A search algorithm for the maximal attractor of a cellular automaton

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Abstract. We present an algorithm which finds the maximal attractor (limit set) of those cellular automata whose maximal attractor is a sofic subshift.

1 Introduction

The maximal attractor (limit set) is one of its most intensively studied structures in cellular automata theory. It is the intersection of all forward images of the state space. A cellular automaton is called stable, if the maximal attractor is attained in a finite number of forward images. In this case, the maximal attractor is a sofic subshift, and its structure can be obtained by an algorithm. However, the problem whether a cellular automaton is stable is undecidable (Culik, Pachl and Yu [2]), the set of stable cellular automata being only recursively enumerable.

The maximal attractor of an unstable cellular automaton may but need not be sofic. In fact, its language complexity may be arbitrarily high. The only constraint is that the complement of its language is recursively enumerable (Culik, Hurd and Yu [1]). There are other constraints of dynamical nature: The maximal attractor contains a homogenous configuration (of the form a^∞) and it is chain-mixing for the shift (see Formenti and Kůrka [3]).

To determine the maximal attractor of a given cellular automaton is not always a simple task. One can estimate the maximal attractor from above by the forward images of the state space, but it is undecidable whether this procedure ever ends. In the present paper we propose a method for the construction of the maximal attractor from bellow. The key role in this procedure is played by signal subshifts (see Kůrka [5]) which are infinite subshifts consisting of weakly periodic configurations of a given period (p, q), *i.e.*, satisfying $F^q\sigma^p(x) = x$. Signal subshifts are of finite type and can be easily computed. All signal subshifts are contained in the maximal attractor.

Our algorithm works for some cellular automata with a finite number of signal subshifts. From them, we build their join - a larger subshift which is still included in the maximal attractor. Then, we construct the forward images of the join, obtaining an increasing sequence of subshifts included in the maximal attractor. If this procedure stops in finite time, we test a special condition of decreasing preimages. A subshift has decreasing preimages, if the preimage of any word not in the language contains a shorter word which is not in the language. This condition is decidable, and a subshift which satifies it already includes whole maximal attractor. While our algorithm is not (and cannot be) universal, it works for a large class of cellular automata.

2 Subshifts and cellular automata

For a finite alphabet A, denote by $A^* := \bigcup_{n\geq 0} A^n$ the set of words over A. The length of a word $u = u_0 \dots u_{n-1} \in A^n$ is denoted by $|u| := n$. The word of zero length is λ. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{Z}}$ the space of A-**configurations**, or doubly-infinite sequences of letters of A equipped with the metric $d(x,y) := 2^{-n}$, where $n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$.

The **shift map** $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i := x_{i+1}$. For any nonzero $u \in A^*$ we have a σ-periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{Z}}$, which is closed and strongly σ-invariant, *i.e.*, $\sigma(\Sigma) = \Sigma$. A subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is **transitive**, if for any words $u, v \in \mathcal{L}(\Sigma)$ there exists $w \in A^*$ such that $uwv \in \mathcal{L}(\Sigma)$.

For a subshift Σ there exists a set $D \subseteq A^*$ of forbidden words such that $\Sigma = \Sigma_D := \{x \in A^{\mathbb{Z}} : \; \forall u \sqsubseteq x, u \notin D\}$. A subshift is uniquely determined by its **language**

$$\mathcal{L}(\Sigma) := \bigcup_{n\geq 0} \mathcal{L}^n(\Sigma), \quad \text{where} \quad \mathcal{L}^n(\Sigma) := \{u \in A^n : \; \exists x \in \Sigma, u \sqsubseteq x\}.$$

If $x \in A^{\mathbb{Z}}$ is a configuration and $I \subseteq \mathbb{Z}$ is an interval, denote by $x|I : I \to A$ the restriction of x to I. The extended language of Σ is

$$\overline{\mathcal{L}}(\Sigma) = \{x|I : \; x \in \Sigma, \quad I \subseteq \mathbb{Z} \quad \text{is an interval}\}.$$

A cellular automaton is a continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which commutes with the shift map, *i.e.*, $F\sigma = \sigma F$. For a cellular automaton F there exists a **local rule** $f : A^{d+1} \to A$ such that $F(x)_i = f(x_{[i-m,i-m+d]})$ for some **memory** $m \in \mathbb{Z}$ and **diameter** $d \geq 0$. The local rule can be extended to a map $f : A^* \to A^*$ by $f(u)_i := f(u_{[i,i+d]})$ for $0 \leq i < |u| - d$. The maximal attractor (limit set) of a cellular automaton F is $\Omega_F = \bigcap_{n\geq 0} F^n(A^{\mathbb{Z}})$.

