Möbius number systems with sofic subshifts

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Abstract. A real Möbius iterative system is an action of a free semigroup of finite words acting via Möbius transformations on the extended real line. Its convergence space consists of all infinite words, such that the images of the Cauchy measure by the prefixes of the word converge to a point measure. A Möbius number system consists of a Möbius iterative system and a subshift included in the convergence space, such that any point measure can be obtained as the limit of some word of the subshift. We give some sufficient conditions on sofic subshifts to form Möbius number systems. We apply our theory to several number systems based on continued fractions.

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1. Introduction

Classical number systems for compact real intervals such as decadic, binary, or binary signed systems can be obtained from contractive iterative systems (see e.g., Barnsley [1] or Edgar [2]). An iterative system $(F_a : X \to X)_{a\in A}$ consists of continuous self-maps of a compact metric space X indexed by a finite alphabet A. In contractive iterative systems, each infinite word $u \in A^{\mathbb{N}}$ determines a unique point $x = \Phi(u)$ which is contained in all images $F_{u_0}F_{u_1}\cdots F_{u_{n-1}}(X)$ of the state space X by the prefixes of u. The range of the symbolic representation Φ is a compact subset of X called the attractor of the system. The method works, however, only for compact subspaces of the real line or complex plane. It is therefore natural to look for number systems on a compactification of the real line such as the extended real line $\overline{\mathbb{R}}$. A natural choice for the mappings are Möbius transformations, since these are the only holomorphic isomorphisms of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. However, since Möbius transformations are surjective, the Barnsley theorem does not work for them.

In Kůrka [8] we have used convergence of measures (instead of convergence of sets), to obtain symbolic representations of the extended real line from iterative Möbius systems. The uniform Haar measure on the unit circle is transferred by the stereographic projection to the Cauchy measure on $\overline{\mathbb{R}}$. We say that an infinite word $u \in A^{\mathbb{N}}$ converges to a number $x \in \overline{\mathbb{R}}$, if the images of the Cauchy measure by the prefixes of u converge to the point measure δ_x. In this case we say that u is a representation of x and write $\Phi(u) = x$. The domain of Φ is the convergence space $\mathbb{X}_F \subseteq A^{\mathbb{N}}$. To get a number system, the range of Φ should be all $\overline{\mathbb{R}}$. Since Φ is usually not continuous on $\mathbb{X}_F$, we look for a subshift $\Sigma \subseteq \mathbb{X}_F$ such that $\Phi(\Sigma) = \overline{\mathbb{R}}$ and Φ is continuous on Σ. In this case we say that (F, Σ) is a Möbius number system.

In [8] we have constructed several Möbius number systems using contractions to vertices of a regular polygon inscribed to the unit circle. In [7] we have obtained some results on topological dynamics of Möbius iterative systems. In the present paper we develop a theory of Möbius number systems with sofic subshifts. In Theorem 9 we characterize those Möbius iterative systems whose range is whole $\overline{\mathbb{R}}$. In the Convergence Theorem 11 we give a sufficient condition on a sofic subshift Σ to satisfy $\Sigma \subseteq \mathbb{X}_F$. In the Surjectivity Theorem 15 we give a sufficient condition for $\Phi(\Sigma) = \overline{\mathbb{R}}$. Finally we show that for Möbius number systems whose coefficients are in a computable field (such as rational numbers), efficient arithmetic algorithms exist. We apply the theory to several systems based on continued fractions.

2. Möbius transformations

An orientation-preserving **real Möbius transformation** (MT) is a self-map of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ of the form $M_{(a,b,c,d)}(x) = (ax+b)/(cx+d)$, where $ad - bc > 0$. Real Möbius transformations act also on the complex upper half-plane $\mathbb{U} := \{z \in \mathbb{C} :\ \Im(z) > 0\}$, where they preserve the hyperbolic metric $ds = dz/\Im(z)$ (see e.g., Katok [3]). The upper half-plane $\mathbb{U}$ is conformally isomorphic to the unit disc $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$ via isomorphisms $\mathbf{d} : \mathbb{U} \to \mathbb{D}$ and $\mathbf{u} : \mathbb{D} \to \mathbb{U}$ given by $\mathbf{d}(z) = (iz+1)/(z+i)$, $\mathbf{u}(z) = (-iz+1)/(z-i)$. The transformation $\mathbf{d}$ maps i to 0 and $\overline{\mathbb{R}}$ to $\partial\mathbb{D} = \{z \in \mathbb{C} :\ |z| = 1\}$. On $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$ we have **disc Möbius transformations**

$$\widehat{M}_{(a,b,c,d)}(z) = \mathbf{d} \circ M_{(a,b,c,d)} \circ \mathbf{u}(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. Disc MT preserve the hyperbolic metric $ds = dz/(1-|z|^2)$. Define the **circle distance** d on $\overline{\mathbb{R}}$ by

$$d(x,y) = \min\{2|\arctan(x) - \arctan(y)|, 2\pi - 2|\arctan(x) - \arctan(y)|\},$$

which is the length of the shortest arc joining $\mathbf{d}(x)$ and $\mathbf{d}(y)$ in $\partial\mathbb{D}$. Open and closed **intervals** are balls $B_r(a) = \{x \in \overline{\mathbb{R}} :\ d(x,a) < r\}$, $\overline{B}_r(a) = \{x \in \overline{\mathbb{R}} :\ d(x,a) \le r\}$. Their lengths are $||\overline{B}_r(a)|| = ||B_r(a)|| = \min\{2r, 2\pi\}$. We define the length $||W||$ of a set $W \subseteq \overline{\mathbb{R}}$ as the length of the shortest interval which contains W. Denote by $\mathcal{I}$ the set of closed intervals. If we regard $\overline{\mathbb{R}}$ as the projective real line (the space of one-dimensional subspaces of $\mathbb{R}^2$) with **homogenous coordinates** $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ and equality $x = y \ \Leftrightarrow\ x_0 y_1 = x_1 y_0$, then the open interval with distinct endpoints a, b can be defined by $(a,b) := \{x \in \overline{\mathbb{R}} :\ (a_0 x_1 - a_1 x_0)(x_0 b_1 - x_1 b_0)(b_0 a_1 - b_1 a_0) > 0\}$. Closed intervals are defined by $[a,b] := (a,b) \cup \{a,b\}$, etc. The **norm** of a Möbius transformation $M = M_{(a,b,c,d)}$ is $||M|| := (a^2+b^2+c^2+d^2)/(ad-bc)$. The **square of the trace** of M is $\mathbf{tr}^2(M) := (a+d)^2/(ad-bc)$, and the **circle derivation** of M is

$$M^\bullet(x) := \lim_{y \to x} \frac{d(M(y), M(x))}{d(y,x)} = |\widehat{M}'(\mathbf{d}(x))| = \frac{(ad-bc)(x^2+1)}{(ax+b)^2 + (cx+d)^2}.$$

Proposition 1 *Let M be a Möbius transformation. Then $||M|| \ge 2$ and*

$$\min\{M^\bullet(x) :\ x \in \overline{\mathbb{R}}\} = \tfrac{1}{2}(||M|| - \sqrt{||M||^2 - 4}),$$
$$\max\{M^\bullet(x) :\ x \in \overline{\mathbb{R}}\} = \tfrac{1}{2}(||M|| + \sqrt{||M||^2 - 4}).$$

Proof: The equation $M^\bullet(x) = 1/\varepsilon$ for x has a solution iff its discriminant is nonnegative, i.e., if $\varepsilon^2 - ||M||\varepsilon + 1 \leq 0$. The minimum and the maximum ε for which this holds are the solutions of $\varepsilon^2 - ||M||\varepsilon + 1 = 0$. □

If $\mathbf{tr}^2(M) > 4$ then M is **hyperbolic**, has a stable fixed point $\mathbf{s} \in \overline{\mathbb{R}}$ with $M(\mathbf{s}) = \mathbf{s}$, $M^\bullet(\mathbf{s}) < 1$, and an unstable fixed point $\mathbf{u} \in \overline{\mathbb{R}}$ with $M(\mathbf{u}) = \mathbf{u}$, $M^\bullet(\mathbf{u}) > 1$. If $\mathbf{tr}^2(M) = 4$ then M is **parabolic** and, unless it is the identity, has a unique fixed point $\mathbf{s} \in \overline{\mathbb{R}}$ with $M^\bullet(\mathbf{s}) = 1$. If $\mathbf{tr}^2(M) < 4$ then M is **elliptic** and has a unique fixed point in $\mathbb{U}$ and no fixed point in $\overline{\mathbb{R}}$. A rotation by α is the transformation $R_\alpha = M_{(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, -\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2})}$, $\widehat{R}_\alpha(z) = e^{i\alpha} \cdot z$.

The values $M(i)$ and $\widehat{M}(0)$ of a Möbius transformation have a probabilistic interpretation. Given a compact metric space X, we denote by $\mathfrak{M}(X)$ the space of Borel probability measures with weak* topology, i.e., $\lim_{n\to\infty} \nu_n = \nu$ iff $\lim_{n\to\infty} \int f d\nu_n = \int f d\nu$ for each continuous function $f : X \to \mathbb{R}$. A continuous function $F : X \to Y$ extends to a continuous function $F_* : \mathfrak{M}(X) \to \mathfrak{M}(Y)$ by $(F_*\nu)(U) = \nu(F^{-1}(U))$. Denote by δ_x the Dirac **point measure** concentrated on x, i.e., $\delta_x(U) = 1$ iff $x \in U$. Measures on $\overline{\mathbb{R}}$ which are absolutely continuous with respect to the Lebesgue measure have **densities**, which are nonnegative functions with unit integral. The **uniform Haar measure** ℓ on $\partial\mathbb{D}$ yields the **Cauchy measure** $\mathbf{u}_*\ell$ on $\overline{\mathbb{R}}$ with density $h(x) = 1/\pi(1+x^2)$. The density $h_{(a,b,c,d)}$ of $(M_{(a,b,c,d)}\mathbf{u})_*\ell$ on $\overline{\mathbb{R}}$ is

$$h_{(a,b,c,d)}(x) = \frac{(ad-bc)/\pi}{(dx-b)^2 + (cx-a)^2} = \frac{\sigma/\pi}{(x-\mu)^2 + \sigma^2}, \text{ where}$$
$$M_{(a,b,c,d)}(i) = \mu + i\sigma = \frac{(ac+bd) + (ad-bc)i}{c^2+d^2}.$$

Thus $h_{(a,b,c,d)}$ is the density of the generalized Cauchy measure with parameters μ and σ which are the real and imaginary parts of $M_{(a,b,c,d)}(i)$. While generalized Cauchy measures have infinite variance and no mean, the parameters μ and σ play a similar role as the mean and variance of the normal distribution. If X_0, X_1 are random variables with generalized Cauchy distributions with parameters μ_0, σ_0 and μ_1, σ_1 respectively, then $X_0 + X_1$ has Cauchy distribution with parameters $\mu_0 + \mu_1$, $\sigma_0 + \sigma_1$. A measure $\mu \in \mathfrak{M}(\partial\mathbb{D})$ can be characterized by its **mean** $\mathcal{E}(\mu) := \int_{\partial\mathbb{D}} z\, d\mu$ which is a complex number in the closed unit disc $\overline{\mathbb{D}}$. For a point measure of $x \in \partial\mathbb{D}$ we have $\mathcal{E}(\delta_x) = x$.

Proposition 2 *If $M = M_{(a,b,c,d)}$ is a real MT and $\widehat{M}$ the corresponding disc MT, then*

$$\mathcal{E}(\widehat{M}\ell) = \widehat{M}(0) = \mathbf{d}(M(i)) = \frac{(d-a)+(b+c)i}{(b-c)+(d+a)i}, \quad |\widehat{M}(0)|^2 = \frac{||M||-2}{||M||+2}.$$

See Kůrka [8] for a proof. A Möbius transformation M is uniquely determined by its **mean** $\widehat{M}(0)$ and its **unit tangent vector** $\widehat{M}'(0)/|\widehat{M}'(0)|$ (see Katok [3]). We shall need several criteria for convergence of generalized Cauchy measures.

Proposition 3 *Let $(M_n : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{n\geq 0}$ be a sequence of MT and $x \in \overline{\mathbb{R}}$. The following conditions are equivalent.*

(1) $\lim_{n\to\infty} M_n(i) = x$.

(2) $\lim_{n\to\infty} \widehat{M}_n(0) = \mathbf{d}(x)$.

(3) $\lim_{n\to\infty}(M_n\mathbf{u})_*\ell = \delta_x$.

(4) For each open interval $I \ni x$*,* $lim_{n\to\infty}||M_n^{-1}(I)|| = 2\pi$.

