Simulation in dynamical systems and Turing machines

Petr Kůrka
Department of Philosophy of Mathematics and Physics,
Faculty of Mathematics and Physics, Charles University
Malostranské náměstí 25, 118 00 Praha 1, Czechia
fax: (422) 532 742, email: kurka@cspguk11

Abstract

We say that a dynamical system g simulates a dynamical system f, if f is a quotient of a subsystem of g. We get a nontrivial theory, when the conjugacies involved are required to be uniformly continuous. In particular $h(g) \geq h(f)$ for the topological entropies of f and g. We show that every dynamical system on a compact space is uniformly simulated by a dynamical system on discontinuum and therefore also by a dynamical system on the unit interval. We investigate uniform simulations of generalized shifts, which are dynamical system version of Turing machines. We prove that S-unimodal maps of the real interval do not simulate generalized shifts.

1 Introduction

The behaviour of a dynamical system is usually elucidated, when a conjugacy to another dynamical system is found. Usually such a conjugacy is required to be a homeomorphism, but sometimes one-to-one or surjective conjugacies are also useful. We speak in this case about subsystems or quotients of a dynamical system in question. Thus for example the Z-shift $\sigma_Z : 2^Z \to 2^Z$ is a subsystem of the Smale's horseshoe, and the N-shift $\sigma_N : 2^N \to 2^N$ is a subsystem of the quadratic system $F_r(x) = rx(1-x)$ for $r > 4$. On the other hand F_4 is a quotient of σ_N and the hyperbolic toral automorphism $(x, y) \mapsto (x+y \bmod 1, x)$ is a quotient of a subsystem of the shift on 3^N (see [3]).

When a dynamical system f is either a subsystem or a quotient of g, then it is in a certain sense simpler. The dynamics of f is completely determined by that of g, so that g can be used for simulation of f. More generally, we can define that g simulates f, if f can be obtained from g by a finite sequence of subsystems or quotients. We show that f is then a quotient of a subsystem of g.

However, when the conjugacies involved are merely supposed to be continuous, then the theory obtained is rather trivial. The complexity of dynamics might be encoded in the conjugacies themselves, rather than in the other dynamical system. To avoid these cases we consider simulations by uniformly continuous mappings. Uniform simulations conserve the topological entropy: When g uniformly simulates f, then $h(g) \geq h(f)$. The converse statement, however, is not true. We show further that each dynamical system on a compact space is uniformly simulated by some dynamical system on discontinuum and therefore also by some dynamical system on unit interval.

We consider then generalized shifts introduced in [7]. They are dynamical systems on discontinuum, which mimic Turing machines. Universal dynamical systems, i.e. systems which simulate any generalized shift, possess high degree of unpredictability and complexity. They are also conjectured to be at the base of dynamics of living systems (see [5]). Moore shows in [7] that each generalized shift is a subsystem of a piecewise linear dynamical system on the unit square. While any generalized shift is also a subsystem of a dynamical system on the real line, Koiran et al show in [4] that in general it is not a subsystem of any piecewise linear dynamical system on unit interval. Their result holds also for systems with negative Schwarzian derivative. We are not able to get this result with our more general concept of simulation, but we prove it for S-unimodal functions. In doing so we introduce a concept of generalized N-shift, which works on the space 2^N in a similar manner as a generalized shift does on 2^Z. We show that each S-unimodal function is uniformly simulated by some generalized N-shift and that no generalized N-shift uniformly simulates the Z-shift.

2 Continuous simulation

In this section we conceive dynamical system as a continuous mapping $f : X \to X$, where X is a topological space. A conjugacy $\varphi : f \Rightarrow g$ from $f : X \to X$ to $g : Y \to Y$ is a continuous mapping $\varphi : X \to Y$, such that $\varphi f = g\varphi$. Thus we work in the category $\mathcal{TOP}$ of topological spaces and continuous mappings (see [6]).

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi\downarrow & & \downarrow\varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

When the conjugacy φ is one-to-one (i.e. a monomorphism in $\mathcal{TOP}$), then g can be used to simulate f. To get the orbit of $x \in X$, just follow the trajectory of $\varphi(x)$ and then find its unique inverse image. Conversely, when φ is surjective (an epimorphism in $\mathcal{TOP}$), then f can be used to simulate g. For a given $y \in Y$ pick up any $x \in X$ with $\varphi(x) = y$ and follow its trajectory. Its image is then the trajectory of y. These two types of simulations can be concatenated. One possibility is that there exists a dynamical system $h_0 : Z_0 \to Z_0$, a surjective conjugacy $\pi_0 : h_0 \Rightarrow f$ and a one-to-one conjugacy $\mu_0 : h_0 \Rightarrow g$. A dual possibility is that there exists a dynamical system $h_1 : Z_1 \to Z_1$, a one-to-one conjugacy $\mu_1 : f \Rightarrow h_1$, and a surjective conjugacy $\pi_1 : g \Rightarrow h_1$. These two cases are equivalent, as next two Propositions show.

