PREFACE

Mathematical theory of dynamical systems originated in Newton mechanics. In Newton theory of planetary motions, the sun and the planets are regarded as mass points - particles with mass but no volume. They are mutually attracted by force inversely proportional to their distance and move according to the Newton law of motion $F = ma$.

Newton mechanics is deterministic. The positions and momenta of all particles at present time determine uniquely their positions and momenta at arbitrary later time and can be obtained by solving a system of differential equations. The problem to find this solution, to express the positions and momenta as functions of time, is the famous n-body problem. If $n = 2$, the two bodies move around their common center of gravity and their relative motion satisfies the Kepler laws. For $n \geq 3$ no simple solution has been found and, starting with Poincaré [**128**], the problem has been studied with geometrical and topological methods.

The key concept in the geometric approach is that of the phase space or **state space**. The state of the system of n particles consists of all their positions and momenta. Since both the position and momentum of a particle is represented by a three-dimensional vector, we need $6n$ real numbers to specify a state of n particles. The state of the system is therefore regarded as a point in an abstract $6n$-dimensional space. The change of state is conceived as movement in the state space and the curve in the state space which represents this movement is called **trajectory**. There exists a function F which assigns to a state x of the system at time 0 the state $F(x,t)$ at time t. Such a function is called a **flow**.

Topological dynamics studies geometrical and topological properties of trajectories. For example, in the movement of planets there are many periodicities. In planetary system the mass of any planet is significantly smaller than that of the sun, and the dominant force which acts on a planet is the attraction to the sun, while attracting forces to other planets are much smaller. The course of a planet around the sun is therefore very close to the Keplerian elliptic motion. After one orbit, the position and momentum of a planet is very close to its initial state and momentum. If the ratio of orbit times of two planets is close to a rational number $\frac{p}{q}$, then one planet makes p orbits in nearly the same time as the other planet makes q orbits, and the state of the system goes very closely to the initial state. The same phenomenon can be observed in whole planetary system. A trajectory repeatedly visits any arbitrarily small neighbourhood of the initial state. Moreover, the time gaps between successive visits are nearly constant or at least bounded (Figure 1 top left). Trajectories with this property are called **almost periodic** and systems whose any trajectory is almost periodic are called **minimal** (Birkhoff [**18, 19**]).

A quite different behaviour exhibit thermodynamic systems. The state of a thermodynamic system is a vector of macroscopic variables like temperature, volume, pressure or concentrations of constituent components. The state is again conceived as a point in a multidimensional state space. An isolated thermodynamic system tends to equilibrium. This is a particular state which is approached by all trajectories. We say that the equilibrium is an **attracting stationary point** (Figure 1 top right).

In a generalized thermodynamics of open systems studied by Prigogine and his school [**132**], the dynamics might be much more complicated than a simple approach to equilibrium. An open thermodynamic system exchanges with its environment both energy and matter and these flows are either constant or in some way controlled. An open thermodynamic system may approach a **limit cycle** in which the state variables oscillate periodically (Figure 1 bottom left). This is the case of Belousov-Zhabotinsky reaction or its mathematical model, the Brusellator.

An attracting fixed point or a limit cycle are examples of an **attractor**. An attractor is a subset of the state space, such that all trajectories which start in its neighbourhood tend to it. An attractor itself is a dynamical system, a **subsystem** of the original system. A dynamical system might have several attractors each with its own **basin of attraction**. In contrast, minimal systems do not have attractors nor subsystems other than themselves.

In even more complex thermodynamic systems, we observe **chaotic attractors** with irregular dynamics. An example is the Lorenz attractor [**101**] developed as a meteorological model (Figure 1 bottom right). Like minimal systems, chaotic systems have trajectories which repeatedly visit any region of the state space. In contrast to minimal systems, however, the time gaps between successive visits are irregular and not bounded. Moreover, not all trajectories have this property. There are many periodic trajectories which visit only small regions of the state space. Another significant property of chaotic systems is their **sensitivity** to initial conditions. If two trajectories start in close initial states, their distance slowly grows and finally the trajectories separate and become independent.

Besides classical mechanics and thermodynamics, dynamical systems have been used in many other sciences, for example in populational biology, economics or sociology. Not all these models are formulated as systems of differential equations. There are also dynamical systems with discrete time, where the time variable takes only integer values. This is natural in description of seasonal phenomena, e.g., numbers of migrating animals which come only for a short season in a year. The essential feature of deterministic dynamics (as opposed to probabilistic dynamics studied in stochastic processes) is that the state at present time determines the state at any future time. Thus there exists a function $F : X \to X$ which assigns to the present state $x \in X$ the state $F(x) \in X$ at time 1. The state at time 2 is then $F(F(x)) = F^2(x)$. Function

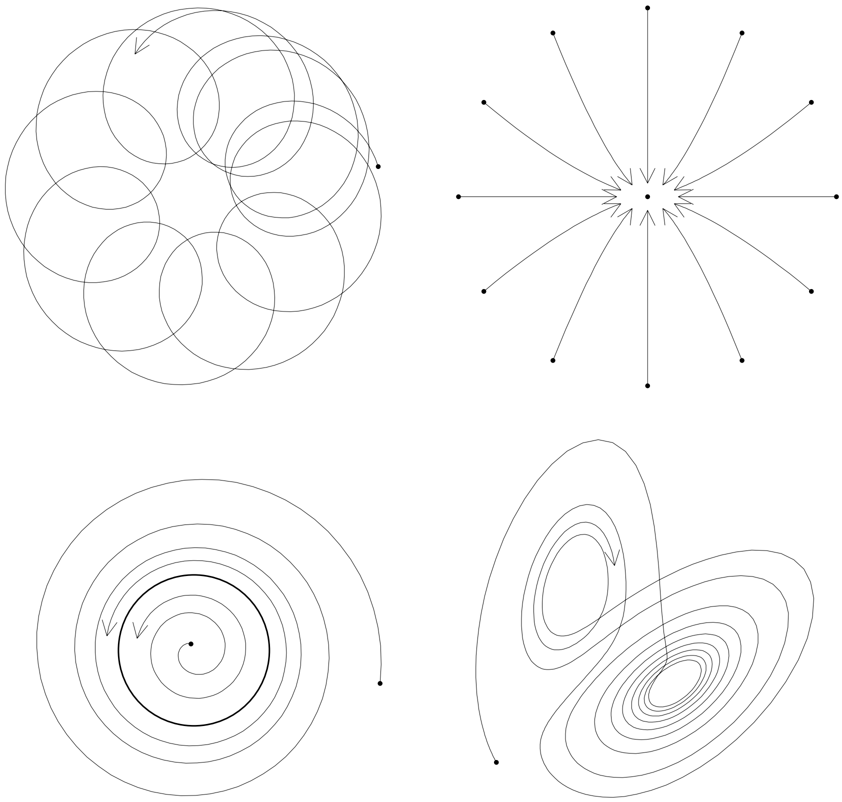

FIGURE 1. Minimal system, attracting state, limit cycle and chaotic system.

$F^2 = F \circ F$ is called the second iteration of F and in general, the n**-th iteration** $F^n : X \to X$ is the composition of F with itself n times. A **trajectory** of a discrete time system is a sequence of states $x_n \in X$ given by the recursive formula

$$x_{n+1} = F(x_n), \quad \text{or} \quad x_n = F^n(x_0).$$

Some systems are deterministic not only with respect to the future but also with respect to the past. From their present state we can determine their state at arbitrary past time. In this case the function F is bijective, and the state at time -1 is $F^{-1}(x)$.