(5) There exists $c > 0$ *and a sequence of intervals* $I_m \ni x$ *such that*

$$\lim_{m\to\infty} ||I_m|| = 0, \quad \textit{and} \quad \forall m, \liminf_{n\to\infty} ||M_n^{-1}(I_m)|| > c.$$

Proof: (1) $\Leftrightarrow$ (2) follows from $\mathbf{d}(i) = 0$.
(2) $\Leftrightarrow$ (3) follows from Proposition 2.
(3) $\Rightarrow$ (4) follows from the definition of convergence of measures.
(4) $\Rightarrow$ (5) is trivial.
(5) $\Rightarrow$ (2): We can assume that I_m are open intervals. If $||M_n^{-1}(I_m)|| > c$, there exists $x \in \overline{\mathbb{R}}$ with $(M_n^{-1})^\bullet(x) \geq c/||I_m||$. By Proposition 1, we get $\lim_{n\to\infty} ||M_n|| = \infty$, and therefore $\lim_{n\to\infty} ||\widehat{M}_n(0)|| = 1$. There exist rotations R_n such that $\lim_{n\to\infty} \widehat{R}_n\widehat{M}_n(0) = \mathbf{d}(x)$, and we get $\lim_{n\to\infty} ||M_n^{-1}(R_n^{-1}(I_m))|| = 2\pi$ by (4). Since $\liminf_{n\to\infty} ||M_n^{-1}(I_m)|| \geq c$, the intervals $R_n^{-1}(I_m)$ and I_m intersect for large n, so $\lim_{n\to\infty} R_n = \mathrm{Id}$ and $\lim_{n\to\infty} \widehat{M}_n(0) = \lim_{n\to\infty} \widehat{R}_n^{-1}\widehat{R}_n\widehat{M}_n(0) = \mathbf{d}(x)$. □

Proposition 4 *Let* $(M_n : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{n\geq 0}$ *be a sequence of Möbius transformations and assume that there exist distinct* $y, z \in \overline{\mathbb{R}}$ *such that* $\lim_{n\to\infty} M_n(y) = \lim_{n\to\infty} M_n(z) = x$*. Then* $\lim_{n\to\infty} M_n(i) = x$.

Proof: Let $M_n = M_{(a_n,b_n,c_n,d_n)}$ and $a_n^2 + b_n^2 + c_n^2 + d_n^2 = 1$. Assume first $x, y, z \in \mathbb{R}$. We have

$$M_n(y) - M_n(z) = \frac{(a_nd_n - b_nc_n)(y - z)}{(c_ny + d_n)(c_nz + d_n)}.$$

Since $|c_ny + d_n|$ and $|c_nz + d_n|$ are bounded away from zero, we get $\lim_{n\to\infty} a_nd_n - b_nc_n = 0$ and $\lim_{n\to\infty}(M_n(y) - M_n(i)) = 0$. If some of the x, y, z are ∞, the proof is similar. □

3. Möbius number systems

For a finite alphabet A, denote by $A^+ := \bigcup_{m>0} A^m$ the set of nonempty words. With the concatenation operation, A^+ is the free semigroup over A. The length of a word $u = u_0 \dots u_{m-1} \in A^m$ is denoted by $|u| := m$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$ subwords of u associated to intervals. We denote by $A^{\mathbb{N}}$ the Cantor space of infinite sequences of letters of A equipped with metric $d_A(u, v) := 2^{-k}$, where $k = \min\{i \geq 0 : u_i \neq v_i\}$. Given $u \in A^n$, $v \in A^m$, denote by $u.v \in A^{\mathbb{N}}$ the **preperiodic word** with preperiod u and period v defined by $(u.v)_i = u_i$ for $i < n$ and $(u.v)_{n+km+i} = v_i$ for $i < m$. We say that $u \in A^+$ is a subword of $v \in A^+ \cup A^{\mathbb{N}}$ and write $u \sqsubseteq v$ if $u = v_{[i,j)}$ for some $i < j$.

The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty subset $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. For a subshift Σ there exists a set $D \subseteq A^+$ of **forbidden words** such that $\Sigma = \Sigma_D := \{x \in A^{\mathbb{N}} : \ \forall u \sqsubseteq x, u \notin D\}$. A subshift Σ is of **finite type** (SFT) of **order** k, if there exists a finite set $D \subseteq A^k$ such that $\Sigma = \Sigma_D$. A subshift is uniquely determined by its **language** $\mathcal{L}(\Sigma) := \{u \in A^+ :\ \exists x \in \Sigma, u \sqsubseteq x\}$. Denote by $\mathcal{L}^n(\Sigma) := \mathcal{L}(\Sigma) \cap A^n$ and $\mathcal{L}_D := \mathcal{L}(\Sigma_D)$.

An **iterative system** is a continuous map $F : A^+ \times X \to X$, or a family of continuous maps $(F_u : X \to X)_{u \in A^+}$ satisfying $F_{uv} = F_u \circ F_v$. An iterative system is determined by its generators $(F_a : X \to X)_{a \in A}$. Assume that $F : A^+ \times X \to X$ is an iterative system, $B \subset A^+$ a finite set and $\mathcal{W} = \{W_b :\ b \in B\}$ a family of subsets of X. We identify a word in $B^+ \cup B^{\mathbb{N}}$ with the concatenation of its letters. In this sense, $B^+ \subset A^+$ and $B^{\mathbb{N}} \subset A^{\mathbb{N}}$. For $u \in B^{n+1}$ set

$$\begin{aligned} W_u &:= W_{u_0} \cap F_{u_0}(W_{u_1}) \cap F_{u_{[0,2)}}(W_{u_2}) \cap \cdots \cap F_{u_{[0,n)}}(W_{u_n}) \\ \Sigma_{\mathcal{W}} &:= \{u \in B^{\mathbb{N}} :\ \forall k, W_{u_{[0,k]}} \neq \emptyset\} \end{aligned}$$

Then $\Sigma_{\mathcal{W}} \subseteq B^{\mathbb{N}}$ is a subshift. By an abuse of notation we identify $\Sigma_{\mathcal{W}}$ with $\{\sigma^n(u) :\ u \in \Sigma_{\mathcal{W}},\ n \geq 0\}$ and regard $\Sigma_{\mathcal{W}}$ also as a subshift of $A^{\mathbb{N}}$.

Proposition 5 *Let $F : A^+ \times X \to X$ be an iterative system, $B \subset A^+$ a finite set and $\mathcal{W} = \{W_b :\ b \in B\}$ a family of subsets of X such that whenever $F_a(W_b) \cap W_a \neq \emptyset$ then $F_a(W_b) \subseteq W_a$. Then $\Sigma_{\mathcal{W}} \subseteq A^{\mathbb{N}}$ is a subshift of finite type.*

Proof: Let $u \in B^{n+1}$ be such that $u_{[i,i+1]} \in \mathcal{L}(\Sigma_{\mathcal{W}})$ for all $i < n$, so $F_{u_i}(W_{u_{i+1}}) \subseteq W_{u_i}$. Then $F_{u_{[0,n)}}(W_{u_n}) \subseteq F_{u_{[0,n-1)}}(W_{u_{n-1}}) \subseteq \cdots \subseteq F_{u_0}(W_{u_1}) \subseteq W_{u_0}$, so $\emptyset \neq F_{u_{[0,n)}}(W_{u_n}) \subseteq W_u$ and $u \in \mathcal{L}(\Sigma_{\mathcal{W}})$. Thus $\Sigma_{\mathcal{W}}$ is a SFT of order 2 as a subshift of $B^{\mathbb{N}}$ and a SFT as a subshift of $A^{\mathbb{N}}$. $\square$

Definition 6 *We say that $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, is a* **Möbius iterative system**, *if all $F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are orientation-preserving Möbius transformations. The* **convergence space** $\mathbb{X}_F \subseteq A^{\mathbb{N}}$ *and the* **symbolic representation** $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ *are defined by*

$$\mathbb{X}_F := \{u \in A^{\mathbb{N}} :\ \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}, \quad \Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(i).$$

Thus $u \in \mathbb{X}_F$ iff the limit $\lim_{n\to\infty} F_{u_{[0,n)}}(i)$ exists, and belongs to $\overline{\mathbb{R}}$. We denote by $\mathbf{s}_u$ the stable fixed point of F_u (provided F_u is not elliptic). We denote by

$$\mathbf{U}_u := \{x \in \overline{\mathbb{R}} :\ F_u^\bullet(x) < 1\},\ \mathbf{V}_u := \{x \in \overline{\mathbb{R}} :\ (F_u^{-1})^\bullet(x) > 1\}$$

the **contracting interval** of F_u and the **expanding interval** of F_u^{-1} respectively. If F_u is a rotation or identity, then $\mathbf{U}_u = \mathbf{V}_u = \emptyset$. Otherwise, both $\mathbf{V}_u$ and $\mathbf{U}_u$ are nonempty open intervals, $F_u(\mathbf{U}_u) = \mathbf{V}_u$, and $||\mathbf{V}_u|| < \pi < ||\mathbf{U}_u||$. The symbolic representation Φ is usually not continuous on $\mathbb{X}_F$. This is why we look for subshifts of $\mathbb{X}_F$, on which Φ is continuous.

Proposition 7 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system.*

(1) For $v \in A^+$, $u \in A^{\mathbb{N}}$ we have $vu \in \mathbb{X}_F$ iff $u \in \mathbb{X}_F$, and $\Phi(vu) = F_v(\Phi(u))$.

(2) For $v \in A^+$ we have $.v = v^\infty \in \mathbb{X}_F$ iff F_v is not elliptic. In this case $\Phi(.v) = \mathbf{s}_v$ is the stable fixed point of F_v.

Proof: (1) is trivial. (2): If F_v is elliptic, then all $F_{v^k}(i)$ lie on a closed curve in $\mathbb{U}$, so $F_{v^k}(i)$ cannot converge to a real number. If F_v is hyperbolic or parabolic, then the stable fixed point $\mathbf{s}_v$ attracts all points of $\mathbb{U}$. $\square$

Theorem 8 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system, $B \subseteq A^+$ a finite set and $\mathcal{W} = \{W_b :\ b \in B\}$ a family of open intervals such that $\overline{\mathcal{W}} := \{\overline{W_b} :\ b \in B\}$ is a cover of $\overline{\mathbb{R}}$ and $\overline{W_b} \subseteq \mathbf{V}_b$ for each $b \in B$. Then $\Sigma_{\overline{\mathcal{W}}} \subseteq \mathbb{X}_F$, $\Phi : \Sigma_{\overline{\mathcal{W}}} \to \overline{\mathbb{R}}$ is continuous and $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective.*

Proof: For $u \in B^+$ denote by $\overline{W}_u := \overline{W_{u_0}} \cap F_{u_0}(\overline{W_{u_1}}) \cap \cdots \cap F_{u_{[0,n)}}(\overline{W_{u_n}})$. There exists an increasing continuous function $\psi : [0, 2\pi] \to [0, \pi]$ such that $\psi(0) = 0$, $0 < \psi(t) < t$ for $t > 0$, and $||F_b(W)|| \leq \psi(||W||)$ for each $b \in B$ and $W \subseteq \overline{\mathbf{U}_b}$ (recall that the length of a set is the length of the shortest interval that contains it - $\overline{W}_u$ are not necessarily intervals). Given $u \in \Sigma_{\overline{\mathcal{W}}}$ (regarded as a subshift of $B^{\mathbb{N}}$), and $m \leq n$ we have $F^{-1}_{u_{[0,m]}}(\overline{W}_{u_{[0,n]}}) \subseteq F^{-1}_{u_{[0,m]}} F_{u_{[0,m)}}(\overline{W_{u_m}}) = F^{-1}_{u_m}(\overline{W_{u_m}}) \subseteq \mathbf{U}_{u_m}$, so

$$\begin{aligned} ||\overline{W}_{u_{[0,n)}}|| &= ||F_{u_0} F^{-1}_{u_0}(\overline{W}_{u_{[0,n)}})|| \leq \psi(||F^{-1}_{u_0}(\overline{W}_{u_{[0,n)}})||) \\ &= \psi(||F_{u_1} F^{-1}_{u_{[0,1]}}(\overline{W}_{u_{[0,n)}})||) \leq \psi^2(||F^{-1}_{u_{[0,1]}}(\overline{W}_{u_{[0,n)}})||) < \cdots \\ &\leq \psi^n(||F^{-1}_{u_{[0,n)}}(\overline{W}_{u_{[0,n)}})||) \leq \psi^n(2\pi). \end{aligned}$$

Since $\psi(t) < t$ and the only fixed point of ψ is zero, we get $\lim_{n\to\infty} ||\overline{W}_{u_{[0,n)}}|| = 0$, so there exists a unique point $x \in \bigcap_n \overline{W}_{u_{[0,n)}}$. We show that $\Phi(u) = x$. Set $c := \min\{d(\overline{W_b}, \overline{\mathbb{R}} \setminus V_b) :\ b \in B\} > 0$. Then for each open interval $I \ni x$ there exists n such that for all $j > n$ we have $||F^{-1}_{u_{[0,j)}}(I)|| \geq \min\{\psi^{-j}(||I||), c\}$. By Proposition 3 we get $u \in \mathbb{X}_F$ and $\Phi(u) = x$, so $\Sigma_{\overline{\mathcal{W}}} \subseteq \mathbb{X}_F$. For $u \in \mathcal{L}(\Sigma_{\overline{\mathcal{W}}})$ we have $\Phi([u]) \subseteq \overline{W}_u$, and since the diameter of $\overline{W}_u$ converges to 0 as $|u|$ goes to infinity, $\Phi : \Sigma_{\overline{\mathcal{W}}} \to \overline{\mathbb{R}}$ is continuous. We show that $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective. Given $x \in \overline{\mathbb{R}}$, there exists $u_0 \in A$ such that $x = x_0 \in \overline{W}_{u_0}$ and $(x_0, x_0 + \varepsilon_0) \subseteq W_{u_0}$ for some $\varepsilon_0 > 0$. There exists u_1 such that $x_1 = F^{-1}_{u_0}(x_0) \in \overline{W_{u_1}}$ and $(x_1, x_1 + \varepsilon_1) \subseteq W_{u_1}$, so $W_{u_{[0,1]}} \neq \emptyset$. We continue by induction, so for each k, $x_k = F^{-1}_{u_{k-1}}(x_{k-1}) \in \overline{W_{u_k}}$ and $(x_k, x_k + \varepsilon_k) \subseteq W_{u_k}$ for some $\varepsilon_k > 0$. Thus $W_{u_{[0,k)}} \neq \emptyset$, $x \in \overline{W_{u_{[0,k)}}}$, so $x = \Phi(u)$ and $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective. □

Using more sophisticated techniques, A.Kazda proves in [4] that even under weaker assumptions $W_b \subseteq \mathbf{V}_b$ we get $\Sigma_{\mathcal{W}} \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is continuous and surjective.