$$
\begin{tikzcd}
X \arrow[r, "f"] & X \\
Z_0 \arrow[u, "\pi_0"] \arrow[r, "h_0"] \arrow[d, "\mu_0"'] & Z_0 \arrow[u, "\pi_0"'] \arrow[d, "\mu_0"] \\
Y \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mu_1"'] & X \arrow[d, "\mu_1"] \\
Z_1 \arrow[r, "h_1"] & Z_1 \\
Y \arrow[u, "\pi_1"] \arrow[r, "g"] & Y \arrow[u, "\pi_1"']
\end{tikzcd}
$$

Proposition 1 *Let $f : X \to X$, $h_0 : Z_0 \to Z_0$, $g : Y \to Y$ be dynamical systems, let $\pi_0 : h_0 \Rightarrow f$ be a surjective conjugacy and $\mu_0 : h_0 \Rightarrow g$ a one-to-one conjugacy. Then there exists a dynamical system $h_1 : Z_1 \to Z_1$, a one-to-one conjugacy $\mu_1 : f \Rightarrow h_1$, and a surjective conjugacy $\pi_1 : g \Rightarrow h_1$.*

Proof: Z_1 together with μ_1 and π_1 is constructed as a pushout of π_0 and μ_0. This means that $\mu_1\pi_0 = \pi_1\mu_0$ and for any morphisms (continuous mappings) $\mu_1' : X \to Z_1'$, $\pi_1' : Y \to Z_1'$, for which $\mu_1'\pi_0 = \pi_1'\mu_0$ there exists a unique morphism $\varphi : Z_1 \to Z_1'$ for which $\mu_1' = \varphi\mu_1$ and $\pi_1' = \varphi\pi_1$. Explicitly, $Z_1 = X \cup (Y - \mu_0(Z_0))$ (disjoint union), μ_1 is the inclusion, and π_1 is defined by $\pi_1(y) = y$ for $y \in Y - \mu_0(Z_0)$, $\pi_1(y) = \pi_0\mu_0^{-1}(y)$ for $y \in \mu_0(Z_0)$. The topology on Z_1 is the coarsest topology, for which both π_1 and μ_1 are continuous. Then μ_1 is one-to-one and π_1 is surjective. Since $\mu_1 f\pi_0 = \mu_1\pi_0 h_0 = \pi_1\mu_0 h_0 = \pi_1 g\mu_0$, there exists a unique $h_1 : Z_1 \to Z_1$, for which $h_1\mu_1 = \mu_1 f$ and $h_1\pi_1 = \pi_1 g$. $\square$

Proposition 2 *Let $f : X \to X$, $h_1 : Z_1 \to Z_1$, $g : Y \to Y$ be dynamical systems, let $\mu_1 : f \Rightarrow h_1$ be a one-to-one conjugacy and $\pi_1 : g \Rightarrow h_1$ a surjective conjugacy. Then there exists a dynamical system $h_0 : Z_0 \to Z_0$, a surjective conjugacy $\pi_0 : h_0 \Rightarrow f$, and a one-to-one conjugacy $\mu_0 : h_0 \Rightarrow g$.*

Proof: Z_0 together with π_0 and μ_0 is constructed as a pullback of π_0 and μ_0. Explicitly $Z_0 = \pi_1^{-1}\mu_1(X) \subseteq Y$, μ_0 is the inclusion and π_0 is defined by $\pi_0(z) = \mu_1^{-1}\pi_1\mu_0(z)$. The topology on Z_0 is the finest topology, for which both π_0 and μ_0 are continuous. We get again the unique morphism h_0 for which $\pi_0 h_0 = f\pi_0$, $\mu_0 h_0 = g\mu_0$. $\square$

Remark 1 *Let π_0, μ_0 be as in Proposition 1, let μ_1, π_1 be their pushout and let π_0', μ_0' be the pullback of μ_1 and π_1. Then Z_0' and Z_0 are isomorphic as sets, but the topology of Z_0' is coarser than that of Z_0. Similarly in the dual case.*

We say that g simulates f, if there is a dynamical system $h_0 : Z_0 \to Z_0$, a surjective conjugacy from h_0 to f and a one-to-one conjugacy from h_0 to g. It follows that simulation is transitive: If g simulates f and h simulates g, then h simulates f. While topological spaces allow symmetric treating of one-to-one and surjective simulations, the notion of continuous simulation seems to be too general. It turns out that for any dynamical system $f : X \to X$ there is a surjective simulation in $g : X \times N \to X \times N$ defined by $g(x, n) = (x, n + 1)$. (Here N is the set of nonnegative integers with the discrete topology). It suffices to take $\pi(x, n) = f^n(x)$ (n-th iteration). Moreover, if X is a subset of the real line, then there is a continuous one-to-one mapping $\mu_0 : X \to [0, 1]$, and a one-to-one conjugacy from g to $h : R^1 \to R^1$ defined by $h(x) = x + 1$. It suffices to take $\mu(x, n) = \mu_0(x) + n$.

$$\begin{tikzcd}
X \arrow[r, "f"] & X \\
X{\times}N \arrow[u, "\pi"] \arrow[r, "g"] \arrow[d, "\mu"'] & X{\times}N \arrow[u, "\pi"'] \arrow[d, "\mu"] \\
R^1 \arrow[r, "h"] & R^1
\end{tikzcd}$$

So it is π rather than g or h, which computes f. To avoid such pathological examples, we require that the simulations be as simple as possible, for example uniformly continuous mappings between metric spaces. Since the category of metric spaces and uniformly continuous mappings is not closed with respect to pushouts, we lose the symmetry of continuous simulations.