3 Sofic subshifts

A subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is **sofic** if its language $\mathcal{L}(\Sigma)$ is regular. Sofic subshifts are usually described by labelled graphs. A **labelled graph** over an alphabet A is a structure $G = (V, E, s, t, l)$, where V is a finite set of vertices, E is a finite set of edges, $s, t : E \to V$ are the **source** and **target maps**, and $l : E \to A$ is a **labelling function**. A finite or infinite word $w \in E^* \cup E^{\mathbb{Z}}$ is a **path** in G if $t(w_i) = s(w_{i+1})$ for all i. The source and target of a finite path $w \in E^n$ are $s(w) := s(w_0)$, $t(w) := t(w_{n-1})$. The **label** of a path is defined by $l(w)_i := l(w_i)$. A subshift Σ is sofic if there exists a labelled graph G such that $\Sigma = \Sigma_G$ is the set of labels of all doubly infinite paths in G. In this case we say that G is a **presentation** of Σ (see e.g. Lind and Marcus [6], or Kitchens [4]). Among all presentations of a sofic subshift there exists a minimal one which corresponds to the minimal deterministic finite automaton which recognizes its language.

A labelled graph $G = (V, E, s, t, l)$ is **connected** if for any two vertices $q, r \in V$ there exists a path $w \in E^*$ from q to r. A **subgraph** of a graph G is a graph $G' = (V', E', s', t', l')$, such that $V' \subseteq V$, $E' = \{e \in E : \; s(e) \in V' \;\&\; t(e) \in V'\}$, and s', t', l' coincide respectively with s, t, l on E'. A **connected component** of G is a subgraph of G which is connected and maximal with this property. The subshift of a connected graph is transitive. Conversely, every transitive sofic subshift $\Sigma \subseteq A^{\mathbb{Z}}$ has a connected presentation.

A subshift Σ is of **finite type** (SFT), if $\Sigma = \Sigma_D$ for some finite set $D \subseteq A^*$ of forbidden words. The order $\mathfrak{o}(\Sigma)$ of a SFT is the length of its longest forbidden word. A configuration $x \in A^{\mathbb{Z}}$ belongs to Σ iff $x_{[i,i+\mathfrak{o}(\Sigma))} \in \mathcal{L}(\Sigma)$ for all $i \in \mathbb{Z}$. Any SFT is sofic: If $p = \mathfrak{o}(\Sigma) - 1$, the **canonical graph** $G = (V, E, s, t, l)$ of Σ is given by $V = \mathcal{L}^p(\Sigma)$, $E = \mathcal{L}^{p+1}(\Sigma)$, $s(u) = u_{[0,p)}$, $t(u) = u_{[1,p]}$ and $l(u) = u_p$.

We say that a labelled graph G is p**-distinguishing**, if for any two paths $v, w \in E^p$ with $t(v) = t(w)$ we have $l(v) = l(w)$. In this case there exist labelling functions $\nu : V \to A^p$ and $\pi : E \to A^{p+1}$. For $q \in V$ we have $\nu(q) = u$ iff there exists a path $v \in E^p$ such that $t(v) = q$ and $l(v) = u$. For $e \in E$ we have $\pi(e) = u$ iff there exists a path $w \in E^{p+1}$ such that $w_p = e$ and $l(w) = u$. Any sofic subshift has a p-distinguishing presentation for any $p > 0$. If $G = (V, E, s, t, l)$ is a presentation of Σ, we define a graph $G' = (V', E', s', t', l')$, where $V' \subseteq E^p$ is the set of paths of G of length p, $E' \subseteq E^{p+1}$ is the set of paths of G of length $p + 1$, t' and s' are the prefix and suffix maps and $l'(u) = l(u_q)$. Then G' is a p-distinguishing presentation of Σ.

Given two sofic subshifts $\Sigma_0, \Sigma_1 \subseteq A^{\mathbb{Z}}$, their union and intersection (provided non-empty) are sofic subshifts. Moreover there exists an algorithm which constructs a presentation of $\Sigma_0 \cup \Sigma_1$ and $\Sigma_0 \cap \Sigma_1$ from those of Σ_0 and Σ_1. It is also decidable whether $\Sigma_0 \subseteq \Sigma_1$. Given a labelled graph G it is decidable whether Σ_G is a SFT (see Lind and Marcus [6], page 94).