Theorem 9 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system.*

(1) If $\overline{\bigcup_{u\in A^+} \mathbf{V}_u} \neq \overline{\mathbb{R}}$, then $\Phi(\mathbb{X}_F) \neq \overline{\mathbb{R}}$.

(2) If $\{\mathbf{V}_u :\ u \in A^+\}$ is a cover of $\overline{\mathbb{R}}$, then $\Phi(\mathbb{X}_F) = \overline{\mathbb{R}}$, there exists a subshift $\Sigma \subseteq \mathbb{X}_F$ on which Φ is continuous, and $\Phi(\Sigma) = \overline{\mathbb{R}}$.

Proof: (1) Assume that $x \in \overline{\mathbb{R}}$ does not belong to the closure of the union of all $\mathbf{V}_u$, so there exists an open interval $I \ni x$ which is disjoint from all $\mathbf{V}_u$. Given $u \in A^{\mathbb{N}}$, then for each n we have $||F^{-1}_{u_{[0,n)}}(I)|| \leq ||I||$. By Proposition 3, $F_{u_{[0,n)}}(i)$ cannot converge to x, so $x \notin \Phi(\mathbb{X}_F)$.

(2): If $\{\mathbf{V}_u :\ u \in A^+\}$ is a cover of $\overline{\mathbb{R}}$, then by compactness there exists a finite set $B \subset A^+$ such that $\{\mathbf{V}_u :\ u \in B\}$ is a cover of $\overline{\mathbb{R}}$. It follows that there exists a family of open intervals $\mathcal{W} = (W_b)_{b\in B}$ such that $\overline{\mathcal{W}} = \{\overline{W_b} :\ b \in B\}$ is a cover of $\overline{\mathbb{R}}$ and $\overline{W_b} \subseteq \mathbf{V}_b$. We apply Theorem 8. □

Definition 10 *We say that* (F, Σ) *is a* **Möbius number system**, *if* $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is a Möbius iterative system and* $\Sigma \subseteq \mathbb{X}_F$ *is a subshift such that* $\Phi : \Sigma \to \overline{\mathbb{R}}$ *is surjective and continuous.*

If $\mathcal{W}$ is a family of intervals satisfying the assumptions of Theorem 8, then both $(F, \Sigma_{\mathcal{W}})$ and $(F, \Sigma_{\overline{\mathcal{W}}})$ are Möbius number systems. We conjecture that under the assumptions of Theorem 9(2) there exists a SFT $\Sigma \subseteq \mathbb{X}_F$ such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective. In our examples in final sections, Σ is usually constructed as an SFT approximation to some $\Sigma_{\mathcal{W}}$. This means that the forbidden words of Σ are chosen among forbidden words of $\Sigma_{\mathcal{W}}$.

4. Convergence Theorem

A **labelled graph** over an alphabet A is a structure $G = (V, E, s, t, h)$, where V is a finite set of vertices, E is a finite set of edges, $s : E \to V$ is a surjective **source map**, $t : E \to V$ is a **target map**, and $h : E \to A$ is a **labelling function**. A finite or infinite word $u \in E^+ \cup E^{\mathbb{N}}$ is a **path** in G if $t(u_i) = s(u_{i+1})$ for all i. The source and target of a finite path $u \in E^n$ are $s(u) := s(u_0)$, $t(u) := t(u_{n-1})$. We denote by $\mathcal{O}_p \subseteq E^{\mathbb{N}}$ the set of all infinite paths with source p. Since the source map is surjective, $\mathcal{O}_p \neq \emptyset$. The **label** $h(u) \in A^+ \cup A^{\mathbb{N}}$ of a path u is defined by $h(u)_i := h(u_i)$. We denote by $\Sigma_{|G|} \subseteq E^{\mathbb{N}}$ the subshift of all paths of G and by $\Sigma_G \subseteq A^{\mathbb{N}}$ the subshift of all their labels. The languages of these subshifts are denoted by $\mathcal{L}_{|G|}$ and $\mathcal{L}_G$. A subshift Σ is **sofic** iff there exists a labelled graph G such that $\Sigma = \Sigma_G$. Each SFT is sofic (see Lind and Marcus [9]). A finite set $P \subset \mathcal{L}_{|G|}$ is a **suffix code** for G, if each long enough finite path of G has a unique suffix which belongs to P.

Alternatively, sofic subshifts are defined by finite automata. A **deterministic finite automaton** (DFA) over an alphabet A is a structure $\mathcal{A} = (V, \delta, \mathbf{i})$, where V is a finite set of states, $\delta : V \times A \to V$ is a partial **transition function**, and $\mathbf{i} \in V$ is the initial state. The function δ is extended to a partial function $\delta : V \times A^+ \to V$ by $\delta(p, ua) = \delta(\delta(p, u), a)$, where the left-hand-side is defined iff the right-hand-side is defined. The language $\mathcal{L}(\mathcal{A})$ of $\mathcal{A}$ consists of words $u \in A^+$ which are **accepted**, i.e., for which $\delta(\mathbf{i}, u)$ is defined. A subshift Σ is sofic iff there exists a DFA which accepts exactly the words of $\mathcal{L}(\Sigma)$. The graph $G = (V, E, s, \delta, h)$ of a DFA $\mathcal{A} = (V, \delta, \mathbf{i})$ is defined by $E = \{(q, a) \in V \times A : \ \delta(q, a) \text{ is defined}\}$, $s(q, a) = q$, and $h(q, a) = a$, so the edges of G are $q \xrightarrow{a} \delta(q, a)$. Then $\mathcal{L}_G = \mathcal{L}(\mathcal{A})$.

Theorem 11 (Convergence theorem) *Let* $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *be a Möbius iterative system and let* $G = (V, E, s, t, h)$ *be a labelled graph. Assume that there exist closed intervals* $\mathcal{I} = \{I_q : \ q \in V\}$ *and a finite suffix code* $P \subset \mathcal{L}_{|G|}$ *such that for each path* $u \in P$ *we have* $I_{t(u)} \subseteq \overline{\mathbf{U}_{h(u)}}$, *and* $F_{h(u)}(I_{t(u)}) \subseteq I_{s(u)}$. *Then* $\Sigma_G \subseteq \mathbb{X}_F$ *and* $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ *is continuous.*

Proof: There exists an increasing continuous function $\psi : [0, 2\pi] \to [0, \pi]$ such that $\psi(0) = 0$, $0 < \psi(t) < t$ for $t > 0$, and for every $u \in P$ and any set $W \subseteq \overline{\mathbf{U}_{h(u)}}$ we have $||F_{h(u)}(W)|| \leq \psi(||W||)$. There exists a constant $d > 0$ such that for every $u \in P$, every suffix v of u and any set $W \subseteq \overline{\mathbf{U}_{h(u)}}$ we have $||F_{h(v)}(W)|| \leq d \cdot ||W||$. Given an infinite path $u \in \Sigma_{|G|}$, there exists by compactness its **parsing** $v \in (A^+)^{\mathbb{N}}$, such that $v_k \in P$ for all $k > 0$, v_0 is a suffix of some $v'_0 \in P$, and for each n there exists j_n such that $v_{[0,n)} = u_{[0,j_n)}$. Set $W_0 := I_{s(v_0)}$, $W_n := F_{h(v_{[0,n)})}(I_{s(v_n)})$. Since $F_{h(v_n)}(I_{t(v_n)}) \subseteq I_{s(v_n)}$ and $t(v_n) = s(v_{n+1})$, we get $W_{n+1} = F_{h(v_{[0,n]})}(I_{t(v_n)}) \subseteq F_{h(v_{[0,n)})}(I_{s(v_n)}) = W_n$. For

$m < n$ we get by induction $F_{h(v_{[m,n)})}(I_{s(v_n)}) \subseteq I_{s(v_m)} = I_{t(v_{m-1})} \subseteq \overline{\mathbf{U}_{h(v_{m-1})}}$ and therefore

$$\begin{aligned}||W_n|| &\le d \cdot ||F_{h(v_{[1,n)})}(I_{s(v_n)})|| \le d \cdot \psi(||F_{h(v_{[2,n)})}(I_{s(v_n)})||) < \cdots \\ &\le d \cdot \psi^{n-1}(||I_{s(v_n)}||)) \le d \cdot \psi^{n-1}(2\pi).\end{aligned}$$

Since the only fixed point of ψ is 0, we have $\lim_{n\to\infty} ||W_n|| = 0$, so there exists a unique $x \in \bigcap_n W_n$. We have $(F_{h(v_{[0,n)})})^{-1}(W_m) = (F_{h(v_{[m,n)})})^{-1}(I_{s(v_m)}) \supseteq I_{s(v_n)}$ for each $m \le n$, so $\liminf_{n\to\infty} ||(F_{h(v_{[0,n)})})^{-1}(W_m)|| \ge c := \min\{||I_q|| : \ q \in V\}$ and therefore $\lim_{n\to\infty} F_{u_{[0,n)}}(i) = x$ by Proposition 3. Thus $h(u) \in \mathbb{X}_F$ and $\Phi(h(u)) = x$, so $\Sigma_G \subseteq \mathbb{X}_F$. We show that $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous. Given $w \in \Sigma_G$, let $u \in \Sigma_{|G|}$ be a path with $h(u) = w$, and let let $v \in (A^+)^{\mathbb{N}}$ be its parsing. For $m > 0$ let n be such that $j_n \le m < j_{n+1}$. Then $\Phi(w) \in F_{h(u_{[0,j_n)})}(\overline{\mathbf{U}_{h(u_{j_n-1})}})$, and

$$||F_{h(u_{[0,j_n)})}(\overline{\mathbf{U}_{h(u_{j_n-1})}})|| \le d\psi^{n-1}(2\pi) \le d\psi^{m/p-1}(2\pi),$$

where $p := \min\{|u| : \ u \in P\}$. Thus the diameter of $\Phi([w_{[0,m)}])$ is at most $2d\psi^{m/p-1}(2\pi)$, so $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous. □

We generalize now Theorem 8 to families indexed by vertices of a labelled graph. Given a Möbius iterative system $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, and a labelled graph $G = (V, E, s, t, h)$, consider a family $\mathcal{W} = (W_p)_{p\in V}$ of subsets of $\overline{\mathbb{R}}$. For $u \in \mathcal{L}^{n+1}_{|G|}$ set

$$\begin{aligned}W_u &:= W_{s(u_0)} \cap F_{h(u_0)}(W_{t(u_0)}) \cap \cdots \cap F_{h(u_{[0,n]})}(W_{t(u_n)}) \\ \Sigma_{\mathcal{W}} &:= \{h(u) : \ u \in \Sigma_{|G|}, \ \forall k, W_{u_{[0,k]}} \neq \emptyset\}.\end{aligned}$$

Then $\Sigma_{\mathcal{W}} \subseteq \Sigma_G$ is a subshift.