3 Uniform simulation

Definition 1 *A dynamical system is a continuous mapping $f : X \to X$ of a metric space $X = (X, \varrho_X)$ into itself.*

Definition 2 *Let $f : X \to X$, $g : Y \to Y$ be dynamical systems. A uniform simulation of f in g is a uniformly continuous surjective mapping $\pi : (Y_\pi, \varrho_Y) \to (X, \varrho_X)$ such that $Y_\pi \subseteq Y$ is a g-invariant subset of Y and $(\forall y \in Y_\pi)(f\pi(y) = \pi g(y))$. We say that f is a subsystem of g if π is a homeomorphism. We say that f is a quotient of g, if $Y_\pi = Y$.*

$$\begin{tikzcd}
X \arrow[r, "f"] & X \\
Y_\pi \arrow[u, "\pi"] \arrow[r, "g"] \arrow[d, "\subseteq"'] & Y_\pi \arrow[u, "\pi"'] \arrow[d, "\subseteq"] \\
Y \arrow[r, "g"] & Y
\end{tikzcd}$$

Proposition 3 *Uniform simulation is transitive.*

Proof: Let $f : X \to X$, $g : Y \to Y$, $h : Z \to Z$ be dynamical systems, let $\pi_1 : Y_1 \to X$, $\pi_2 : Z_2 \to Y$ be uniform simulations of f in g and g in h respectively, with domains $Y_1 \subseteq Y$ and $Z_2 \subseteq Z$. Let $Z_0 = \pi_2^{-1}(Y_1) \subseteq Z_2$ and $\pi_0(z) = \pi_1\pi_2(z)$. Then π_0 is a uniform simulation of f in h. $\square$

Remark 2 *If $\varphi : f \Rightarrow g$ is a one-to-one conjugacy, then g uniformly simulates f, if the inverse mapping $\pi = \varphi^{-1} : (\varphi(X), \varrho_Y) \to (X, \varrho_X)$ is uniformly continuous.*

Definition 3 *Let $f : X \to X$ be a dynamical system, $\varepsilon > 0$, and n a positive integer. A set $S \subseteq X$ is (ε, n)-separated, if for each different $x, y \in S$ there exists $i < n$ such that $\varrho_X(f^i(x), f^i(y)) \geq \varepsilon$. Let $h(f, \varepsilon, n)$ be the maximal size of an (ε, n)-separated set. The topological entropy $h(f)$ of f is defined by*

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon), \quad \textit{where} \quad h(f, \varepsilon) = \limsup_{n \to \infty} \frac{\ln h(f, \varepsilon, n)}{n}$$

Proposition 4 *If g uniformly simulates f, then $h(g) \geq h(f)$.*

Proof: For $0 < \varepsilon < diam(X)$ let $\delta(\varepsilon) > 0$ be the maximal number with the property $(\forall y, y' \in Y_\pi)(\varrho_Y(y, y') < \delta(\varepsilon) \Rightarrow \varrho_X(\pi(y), \pi(y')) < \varepsilon)$. Let $S \subseteq X$ be a (ε, n)-separated set. Then there exists a set $T \subseteq Y_\pi$ such that $\pi(T) = S$, π is one-to-one on T and $||T|| = ||S||$. Clearly T is a $(\delta(\varepsilon), n)$-separated set for $g : Y \to Y$. It follows $h(g, \delta(\varepsilon), n) \geq h(f, \varepsilon, n)$ and $h(g, \delta(\varepsilon)) \geq h(f, \varepsilon)$. Since $h(f, \varepsilon)$ is nonincreasing in ε and $\delta(\varepsilon)$ is nondecreasing in ε, there is $h(g) \geq h(f)$. $\square$

4 Dynamical systems on discontinuum

Most well-known examples of simulation arise in symbolic dynamic and involve dynamical systems on discontinuum. Topologically, discontinuum is a countable product of finite discrete spaces, it is metrizable and homeomorphic to the Cantor middle thirds set. Usually it is constructed as 2^Z or 2^N where Z and

N are the sets of integers and nonnegative integers respectively, and $2 = \{0, 1\}$ is a two-point set. The space 2^Z of doubly infinite bit strings $x = ...x_{-1}x_0x_1...$, $x_i \in 2$, is equipped with the metric

$$\varrho_Z(x,y) = 2^{-n} \text{ where } n = \inf\{i \geq 0 | x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

The space 2^N of infinite bit strings $x = x_0x_1...$ is equipped with the metric

$$\varrho_N(x,y) = 2^{-n} \text{ where } n = \inf\{i \geq 0 | x_i \neq y_i\}$$

The discontinuum possesses a kind of universality, as the next Theorem shows.