Proposition 1 *Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a sofic subshift and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a cellular automaton with local rule $f : A^{d+1} \to A$. Then $F(\Sigma)$ and $F^{-1}(\Sigma)$ are sofic subshifts, and there exists an algorithm which constructs their graphs from the local rule f and a graph of Σ.*

Proof. Let $G = (V, E, s, t, l)$ be a presentation of Σ. Let V_0 be the set of paths of G of length d, and let E_0 be the set of paths of G of length $d+1$. define the source and target maps by $s_0(u) = u_{[0,d)}$, $t_0(u) = u_{[1,d]}$. Then $G_0 = (V_0, E_0, s_0, t_0, f)$ is a presentation of $F(\Sigma)$.
Set $E_1 = \{(e, u) \in E \times A^{d+1} : \ l(e) = f(u)\}$, and define s_1, t_1 by $s_1(e, u) = (s(e), u_{[0,d)})$, $t_1(e, u) = (t(e), u_{[1,d]})$. Finally set $V_1 = s_1(E_1) \cup t_1(E_1) \subseteq V \times A^d$ and define the labelling function by $l_1(e, u) = u_d$. Then $G_1 = (V_1, E_1, s_1, t_1, l_1)$ is a presentation of $F^{-1}(\Sigma)$.

4 Join of subshifts

Definition 1 *Given an integer $c \geq 0$, the* c**-join** $\Sigma_0 \overset{c}{\vee} \Sigma_1$ *of subshifts $\Sigma_0, \Sigma_1 \subseteq A^{\mathbb{Z}}$ consists of all configurations $x \in A^{\mathbb{Z}}$ such that either $x \in \Sigma_0 \cup \Sigma_1$, or there exist integers b, a such that $b - a \geq c$, $x_{(-\infty,b)} \in \overline{\mathcal{L}}(\Sigma_0)$, and $x_{[a,\infty)} \in \overline{\mathcal{L}}(\Sigma_1)$.*

Proposition 2 *The c-join of two subshifts is a subshift and the operation of c-join is associative. A configuration $x \in A^{\mathbb{Z}}$ belongs to $\Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n$ iff there exist integers $k > 0$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and intervals $I_1 = (a_1, b_1), I_2 = [a_2, b_2), \ldots, I_k = [a_k, b_k)$ such that $a_1 = -\infty$, $b_k = \infty$, $a_j < a_{j+1}$, $b_j < b_{j+1}$, $b_j - a_{j+1} \geq c$, and $x|I_j \in \overline{\mathcal{L}}(\Sigma_{i_j})$.*

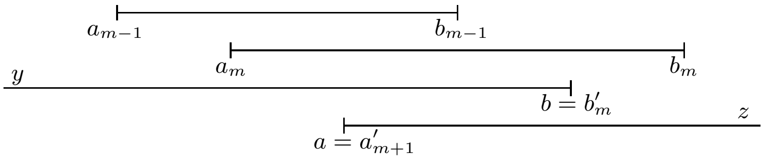

Fig. 1. Associativity of the join.

Proof. It is clear that the c-join of two subshifts is a subshift. We prove by induction the formula for $\Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n$. Let $x \in (\Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_{n-1}) \overset{c}{\vee} \Sigma_n$, so there exists $y \in \Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_{n-1}$, $z \in \Sigma_n$ and integers a, b such that $x_{(-\infty,b)} = y_{(-\infty,b)}$, $x_{[a,\infty)} = z_{[a,\infty)}$ and $b - a \geq c$. By the induction hypothesis there exist integers k and intervals $I_1, \ldots I_k$ such that $y|I_j \in \overline{\mathcal{L}}(\Sigma_{i_j})$. Let m be the unique index such that $b_{m-1} < b \leq b_m$ (see Figure 1). Set $i'_j := i_j$ for $j \leq m$ and $i'_{m+1} = n$. For $k \leq m+1$ define intervals $I'_j := [a'_j, b'_j)$ by

$$a'_j := \begin{cases} a_j & \text{for} \quad j \leq m \\ \max\{a, a_m + 1\} & \text{for} \quad j = m+1 \end{cases}, \quad b'_j := \begin{cases} b_j & \text{for} \quad j < m \\ b & \text{for} \quad j = m \\ \infty & \text{for} \quad j = m+1 \end{cases}$$

Then clearly $x|I'_j \in \overline{\mathcal{L}}(\Sigma_{i'_j})$ and $b'_j - a'_{j+1} = b_j - a_{j+1} \geq c$ for $j < m$. If $a'_{m+1} = a_m + 1$, then $b'_m - a'_{m+1} = b - a_m - 1 \geq b_{m-1} - a_m \geq c$. If $a'_{m+1} = a$, then $b'_m - a'_{m+1} = b - a \geq c$. Similarly it can be shown that the formula holds for $x \in \Sigma_1 \overset{c}{\vee} (\Sigma_2 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n)$ provided it holds for $z \in \Sigma_2 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n$. This proves associativity. $\square$