Proposition 12 *Let $\mathcal{W} = (W_p)_{p\in V}$ be a family of subsets of $\overline{\mathbb{R}}$ and assume that $F_{h(e)}(W_{t(e)}) \subseteq W_{s(e)}$ for each edge $e \in E$. Then $\Sigma_{\mathcal{W}} = \Sigma_G$.*

Proof: If $u \in \mathcal{L}^{n+1}_{|G|}$, then

$$F_{h(u_{[0,n]})}(W_{t(u_n)}) \subseteq F_{h(u_{[0,n-1]})}(W_{t(u_{n-1})}) \subseteq \cdots \subseteq F_{h(u_0)}(W_{t(u_0)}) \subseteq W_{s(u_0)},$$

so $\emptyset \neq F_{h(u_{[0,n]})}(W_{t(u_n)}) \subseteq W_{u_{[0,n]}}$ and $h(u) \in \mathcal{L}^{n+1}(\Sigma_{\mathcal{W}})$. □

Theorem 13 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system, $G = (V, E, s, t, h)$ a labelled graph, and $\mathcal{W} = (W_p)_{p\in V}$ a family of open intervals such that $\overline{W}_{s(e)} \subseteq \mathbf{V}_{h(e)}$ for each edge $e \in E$, $\overline{W_p} \subseteq \bigcup\{F_{h(e)}(\overline{W_{t(e)}}) : \ s(e) = p\}$ for each vertex $p \in V$, and $\overline{\mathcal{W}} = \{\overline{W_p} : \ p \in V\}$ is a cover of $\overline{\mathbb{R}}$. Then $\Sigma_{\overline{\mathcal{W}}} \subseteq \mathbb{X}_F$, $\Phi : \Sigma_{\overline{\mathcal{W}}} \to \overline{\mathbb{R}}$ is continuous, and $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective.*

Proof: We generalize the proof of Theorem 8. If $m \le n$, then

$$F^{-1}_{h(u_{[0,m]})}(\overline{W}_{u_{[0,n]}}) \subseteq F^{-1}_{h(u_{[0,m]})} F_{h(u_{[0,m)})}(\overline{W_{t(u_{m-1})}}) = F^{-1}_{h(u_m)}(\overline{W_{s(u_m)}}) \subseteq \mathbf{U}_{h(u_m)},$$

so for each $u \in \Sigma_{\overline{\mathcal{W}}}$ we have $\lim_{n\to\infty} ||\overline{W}_{u_{[0,n)}}|| = 0$ and $\Phi : \Sigma_{\overline{\mathcal{W}}} \to \overline{\mathbb{R}}$ is continuous. Given $x \in \overline{\mathbb{R}}$ there exists $p \in V$ such that $x = x_0 \in \overline{W_p}$ and $(x_0, x_0 + \varepsilon_0) \subseteq W_p$ for some $\varepsilon_0 > 0$. There exists an edge u_0 with source p such that $x_1 = F^{-1}_{h(u_0)}(x_0) \in \overline{W_{t(u_0)}}$ and $(x_1, x_1 + \varepsilon_1) \subseteq W_{t(u_0)}$ for some $\varepsilon_1 > 0$. By induction there exists an edge u_k with source $t(u_{k-1})$ such that $x_k = F^{-1}_{h(u_{k-1})}(x_{k-1}) \in \overline{W_{t(u_k)}}$ and $(x_k, x_k + \varepsilon_k) \subseteq W_{t(u_k)}$ for some $\varepsilon_k > 0$. Then $u \in \Sigma_{\mathcal{W}}$ and $x = \Phi(u)$, so $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective. □

5. Surjectivity theorems

We say that a continuous surjective map $\Phi : X \to Y$ has the **extension property**, if for any continuous map $\varphi : X \to Y$ there exists a continuous map $F : X \to X$ such that $\Phi \circ F = \varphi$. In this case, any continuous map $G : Y \to Y$ can be lifted to a continuous map $F : X \to X$ such that $\Phi \circ F = G \circ \Phi$. By a theorem of Weihrauch (see Weihrauch [12], Theorem 3.2.11, page 70 or Kůrka [6] Theorem 3.8, page 110), for every compact metric space Y and any Cantor space X there exists a continuous surjection $\Phi : X \to Y$ with the extension property. We say that a Möbius number system (F, Σ) is **redundant**, if the symbolic representation $\Phi : \Sigma \to \overline{\mathbb{R}}$ has the extension property.

Given a Möbius number system (F, Σ), define the **cylinder of a word** $u \in \mathcal{L}(\Sigma)$ by $\Phi([u])$, where $[u] := \{v \in \Sigma : v_{[0,|u|)} = u\}$ is the symbolic cylinder of u. Define the **cylinder of a vertex** $p \in V$ by $[p]_\Phi = \Phi(h(\mathcal{O}_p))$. The cylinders of vertices satisfy the conditions of both Theorem 11 and 13, i.e., $[p]_\Phi = \bigcup\{F_{h(e)}([t(e)]_\Phi) : s(e) = p\}$. However, $[p]_\Phi$ are in general not intervals. Next theorems give conditions which imply that $[p]_\Phi$ are intervals which cover $\overline{\mathbb{R}}$. We apply them usually to the graphs of DFA, in which $[\mathbf{i}]_\Phi = \overline{\mathbb{R}}$. A **selector** for a graph $G = (V, E, s, t, h)$ is a map $K : V \to E$ which selects at each vertex an outgoing edge, i.e., $s(K(p)) = p$. A selector K determines for each $p \in V$ a unique eventually periodic path $K^p \in \mathcal{O}_p$ defined by $K^p_0 = K(p)$, $K^p_{i+1} = K(t(K^p_i))$.

Theorem 14 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system, let $G = (V, E, s, t, h)$ be a labelled graph such that $\Sigma_G \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous. Then there exists a system of closed intervals $\mathcal{J} = \{J_p : p \in V\}$ and selectors $L, R : V \to E$ such that for each $p \in V$ we have*

(1) $[p]_\Phi \subseteq J_p$.

(2) Either $J_p = [\Phi(h(L^p)), \Phi(h(R^p))]$, *or* $J_p = \overline{\mathbb{R}}$.

(3) $\lim_{n\to\infty} ||F_{h(u_{[0,n)})}(J_{t(u_{[0,n)})})|| = 0$ *for each* $u \in \Sigma_{|G|}$.

Proof: Denote by V_0 the set of all vertices $p \in V$, such that there exist arbitrarily long paths with target p. If $p \in V_0$, and $t(u) = p$, then $F_u([p]_\Phi) \subseteq \Phi[h(u)]$. By the continuity of Φ, the diameter of this set goes to 0 as the length of u goes to infinity. It follows that there exist unique $a_p, b_p \in [p]_\Phi$ such that $[p]_\Phi \subseteq [a_p, b_p]$ and the diameter of $F_u([a_p, b_p])$ goes to zero as the length of u goes to infinity. Since for each $p \in V_0$ we have $[p]_\Phi = \bigcup\{F_{h(e)}([q]_\Phi) : p \xrightarrow{e} q\}$, there exists an edge $e = L(p) \in E$ such that $p \xrightarrow{e} q$ and $a_p = F_{h(e)}(a_q)$. Analogously there exists an edge $e' = R(p) \in E$ with source p, target q' and $b_p = F_{h(e')}(b_{q'})$. Thus L, R are selectors on V_0 and we have their paths L^p, R^p. For a given vertex $p \in V_0$ let m, n be the preperiod and period of L^p and set $q = t(L^p_{[0,m)}) = t(L^p_{[0,m+n)})$, $u := h(L^p_{[0,m)})$, $v = h(L^p_{[m,m+n)})$, so $h(L^p) = u.v$. By Proposition 7, $\Phi(h(L^p)) = F_u(\mathbf{s}_v)$, where $\mathbf{s}_v$ is the stable fixed point of F_v. For every $k \geq 0$ we have $a_p = F_u(a_q) = F_{uv^k}(a_q) \in \Phi([uv^k])$,

$\Phi(h(L^p)) = F_{uv^k}(\mathbf{s}_v) \in \Phi([uv^k])$. Since the diameter of $\Phi([uv^k])$ converges to zero as k goes to infinity, we get $\Phi(h(L^p)) = a_p$. Similarly we prove $\Phi(h(R^p)) = b_p$. We now extend the selectors L, R to whole V by induction. Suppose that $p \in V$ is a vertex, such that for all edges e with $s(e) = p$, $L(t(e))$ and $R(t(e))$ are already defined. Denote by $J'_p := \bigcup\{F_{h(e)}(J_{t(e)}) : \; s(e) = p\}$. If $J'_p = \overline{\mathbb{R}}$, then we set $J_p = \overline{\mathbb{R}}$ and define $L(p), R(p)$ arbitrarily. If $J'_p \neq \overline{\mathbb{R}}$, then there exist distinct $a_p, b_p \in J'_p$ such that $J'_p \subseteq [a_p, b_p]$ and we set $J_p := [a_p, b_p]$. There exist edges $L(p), R(p)$ with source p and targets q, r such that $a_p = F_{h(L(p))}(\Phi(h(L^q)) = \Phi(h(L^p))$, $b_p = F_{h(R(p))}(\Phi(h(R^r)) = \Phi(h(R^p))$. To prove (3), note that for each path u of $\Sigma_{|G|}$ there exists n_0 such that $t(u_n) \in V_0$ for each $n \geq n_0$, so $\lim_{n\to\infty} ||F_{h(u_{[0,n]})}(J_{t(u_n)})|| = 0$. □

If L, R are selectors from Theorem 14, then for each selector K and for each $p \in V$ we have $\Phi(h(K^p)) \in [\Phi(h(L^p)), \Phi(h(R^p))]$. Since there is only a finite number of selectors for a given labelled graph, the left and right selectors L, R from Theorem 14 can be found effectively.

Theorem 15 (Surjectivity theorem) *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system, let $G = (V, E, s, t, h)$ be a labelled graph such that $\Sigma_G \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous. Let $\mathcal{J} = \{J_p : \; p \in V\}$ be intervals and L, R selectors from Theorem 14. Assume that the intervals J_p cover $\overline{\mathbb{R}}$ and that for each p, the intervals $\{F_{h(e)}(J_q) : \; p \xrightarrow{e} q\}$ cover J_p. Then $\Phi(\Sigma_G) = \overline{\mathbb{R}}$ and $[p]_\Phi = J_p$ for each $p \in V$. If moreover the open intervals J_p° cover $\overline{\mathbb{R}}$ and if $\{F_{h(e)}(J_q^\circ) : \; p \xrightarrow{e} q\}$ is a cover of J_p°, then $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ has the extension property.*

The proof is analogous to the proof of Theorem 13. Alternatively, we can prove the surjectivity using smaller intervals than J_p. This is useful when the endpoints of J_p are irrational, and arithmetical algorithms can be simplified when we replace them by intervals with rational endpoints.

Theorem 16 *Let $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system, G a labelled graph such that $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous. Let $\mathcal{W} = (W_p)_{p \in V}$ be a family of intervals such that $\overline{\mathcal{W}} = \{\overline{W_p} : \; p \in V\}$ is a cover of $\overline{\mathbb{R}}$, $\overline{W_p} \subseteq J_p$, and $\overline{W_p} \subseteq \bigcup\{F_{h(e)}(\overline{W_{t(e)}}) : \; s(e) = p\}$ for each vertex $p \in V$. Then $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ is surjective.*

Proposition 17 *Let (F, Σ) be a Möbius number system with sofic subshift Σ, let $\mathcal{A} = (V, \delta, \mathbf{i})$ be a DFA for $\mathcal{L}(\Sigma)$ and assume that the system $\mathcal{J} = \{J_p : \; p \in V\}$ of intervals constructed in Theorem 14 satisfies the conditions of Theorem 15. Then $J_\mathbf{i} = \overline{\mathbb{R}}$ and $\Phi([u]) = F_u(J_{\delta(\mathbf{i},u)})$ for each $u \in \mathcal{L}(\Sigma)$.*

Proof: For $u \in \mathcal{L}(\Sigma)$ and $v \in A^\mathbb{N}$ we have

$$\begin{aligned}\Phi([u]) &= \{\Phi(uv) : \; uv \in \Sigma\} = \{F_u(\Phi(v)) : \; v \in h(\mathcal{O}_{\delta(\mathbf{i},u)})\} \\ &= F_u([\delta(\mathbf{i}, u)]_\Phi) = F_u(J_{\delta(\mathbf{i},u)}).\end{aligned}$$

□

6. Arithmetical algorithms

Definition 18 *Let (F, Σ) be a Möbius number system with the symbolic representation $\Phi : \Sigma \to \overline{\mathbb{R}}$. We say that $\mathcal{E} : \overline{\mathbb{R}} \to \Sigma$ is a* **number expansion map***, if $\Phi\mathcal{E}(x) = x$ for each $x \in \overline{\mathbb{R}}$. We say that $\mathcal{E} : \mathcal{I} \to \mathcal{L}(\Sigma)$ is an* **interval expansion map***, if $I \subseteq \Phi([\mathcal{E}(I)])$ for each closed interval $I \in \mathcal{I}$.*

We say that $u \in \Sigma$ is an expansion of $x \in \overline{\mathbb{R}}$ if $\Phi(u) = x$. The expansion of a number can be conceived as the label of a path in the infinite **expansion graph** defined as follows. Let G be the graph of a DFA for $\mathcal{L}(\Sigma)$, and let $\mathcal{W} = \{W_p : \ p \in V\}$ be a family of intervals which satisfies the conditions of Theorem 16. The vertices of the expansion graph are pairs (x, p) where $p \in V$ and $x \in W_p$. We have a labelled edge $(x, p) \xrightarrow{a} (y, q)$, if $p \xrightarrow{a} q$ in G, and $y = F_a^{-1}(x)$ (this implies $x \in F_a(W_q)$). The expansion of a number x is the label of any infinite path with the source vertex $(x, \mathbf{i})$. If we have a selector K for the expansion graph (defined for example by a linear preference order on outgoing edges of a given vertex of G), then we have a number expansion map $\mathcal{E}$, where $\mathcal{E}(x)$ is the label of the path with source $(x, \mathbf{i})$ selected by K.