Theorem 1 *Every dynamical system $f : X \to X$ on a compact space X is a quotient of some dynamical system $g : 2^N \to 2^N$.*

Proof: It is a well-known fact, that each compact space is a quotient of the discontinuum. Not every projection, however, need be a conjugacy for f. Let us first construct a sequence of finite open covers $\mathcal{V}^{(i)}$ of X, such that $diam(\mathcal{V}^{(i)}) < 2^{-i}$ (i.e. $(\forall V \in \mathcal{V}^{(i)})(\forall x, y \in V)(\varrho(x,y) < 2^{-i})$, and $(\forall V \in \mathcal{V}^{(i)})(\exists V_1, V_2 \in \mathcal{V}^{(i-1)})(V \subseteq V_1, f(V) \subseteq V_2)$. To construct $\mathcal{V}^{(0)}$ just choose an open cover of X with diameter less than 1, and select a finite subcover. Suppose that $\mathcal{V}^{(i-1)}$ has been constructed. For each $x \in X$ let V_x be an open neighbourhood of x such that $diam(V_x) < 2^{-i}$, and there exist $V_1, V_2 \in \mathcal{V}^{(i-1)}$ such that $V_x \subseteq V_1 \cap f^{-1}(V_2)$. Let $\mathcal{V}^{(i)}$ be a finite subcover of $\{V_x | x \in X\}$. We construct now a sequence of positive integers $k_0, k_1, ...$ and for each finite sequence $u_0, ..., u_{n-1}$ of binary words with lengths $|u_i| = k_i$ a nonempty closed set $A_{u_0...u_{n-1}} \subseteq X$. Let $2^{k_0} \geq ||\mathcal{V}^{(0)}||$ (number of elements), and define $\{A_{u_0} | (|u_0| = k_0)\} = \{\overline{V} | V \in \mathcal{V}^{(0)}\}$ as systems of closures of all elements of $\mathcal{V}^0$ (with possible repetitions). Suppose we have defined $A_{u_0,...u_{n-1}}$ for all $u_0, ...u_{n-1}$ with lengths $k_0, ...k_{n-1}$. Let k_n be such that for all such $u_0, ...u_{n-1}$ we have $2^{k_n} \geq ||\{V \in \mathcal{V}^{(n)} | \overline{V} \cap A_{u_0...u_{n-1}} \neq 0\}||$. For given $u_0, ..., u_{n-1}$ define

$$\{A_{u_0...u_n} | (|u_n| = k_n)\} = \{A_{u_0...u_{n-1}} \cap \overline{V} | V \in \mathcal{V}^{(n)}, A_{u_0...u_{n-1}} \cap \overline{V} \neq 0\}.$$

Then for each sequence $u_0, ..., u_n$ we have $A_{u_0...u_n} \subseteq A_{u_0...u_{n-1}}$, and there is a sequence $v_0, ..., v_{n-1}$ such that $f(A_{u_0...u_n}) \subseteq A_{v_0...v_{n-1}}$. Denote $g(u_1...u_n) = v_1...v_{n-1}$. For $u \in 2^N$ define now $A_u = \cap\{A_{u_0...u_{n-1}} | u_0...u_{n-1} \sqsubseteq u\}$ and $g(u) = \lim\{g(u_0...u_{n-1}) | u_0...u_{n-1} \sqsubseteq u\}$ (here $\sqsubseteq$ means initial substring). Then A_u is a nonempty closed set, and since its diameter is zero, it contains unique point $\pi(u) \in A_u$. It is easy to see that $\pi : 2^N \to X$ and $g : 2^N \to 2^N$ are continuous and $f\pi = \pi g$. $\square$

Theorem 2 *Every dynamical system $f : X \to X$ on a compact space X is uniformly simulated by a dynamical system on the real unit interval.*

Proof: We use the map $\xi : 2^N \to [0, 1]$ defined by $\xi(x) = \sum_{i=0}^{\infty} 2x_i 3^{-i-1}$ and extend f in a linear manner on the whole unit interval. $\square$

5 Generalized shifts

Moore defines in [7] a class of dynamical systems on 2^Z, which he calls generalized shifts (see also [2, 4]). They are essentially Turing machines, whose inner state is written on the tape. In contrast to Turing machines, the input on the tape might be infinite. A generalized shift inspects a finite portion of the tape, changes it according to a given table and shifts the tape left or right according to another table.

Definition 4 *1. A function $g : 2^Z \to S$ to a finite set S is finitely determined, if there exists $r \geq 0$ such that for any $x, y \in 2^Z$ if $\varrho_Z(x,y) < 2^{-r}$, then $g(x) = g(y)$. We call the interval $(-r, r) \subseteq Z$ the domain of dependence of g. If g is finitely determined, then $g(x) = \overline{g}(x_{-r+1}, ..., x_{r-1})$ for some function $\overline{g}$ of $2r-1$ binary variables.*
2. A function $f : 2^Z \to 2^Z$ is finitely determined, if each its component $f_i : 2^Z \to 2$ is finitely determined and for all but finitely many indices i this component is the projection, i.e. $f_i(x) = (f(x))_i = x_i$.
3. A function $T : 2^Z \to 2^Z$ is a generalized shift, if $T(x) = \sigma_Z^{g(x)}(f(x))$, where $g : 2^Z \to \{-1, 0, 1\}$, and $f : 2^Z \to 2^Z$ are finitely determined.