Proposition 3 *Let $\Sigma_0, \Sigma_1 \subseteq A^{\mathbb{Z}}$ be sofic subshifts and $c \geq 0$. Then $\Sigma_0 \overset{c}{\vee} \Sigma_1$ is a sofic subshift and there exists an algorithm which constructs its presentation from those of Σ_i.*

Proof. Let $G_i = (V_i, E_i, s_i, t_i, l_i)$ be c-distinguishing presentations of Σ_i, and assume that $V_0 \cap V_1 = \emptyset$ and $E_0 \cap E_1 = \emptyset$. We have labelling functions $\pi_i : E_i \to A^{c+1}$. Set $V = V_0 \cup V_1$,

$$E = E_0 \cup E_1 \cup \{(e_0, e_1) \in E_0 \times E_1 : \ \pi_0(e_0) = \pi_1(e_1)\}.$$

The source, target and label maps extend s_i, t_i, l_i. For the new edges we have $s(e_0, e_1) = s_0(e_0)$, $t(e_0, e_1) = t_1(e_1)$, $l(e_0, e_1) = l_0(e_0) = l_1(e_1)$. Then $\Sigma_G = \Sigma_0 \overset{c}{\vee} \Sigma_1$. $\square$

5 Signal subshifts

Definition 2 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. A configuration $x \in A^{\mathbb{Z}}$ is* **weakly periodic***, if $F^q\sigma^p(x) = x$ for some $q > 0$ and $p \in \mathbb{Z}$. We call (p, q) the* **period** *of x and p/q its speed. Let $\Sigma_{(p,q)} := \{x \in A^{\mathbb{Z}} : F^q\sigma^p(x) = x\}$ be the set of all weakly periodic configurations with period (p, q). A* **signal subshift** *is any infinite $\Sigma_{(p,q)}$.*

Observe that $\Sigma_{(p,q)}$ is closed and σ-invariant, so it is a subshift provided it is nonempty. Moreover, $\Sigma_{(p,q)}$ is F-invariant and $F : \Sigma_{(p,q)} \to \Sigma_{(p,q)}$ is bijective, so $\Sigma_{(p,q)} \subseteq \Omega_F$. If $\Sigma_{(p,q)}$ is finite, it consists only of σ-periodic configurations.

Proposition 4 *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with diameter d and memory m, so $F(x)_i = f(x_{[i-m,i-m+d]})$.*

(1) If $\Sigma_{(p,q)}$ is nonempty, then it is a subshift of finite type.
(2) If $\Sigma_{(p,q)}$ is infinite, then $m - d \le p/q \le m$.
(3) If $p_0/q_0 < p_1/q_1$, then $\Sigma_{(p_0,q_0)} \cap \Sigma_{(p_1,q_1)} \subseteq \{x \in A^{\mathbb{Z}} : \sigma^p(x) = x\}$, where $p = q(\frac{p_1}{q_1} - \frac{p_0}{q_0})$ and $q = \mathrm{lcm}(q_0, q_1)$ (the least common multiple).

Proof. (1) Set $D := \{u \in A^{dq+1} : f^q(u) \ne u_{mq-p}\}$. Then $\Sigma_{(p,q)} = \Sigma_D$.

(2) If $x \in \Sigma_{(p,q)}$, then $x_i = f^q(x_{[i+p-mq,i+p-mq+dq]})$. If $p-mq+dq < 0$, then there exists a function $g : A^{mq-p} \to A^{mq-p}$ such that $x_{[i+p-mq+1,i]} = g(x_{[i+p-mq,i-1]})$ for every $i \in \mathbb{Z}$. This is possible only if x is σ-periodic. Moreover, the period of x is bounded, so $\Sigma_{(p,q)}$ is finite. The proof is similar for $p > mq$.

(3) Set $p_2 = qp_0/q_0$, $p_3 = qp_1/q_1$. If $x \in \Sigma_{(p_0,q_0)} \cap \Sigma_{(p_1,q_1)}$, then $\sigma^{-p_2}(x) = F^q(x) = \sigma^{-p_3}(x)$, so $\sigma^{p_2-p_3}(x) = x$. □

A positively expansive CA has no signal subshifts (see [5] for a proof). The shift cellular automaton $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ has the unique signal subshift $\Sigma_{(-1,1)} = A^{\mathbb{Z}}$. A cellular automaton with infinitely many signal subshifts with infinitely many speeds has been constructed in [5].