In a similar manner, an interval expansion map can be obtained from the **interval expansion graph** whose vertices are (I, p), where $p \in V$ and $I \subseteq W_p$ is an interval. We have a labelled edge $(I, p) \xrightarrow{a} (J, q)$, if $p \xrightarrow{a} q$ in G, and $J = F_a^{-1}(I)$. If the length of I is too large then there are no edges with source (I, p), so each path in the interval expansion graph is finite.

To obtain algorithms for arithmetical operations, we need Möbius transformations with coefficients in a **computable field**, i.e., in a countable subfield of $\mathbb{R}$ whose arithmetical operations are recursive. The field $\mathbb{Q}$ of rational numbers is computable. If $\mathcal{K}$ is a computable field, and if $x_1, \ldots, x_n \in \mathcal{K}$ are positive, then $\mathcal{K}(\sqrt{x_1}, \ldots, \sqrt{x_n})$ (the smallest subfield of $\mathbb{R}$ which contains $\mathcal{K}$ and all $\sqrt{x_i}$) is a computable field. Given a computable field $\mathcal{K}$, denote by $\overline{\mathcal{K}} = \mathcal{K} \cup \{\infty\}$, and by $\mathcal{I}_{\mathcal{K}}$ the set of closed intervals with endpoints in $\overline{\mathcal{K}}$. The sum of two intervals $I, J \in \mathcal{I}_{\mathcal{K}}$ is defined by $I + J = \{z \in \overline{\mathbb{R}} : \exists x \in I, \exists y \in J, z = x + y\}$ (we have $a + \infty = \infty$ for $a \neq \infty$ while $\infty + \infty$ is undefined).

Assume that (F, Σ) is a Möbius number system with a sofic subshift Σ, such that the coefficients of F_a belong to a computable field, and let $\mathcal{A} = (V, \delta, \mathbf{i})$ be a DFA for Σ with the system of intervals $\mathcal{J} = \{J_p : \ p \in V\}$ from Theorem 15. The endpoints of J_p are solutions of quadratic equations, so they belong to a possibly larger computable field $\mathcal{K}$. Then we have arithmetical algorithms analogous to those described in Vuillemin [11] or Kornerup and Matula [5].

For each $u \in \mathcal{L}(\Sigma)$, the cylinder $\Phi([u]) = F_u(J_{\delta(\mathbf{i},u)})$ has endpoints in $\overline{\mathcal{K}}$ and can be algorithmically computed. The expansion $\mathcal{E}(I)$ of an interval $I \in \mathcal{I}_{\mathcal{K}}$ can be algorithmically computed using a selector for the interval expansion graph. The sum of $u, v \in \mathcal{L}(\Sigma)$ is then $\mathcal{E}(\Phi([u]) + \Phi([v]))$. Alternatively, instead of J_p we can use a family of intervals which satisfies the conditions of Theorem 16. The algorithm also works in an on-line manner for infinite words $u, v \in \Sigma$, whose sum is computed in an infinite loop and written to the output word $w \in A^{\mathbb{N}}$: in step n the algorithm reads u_{n-1} and v_{n-1} and computes the expansion $w^{(n)} := \mathcal{E}(\Phi([u_{[0,n)}]) + \Phi([v_{[0,n)}]))$, so $w^{(n)}$ is a prefix of $w^{(n+1)}$. It may happen that $\lim_{n\to\infty} |w^{(n)}|$ is finite, for example when $\Phi(u) = \Phi(v) = \infty$. If not, then $w = \lim_{n\to\infty} w^{(n)} \in A^{\mathbb{N}}$ and $\Phi(w) = \Phi(u) + \Phi(v)$. We have similar algorithms for other arithmetic operations and also conversion algorithms between different Möbius number systems.

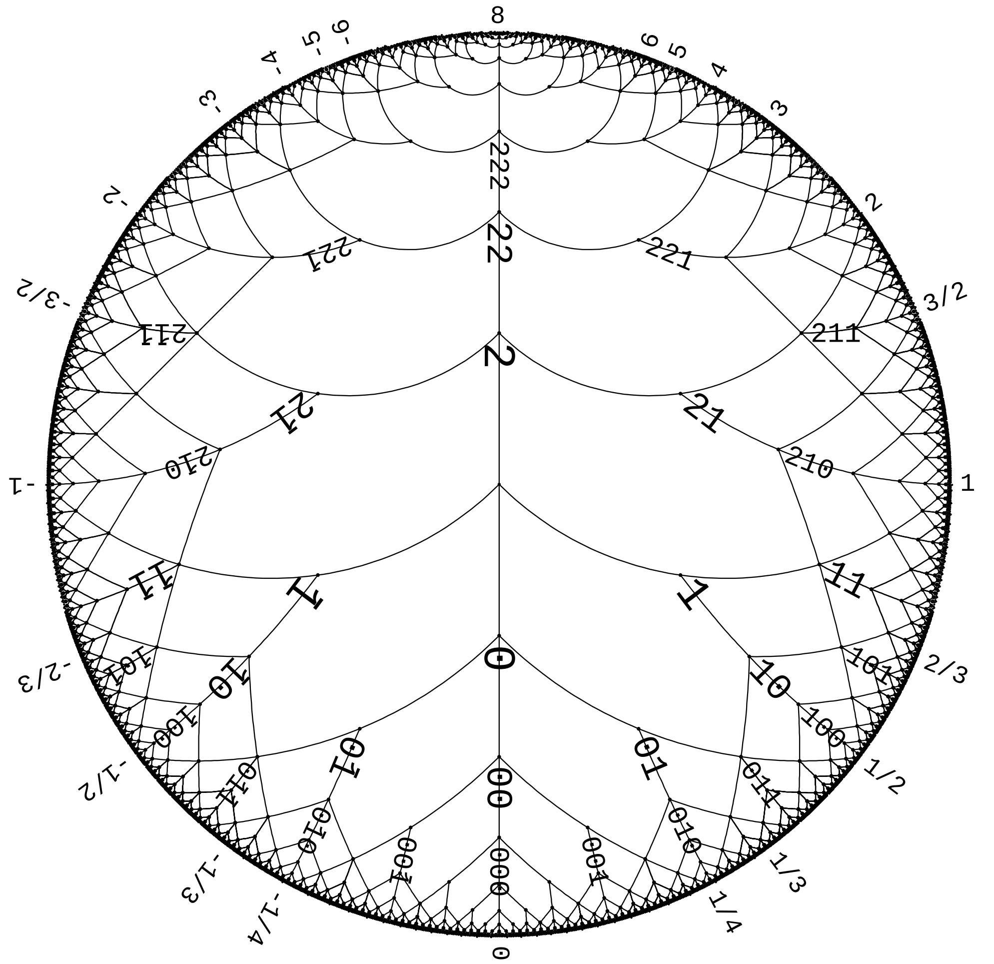


Figure 1. Means of the binary signed system (BSS)

7. Binary signed system

The classical binary signed system uses transformations $(x-1)/2$, $x/2$, $(x+1)/2$, which are contractive on the attractor $[-1, 1]$. To get a Möbius number system, we add the transformation $2x$. The number of 2's at the beginning of a word corresponds to the placement of the binary point.

Example 1 *The* **Möbius binary signed system** *(BSS) consists of the alphabet $A = \{\overline{1}, 0, 1, 2\}$, transformations*

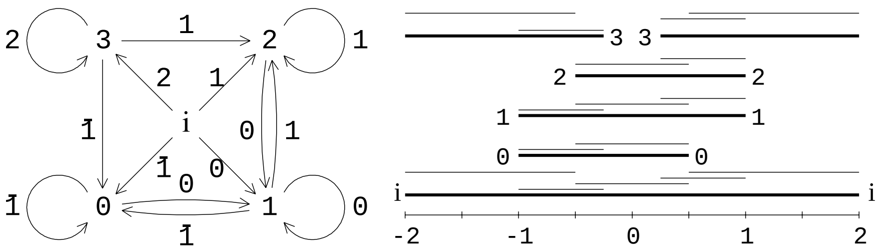
$$F_{\overline{1}}(x) = (-1+x)/2,\ F_0(x) = x/2,\ F_1(x) = (1+x)/2,\ F_2(x) = 2x,$$

and the subshift Σ_D with forbidden words $D = \{20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1\}$.

The means $\widehat{F}_u(0)$ of words $u \in \mathcal{L}(\Sigma_D)$ can be seen in Figure 1. For each Möbius transformation F there exists a family of Möbius transformations $(F^t)_{t\in\mathbb{R}}$ such that $F^0 = \mathrm{Id}$, $F^1 = F$, and $F^{t+s} = F^t F^s$. In Figure 1, each mean $\widehat{F}_{ua}(0)$ is joined to

$\widehat{F}_u(0)$ by the curve $(\widehat{F}_u(\widehat{F}_a^t(0)))_{0\le t\le 1}$. The labels $u \in A^+$ at $\widehat{F}_u(0)$ are written in the direction of the unit tangent vectors $\widehat{F}'_u(0)/|\widehat{F}'_u(0)|$.

$t(u)$	$h(u)$	$\overline{\mathbf{U}_{h(u)}}$	$I_{t(u)}$	$F_{h(u)}(I_{t(u)})$	$s(u)$
0	$\overline{1}$	$[-3,1]$	$[-1,\frac{1}{2}]$	$[-1,-\frac{1}{4}]$	$0,1,3$
1	0	$[-\sqrt{2},\sqrt{2}]$	$[-1,1]$	$[-\frac{1}{2},\frac{1}{2}]$	$0,1,2$
2	1	$[-1,3]$	$[-\frac{1}{2},1]$	$[\frac{1}{4},1]$	$1,2,3$
3	2222	$[\frac{1}{4},-\frac{1}{4}]$	$[\frac{1}{4},-\frac{1}{4}]$	$[4,-4]$	3



$t(e)$	$h(e)$	L,R	$h(L^{t(e)})$	$h(R^{t(e)})$	$J_{t(e)}$	$F_{h(e)}(J_{t(e)})$
0	$\overline{1}$	$0,1$	$.\overline{1}$	01.1	$[-1,\frac{1}{2}]$	$[-1,-\frac{1}{4}]$
1	0	$0,2$	$\overline{1}.\overline{1}$	1.1	$[-1,1]$	$[-\frac{1}{2},\frac{1}{2}]$
2	1	$1,2$	$0\overline{1}.\overline{1}$	$.1$	$[-\frac{1}{2},1]$	$[\frac{1}{4},1]$
3	2	$2,0$	$10\overline{1}.\overline{1}$	$\overline{1}01.1$	$[\frac{1}{4},-\frac{1}{4}]$	$[\frac{1}{2},-\frac{1}{2}]$

Figure 2. Convergence and surjectivity in BSS

For each $u \in \mathcal{L}(\Sigma_D)$, the transformation F_u can be written as

$$F_u(x) = 2^n \left(s_0 + \frac{1}{2}\left(s_1 + \frac{1}{2}\left(s_2 + \cdots + \frac{1}{2}\left(s_{k-1} + \frac{x}{2}\right)\cdots\right)\right)\right)$$
$$= \sum_{i=0}^{k-1} 2^{n-i} s_i + 2^{n-k} x$$

for some $n, k \ge 0$, $s_i \in \{-1, 0, 1\}$, $s_0 \neq 0$, and $s_i s_{i+1} \neq -1$. This includes the case $k = 0$ when $F_u(x) = 2^n x$. For each $u \in \Sigma_D \setminus \{.2\}$ and $x \neq \infty$ we get $\lim_{k\to\infty} F_{u_{[0,k)}}(x) = \sum_{i=0}^{\infty} 2^{n-i} s_i$, so $\Phi(u) = \sum_{i=0}^{\infty} 2^{n-i} s_i$ by Proposition 4. Clearly $\Phi(.2) = \infty$, so $\Sigma_D \subseteq \mathbb{X}_F$.

