Entropy of Turing machines with moving head

Petr Kůrka
Center for Theoretical Study,
Academy of Sciences and Charles University in Prague,
Jilská 1, CZ-11000 Praha 1, Czechia

In Kůrka [1] the last theorem says that a Turing machine with moving head (TMH) has the zero topological entropy, but the proof is not correct. I give here a correct proof. Given a Turing machine with alphabet A, set of inner states Q and the transition function $\delta = (\delta_Q, \delta_A, \delta_Z) : Q \times A \to Q \times A \times \{-1, 0, 1\}$, we have the sofic subshift

$$X_1 = \{x \in B^{\mathbb{Z}} : \ |\{i \in \mathbb{Z} : \ x_i \in Q\}| \leq 1\}$$

of all configurations which contain at most one head. The TMH determined by δ is a cellular automaton $F : X_1 \to X_1$ (see [1] for the definition of F).

Given an initial configuration $x \in A^{\mathbb{Z}}$ and an interval $I \subseteq \mathbb{Z}$ of width $|I| = d$, we consider rectangles $I \times [kd^2, (k+1)d^2)$ in the space-time diagram. Since there are at most d^2 heads in such a rectangle, one of the d columns of I contains at most d heads. We say that an I-cut is a collection of these columns, one for each rectangle $I \times [kd^2, (k+1)d^2)$. Formally an I-cut of x is a sequence of integers $(j_k \in I)_{k \geq 0}$ such that for each $k \geq 0$ we have $|\{i \in [kd^2, (k+1)d^2) : \ F^i(x)_{j_k} \in Q\}| \leq d$. Clearly, for each x and I, an I-cut exists. Given arbitrary finite interval $J \subseteq \mathbb{Z}$ we estimate the entropy of the column subshift $\Sigma_J = \{u \in (B^{|J|})^{\mathbb{N}} : \ \exists x \in X, \forall i, u_i = F^i(x)_J\} \subseteq (B^{|J|})^{\mathbb{N}}$. Given $d > 0$, let I_0, I_1 be the columns of width d which are left and right neighbours of J (and disjoint with J). A word of $\mathcal{L}(\Sigma_J)$ of length nd^2 obtained from $x \in X$ is determined by the top row $(x_J) \in B^{|J|}$ and by the values of cuts of length nd^2 through intervals I_0, I_1. Here we define the value of a cut j_k in the rectangle $I \times [kd^2, (k+1)d^2)$ as consisting of the value of the top row $F^{kd^2}(x)_I$ and of the values $\{F^i(x)_{j_k} : \ kd^2 \leq i < (k+1)d^2\}$ of the j_k-th column. The number of values of all possible cuts through $I \times [kd^2, (k+1)d^2)$ is bounded by the product of the number $|B|^d$ of values of the top row, number d of possible columns j_k, number $(d^2)^d$ of possible d-tuples $i \in [kd^2, (k+1)d^2)$ with $F^i(x)_{j_k} \in Q$, number $|Q|^d$ of d-tuples the states of the head at these positions, and the number $|A|^d$ of d-tuples of the scanned letters, i.e., by $|B|^d \cdot d^{2d+1} \cdot |Q|^d \cdot |A|^d$. For the number $P(nd^2)$ of words of $\mathcal{L}(\Sigma_J)$ of length nd^2 we get

$$\begin{aligned}
P(nd^2) &\leq |B|^{|J|} \cdot (|B|^d \cdot d^{2d+1} \cdot |Q|^d \cdot |A|^d)^{2n} \\
\frac{\ln P(nd^2)}{nd^2} &\leq \frac{|J| \cdot \ln |B|}{nd^2} + \frac{2\ln(|B| \cdot |Q| \cdot |A|)}{d} + \frac{2(2d+1)\ln d}{d^2} \\
H_{top}(\Sigma_J) &\leq \frac{2\ln(|B| \cdot |Q| \cdot |A|)}{d} + \frac{2(2d+1)\ln d}{d^2}
\end{aligned}$$

Since this holds for arbitrary $d > 0$, we get $H_{top}(\Sigma_J) = 0$ for each finite interval $J \subseteq \mathbb{Z}$ and it follows that the topological entropy of TMH $F : X_1 \to X_1$ is zero.

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References

[1] Petr Kůrka: On topological dynamics of Turing machines. Theoretical Computer Science 174 (1997) 203-216