Theorem 5 *Consider a cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and the signal subshifts $\Sigma_{(p_1,q_1)}, \ldots, \Sigma_{(p_n,q_n)}$ with decreasing speeds,* i.e., *$p_i/q_i > p_j/q_j$ for $i < j$. Set $q := \mathrm{lcm}\{q_1, \ldots q_n\}$ (the least common multiple). There exists $c \ge 0$ such that for $\Sigma := \Sigma_{(p_1,q_1)} \stackrel{c}{\vee} \cdots \stackrel{c}{\vee} \Sigma_{(p_n,q_n)}$ we have $\Sigma \subseteq F^q(\Sigma)$ and therefore $\Sigma \subseteq \Omega_F$.*

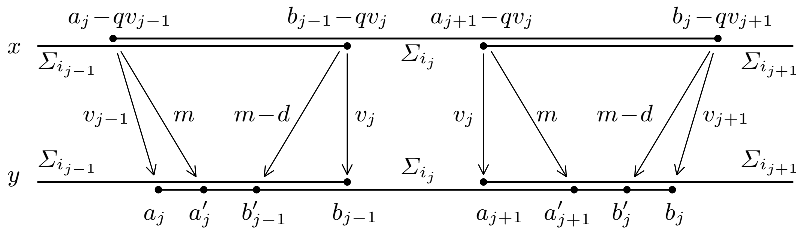

Fig. 2. Preimage of a configuration in the join.

Proof. Let m be the memory of F and d its diameter. If $i < j$ then $m - d \leq \frac{p_j}{q_j} < \frac{p_i}{q_i} \leq m$, so $0 < \frac{p_i}{q_i} - \frac{p_j}{q_j} \leq d$. Set

$$c := \max\left\{ q\left(\frac{p_j}{q_j} - \frac{p_i}{q_i} + d\right) : \ i < j \ \& \ \Sigma_i \cap \Sigma_j \neq \emptyset \right\} \geq 0$$

Let $y \in \Sigma$, $1 \leq i_1 < i_2 \cdots < i_k \leq n$ and let $I_1 = (a_1, b_1], \dots, I_k = [a_k, b_k)$ be intervals such that the restrictions $y|I_j$ belong to $\overline{\mathcal{L}}(\Sigma_{(p_{i_j}, q_{i_j})})$. Let $v_j := p_{i_j}/q_{i_j}$ be the speed of the i_j-th signal. The configurations in $\Sigma_{i_j} \cap \Sigma_{i_{j+1}}$ are σ-periodic with period $n_j := q(v_j - v_{j+1})$. Let u be the prefix of $y|I_j$ of length n_{j-1} and let v be the suffix of $y|I_j$ of length n_j. Since $c \geq n_{j-1}$ and $c \geq n_j$, we get $y^{(j)} = u^\infty(y|I_j)v^\infty \in \Sigma_{(p_{i_j}, q_{i_j})}$ and $F^q\sigma^{qv_j}(y^{(j)}) = y^{(j)}$. Set $J_j = [a_j - qv_{j-1}, b_j - qv_{j+1})$. For the endpoints of these intervals we get $(b_j - qv_{j+1}) - (a_{j+1} - qv_j) = b_j - a_{j+1} + qn_j \geq c$. There exists a unique configuration x such that $x|J_j = \sigma^{qv_j}(y^{(j)})|J_j$. We show $F^q(x) = y$. Set $a'_j := a_j + q(m - v_{j-1})$, $b'_j := b_j + q(m - d - v_{j+1})$ (see Figure 2). For $a'_j \leq k \leq b'_j$ we have

$$\begin{aligned} F^q(x)_k &= f^q(x_{[k-qm, k-qm+qd]}) = f^q(\sigma^{qv_j}(y^{(j)})_{[k-qm,k-qm+qd]}) \\ &= F^q\sigma^{qv_j}(y^{(j)})_k = (y^{(j)})_k = y_k \end{aligned}$$

We have $b'_j - a'_{j+1} = b_{j-1} - a_j - q(v_{j+1} - v_j + d) \geq c - c = 0$, so the intervals $[a'_j, b'_j)$ cover whole $\mathbb{Z}$, and $F^q(x) = y$. Thus $\Sigma \subseteq F^q(\Sigma)$. It follows that for any $k > 0$ we have $\Sigma \subseteq F^{kq}(\Sigma) \subseteq F^{kq}(A^{\mathbb{Z}})$, so $\Sigma \subseteq \Omega_F$. $\square$