The graph G of a DFA for $\mathcal{L}(\Sigma_D)$ can be seen in Figure 2 center left. The subgraph G_0 of G with vertices $V_0 = \{0, 1, 2, 3\}$ yields Σ_D as well: $\Sigma_D = \Sigma_G = \Sigma_{G_0}$. The continuity of Φ can be shown by the Convergence theorem 11 applied to the graph G_0. We take the suffix code consisting of all paths whose labels are in the set $P := \{\overline{1}, 0, 1, 2222\}$, and intervals $(I_p)_{p\in V_0}$ in Figure 2 top. The selectors, their paths and cylinders of vertices can be seen in Figure 2 bottom and in Figure 2 center right. Since the interiors of J_p satisfy the assumptions of Surjectivity theorem 15, the system is surjective and redundant.

8. Regular continued fractions

Regular continued fractions are based on iterations of transformations $1 + x$ and $1/x$. Since the transformation $1/x$ is orientation-reversing, we use rather the orientation

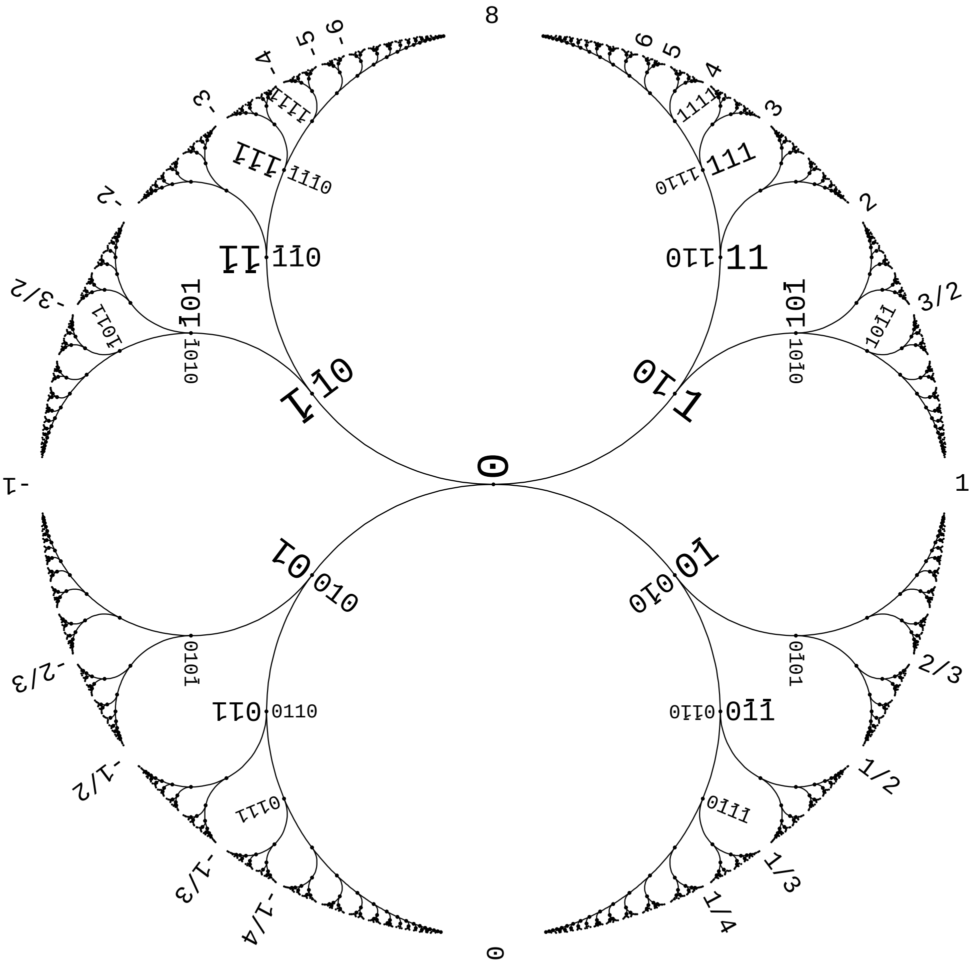


Figure 3. Means in regular continued fractions (RCF)

preserving transformation $F_0(x) = -1/x$ which corresponds to the rotation $\widehat{F}_0(z) = -z$ of the unit circle by π. It is then natural to allow as partial quotients also negative numbers. To expand a positive number x, we repeatedly subtract 1 till we get into the interval $[0, 1)$. Then we apply F_0, getting a negative number less than -1. We repeatedly add 1 till we get into the interval $(-1, 0]$, apply F_0 and repeat the process.

Example 2 *The* **Möbius system of regular continued fraction** *(RCF, Figure 3) consists of the alphabet $A = \{\overline{1}, 0, 1\}$, transformations*

$$F_{\overline{1}}(x) = -1 + x,\ F_0(x) = -1/x,\ F_1(x) = 1 + x,$$

and the subshift Σ_D with forbidden words $D = \{00, 1\overline{1}, \overline{1}1, 101, \overline{1}0\overline{1}\}$.

The expansion procedure for RCF is reflected in the family of intervals $\mathcal{W} = \{W_b :\ b \in B\}$, where $B = \{\overline{1}, 0\overline{1}, 01, 1\}$, $W_{\overline{1}} = (\infty, -1)$, $W_{01} = (-1, 0)$, $W_{0\overline{1}} = (0, 1)$, and $W_1 = (1, \infty)$. Using Proposition 5 we get $\Sigma_{\mathcal{W}} = \Sigma_D$. We have $W_b \subseteq \mathbf{V}_b$, since $\mathbf{V}_{\overline{1}} = (\infty, -\frac{1}{2})$, $\mathbf{V}_{01} = (-2, 0)$, $\mathbf{V}_{0\overline{1}} = (0, 2)$, $\mathbf{V}_1 = (\frac{1}{2}, \infty)$. While the conditions of

Theorem 8 do not apply, a stronger theorem of A.Kazda [4] shows that $\Phi : \Sigma_D \to \overline{\mathbb{R}}$ is continuous and surjective. This can be also proved using Theorems 11 and 15. Alternatively, we can use the theory of continued fractions. For each $u \in \mathcal{L}(\Sigma_D)$,the transformation F_u can be written as

$$F_u(x) = F_1^{a_0} F_0 F_1^{a_1} \cdots F_0 F_1^{a_n}(x) = a_0 - \frac{1|}{|a_1} - \cdots - \frac{1|}{|a_n + x}$$

where $a_i \in \mathbb{Z}$, $a_i a_{i+1} \leq 0$ and $a_i \neq 0$ for $i > 0$. Thus we obtain a continued fraction whose partial quotients $(-1)^i a_i$ are either all positive or all negative. Each rational number has exactly two expansions of the form $u.1$, and $v.\overline{1}$, while each irrational number has a unique expansion. The system is surjective but not redundant.

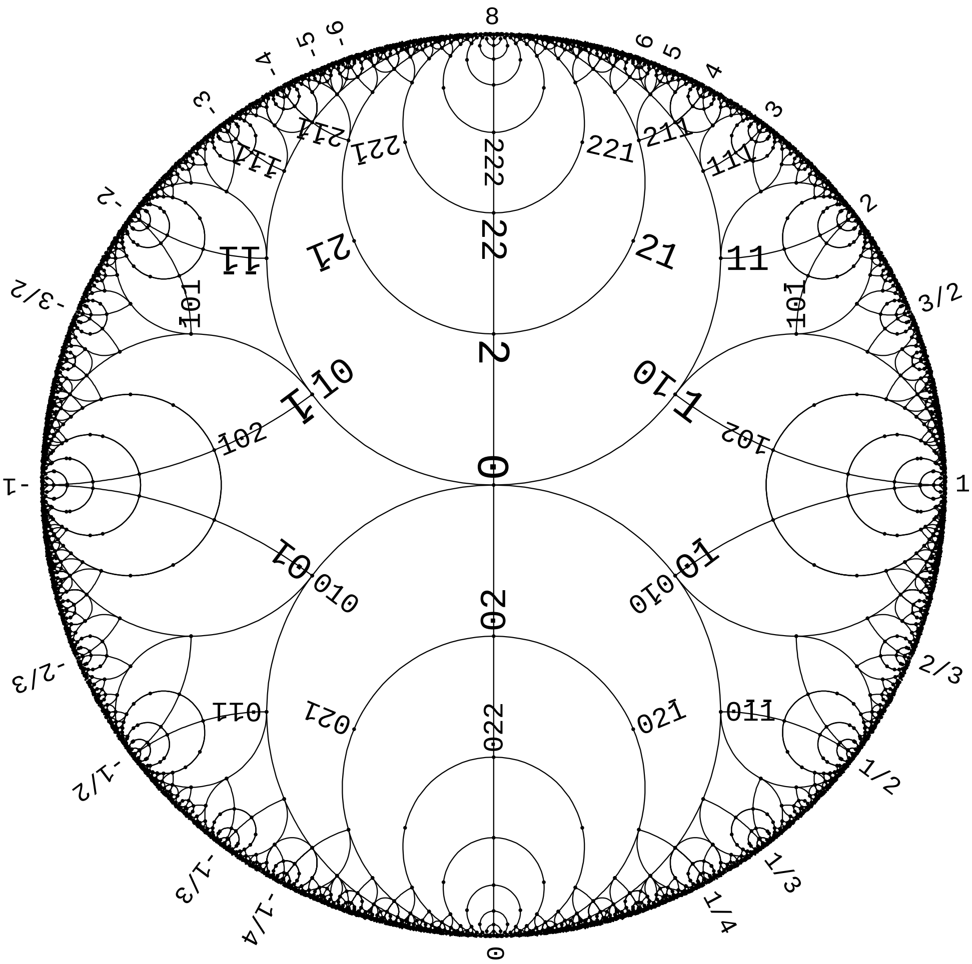


Figure 4. Means in the Binary continued fractions (BCF)

9. Binary continued fractions

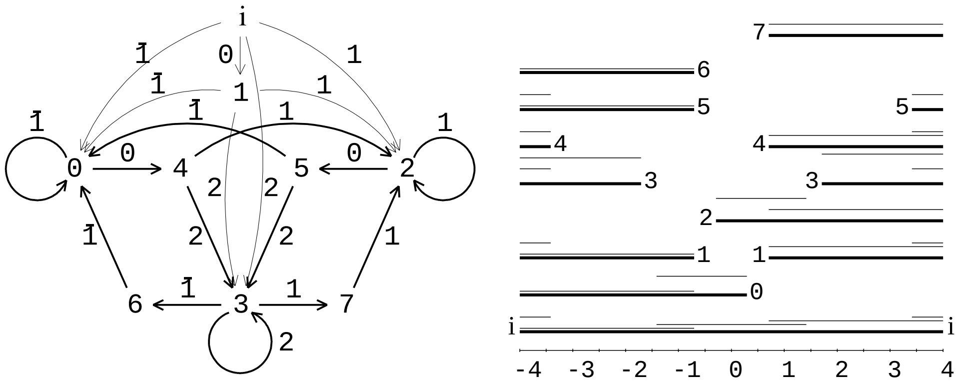
The convergence in regular continued fractions is quite slow, so we add the transformation $F_2(x) = 2x$ to make it faster. Several sofic subshifts can be constructed which yield a Möbius number systems with these transformations.

Example 3 *The* **Möbius system of binary continued fraction** *(BCF, Figure 4) consists of the alphabet* $A = \{\overline{1}, 0, 1, 2\}$*, transformations*

$$F_{\overline{1}}(x) = -1 + x,\ F_0(x) = -1/x,\ F_1(x) = 1 + x,\ F_2(x) = 2x,$$

and subshift Σ_D *with forbidden words* $D = \{00, 1\overline{1}, \overline{1}1, 101, \overline{1}0\overline{1}, \overline{1}2, 12, 20, 210, 2\overline{1}0\}$.