There are many similarities between discrete time and continuous time dynamical systems. The concepts of almost periodicity, minimality, chaoticity or attractors can be studied in both classes by similar methods. However, discrete time systems are conceptually simpler and they can be more easily visualized, especially if the state space is only one-dimensional.

For this reason we start in Chapter 1 with a particular one-dimensional discrete time dynamical system called logistic or **quadratic system**, which originated in population biology. The system is given by a very simple quadratic equation. Despite the simplicity of its definition, it displays very wide range of different dynamic behaviours ranging from dynamics dominated by a single attracting periodic orbit to chaotic dynamics. Between these two cases there is the Feigenbaum system characterized by the presence of a minimal subsystem which attracts nearly all orbits. We show how these systems can be understood via symbolic systems, the binary shift in the case of chaotic system and adding machine in the case of Feigenbaum system. Then we treat rotations of the circle as examples of minimal systems and a nice application of this theory in spiral phyllotaxis.

In Chapter 2 we present topological dynamics based on **dynamical relations** of Akin [**3**]. Dynamical system is defined as a pair (X, F), where X is a compact metric space and $F : X \to X$ is a continuous map. The abstract setting of compact metric spaces leads to an elegant theory. The key assumption of compactness is necessary to get interesting theorems. Dynamical relations are natural starting point for the study of minimal, transitive and chain-transitive systems. The concepts of almost periodic, recurrent, nonwandering, and chain-recurrent point come also very naturally in this setting. Next come the related concepts of equicontinuity and sensitivity, and structural theorems which show their connections to minimality. Minimal equicontinuous systems are studied with the help of spectral theory. We consider the space $\mathbf{C}(X, \mathbb{C})$ of complex-valued functions defined on the state space X. A dynamical system induces a linear transformation on $\mathbf{C}(X, \mathbb{C})$ and the eigenvalues of this transformation characterize the dynamical system completely.

Then we study attractors and their basins. The theory of attractors relies on the concept of chain relation. An attractor is completely determined by the chain-recurrent points which it contains. The order on the set of chain-recurrent points yields an order on the family of attractors. Finally we study dynamical systems from the point of view of information sources. The amount of information which a dynamical system generates per step is called topological entropy.

In Chapter 3 we treat the powerful method of **symbolic dynamics** due to Morse and Hedlund [**111, 112**]. The state of a symbolic dynamical system is an infinite sequence of symbols and the dynamics is defined by combinatorial operations with these symbols. The state space of a symbolic dynamical system can be represented as a subspace of the Cantor middle thirds set. Despite the simplicity of their spaces, symbolic dynamical systems illustrate a wide range of behaviour and are in some sense universal compact dynamical systems. Any dynamical system can be obtained as a factor of a symbolic dynamical system. This means that a symbolic system provides a tractable model for the system in question. Any trajectory of an original system is represented by a trajectory or several trajectories in the symbolic system.

A special class of symbolic dynamical systems are subshifts - subsystems of a particularly simple shift dynamical system. A subshift can be specified combinatorially by its language. The simplest class of subshifts are subshifts of finite type and sofic subshifts. They are usually chaotic and can be described using the graph theory and the theory of finite automata. Their topological entropy can be evaluated using the Perron-Frobenius theory of nonnegative matrices.

In Chapter 4 we study minimal symbolic systems. Equicontinuous minimal systems are well understood. They are adding machines, or rotations on a p-adic group. In contrast, minimal subshifts come in many different types and no general classification scheme is available. We consider three special classes of minimal subshifts. The structure of substitutive subshifts is quite regular. They have remarkable dynamical properties, although these properties vary little. Sturmian subshifts are symbolic extensions of rotations of the circle. They are associated to irrational numbers $\alpha \in [0,1)$ via their continued fraction expansion. Toeplitz subshifts are associated to quasi-periodic symbolic sequences which return periodically into any neighbourhood of the initial state. Toeplitz subshifts themselves come in many different types and provide many counterexamples which illustrate relations between dynamical concepts.

In Chapter 5 we study cellular automata as a particular class of symbolic dynamical systems. Cellular automata developed independently of the mainstream of dynamical systems. They originated in computer science as models of self-reproduction and universal computation and their properties in many respects differ from those encountered in differential dynamics. However, their study by methods of topological dynamics is very fruitful and brings some very interesting and unusual examples into this theory.

The Appendix gives some mathematical background, in particular the basic theory of compact metric spaces. While we restrict ourselves to dynamical systems on compact metric spaces, occasionally we have to resort to the larger categories of compact Hausdorff spaces or uniform spaces. A special topic is the Perron-Frobenius spectral theory of nonnegative matrices. We need it for the computation of topological entropy of sofic systems and also for the computations of frequencies of words in substitutive (minimal) subshifts.

The book assumes basic mathematical knowledge at the level of elementary courses of calculus and linear algebra. However, from chapter two onwards, it requires a certain level of abstract mathematical thinking and some prior knowledge of the theory of metric spaces would be helpful. I hope that the book can be used as a textbook on both pregraduate and graduate level. In fact, I have used the material collected in this book in a series of courses given at Charles University in Prague and also in courses at Universidad de Chile in Santiago de Chile and Universidad Católica del Norte in Antofagasta. I thank all my students for patience and valuable comments.

During past ten years I profited from fruitful conversations and cooperations with many colleagues and friends. These include in particular Pierre Arnoux, Bohuslav Balcar, François Blanchard, Mike Boyle, Tomasz Downarowicz, Enrico Formenti, Elie Glasner, Bernard Host, late Anzelm Iwanik, Alejandro Maass, Pierre Tisseur, Sandro Vaienti and Benjamin Weiss. I should like to thank them all for inspiring discussions and encouragement.

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Petr Kůrka

CHAPTER 1

DYNAMICAL SYSTEMS

In dynamics we study how things change with time. Topological dynamics deals with deterministic systems. We suppose that at any time, the system in question can be characterized by its state, usually a vector of real numbers. If we know the state of the system at present time, we can determine its state at arbitrary later time, sometimes even at arbitrary past time. Most examples of dynamical systems are obtained from differential equations or systems of differential equations.

1.1. Differential equations

An autonomous differential equation of first order is an equation of the form

$$f'(t) = \varphi(f(t)), \quad f(0) = x.$$

Here $f(t)$ is the unknown function, $\varphi : \mathbb{R} \to \mathbb{R}$ is a given real function and $x \in \mathbb{R}$ is an initial condition, the state of the system at time 0. If $\varphi(x) = 0$, there is a constant solution $f(t) = x$. We say that x is a **stationary point**. If $\varphi(x) \neq 0$ and if φ is continuous, then it is nonzero in an open interval containing x. Using the substitution $y = f(t)$ we can integrate this equation.

$$t = \int_0^t dt = \int_0^t \frac{f'(t)\,dt}{\varphi(f(t))} = \int_{f(0)}^{f(t)} \frac{dy}{\varphi(y)} = [G(y)]_x^{f(t)} = G(f(t)) - G(x)$$

Here $G(y)$ is a primitive function to $1/\varphi(y)$, i.e., $G'(y) = 1/\varphi(y)$. Since φ does not change sign on the interval considered, G is monotone (increasing or decreasing), and has an inverse. We get a solution

$$f(t) = G^{-1}(G(x) + t)$$

defined on an interval (t_1, t_2) which contains 0. We say that a differential equation defines a dynamical system, if this solution is defined for all positive t and if it is unique.

