of the present paper is the study of the recurrence dimension for minimal sets, just to show that it is a non-trivial topological invariant. The variety of behaviors of this dimension shows that it captures fine combinatorial structures of the underlying symbolic systems. The role of this dimension for systems with higher chaotic behavior, partially disclosed in [8] is a very promising field of research.

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