$t(u)$	$h(u)$	$\overline{\mathbf{U}_{h(u)}} = I_{t(u)}$	$F_{h(u)}(I_{t(u)})$	$s(u)$
0	$\overline{1}\overline{1}$	$[\infty, 1]$	$[\infty, -1]$	$0, 3, 5, 6$
2	11	$[-1, \infty]$	$[1, \infty]$	$2, 3, 4, 7$
3	2	$[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]$	$[\sqrt{2}, -\sqrt{2}]$	$3, 4, 5$
4	$\overline{1}\overline{1}0$	$[0, -1]$	$[\infty, -1]$	$0, 3, 6$
5	110	$[1, 0]$	$[1, \infty]$	$2, 3, 7$
6	$2\overline{1}$	$[2 + \frac{\sqrt{10}}{2}, 2 - \frac{\sqrt{10}}{2}]$	$[2 + \sqrt{10}, 2 - \sqrt{10}]$	$3, 4, 5$
7	21	$[-2 + \frac{\sqrt{10}}{2}, -2 - \frac{\sqrt{10}}{2}]$	$[-2 + \sqrt{10}, -2 - \sqrt{10}]$	$3, 4, 5$



$t(e)$	$h(e)$	L, R	$h(L^{t(e)})$	$h(R^{t(e)})$	$J_{t(e)}$	$F_{h(e)}(J_{t(e)})$
0	$\overline{1}$	$0, 4$	$.\overline{1}$	$.02\overline{1}\overline{1}$	$[\infty, 1 - \frac{\sqrt{2}}{2}]$	$[\infty, -\frac{\sqrt{2}}{2}]$
1	0	$2, 0$	1.0211	$\overline{1}.02\overline{1}\overline{1}$	$[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]$	$[-\sqrt{2}, \sqrt{2}]$
2	1	$5, 2$	$.0211$	$.1$	$[\frac{\sqrt{2}}{2} - 1, \infty]$	$[\frac{\sqrt{2}}{2}, \infty]$
3	2	$7, 6$	$.1102$	$.\overline{1}\overline{1}02$	$[\frac{\sqrt{2}}{2} + 1, -\frac{\sqrt{2}}{2} - 1]$	$[2 + \sqrt{2}, -2 - \sqrt{2}]$
4	0	$2, 3$	1.0211	$.2\overline{1}\overline{1}0$	$[\frac{\sqrt{2}}{2}, -2 - \sqrt{2}]$	$[-\sqrt{2}, 1 - \frac{\sqrt{2}}{2}]$
5	0	$3, 0$	$.2110$	$\overline{1}.02\overline{1}\overline{1}$	$[2 + \sqrt{2}, -\frac{\sqrt{2}}{2}]$	$[\frac{\sqrt{2}}{2} - 1, \sqrt{2}]$
6	$\overline{1}$	$0, 0$	$\overline{1}.\overline{1}$	$.\overline{1}02\overline{1}$	$[\infty, -\frac{\sqrt{2}}{2}]$	$[\infty, -\frac{\sqrt{2}}{2} - 1]$
7	1	$2, 2$	$.1021$	1.1	$[\frac{\sqrt{2}}{2}, \infty]$	$[\frac{\sqrt{2}}{2} + 1, \infty]$

Figure 5. Convergence and surjectivity in BCF

The convergence and surjectivity is shown in Figure 5. The graph G of a DFA for Σ_D is in Figure 5 center left. Note that the subgraph with vertices $\{0, 4, 5, 2\}$ defines the subshift of RCF. For the convergence of BCF we use the subgraph with vertices

$V_0 = \{0, 2, 3, 4, 5, 6, 7\}$ and the suffix code consisting of all paths with labels in the set $P = \{\overline{1}\overline{1}, 11, 2, \overline{1}\overline{1}0, 110, 2\overline{1}, 21\}$. The intervals $I_{t(u)} = \overline{\mathbf{U}_{h(u)}}$ satisfy the assumptions of the Convergence theorem 11 (Figure 5 top), so $\Sigma_D \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_D \to \overline{\mathbb{R}}$ is continuous. The selectors, their paths and cylinders of vertics are in Figure 5 bottom and in Figure 5 center right. The system is surjective and redundant. There exist smaller subshifts which avoid long chains of slowly converging parabolic transformations $F_{\overline{1}}$ and F_1. If 1^3 and $\overline{1}^3$ are added to D, the resulting system is surjective but no more redundant. If 1^4 and $\overline{1}^4$ are added to D, the resulting system is surjective and redundant. The BCF system can be regarded as an SFT approximation to $\Sigma_{\mathcal{W}}$ from Proposition 19.

Proposition 19 *Let (F, Σ_D) be the BCF system (Definition 3). Consider the open cover $\mathcal{W} = (W_a)_{a \in A}$, where $W_{\overline{1}} = (-c, -b)$, $W_0 = (-a, a)$ $W_1 = (b, c)$, $W_2 = (d, -d)$, and*

$$\tfrac{\sqrt{2}}{2} < b < 1,\; b < a < \min\{1, \tfrac{2b-1}{2-2b}\},\; 2a + 2 < d < \tfrac{1}{1-b},\; d < c < \min\{d + 1, \tfrac{1}{1-b}\}.$$

Then $\Sigma_{\mathcal{W}} \subset \Sigma_D$.

Note that if $c - n < b$, then 1^n and $\overline{1}^n$ are forbidden words in $\Sigma_{\mathcal{W}}$.

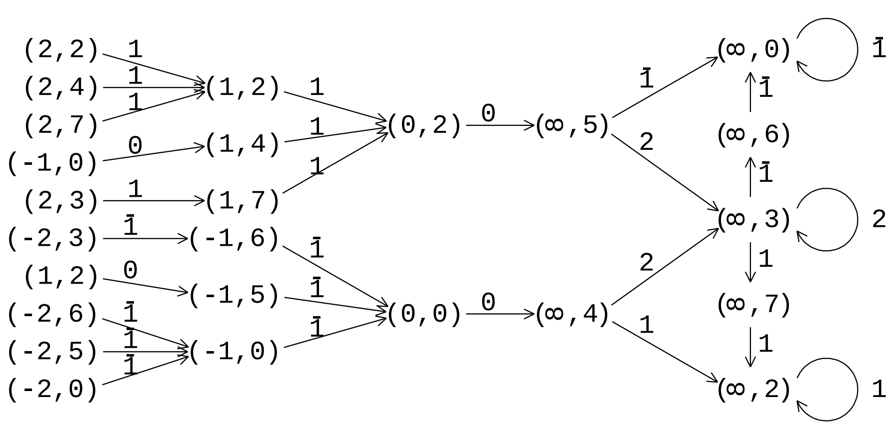


Figure 6. Number expansion graph of BCF

Theorem 20 *In BCF, each expansion of each rational number is preperiodic with period length 1 and has the form $u.a$, where $a \in \{\overline{1}, 1, 2\}$.*

Proof: We use the matrices of F_a^{-1} in the form

$$F_{\overline{1}}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\; F_0^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\; F_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\; F_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Consider the expansion graph with vertices (x_0, x_1, p), where $(x_0, x_1) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ are homogenous integer coordinates, and $x_0/x_1 \in J_p$. We have an edge $(x, p) \xrightarrow{a} (y, q)$ if $y = F_a^{-1} \cdot x$ (here x, y are viewed as column vectors). Define a Lyapunov function $f : \mathbb{Z}^2 \to \mathbb{N}$ by $f(x_0, x_1) := \max\{|x_0|, |x_1|\}$. We show that $f(y) \le f(x)$ whenever $(x, p) \xrightarrow{a} (y, q)$. If $a = 0$ then $f(y) = f(x)$. If $a \in \{\overline{1}, 1\}$, then $|x_0/x_1| \ge \sqrt{2}/2$ and we distinguish two cases. If $|x_0/x_1| \geq 1$, then $f(y) = \max\{|x_1|, |x_0| - |x_1|\} \leq |x_0| = f(x)$. If $\sqrt{2}/2 \leq |x_0/x_1| \leq 1$, then $f(y) = \max\{|x_1|, |x_1| - |x_0|\} \leq |x_1| = f(x)$. If $a = 2$ then $|x_0/x_2| \geq 2 + \sqrt{2}$ and $f(y) = f(x_0, 2x_1) = |x_0| = f(x)$. Since there is only a finite number of vertices with a bounded value of $f(x)$, each infinite path in the expansion graph contains a cycle, and the function $f(x)$ is eventually constant along the path. The cycle cannot contain 11 or $\overline{11}$: If $(x,p) \xrightarrow{1} (x-1, q) \xrightarrow{1} (x-2, r)$, then $x - 1 \geq \sqrt{2}/2$ and $f(x-1) < f(x)$. If the cycle contains $(x, 2) \xrightarrow{0} (-\frac{1}{x}, 5) \xrightarrow{\bar{1}} (\frac{x-1}{x}, 0) \xrightarrow{0} (\frac{x}{1-x}, 4) \xrightarrow{1} (\frac{2x-1}{1-x}, 2)$, then $x = (2x-1)/(1-x)$, so x is irrational. Thus the only possibilities for the cycle are $(\infty, p) \xrightarrow{a} (\infty, p)$ with $a \in \{\bar{1}, 1, 2\}$ and $p \in \{0, 2, 3\}$ (see Figure 6). $\square$

x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$
0/1	0	1/4	$02\overline{11}0$	1/3	$0\bar{1}^30$	2/5	$0\overline{11}0110$
1/2	$0\overline{11}0$	3/5	$0\bar{1}010\overline{11}0$	2/3	$0\bar{1}0110$	3/4	$0\bar{1}01^30$
1/1	10	4/3	$10\bar{1}^30$	3/2	$10\overline{11}0$	5/3	$10\bar{1}0110$
2/1	110	5/2	$110\overline{11}0$	3/1	1^30	4/1	2110

Figure 7. Expansions of Farey fractions in BCF according to Proposition 19 with $a = \frac{4}{5}$, $b = \frac{3}{4}$, $c = \frac{19}{5}$, $d = \frac{15}{4}$. Here $u = \mathcal{E}(x)$ stands for $u.2$

10. Compressed continued fractions

Another subshift which works for the transformations of the BCF system from Example 3 has forbidden words

$$D = \{00, 1\bar{1}, \bar{1}1, 20, 12, \bar{1}2, 11, \overline{11}, 101, \bar{1}0\bar{1}, 10\bar{1}, \bar{1}01, 1021, \bar{1}02\bar{1}\, 102\bar{1}, \bar{1}021\}.$$

In this subsfift, 1 and $\bar{1}$ are always followed by 022, so we can combine them into single digits 1022, $\bar{1}022$ which yield transformations $\pm 1 - 1/4x$. These two transformations are conjugated to simpler transformations $-1/(x \pm 2)$.

Example 4 *The* **Möbius system of compressed continued fractions** *(CCF, Figure 8) consists of the alphabet $A = \{\bar{1}, 0, 1, 2\}$, transformations*

$$F_{\bar{1}}(x) = 1/(-2-x),\ F_0(x) = x/2,\ F_1(x) = 1/(2-x),\ F_2(x) = 2x,$$

and the subshift Σ_D with forbidden words $D = \{\bar{1}2, 02, 12, 20\}$

Transformations $F_{\bar{1}}$ and F_1 are parabolic with fixed points -1 and 1 respectively. To prove the convergence, we use the prefix code $P := \{\bar{1}, 0, 1, 2222\}$ and intervals $I_2 = [\frac{1}{4}, -\frac{1}{4}]$, $I = I_{\bar{1}} = I_0 = I_1 = [-1, 1]$ (see Figure 9 top). In fact I is the attractor of the subsystem with alphabet $B := \{\bar{1}, 0, 1\}$. The surjectivity is shown in Figure 9 bottom. The convergence can be proved also by the continued fraction theory. For each $u \in \mathcal{L}(\Sigma_D)$, F_u can be written as

$$\begin{aligned} F_u(x) &= F_0^{n_0-1} F_{s_0} F_0^{n_1-2} F_{s_1} \cdots F_0^{n_k-2} F_{s_k}(x) \\ &= 2^{1-n_0}/(2s_0 - 2^{2-n_1}/(2s_1 - \cdots - 2^{2-n_k}/(2s_k - x)\cdots) \end{aligned}$$