Example 1.1 **(Exponential growth)**. —

$$f'(t) = af(t), \quad f(0) = x$$

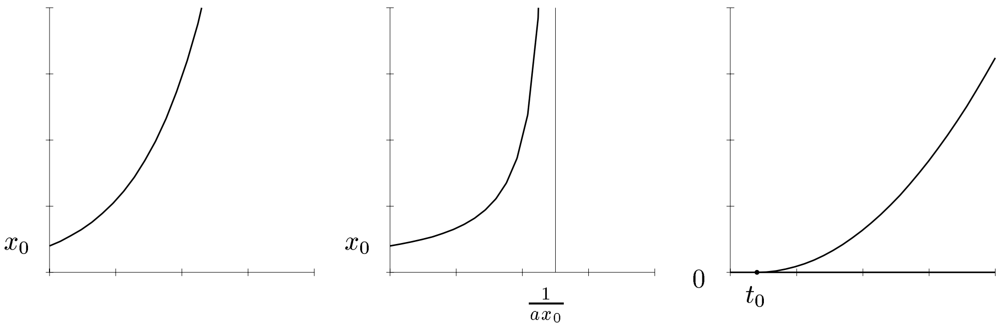

FIGURE 1. Exponential, hyperbolic and quadratic growths.

This is a model of growth of a population with constant growth a. There is a unique solution

$$f(t) = xe^{at}, \quad t \in (-\infty, +\infty)$$

defined for all times t.

Example 1.2 **(Hyperbolic growth).** —

$$f'(t) = af^2(t), \quad f(0) = x.$$

There is a unique solution

$$f(t) = \frac{1}{\frac{1}{x} - at}.$$

If $x > 0$, the solution is defined in the interval $(-\infty, \frac{1}{ax})$ and tends to infinity as t tends to $\frac{1}{ax}$. This phenomenon is called overflow. The solution cannot be continued beyond a certain time.

Example 1.3 **(Quadratic growth).** —

$$f'(t) = a\sqrt{f(t)}, \quad f(0) = x \geq 0.$$

For $x > 0$ there exists a unique solution $f(t) = (\frac{at}{2} - \sqrt{x})^2$. For $x = 0$ there exists an infinite number of solutions. Any $t_0 \geq 0$ yields a solution

$$f(t) = \begin{cases} 0 & \text{for } t \leq t_0, \\ \frac{a^2(t-t_0)^2}{4} & \text{for } t \geq t_0. \end{cases}$$

1.2. Flows and iterations

Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is a real function, such that the differential equation

$$f'(t) = \varphi(f(t)), \quad f(0) = x$$

has for any $x \in \mathbb{R}$ a unique solution defined for all $t \in \mathbb{R}$. Let $F(x,t)$ be the solution with initial condition x, so

$$\frac{\partial F(x,t)}{\partial t} = \varphi(F(x,t)), \quad F(x,0) = x.$$

For example, for the exponential growth model, $F(x,t) = xe^{at}$. Consider now, for some fixed x, t, functions

$$F_1(s) = F(F(x,t),s), \quad F_2(s) = F(x,t+s).$$

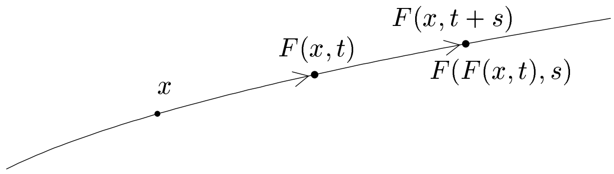

FIGURE 2. A flow

Both F_1 and F_2 are solutions with initial condition $F_1(0) = F_2(0) = F(x,t)$, so they must be equal (Figure 2). This equality can be expressed in a more succinct form. For a real number t, denote by $F^t : \mathbb{R} \to \mathbb{R}$ the mapping defined by $F^t(x) = F(x,t)$. F^t is a transformation of $\mathbb{R}$ which describes how the states of the system change in time t. If x is the state at time 0, $F^t(x)$ is the state at time t. We get

$$F^0(x) = x, \quad F^s(F^t(x)) = F^{s+t}(x),$$

or

$$F^0 = \mathrm{Id}, \quad F^s \circ F^t = F^{s+t}.$$

A function of two variables satisfying these two properties is called a **flow**. Not every flow is obtained from a differential equation. For example, any bijective continuous function $G : \mathbb{R} \to \mathbb{R}$ (not necessarily differentiable) yields a flow $(F^t : \mathbb{R} \to \mathbb{R})_{t\in\mathbb{R}}$ defined by

$$F^t(x) = G^{-1}(G(x) + t).$$

In addition to these continuous time flows there are also flows with discrete time which originate if we observe the system only at discrete time instants indexed by integers. If the system is deterministic, its state x_n at time n uniquely determines its state x_{n+1} at time $n+1$, so there exists a mapping F, such that $x_{n+1} = F(x_n)$. The sequence $(x_n)_{n\geq 0}$ is called the **trajectory** of x_0. We can write $x_n = F^n(x_0)$, where F^n is the composition of F with itself n times. It is called the n**-th iteration** of F.

$$F^n = \overbrace{F \circ \cdots \circ F}^{n}, \quad F^0(x) = x, \quad F^{n+1}(x) = F^n(F(x))$$

For the system $(F^n)_{n\geq 0}$ of iterations of a function F we get similar identities as for continuous time flows:

$$F^0 = \mathrm{Id}, \quad F^{n+m} = F^n \circ F^m$$

These two cases can be generalized to flows indexed by time monoids $T \subseteq \mathbb{R}$ acting on subspaces $X \subseteq \mathbb{R}^p$. Points of X are called **states**.

Definition 1.4. — A **flow** is a triple (X, T, F), where
(1) $X \subseteq \mathbb{R}^p$ is a set of states,
(2) $T \subseteq \mathbb{R}$ is a set of times, such that $0 \in T$ and for any $s,t \in T$, $s+t \in T$,

(3) $F : X \times T \to X$ is a continuous function satisfying

$$F^0 = \mathrm{Id}, \quad F^s \circ F^t = F^{s+t}$$

If $T = [0, +\infty)$, we get a flow with continuous time, which is deterministic with respect to the future. If $T = (-\infty, +\infty)$, the flow is deterministic with respect to both the past and the future. In this case, each F^t is bijective, since $F^t \circ F^{-t} = \mathrm{Id}$. If $T = \mathbb{N} = \{0, 1, 2, \ldots\}$, we get a flow with discrete time which is deterministic with respect to the future. The flow is given by a continuous function $F^1 : X \to X$, since F^n is its n-th iteration. If $T = \mathbb{Z}$, the flow is deterministic with respect to both the past and the future. It is again given by F^1, but F^1 has to be a homeomorphism, i.e., it should have a continuous inverse F^{-1}.

From now on we work with discrete time flows which we call dynamical systems. Their state space is usually a subset of an Euclidean space or, more generally, a metric space. This is a structure, in which the **distance** $d(x, y) \geq 0$ of any pair of points $x, y \in X$ is defined (see Section A.2).