6 Decreasing preimages

Definition 3 *Let $f : A^{d+1} \to A$ be a local function of a cellular automaton. We say that a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ has m-***decreasing preimages***, if for each $u \in A^* \setminus \mathcal{L}(\Sigma)$, each $v \in f^{-m}(u)$ contains as a subword a word*

$w \in A^* \setminus \mathcal{L}(\Sigma)$ such that $|w| < |u|$ (the condition is satisfied trivially if $f^{-m}(u) = \emptyset$). We say that Σ has **decreasing preimages**, *if it has m-decreasing preimages for some $m > 0$.*

Example 1 *The maximal attractor of the ECA128 $F(x)_i = x_{i-1}x_ix_{i+1}$ has decreasing preimages.*

Proof. We have $\Omega_F = \{x \in A^{\mathbb{Z}} :\ \forall n > 0, 10^n1 \not\sqsubseteq x\}$. If $u \in A^* \setminus \mathcal{L}(\Omega_F)$, then it contains 10^n1 as a subword, and each preimage v of u contains $10^{n-2}1$ as a subword. □

Example 2 *There exists a cellular automaton such that $\Omega_F(A^{\mathbb{Z}})$ does not have decreasing preimages.*

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Fig. 3. Nondecreasing preimages.

Proof. The alphabet is $A = \{0, 1, 2, 3\}$, $d = 2$, $m = -1$, and the transition table $f : A^3 \to A$ is given by

$$x02 : 1,\ x03 : 1,\ x12 : 2,\ x13 : 2,\ 02x : 0,\ 12x : 0,$$

where $x \in A$ and the first applicable rule is used, otherwise the letter is unchanged. The letter 3 is stationary, and the binary value of a word $u \in \{0, 1, 2\}^*$ between two threes increases by one every two time steps out of three. Let x be a configuration such that $x_{(-\infty,0]} = 0^\infty 3$. Then for each n, the sequence $f^i(x)_{[-n,0]}$ is eventually periodic with preperiod $n-1$ and period $p_n = 3 \cdot 2^{n-1}$, so $F^{n-1+p_n}(x)_{[-n,0]} = F^{n-1}(x)_{[-n,0]}$ (see Figure 3). It follows that $F^i(x)_{[-n,0]} \in \mathcal{L}(\Omega_F)$ for $i \geq n-1$. On the other hand, if $u \in \{0, 1, 2\}^n \setminus \{2^n\}$, then $3u3 \notin \mathcal{L}(\Omega_F)$. In particular, if $n > 1$ then no word $3F^i(x)_{[-n,0]}$ belongs to $\mathcal{L}(\Omega_F)$. Thus $u = 3F^{n-1}(x)_{[-n,0]} \notin \Omega_F$, but for $m < p_n$ we have $v = 3^{m+1}F^{n-1+p_n-m}(x)_{[-n,0]}3^m \in f^{-m}(u)$ and each subword of v of length $|u| = n + 2$ belongs to Ω_F. □

Proposition 6 *If a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ has decreasing preimages, then $\Omega_F \subseteq \Sigma$.*

Proof. By the assumption, if $u \in A^* \setminus \mathcal{L}(\Sigma)$, then each $v \in f^{-m|u|}$ contains as a subword $w \in A^* \setminus \mathcal{L}(\Sigma)$ with $|w| = 0$. This is a contradiction, since $w = \lambda \in \mathcal{L}(\Sigma)$. Thus $f^{-m|u|}(u)$ is empty and $u \notin \mathcal{L}(\Omega_F)$. Thus $\mathcal{L}(\Omega_F) \subseteq \mathcal{L}(\Sigma)$ and $\Omega_F \subseteq \Sigma$. □

Proposition 7 *There exists an algorithm, which decides whether for a given cellular automaton and given $m > 0$, a given sofic subshift has m-decreasing preimages.*

Proof. By the assumption both $\mathcal{L}(\Sigma)$ and $\mathcal{L}(F^{-m}(\Sigma))$ are regular languages. It follows that the language

$$L = \{v \in A^* : \; f^m(v) \notin \mathcal{L}(\Sigma), \; \forall k \leq md+1, v_{[k,k+|v|-md-1)} \in \mathcal{L}(\Sigma)\}$$

is regular too and we can construct its recognizing finite automaton from that of $\mathcal{L}(\Sigma)$. Since L is empty iff Σ has m-decreasing f-preimages, we get the deciding procedure. □