Here $\sigma_Z^1 = \sigma_Z$ is the shift defined by $\sigma_Z(x)_i = x_{i+1}$, σ_Z^{-1} is its inverse, and σ_Z^0 is the identity. A generalized shift is a continuous mapping. If $n \geq r$ and $\varrho_Z(x,y) < 2^{-n}$, then $\varrho_Z(T(x), T(y)) < 2^{-n+1}$. In sharp contrast to more conventional dynamical systems, generalized shifts have no asymptotically stable periodic points. Their topological entropy is at most $\ln 2$.

Theorem 3 *A generalized shift has no asymptotically stable periodic point. It might have marginally stable periodic points.*

Proof: Let x be a periodic point with period n, i.e. $T^n(x) = x$, and denote $a = g(x) + g(T(x)) + ... + g(T^{n-1}(x))$, $b = |g(x)| + |g(T(x))| + ... + |g(T^{n-1}(x))|$. Thus a is the total shift of the tape after n iterations and b is an upper bound of the maximum shift during the period. Suppose first that $a = 0$. Then the tape returns exactly to its initial configuration and position, and during the cycle it inspects or changes only locations x_i with $|i| < r + b$. It follows, that if $\varrho_Z(x,y) < 2^{-r-b}$, then y is also periodic point with period n and does not converge to x. Thus x is marginally stable in this case. Suppose now, that $a \neq 0$. Then with a possible exception of a finite region of the tape, x is a periodic sequence with period $|a|$, i.e. a periodic point of the shift map. Denote $U = \{y \in X | \varrho_Z(y,x) < 2^{-r-2b}\}$. Then for each neighbourhood V of x there exists $y \in V$, such that $T^{kn}(y) \notin U$ for some k. Therefore x is an unstable periodic point. □

Proposition 5 *The topological entropy of a generalized shift is at most* $\ln 2$. *The identity has topological entropy 0 and the shift has topological entropy* $\ln 2$.

Proof: If T is a generalized shift, then $h(T, 2^{-n}, m) \leq 2^{2n+m+1}$, so $h(T, 2^{-n}) \leq \ln 2$ and therefore $h(T) \leq \ln 2$.

6 Unimodal maps and generalized N-shifts

As Theorem 2 shows, there exist dynamical systems on the unit interval, which simulate generalized shifts. Not all maps on the unit interval, however, have this property. We show that S-unimodal functions do not simulate the Z-shift. In the proof we use a concept of generalized N-shift. This is a dynamical system on 2^N acting similarly as generalized shift does on 2^Z. However, generalized N-shifts correspond rather to finite automata, than to Turing machines. For $k = 0, 1, 2, 3$ define $\sigma_N^k : 2^N \to 2^N$ by

$$\begin{aligned}
&\sigma_N^0(x) = x, \\
&\sigma_N^1(x)_i = x_{i+1}, \\
&\sigma_N^2(x)_i = x_{i-1} \quad \text{for} \quad i > 0, \quad \sigma_N^2(x)_0 = 0, \\
&\sigma_N^3(x)_i = x_{i-1} \quad \text{for} \quad i > 0, \quad \sigma_N^3(x)_0 = 1
\end{aligned}$$

Definition 5 *A dynamical system $T : 2^N \to 2^N$ is a generalized N-shift, if $T(x) = \sigma_N^{g(x)}(f(x))$, where $g : 2^N \to \{0, 1, 2, 3\}$, and $f : 2^N \to 2^N$ are finitely determined. (Finitely determined functions are defined analogously as for Z.)*

We use the notion of S-unimodal function on a real interval of [1] (we denote $[a, b] = [f(1), 1]$). A function $f : [a, b] \to [a, b]$ is S-unimodal, if it is surjective, has negative Schwarzian derivative, $f(b) = a$, and there exists a unique point $c \in (a, b)$ with $f'(c) = 0$, where $f(c) = b$ and $f''(c) < 0$. We use the calculus of itineraries with a minor modification that itineraries do not contain the letter C. When an iteration hits c, then both values 0 and 1 are admitted instead of C, so the itinerary is not determined uniquely. Denote $I_0 = [a, c]$, $I_1 = [c, b]$, 2^* the set of finite bit strings. For $s \in 2^N \cup 2^*$ let $J(s) = \{x \in I | (\forall i < |s|)(f^i(x) \in I_{s_i}\}$. ($|s|$ is the length of s.) Then $J(s)$ is a closed interval (possibly empty), if $s \sqsubseteq t$ (initial string), then $J(t) \subseteq J(s)$ and for $s \in 2^N$ we have $J(s) = \cap\{J(t) | t \in 2^*,\ t \sqsubseteq s\}$.