Definition 1.5. — A **dynamical system** is a pair (X, F), where X is a metric space called the **state space** and $F : X \to X$ is a continuous mapping.

1.3. Fixed points and linearization

Example 1.6 **(Linear system)**. — $(\mathbb{R}, L_a)$ where $a \in \mathbb{R}$ and $L_a(x) = ax$.

Each $a \in \mathbb{R}$ yields a different linear system, so $(\mathbb{R}, L_a)_{a \in \mathbb{R}}$ is a family of dynamical systems parametrized by real numbers a. Iterations of a linear system can be written explicitly as $L_a^n(x) = a^n x$, so the trajectories are geometric sequences. If $|a| > 1$ and $x \neq 0$, then $L_a^n(x)$ diverges to $\pm\infty$. If $|a| < 1$, then any trajectory converges to zero. We say that zero is a **fixed point**, i.e., $L_a(0) = 0$.

If $|a| \leq 1$, then 0 is **stable**. If the initial state is close to zero, whole trajectory stays close to zero. If $|a|$ is even strictly smaller than 1, any trajectory approaches zero, and we say that 0 is an **attracting** fixed point. These concepts can be defined in general. We say that a fixed point is stable, if the trajectories do not move away from it. Given any $\varepsilon > 0$, the trajectories of points sufficiently close to the fixed point do not leave the ball of radius ε around the fixed point. If in addition, they approach the fixed point, we say that the fixed point is attracting.

Denote by $d(x, y)$ the distance of points x and y. The **open ball** with center x and radius $\delta > 0$ is the set of points whose distance from x is smaller than δ

$$B_\delta(x) = \{y \in X : d(y, x) < \delta\}.$$

Definition 1.7. — Let (X, F) be a dynamical system.

(1) A point $x \in X$ is a **fixed point**, if $F(x) = x$.

(2) A fixed point $x \in X$ is **stable** (Figure 3 left), if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall n \geq 0, d(F^n(y), x) < \varepsilon.$$

(3) A fixed point $x \in X$ is **attracting** (Figure 3 right), if it is stable and moreover

$$\exists \delta > 0, \forall y \in B_\delta(x), \lim_{n \to \infty} d(F^n(y), x) = 0.$$

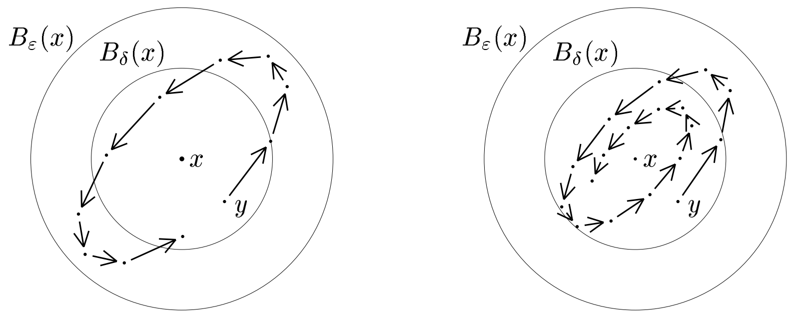

FIGURE 3. Stable and attracting fixed points

For any $a \in \mathbb{R}$, 0 is a fixed point of $(\mathbb{R}, L_a)$, and if $a \neq 1$, it is the only fixed point. Its stability depends on a.

$$|a| < 1 \quad \Rightarrow \quad 0 \text{ is attracting}$$
$$|a| = 1 \quad \Rightarrow \quad 0 \text{ is stable but not attracting}$$
$$|a| > 1 \quad \Rightarrow \quad 0 \text{ is not stable}$$

The trajectories of a dynamical system on the real line can be visualized in a simple geometric manner (Figure 4). From a point x_0 on the x axis we draw a vertical line till it intersects the graph of the function in a point with coordinates (x_0, x_1). From this point we draw a horizontal line till it intersects the diagonal in the point (x_1, x_1). We continue vertically to (x_1, x_2) on the graph and horizontally to (x_2, x_2) on the diagonal, etc.

The fixed points of $(\mathbb{R}, F)$ are intersections of the graph of $y = F(x)$ with the diagonal $y = x$. If we replace the function F in a neighbourhood of its fixed point x by its tangent, and if the slope of this tangent is neither 1 nor -1, the stability properties of the fixed point do not change.

***Theorem 1.8* (Differential criterion of stability).** — *Let $F : \mathbb{R} \to \mathbb{R}$ be a real function which has a derivative in a fixed point $x = F(x)$.*
(1) If $|F'(x)| < 1$ then x is attracting.
(2) If $|F'(x)| > 1$ then x is not stable.

Proof. —
(1) Suppose that $|F'(x)| < 1$ and pick some r with $|F'(x)| < r < 1$. By the definition of derivative as a limit, there exists $\delta > 0$, such that for any $y \in \mathbb{R}$,

$$0 \neq |y - x| < \delta \;\Rightarrow\; \left|\frac{F(y) - x}{y - x}\right| < r \;\Rightarrow\; |F(y) - x| < r|y - x| < \delta.$$

By induction we obtain $|F^n(y) - x| < r^n\delta < \delta$, so x is stable and attracting.

(2) Suppose that $|F'(x)| > 1$ and pick some r with $|F'(x)| > r > 1$. There exists $\delta > 0$ such that for any y, $0 \neq |y - x| < \delta \Rightarrow |F(y) - x| > r|y - x|$. Thus for some n, $|F^n(y) - x| \geq \delta$, so x is not stable. □

The fixed point x in Figure 4 top left is attracting since $0 < F'(x) < 1$. The fixed point in Figure 4 top right is not stable since $F'(x) < -1$. If the derivative of F at

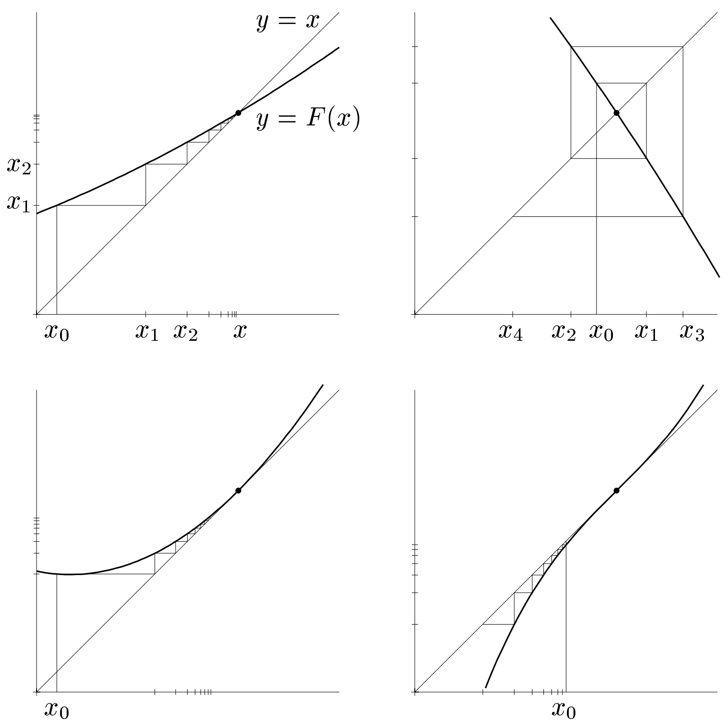

FIGURE 4. Differential criterion for stability

a fixed point is exactly 1, nothing can be deduced about its stability. If $F''(x) > 0$, x attracts trajectories which start at $x_0 < x$ but repels trajectories which start at $x_0 > x$, so x is not stable (Figure 4 bottom left). If $F''(x) = 0$ and $F'''(x) < 0$, then x is attracting, but if $F'''(x) > 0$, it is not stable (Figure 4 bottom right).