Corollary 8 *Let F be a cellular automaton, let $\Sigma_1, \ldots, \Sigma_n$ be signal subshifts with decreasing speeds, and set $q := \mathrm{lcm}(q_1, \ldots, q_n)$. If $F^{kq}(\Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n)$ has decreasing preimages for some k, c, then $\Omega_F = F^{kq}(\Sigma_1 \overset{c}{\vee} \cdots \overset{c}{\vee} \Sigma_n)$.*

Proposition 9 *The set of cellular automata whose maximal attractor is a sofic subshift with decreasing preimages is recursively enumerable.*

Proof. Generate successively all sofic subshifts, verify whether they are strongly invariant and whether they have decreasing preimages. □

While the algorithm based on Proposition 9 is rather time-consuming, and would not give practical results, there is a faster algorithm based on signal subshifts and their join. Given a local rule f with diameter d and memory m, consider the procedure `Omega(f,m,n)` which performs the following steps:

`procedure Omega(f,m,n)`

1. Construct all signal subshifts with periods (p_i, q_i) such that $q_i \leq n$, and $(m-d)q_i \leq p_i \leq mq_i$. Denote by q the least common multiple of q_i.
2. Order the signal subshifts obtained in step 1 by decreasing speeds and construct their c-join Σ, where c is given in the proof of Theorem 5.

3. Construct $F^q(\Sigma), F^{2q}(\Sigma), \ldots$ and test whether $F^{kq}(\Sigma) = F^{(k+1)q}(\Sigma)$.
4. If step 3 ends with $F^{kq}(\Sigma) = F^{(k+1)q}(\Sigma)$, verify whether $F^{kq}(\Sigma)$ has decreasing preimages. If so, $\Omega_F = F^{kq}(\Sigma)$ has been found.

The procedure `Omega(f,m,n)` may fail to give a result if it repeats indefinitely step 3, or if step 4 gives the negative result. To get a procedure not depending on n, one can perform concurrently procedures `Omega(f,m,1)`, `Omega(f,m,2)`..., so that by time $n(n+1)/2$, n steps of `Omega(f,m,1)` have been performed, $n-1$ steps of `Omega(f,m,2)`, etc. The algorithm ends whenewer one of the procedures `Omega(f,m,n)` stops, otherwise it runs indefinitely. The implementation of this algorithm is currently under progress.

7 Examples

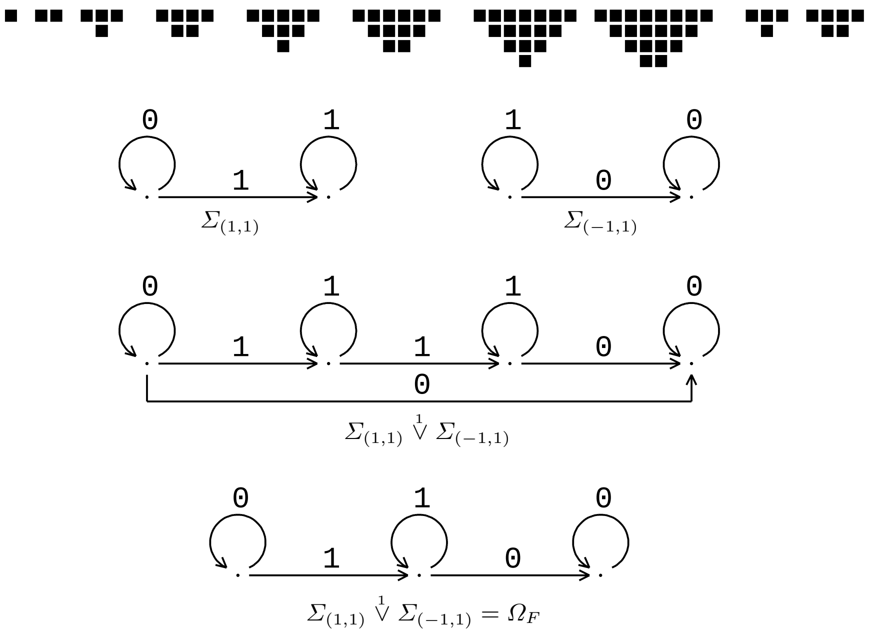

Fig. 4. ECA 128 and its signal subshifts.

Example 3 (ECA 128) *The product rule* $F(x)_i = x_{i-1}x_ix_{i+1}$.