Proposition 6 *Every S-unimodal function f with no stable periodic point is uniformly simulated by σ_N.*

Proof: By Theorem II.5.4 of [1], f has no homtervals and therefore for each $s \in 2^N$ $J(s)$ contains at most one point. Denote $\Sigma_f = \{2^N | J(s) \neq 0\}$ and define $\pi : \Sigma_f \to I$ by $\pi(s) \in J(s)$. Clearly Σ_f is a σ_N-invariant set and $\pi\sigma_N = f\pi$. Let us show that π is continuous. If U is a neighbourhood of $\pi(s)$, then there exists $t \in 2^*$, $t \sqsubseteq s$ such that $J(t) \subseteq U$. Then for all $u \in \Sigma_f$ with $t \sqsubseteq u$ we have $\pi(u) \in J(t)$. To show that π is uniformly continuous, we prove that Σ_f is closed and therefore compact. Let $s^{(j)} \in \Sigma_f$ be a sequence converging to some $s \in 2^N$. Then for each $t \in 2^*$ for which $t \sqsubseteq s$, there exists j with $t \sqsubseteq s^{(j)}$ and therefore $J(t)$ is a nonempty set. Since I is compact, $J(s)$ is nonempty too. $\Box$.

Lemma 1 *Let $f : [a,b] \to [a,b]$ be an increasing function with negative Schwarzian derivative and with a stable fixed point d, such that $f^i(x) \to d$ for all $x \in (a,b)$. Then f is a quotient of the N-shift.*

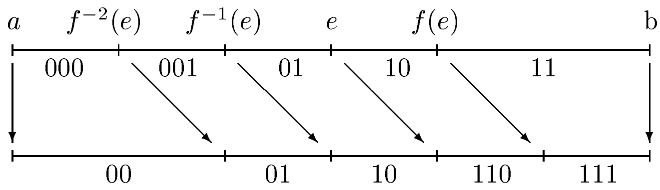
Proof: We must distinguish whether d is an inner point of $[a,b]$, and whether the end points a, b are fixed points or not. Suppose first $d = b$, and $f(a) = a$, $f(b) = b$. Since f has negative Schwarzian derivative, it has no fixed points in (a,b), and its graph is above the diagonal. Consider a generalized N-shift T defined by the table

$$\begin{array}{ll} T(00s_2s_3.....) & = 0s_2s_3... \\ T(01s_2s_3.....) & = 10s_2s_3... \\ T(1s_1s_2s_3...) & = 11s_1s_2s_3... \end{array}$$

Pick up any $e \in (a,b)$. We construct π first on $\{s \in 2^N | s_0 = 1, s_1 = 0\}$, which is mapped onto $[e, f(e)]$. At other points, π is determined uniquely:

$$\begin{array}{ll} \pi(10s_2s_3........) & = e + (f(e) - e) \sum_{i=2}^{\infty} s_i 2^{-i+1} \\ \pi(1^{n+1}0s_{n+2}...) & = f\pi(1^n 0 s_{n+2}...)\ , \ \ n > 0 \\ \pi(01s_2s_3........) & = f^{-1}\pi(10s_2s_3...) \\ \pi(0^{n+1}1s_{n+2}...) & = f^{-1}\pi(0^n 1 s_{n+2}...)\ , \ \ n > 0 \\ \pi(00000.........) & = a \\ \pi(11111.........) & = b \end{array}$$

It is easy to see that $\pi : 2^N \to [a,b]$ is continuous and $\pi T = f\pi$. In other cases we proceed similarly. □



Proposition 7 *Every S-unimodal function with a single stable periodic orbit is uniformly simulated by a generalized N-shift.*

Proof: Suppose first for simplicity that the stable periodic orbit does not contain the critical point. Let p be a stable periodic point, and let n be the first integer with $f^n(p) = p$ and $(f^n)'(p) > 0$. Then p has unique periodic itinerary $\bar{t}$ which is infinite repetition of a bit string $t = t_0...t_{n-1}$. Because p is stable, $J(\bar{t})$ is an interval of positive length and f^n restricted to $J(\bar{t})$ satisfies the conditions of the preceding Lemma. We have therefore a generalized N-shift T_0 and a surjective conjugacy $\pi_0 : T_0 \Rightarrow f^n | J(\bar{t})$. Denote Σ_f the set of sequences $s \in 2^N$ with the property that either $J(s) \neq 0$, or there exists $s' \in 2^N$ with $J(s') \neq 0$, which has with s a common initial part containing a cyclic permutation of t. A generalized N-shift T is defined by the table

$$\begin{array}{ll}
T(s) = \sigma_N(s) & \text{if } s_0 \ldots s_{n-1} \text{ is not a cyclic permutation of } t \\
T(s) = s_1 \ldots s_{n-1} s_0 s_n s_{n+1} \ldots & \text{if } s_0 \ldots s_{n-1} \text{ is a cyclic permutation of } t \not\sqsubseteq s \\
T(s) = s_1 \ldots s_{n-1} s_0 T_0(s_n s_{n+1} \ldots) & \text{if } s_0 \ldots s_{n-1} = t
\end{array}$$