Linear systems are also considered in the n-dimensional Euclidean space $\mathbb{R}^n$. They have form $(\mathbb{R}^n, L_A)$ where $(A_{ij})_{i,j<n}$ is a square $(n \times n)$-matrix and $L_A(x) = A \cdot x$ is the multiplication of A with vector $x = (x_i)_{i<n} \in \mathbb{R}$ given by

$$L_A(x)_i = \sum_{j=0}^{n-1} A_{ij} \cdot x_j.$$

For any linear system, $0 = (0, \ldots, 0)$ is a fixed point. Its stability depends on the eigenvalues of A (see Section A.9). If the absolute values of all eigenvalues are less than 1, then 0 is attracting. If the absolute value of at least one eigenvalue is larger than 1, then 0 is not stable. The differential criterion of stability (Theorem 1.8) generalizes to n-dimensional dynamical systems $(\mathbb{R}^n, F)$. Let x be a fixed point of F and assume that F is differentiable at x with Jacobian matrix

$$A_{ij} = \frac{\partial F_i(x)}{\partial x_j}.$$

If all eigenvalues of A have absolute value less than 1, then x is an attracting fixed point. If at least one eigenvalue of A has absolute value larger than 1, then x is not stable (see Palis and de Melo [**119**]).

1.4. Quadratic system

Many dynamical systems are obtained as approximations of differential equations. If $f(t)$ is a solution of a differential equation $f'(t) = \varphi(f(t))$, then for a time interval of small length ε,

$$f(t+\varepsilon) \approx f(t) + \varepsilon\varphi(f(t)).$$

If we define a function F by $F(x) = x + \varepsilon\varphi(x)$, we get $f(n\varepsilon) \approx F^n(f(0))$ for small ε and small n. For some differential equations this approximation holds globally for all n. More often, however, the approximation holds only for small n, and its qualitative behaviour may be quite different. An interesting simple example with quite complicated behaviour comes from population biology. It has been studied by Ulam and von Neumann [**153**] and widely popularized by May [**107**]. For a detailed treatment see Collet and Eckmann [**37**].

1.4.1. Logistic growth. — Consider a population, with rate of growth a, confined to a territory with capacity K. If the population is much smaller than K, it grows exponentially with rate a. If its size approaches K, its growth rate decreases. The simplest model of this kind is the logistic equation.

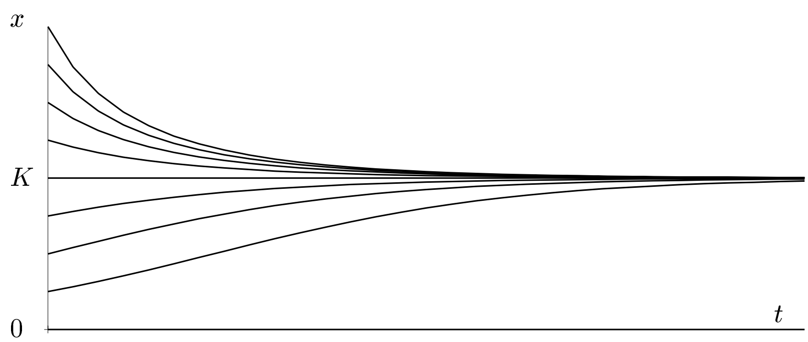

FIGURE 5. Logistic growth

***Example 1.9* (Logistic growth).** —

$$f'(t) = af(t)\left(1 - \frac{f(t)}{K}\right), \quad f(0) = x.$$

The solution is **logistic flow** (Figure 5)

$$G(x,t) = \frac{x \cdot e^{at}}{1 + (e^{at} - 1)\frac{x}{K}}$$

defined for $x, t \geq 0$. There are two stationary points: 0 which is not stable and K which is attracting. From any positive state $x > 0$, the population always attains the capacity K, i.e., $\lim_{t\to\infty} G(x,t) = K$. Discrete time approximation system is given by

$$F(x) = x + \varepsilon a x\left(1 - \frac{x}{K}\right) = rx(1 - cx)$$

where $r = 1 + \varepsilon a$ and $c = \frac{\varepsilon a}{(1+\varepsilon a)K}$. If we multiply both sides of the equation $x_{n+1} = rx_n(1 - cx_n)$ by c, and if we make a linear transformation $cx_n = y_n$, we get $y_{n+1} =$

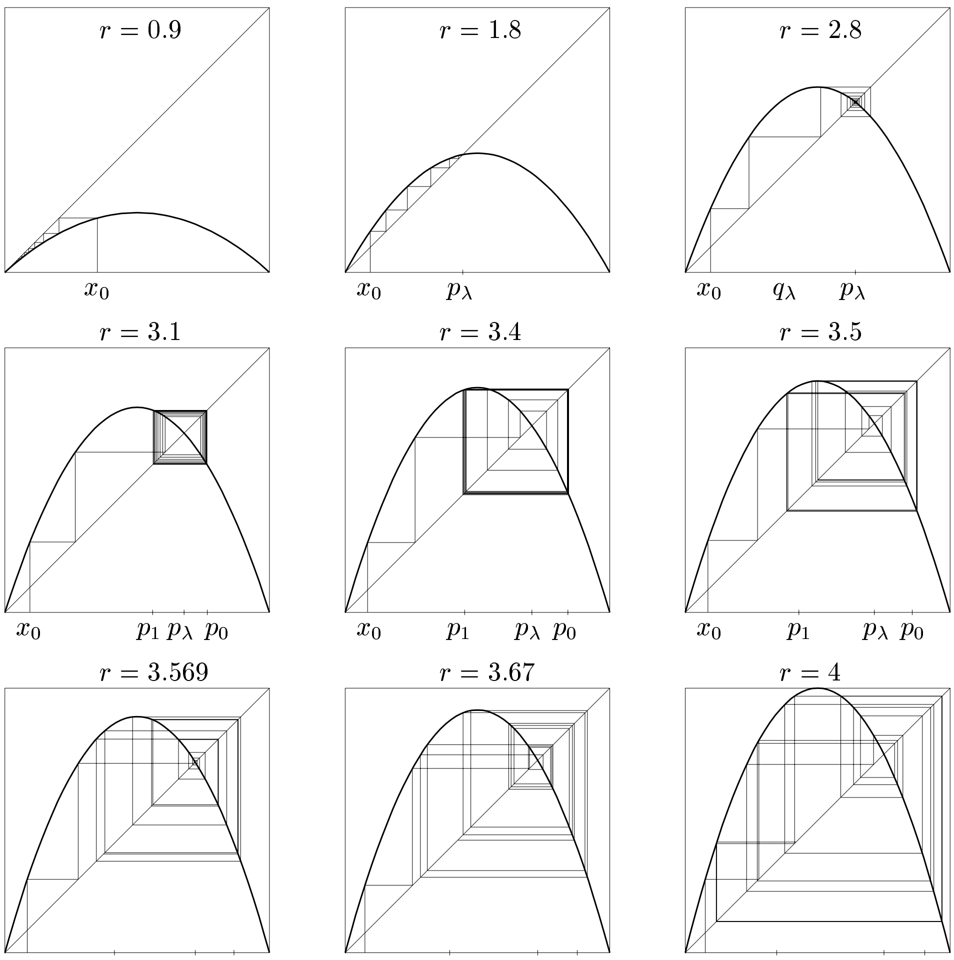

FIGURE 6. Quadratic system

$ry_n(1-y_n)$. If $0 \le r \le 4$, the function $Q_r(y) = ry(1-y)$ maps the closed interval $\mathbb{I} = [0,1]$ to itself. If $a > 0$ then $r > 1$ and the system has a fixed point $p_\lambda = 1 - \frac{1}{r}$ which corresponds to the fixed point K of the differential equation.