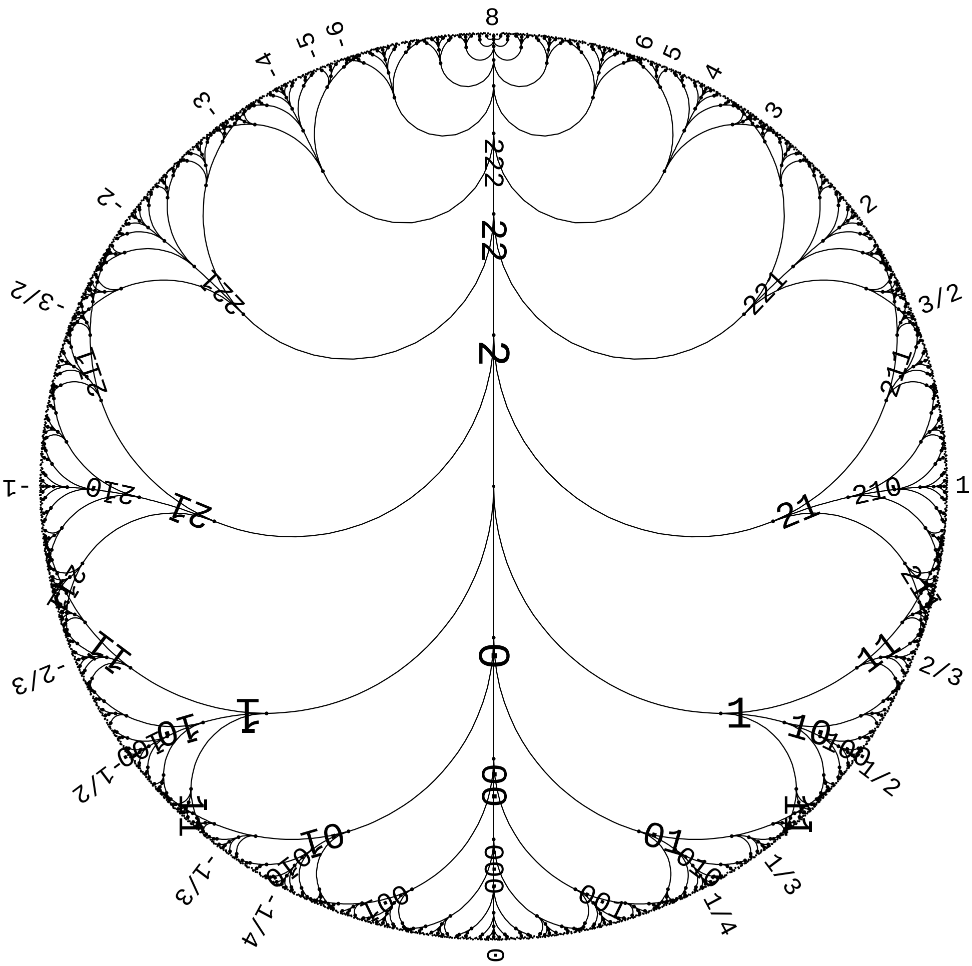


Figure 8. Means of Compressed continued fractions (CCF)

$$= \frac{s_0 2^{-n_0}|}{|1} - \frac{s_0 s_1 2^{-n_1}|}{|1} - \cdots - \frac{s_{k-1} s_k 2^{-n_k}|}{|1 - x s_k/2}$$
$$= \frac{1|}{|s_0 2^{m_0}} - \frac{1|}{|s_1 2^{m_1}} - \cdots - \frac{1|}{|s_k 2^{m_k} - 2^{m_k - 1} x}$$

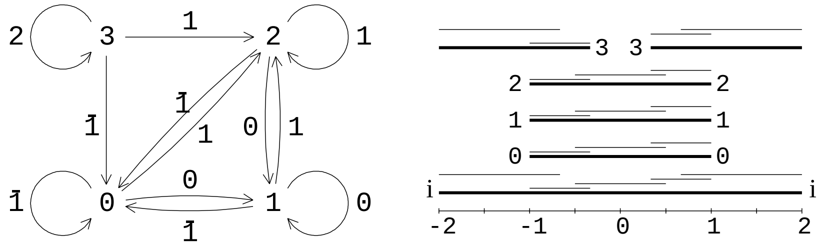
Here $s_i \in \{-1, 1\}$, $n_0 \in \mathbb{Z}$ and $n_i \geq 2$ for $i > 0$. The integers $m_i \in \mathbb{Z}$ are defined by $m_0 = n_0$, $m_{i+1} = n_{i+1} - m_i$, so $m_i + m_{i+1} \geq 2$. The partial numerators and denominators satisfy either $a_i = s_{i-1} s_i 2^{-n_i}$, $b_i = 1$, $|a_i| \leq \frac{1}{4}$, or $a_i = 1$, $b_i = s_i 2^{m_i}$, $|b_i b_{i+1}| \geq 4$. This class of continued fractions converges by a Theorem of Pringsheim (see Satz 27 in page 259 of Perron [10]).

Theorem 21 *In CCF, each expansion of each rational number is eventually periodic with period length* 1.

Proof: We use the matrices of F_a^{-1} in the standard form

$$F_{\overline{1}}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix},\ F_0^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ F_1^{-1} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix},\ F_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$t(u)$	$h(u)$	$\overline{\mathbf{U}}_{h(u)}$	$I_{t(u)}$	$F_{h(u)}(I_{t(u)})$	$s(e)$
0	$\overline{1}$	$[-1,\infty]$	$[-1,1]$	$[-1,-\frac{1}{3}]$	$0,1,3$
1	0	$[-\sqrt{2},\sqrt{2}]$	$[-1,1]$	$[-\frac{1}{2},\frac{1}{2}]$	$0,1,2$
2	1	$[\infty,1]$	$[-1,1]$	$[\frac{1}{3},1]$	$1,2,3$
3	2222	$[\frac{1}{4},-\frac{1}{4}]$	$[\frac{1}{4},-\frac{1}{4}]$	$[4,-4]$	3



$t(e)$	$h(e)$	L,R	$h(L^{t(e)})$	$h(R^{t(e)})$	J_q	$F_{h(e)}(J_{t(e)})$
0	$\overline{1}$	$0,2$	$.\overline{1}$	1.1	$[-1,1]$	$[-1,-\frac{1}{3}]$
1	0	$0,2$	$\overline{1}.\overline{1}$	1.1	$[-1,1]$	$[-\frac{1}{2},\frac{1}{2}]$
2	1	$0,2$	$\overline{1}.\overline{1}$	$.1$	$[-1,1]$	$[\frac{1}{3},1]$
3	2	$2,0$	$1\overline{1}.\overline{1}$	$\overline{1}1.1$	$[\frac{1}{3},-\frac{1}{3}]$	$[\frac{2}{3},-\frac{2}{3}]$

Figure 9. Convergence and surjectivity in CCF

Given a rational number p/q and $u \in \{\overline{1},0,1\}^+$, then $2^n u \in \Sigma_D$ is an expansion of p/q iff there exists a sequence $(p_i,q_i) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $(p_0,q_0) = (p, q\cdot 2^n)$, $(p_{i+1},q_{i+1}) = F_{u_i}^{-1} \cdot (p_i,q_i)$ and $|p_i| \le |q_i|$. We show that $|q_i|$ is a nonincreasing sequence of integers. If $u_i = 1$ or $u_i = \overline{1}$ and $|p_i| < |q_i|$, then $|q_{i+1}| = |p_i| < |q_i|$. If $u_i = 0$ then $q_{i+1} = q_i$, but the number of consecutive 0 is finite, unless $p_i = 0$. For each i there exists $j \ge i$ such that $|q_j| = 1$ and therefore $|p_i| \le 1$. The numbers -1, 0, 1 are fixed points of the expansion algorithm. □

The expansion graph for rational numbers can be seen in Figure 10. Here $p/q \xrightarrow{a} p'/q'$ means $(p',q') = F_a^{-1} \cdot (p,q)$. Note that for $u \in \{\overline{1},0,1\}^n$, the endpoints of $\Phi([u])$ (see Figure 11) all occur in the last n columns of the number expansion graph. In CCF, both F_1 and $F_{\overline{1}}$ are parabolic transformations, so the convergence of 1^∞ and $\overline{1}^\infty$ is quite slow (see Figure 8). If we forbid also 1111 and $\overline{1111}$, the resulting system is still surjective and redundant. It follows that in CCF, each rational number has an expansion of the form $u.0$.

There is also a family of subshifts of CCF which are based on the avoidance of the vicinities of 1 and -1, where the convergence of $F_{\overline{1}}$ and F_1 is slow. Let us replace the interval $I = [-1,1]$ by an interval $(-c,c)$, where $c \le 1$. Using the graph of a DFA in Figure 9, we get a system of intervals $\mathcal{W} = (W_a)_{a=\mathbf{i},0,1,2,3}$, where $W_{\mathbf{i}} = \overline{\mathbb{R}}$, $W_0 = W_1 = W_2 = (-c,c)$, $W_3 = (\frac{1}{c+2}, -\frac{1}{c+2})$. If $\frac{1}{c+2} < \frac{c}{2}$, i.e., if $c > \sqrt{3}-1 \doteq 0.732$, then $\mathcal{W}$ satisfies the assumptions of Theorem 16, and expansion algorithms can be based on it. For $c = (5-\sqrt{7})/3 \doteq 0.785$, which is the stable fixed point of F_{1110}, we get $\Sigma_{\mathcal{W}} = \Sigma_{\{\overline{1}2,02,12,20,1111,\overline{1111}\}}$. Another variant of the CCF system is in Example 5.

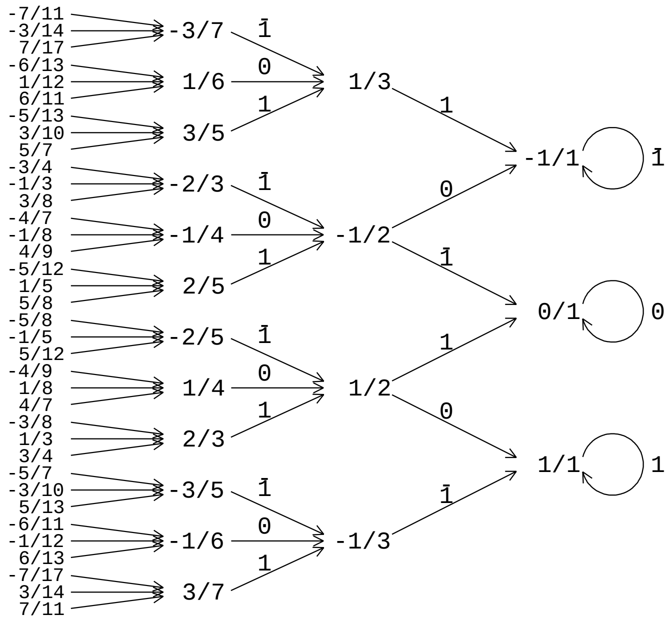
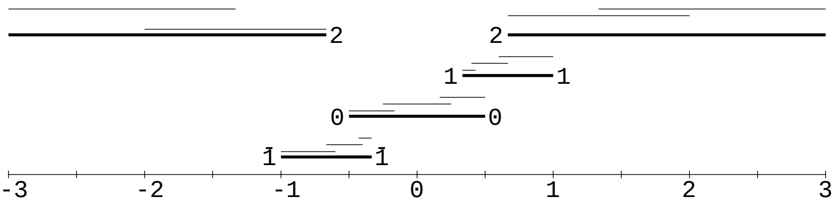


Figure 10. Rational number expansion graph in CCF



u	$\Phi([u])$	u	$\Phi([u])$	u	$\Phi([u])$	u	$\Phi([u])$
$\bar{1}$	$[-1,-\frac{1}{3}]$	0	$[-\frac{1}{2},\frac{1}{2}]$	1	$[\frac{1}{3},1]$	2	$[\frac{2}{3},-\frac{2}{3}]$
$\overline{11}$	$[-1,-\frac{3}{5}]$	$\bar{1}0$	$[-\frac{2}{3},-\frac{2}{5}]$	$\bar{1}1$	$[-\frac{3}{7},-\frac{1}{3}]$	$0\bar{1}$	$[-\frac{1}{2},-\frac{1}{6}]$
00	$[-\frac{1}{4},\frac{1}{4}]$	01	$[\frac{1}{6},\frac{1}{2}]$	$1\bar{1}$	$[\frac{1}{3},\frac{3}{7}]$	10	$[\frac{2}{5},\frac{2}{3}]$
11	$[\frac{3}{5},1]$	$2\bar{1}$	$[-2,-\frac{2}{3}]$	21	$[\frac{2}{3},2]$	22	$[\frac{4}{3},-\frac{4}{3}]$

Figure 11. Cylinders of words in CCF

x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$	x	$\mathcal{E}(x)$
$0/1$		$1/4$	01	$1/3$	011	$2/5$	$1\bar{1}$
$1/2$	1	$3/5$	1011	$2/3$	11	$3/4$	201^3
$1/1$	21	$4/3$	211	$3/2$	2201^3	$5/3$	$221\bar{1}1$
$2/1$	221	$5/2$	$2211\bar{1}$	$3/1$	$2^3 01^3$	$4/1$	$2^3 1$

Figure 12. Expansions of Farey fractions in CCF, $c = 3/4$. Here $u = \mathcal{E}(x)$ stands for $u.0$

Example 5 *The Möbius number system with alphabet* $A = \{\bar{1}, 0, 1, 2\}$*, transformations* $F_{\bar{1}}(x) = 1/(-2-x)$*,* $F_0(x) = x/2$*,* $F_1(x) = 1/(2-x)$*,* $F_2(x) = -1/2x$*, and the* *subshift Σ_D with forbidden words $D = \{\overline{1}2, 02, 12, 22\}$ is surjective and redundant.*

11. Conclusions

The algorithms for both BCF and CCF are efficient if the integers involved are represented in the integer binary signed system. A word $u \in \{\overline{1}, 0, 1\}^+$ represents the number $\sum_{k<|u|} 2^k u_k$. During the expansion and evaluation algorithms, the integer matrices F_u are updated by the right multiplication $F_{ua} := F_u \cdot F_a$. Due to the presence of column $(1, 0)$ or $(0, 1)$ in each F_a, two of the entries of F_u are kept and only moved. Thus these two systems offer a reasonable choice for the implementation of computer arithmetics.

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