We define now the mapping $\pi : \Sigma_f \to [a,b]$. If $s \in \Sigma_f$ does not contain a permutation of t, then $J(s) \neq 0$, and by Proposition II.5.7 of [1], $J(s)$ contains the single point. So we define $\pi(s) \in J(s)$. Let $f_0^{-1} : [f(a), b] \to [a,c]$, $f_1^{-1} : [a,b] \to [c,b]$ be inverse functions to f. We define π by induction on all sequences, which start with a cyclic permutation of t:

$$\begin{array}{ll}
\pi(t_0 \ldots t_{n-2} t_{n-1} s_n s_{n+1} \ldots) & = \pi_0(s_n s_{n+1} \ldots), \\
\pi(t_{n-1} t_0 \ldots t_{n-2} s_n s_{n+1} \ldots) & = f_{t_{n-1}}^{-1} \pi(t_0 \ldots t_{n-1} s_n s_{n+1} \ldots), \\
\ldots & \\
\pi(t_1 \ldots t_{n-1} t_0 s_n s_{n+1} \ldots) & = f_{t_1}^{-1} \pi(t_2 \ldots t_1 s_n s_{n+1} \ldots)
\end{array}$$

At last define by induction $\pi(s) = f_{s_0}^{-1}(\pi(s_1 s_2 \ldots))$, if s contains a cyclic permutation of t, but does not start with it. It follows that when $\pi(s) = x$, then the initial part of s, which does not contain any cyclic permutation of t, is also an initial part of an itinerary of x. Let us show that π is continuous. Let $\pi(s) = x$ and U be a neighbourhood of x. Suppose first that s does not contain any cyclic permutation of t. Then there exists $u \in 2^*$, such that $u \sqsubseteq s$, $J(u) \subseteq U$, and for any $v \in 2^N$ for which $u \sqsubseteq v$, we have $\pi(v) \in J(u)$. If s does contain a cyclic permutation of t, then the continuity of π in s follows from the continuity of π_0. Next we show that Σ_f is closed. Let $s^{(j)} \in \Sigma_f$ be a sequence converging to some $s \in 2^N$. If s does not contain any cyclic permutation of t, then the proof is the same as in the preceding Proposition. If it does, then there exists $s^{(j)}$, which shares with s the initial part containing t, so again $s \in \Sigma_f$. We now modify the above proof for the case that the stable periodic orbit contains the critical point. If $f^i(p) = c$ for some $0 \leq i < n$, then $(f^n)'(p) = 0$. Since $f''(c) < 0$, there exists $\varepsilon > 0$ such that either $(f^n)'(x) > 0$ for $x \in (p - \varepsilon, p)$, or $(f^n)'(x) > 0$ for $x \in (p, p + \varepsilon)$. In the former case $f^{nk}(x) \to p$ for $x \in (p - \varepsilon, p)$ and there exists $t = t_0, \ldots, t_{n-1}$ such that $(p - \varepsilon, p) \subseteq J(\bar{t})$. In the latter case $f^{nk}(x) \to p$ for $x \in (p, p + \varepsilon)$ and there exists $t = t_0, \ldots, t_{n-1}$ such that $(p, p + \varepsilon) \subseteq J(\bar{t})$. We proceed now with the proof with this sequence t. $\square$

Theorem 4 *Every S-unimodal function is uniformly simulated by a generalized N-shift.*

Proof: By Corollary II.4.2 of [1] every S-unimodal function has at most one periodic orbit, so the preceding two Propositions apply. $\square$

We show now that generalized N-shifts do not simulate Z-shift. We use an information-theoretic argument: The entropy of σ_Z is one, it produces one bit of information every time step. If a generalized N-shift would simulate it, it would have to produce also one bit of information every time step and at the same time keep this information stored, which is not possible. If $v : X \to S$ is a random variable defined on a probability space $(X, \mathcal{A}, \mu)$ with values in a finite set S, then its entropy is $H(v) = -\sum_{a \in S} \mu(v^{-1}(a)) \log_2 \mu(v^{-1}(a))$. If v, w are finitely valued random variables, then their conditional entropy is $H(v|w) = H(v, w) - H(w)$.