***Example 1.10* (Quadratic system).** — $(\mathbb{I}, Q_r)$, where $\mathbb{I} = [0,1]$, $r \in [0,4]$ and

$$Q_r(x) = rx(1-x).$$

This is a family of dynamical systems. For any $r \in [0,4]$ we get distinct system $(\mathbb{I}, Q_r)$ (Figure 6). The quadratic equation $Q_r(x) = x$ has two solutions $x = 0$ and $x = 1 - \frac{1}{r}$.

1.4.2. Attracting fixed points. — If $0 < r \le 1$, then 0 is the only fixed point in $\mathbb{I}$. The derivative of Q_r is

$$Q_r'(x) = r - 2rx,$$

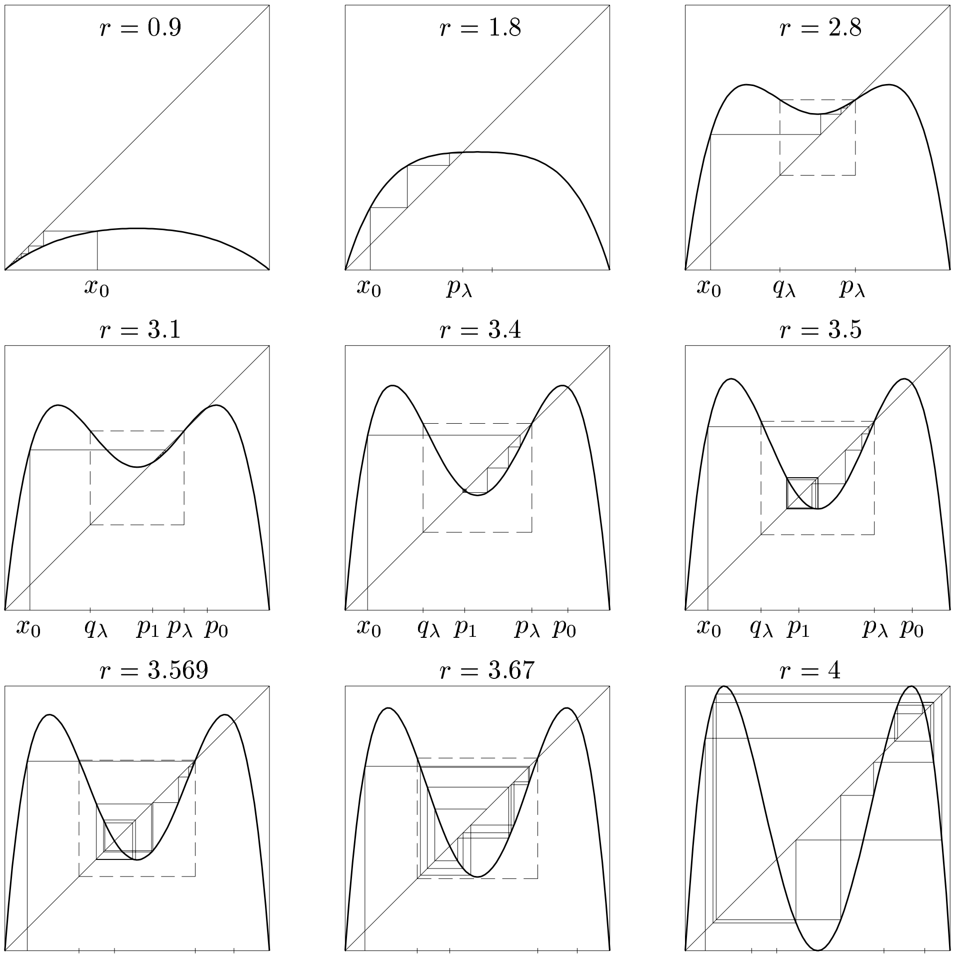

FIGURE 7. Second iteration of the quadratic system

so $Q_r'(0) = r < 1$ and 0 is an attracting fixed point. Moreover, any trajectory converges to zero.

Proposition 1.11. — *If $0 \le r \le 1$, then for any $x \in \mathbb{I}$, $\lim\limits_{n\to\infty} Q_r^n(x) = 0$.*

Proof. — For any $x > 0$, $Q_r(x) < x$, so $(Q_r^n(x))_{n\ge 0}$ is a decreasing sequence which has a limit. If $y = \lim_{n\to\infty} Q_r^n(x)$, then $Q_r(y) = \lim_{n\to\infty} Q_r^{n+1}(x) = y$, so the limit y is a fixed point. But the only fixed point is $y = 0$. (Figure 6 top left). □

If $1 < r$, the fixed point 0 ceases to be stable, and there exists another fixed point

$$p_\lambda = 1 - \frac{1}{r}.$$

Since $Q_r'(p_\lambda) = r - 2rp_\lambda = 2 - r$, p_λ is attracting for $1 < r < 3$. We show that p_λ attracts all points with the exception of 0 and 1. The proof is slightly more complicated, since for $r > 2$, the trajectories oscillate around p_λ (Figure 6 top right).

For this reason we consider the second iteration (Figure 7)

$$Q_r^2(x) = r \cdot rx(1-x) \cdot (1 - rx(1-x)).$$

For $r > 2$, the second iteration has an **invariant interval** $J = [q_\lambda, p_\lambda]$, where $q_\lambda = 1 - p_\lambda = 1/r$. This means that if $x \in J$ then $Q_r^2(x) \in J$, or $Q_r^2(J) \subseteq J$. The graph of the second iteration is within the dashed rectangle (Figure 7). Thus (J, Q_r^2) is also a dynamical system. Note that it has similar shape to Q_r but it is inverted.

Proposition 1.12. — *If* $1 < r \leq 3$, *then for any* $x \in (0,1)$, $\lim_{n\to\infty} Q_r^n(x) = p_\lambda$.

Proof. — If $r \leq 2$, then $p_\lambda \leq \frac{1}{2}$ (Figure 6 top center). We have

$$0 < x < p_\lambda \quad \Rightarrow \quad 0 < x < Q_r(x) < p_\lambda.$$

If $x < p_\lambda$, then $(Q_r^n(x))_{n\geq 0}$ is an increasing sequence, so it has a limit which must be the fixed point p_λ. If $p_\lambda < x \leq \frac{1}{2}$, then the trajectory is a decreasing sequence whose limit is p_λ. If $\frac{1}{2} \leq x < 1$, then $0 < Q_r(x) \leq \frac{1}{2}$, so $(Q_r^n(x))_{n\geq 0}$ also tends to p_λ.