We have two (nontransitive) signal subshifts (see Figure 4)

$$\Sigma_{(1,1)} = \{x \in A^{\mathbb{Z}} :\ 10 \not\sqsubseteq x\}, \quad \Sigma_{(-1,1)} = \{x \in A^{\mathbb{Z}} :\ 01 \not\sqsubseteq x\}.$$

Their intersection is the finite subshift $\{0^\infty, 1^\infty\}$. The maximal attractor is constructed in Figure 4. In the first row, 1-distinguishing presentations for $\Sigma_{(1,1)}$ and $\Sigma_{(-1,1)}$ are constructed. Their join is constructed in the second row. In the third row, the minimal presentation of $\Sigma_{(1,1)} \stackrel{1}{\vee} \Sigma_{(-1,1)}$ is given. As it has decreasing preimages, it equals Ω_F.

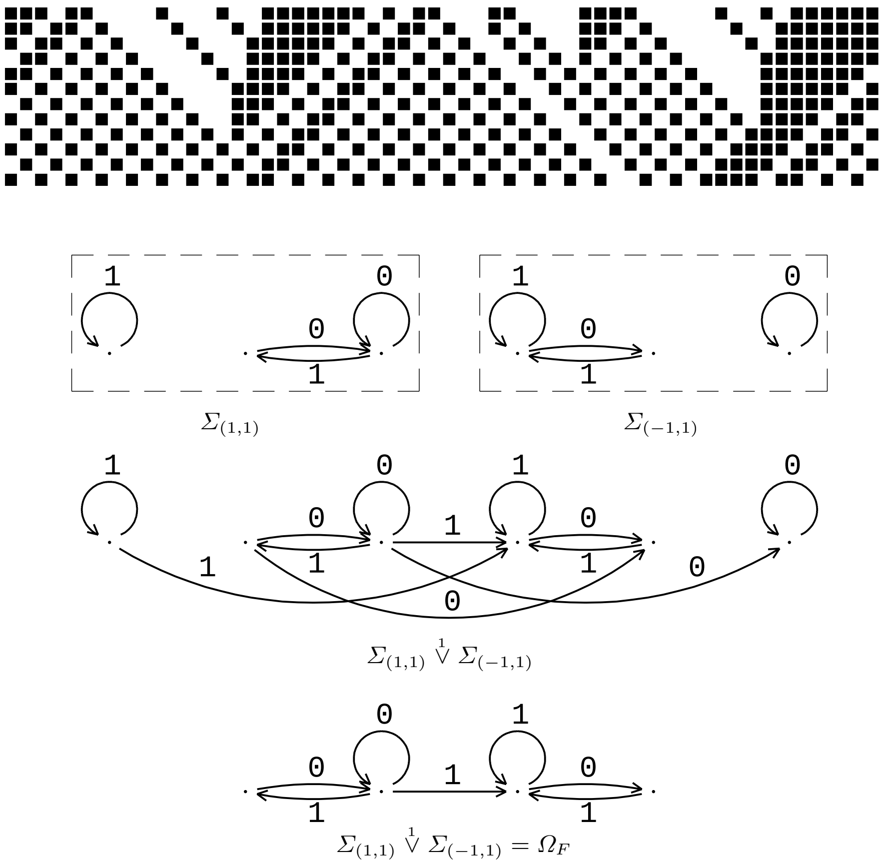

Fig. 5. ECA 184 and its signal subshifts.

Example 4 (ECA 184) *The traffic rule* $F(x)_i = 1$ *iff* $x_{[i-1,i]} = 10$ *or* $x_{[i,i+1]} = 11$.

We have two signal subshifts (see Figure 5) representing holes and jams

$$\Sigma_{(1,1)} = \{x \in A^{\mathbb{Z}} :\ 11 \not\sqsubseteq x\} \cup \{1^\infty\}, \quad \Sigma_{(-1,1)} = \{x \in A^{\mathbb{Z}} :\ 00 \not\sqsubseteq x\} \cup \{0^\infty\}.$$

Their intersection is the finite subshift $\{0^\infty, 1^\infty, (01)^\infty, (10)^\infty\}$. There is one more (nontransitive) signal subshift $\Sigma_{(0,1)} = \{x \in A^{\mathbb{Z}} :\ \forall i < j, x_i \leq x_j\}$. The maximal attractor is constructed in Figure 5. In the first row, 1-distinguishing presentations for $\Sigma_{(1,1)}$ and $\Sigma_{(-1,1)}$ are constructed. Their join is constructed in the second row. In the third row, the minimal presentation of $\Sigma_{(1,1)} \overset{1}{\vee} \Sigma_{(-1,1)}$ is given. As it has decreasing preimages, it equals Ω_F.

Acknowledgments

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