Theorem 5 *No generalized N-shift uniformly simulates the Z-shift.*

Proof: Let T be a generalized N-shift. Suppose that there is a T-invariant subset $X \subseteq 2^N$ and a uniformly continuous surjective mapping $\pi : X \to 2^Z$. with $(\forall x \in X)(\pi T(x) = \sigma_Z \pi(x))$. Let $\mathcal{A}_Z$ be the standard σ-algebra on 2^Z generated by cylinders $\{x \in 2^Z | x_i = 0\}$ and μ_Z the standard measure on $(2^Z, \mathcal{A}_Z)$ defined by $\mu_Z\{x \in 2^Z | x_i = 0\} = 1/2$. Denote $\mathcal{A}_X = \{A \cap X | A \in \mathcal{A}_N\}$ the σ-algebra on X obtained from the restriction of the standard σ-algebra $\mathcal{A}_N$ on 2^N. Define another σ-algebra $\mathcal{B} = \{\pi^{-1}(A) | A \in \mathcal{A}_Z\}$ on X and a measure μ on $(X, \mathcal{B})$ by $\mu(\pi^{-1}(A)) = \mu_Z(A)$, $A \in \mathcal{A}_{\mathcal{Z}}$. Since π is uniformly continuous, $\mathcal{B} \subseteq \mathcal{A}_X$. It is easy to see, that the measure μ can be extended to the whole $\mathcal{A}_X$. In the sequel we consider entropies of finitely valued random variables on the probability space $(X, \mathcal{A}_X, \mu)$. (Warning: μ in general differs from the measure inherited from 2^N.) Since π is a measure preserving transformation, $\pi(x)_i$, $i \in Z$ are mutually independent random variables with uniform distribution. Since π is uniformly continuous, there is an increasing sequence j_i such that

$$\varrho_N(x, y) < 2^{-j_i} \Rightarrow \varrho_Z(\pi(x), \pi(y)) < 2^{-i}$$

We can suppose that $j_i > r$, where r is the length of the domain of dependence of the generalized N-shift T. Then $\pi(x)_0$ depends only on $x_0, ..., x_{j_0-1}$ and therefore $H(\pi(x)_0 | x_0, ..., x_{j_0-1}) = 0$. Similarly $\pi(x)_1 = \sigma_Z \pi(x)_0 = \pi T(x)_0$ depends only on $T(x)_0, ..., T(x)_{j_0-1}$. Since $r < j_0$, this depends only on $x_0, ..., x_{j_0}$, so $H(\pi(x)_1 | x_0, ..., x_{j_0}) = 0$. We have also $H(\pi(x)_{-1} | x_0, ..., x_{j_1-1}) = 0$. Let us pick some $m > j_{j_0}$. Then, because of functional dependence, $H(\pi(x)_{-j_0}, ..., \pi(x)_{m-j_0} | x_0, ..., x_{m-1}) = 0$. However, $\pi(x)_{-j_0}, ..., \pi(x)_{m-j_0}$ are mutually independent random variables with uniform distribution, so their entropy is $m + 1$, while the entropy of $x_0, ..., x_{m-1}$ is at most m. Thus we have $H(\pi(x)_{-j_0}, ..., \pi(x)_{m-j_0} | x_0, ..., x_{m-1}) = H(\pi(x)_{-j_0}, ..., \pi(x)_{m-j_0}, x_0, ..., x_{m-1}) - H(x_0, ..., x_{m-1}) \geq m + 1 - m = 1$, which is a contradiction. $\Box$

$$\begin{tikzcd}
2^Z \arrow[r, "\sigma_Z"] & 2^Z \\
X \arrow[u, "\pi"] \arrow[r, "T"] \arrow[d, "\subseteq"'] & X \arrow[u, "\pi"'] \arrow[d, "\subseteq"] \\
2^N \arrow[r, "T"] & 2^N
\end{tikzcd}$$

Theorem 6 *No S-unimodal function uniformly simulates the Z-shift.*

This follows from the preceding two Theorems and from the transitivity of uniform simulation.

Remark 3 *If the kneading sequence of an S-unimodal mapping is different from $1\overline{0}$, then its topological entropy is less than $\ln 2$ and therefore cannot uniformly simulate the Z-shift with topological entropy $\ln 2$. This is a simpler proof of the above Theorem. The case of S-unimodal mapping with kneading sequence $1\overline{0}$ shows that if $h(g) \geq h(f)$, then g need not necessarily simulate f.*

References

[1] Collet,P.,Eckmann,J.P.: Iterated Maps on the Interval as Dynamical Systems. Birkhauser, Basel 1980.

[2] Crutchfield,J.M., Young,K.: Computation at the onset of chaos. in: Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity, vol VIII (W.H.Zurek, ed.), Addison Wesley 1990.

[3] Devaney,R.L.: An Introduction to Chaotic Dynamical Systems. Addison-Wesley, Redwood City 1989.

[4] Koiran,P.,Cosnard,P.,Garzon,M.: Computability with low-dimensional dynamical systems. Rapport n 92-31, Ecole Normal Superieure de Lyon, 1992.

[5] Langton,Ch.G.: Life at the Edge of Chaos. in: Artificial Life II. SFI Studies in the Sciences of Complexity, vol. X (Ch.G.Langton, Ch.Taylor, J.D.Farmer, S.Rasmunsen, eds.) Addison-Wesley, Redwood City 1992.

[6] Mac Lane,S.: Categories for the Working Mathematician. Springer-Verlag, Berlin 1971.

[7] Moore,Ch.: Unpredictability and undecidability in dynamical systems. Physical Review Letters 64(20), 2354-2357, 1990.

[8] Moore,Ch.: Generalized shifts: unpredictability and undecidability in dynamical systems. Nonlinearity 4, 199-230, 1991.