If $2 < r \leq 3$ (Figure 7 top right), then for $q_\lambda = 1 - p_\lambda$ we have

$$q_\lambda \leq x \leq p_\lambda \;\Rightarrow\; \tfrac{1}{2} < p_\lambda \leq Q_r(x) \leq Q_r(\tfrac{1}{2}) \;\Rightarrow\; q_\lambda \leq Q_r^2(\tfrac{1}{2}) \leq Q_r^2(x) \leq p_\lambda,$$

so $J = [q_\lambda, p_\lambda]$ is invariant for Q_r^2, and $Q_r^2(x) > x$ for $x \in [q_\lambda, p_\lambda)$. If $x \in (0,1)$, then $Q_r^n(x) \in [\frac{1}{2}, p_\lambda]$ for some n and $Q_r^n(x) < Q_r^{n+2}(x) < \cdots$ is an increasing sequence which converges to p_λ. □

If $r > 3$, a **periodic orbit** appears. It consists of two points p_0, p_1 with $Q_r(p_0) = p_1$ and $Q_r(p_1) = p_0$ (Figure 6 center left). We say that p_0 and p_1 are **periodic points** with period 2. They can be obtained as solutions of the equation $Q_r^2(x) = x$. This is an equation of degree four. Since we know two of its solutions 0 and p_λ, we can reduce it to a quadratic equation with solutions

$$p_{0,1} = \left(1 + r \pm \sqrt{(r+1)(r-3)}\right)/2r.$$

The concept of stability can be generalized to periodic points. A periodic point of a system (X, F) is a fixed point of some iteration F^p.

Definition 1.13. — Let (X, F) be a dynamical system.

(1) A point $x \in X$ is **periodic**, if there exists $p > 0$ with $F^p(x) = x$. The least p with this property is called the **period** of x.

(2) The orbit of a periodic point with period $p > 0$ is the set

$$\mathcal{O}(x) = \{F(x), \ldots, F^p(x)\}.$$

(3) A point $x \in X$ is **eventually periodic**, if $F^m(x)$ is periodic for some $m \geq 0$.

(4) A point is **preperiodic**, if it is eventually periodic but not periodic.

(5) A periodic point with period p is stable or attracting, if it is stable or attracting as a fixed point of F^p.

A fixed point is a periodic point with period 1. An example of a preperiodic point is 1, since $Q_r(1) = 0$ is a fixed point. For $r > 1$ there is another preperiodic point $q_\lambda = \frac{1}{r}$, as $Q_r(\frac{1}{r}) = p_\lambda$. Periodic points of period 2 are fixed points of the second iteration (Figure 7). The differential criterion of stability (Theorem 1.8) can also be applied to periodic points. We have

$$(F^2)'(p_0) = F'(F(p_0))F'(p_0) = F'(p_1)F'(p_0) = F'(p_1)F'(F(p_1)) = (F^2)'(p_1).$$

Thus p_0 is stable iff p_1 is stable. Stability is a property of whole orbit $\mathcal{O}(p_0) = \mathcal{O}(p_1) = \{p_0, p_1\}$.

If we evaluate the derivative of Q_r^2 at the points p_0 and p_1, we find that these points are attracting for $3 < r \leq \sqrt{6}+1 \doteq 3.4495$ (Figure 6 center left and center). If r goes beyond $\sqrt{6}+1$, the points p_0, p_1 loose their stability and there appears a periodic orbit of period 4 (Figure 6 center right). This can be seen in the graph of the second iteration (Figure 7). The interval $J = [q_\lambda, p_\lambda]$ is invariant if $r > 2$ and if $Q_r^2(\frac{1}{2}) \geq q_\lambda$. This is satisfied for $r < 3.67$, and as r increases from 2 to 3.67, Q_r^2 restricted to J has geometric form similar to that of Q_r, except inverted. The fixed point p_1 of Q_r^2 inside (q_λ, p_λ) looses its stability at $r = \sqrt{6}+1$ and an attracting periodic orbit of Q_r^2 of period 2, i.e., an attracting periodic orbit of Q_r of period 4 appears (Figure 7 center right). Similarly, Q_r^4 has a smaller invariant interval on which it has a geometric form similar to that of Q_r on $\mathbb{I}$.

1.4.3. Bifurcation diagram. — As r increases from 0 to 4, we obtain an increasing sequence of parameter values $r_0 = 1$, $r_1 = 3$, $r_2 = \sqrt{6}+1$, etc., such that for $r \in (r_i, r_{i+1}]$, Q_r has an attracting periodic orbit of period 2^i and unstable periodic orbits of all periods 2^j with $j < i$. We say that at r_i there occurs a **period doubling bifurcation**. The periodic orbit of period 2^{i-1} looses its stability and an attracting periodic orbit of period 2^i appears. The sequence $(r_i)_{i\geq 0}$ has a limit $\lim_{i\to\infty} r_i = r_\infty \doteq 3.56994\ldots$. At r_∞, the attracting orbit becomes infinite, and the shape of Q_r^2 on J is nearly the same as that of Q_r on $\mathbb{I}$ (Figure 7 bottom left). The quadratic system at r_∞ is called Feigenbaum system. We shall study it in section 1.7.

Beyond r_∞ dynamics which may be much more complex appears. This situation is summarized in a **bifurcation diagram** in Figure 8. The diagram shows how the behaviour of the system $(\mathbb{I}, Q_r)$ depends on r. For a given r we pick at random an initial state x_0 and compute the trajectory $(x_n)_{n\geq 0}$. We wait till the sequence is stabilized and then display some of its elements. If $(\mathbb{I}, Q_r)$ has an attracting periodic

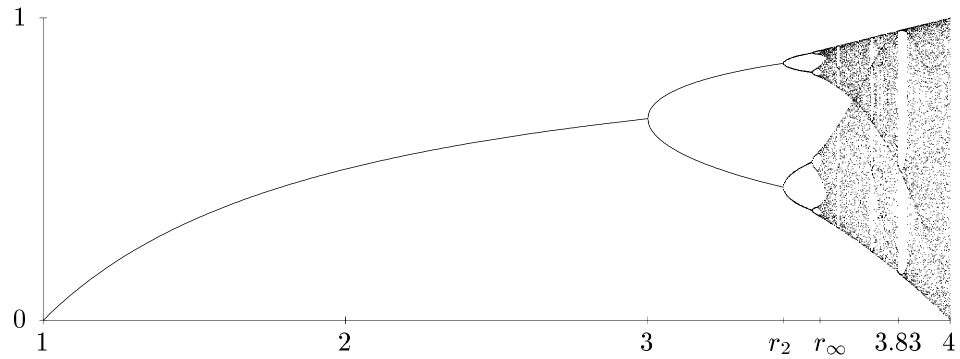

FIGURE 8. Bifurcation diagram

orbit, only the points which are close to this orbit are shown. For other values of r, the stabilized part of a trajectory densely fills an interval, or several intervals.

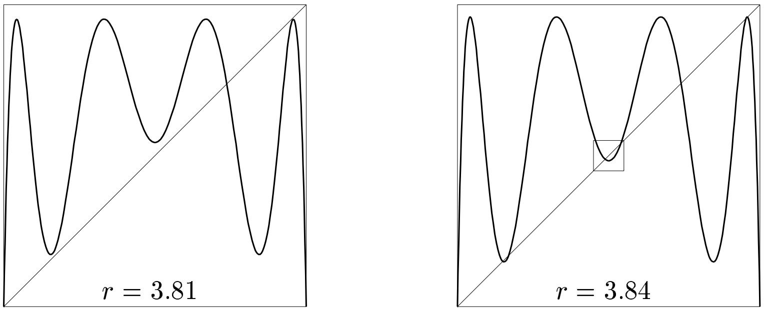

FIGURE 9. Third iteration of the quadratic system

Note that beyond r_∞ there appear other parameters with an attracting period orbit. They can be seen as "windows" in the bifurcation diagram. One such window is at $r = \sqrt{8} + 1 \doteq 3.83$ and it corresponds to an attracting orbit of period 3. This periodic orbit appears exactly at $r = \sqrt{8}+1$ and immediately splits into two periodic orbits of period 3, one of them being attracting (Figure 9). Later this periodic orbit looses its stability and an attracting periodic orbit of period 6 appears followed by a succession of attracting periodic orbits of periods $3 \cdot 2^n$.

1.5. Chaotic systems

While for many parameters r, the behaviour of the quadratic system $(\mathbb{I}, Q_r)$ is hard to analyze, in the case $r = 4$ it can be understood using a particular dynamical system whose states are points of a circle. A **circle** may be represented as the unit circle in the complex plane, i.e., the set of complex numbers with absolute value 1. Such a complex number can be written as $z = e^{2\pi i x}$, where $0 \leq x < 1$. It is convenient to represent points of a circle in this manner by numbers in semiclosed interval $\mathbb{T} = [0, 1)$. Thus points 1, i, -1, $-i$ of the complex plane are represented by numbers 0, $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ respectively (Figure 10). Define the distance between two points as the length of the shortest of the two arcs which join them. This can be written as

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

For example $d(\frac{1}{8}, \frac{1}{4}) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$ and $d(\frac{7}{8}, \frac{1}{8}) = 1 - |\frac{7}{8} - \frac{1}{8}| = \frac{1}{4}$. There is a continuous function $\mathrm{mod}_1 : \mathbb{R} \to \mathbb{T}$ given by $\mathrm{mod}_1(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the unique integer which satisfies $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ (integer part of x). More generally, for $a > 0$ we have the **modulo function** $\mathrm{mod}_a : \mathbb{R} \to [0, a)$ given by $\mathrm{mod}_a(x) = x - a\lfloor \frac{x}{a} \rfloor$.

***Example 1.14* (Angle doubling system).** — $(\mathbb{T}, D)$, where $D(x) = \mathrm{mod}_1(2x)$.

$$D(x) = \begin{cases} 2x & \text{for } x < \frac{1}{2} \\ 2x - 1 & \text{for } x \geq \frac{1}{2} \end{cases}$$

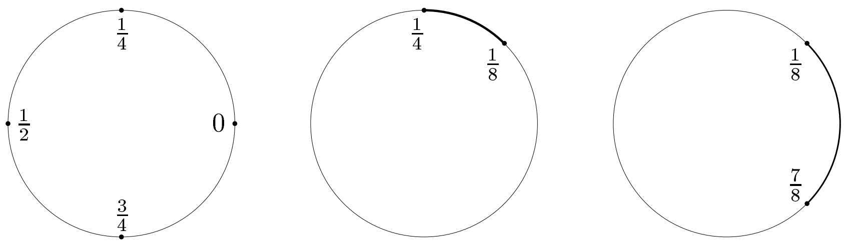

FIGURE 10. The circle

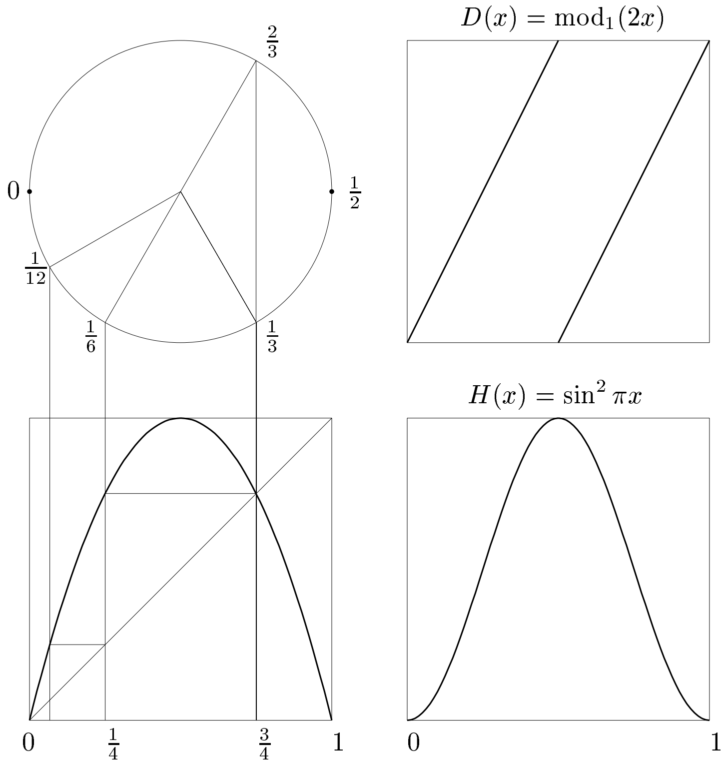

FIGURE 11. The angle doubling system

The graph of D is in Figure 11 top right. The apparent discontinuity at $\frac{1}{2}$ does not really exist as 1 is identified with 0. Note also that in the representation of the circle in the complex plane, D corresponds to function z^2. If $z = e^{2\pi ix}$, then $z^2 = e^{4\pi ix} = e^{2\pi iD(x)}$.

If we project the circle $\mathbb{T}$ to the unit interval $\mathbb{I}$ in such a way that 0 is projected to 0 and $\frac{1}{2}$ to 1, the trajectories of $(\mathbb{T}, D)$ project to trajectories of $(\mathbb{I}, Q_4)$. This can be seen in Figure 11 left (Note that the circle is rotated; now the point 0 is on the left hand side.) The projection is a surjective function $H(x) = \sin^2 \pi x$ (Figure 11 bottom right). For example the trajectory of the point $\frac{1}{12}$ in $(\mathbb{T}, D)$ is the eventually periodic sequence $\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \ldots$. Its projection is the sequence

$$H(\tfrac{1}{12}) = \tfrac{1}{4}(2-\sqrt{3}), \quad H(\tfrac{1}{6}) = \tfrac{1}{4}, \quad H(\tfrac{1}{3}) = \tfrac{3}{4}, \quad H(\tfrac{2}{3}) = \tfrac{3}{4}, \ldots$$

We have

$$Q_4 \circ H(x) = 4\sin^2 \pi x(1-\sin^2 \pi x) = (2\sin \pi x \cos \pi x)^2 = \sin^2 2\pi x = H \circ D(x),$$

so the following diagram commutes.

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{D} & \mathbb{T} \\ H\downarrow & & \downarrow H \\ \mathbb{I} & \xrightarrow{Q_4} & \mathbb{I} \end{array}$$

We say that $H : (\mathbb{T}, D) \to (\mathbb{I}, Q_4)$ is a **factor map** and $(\mathbb{I}, Q_4)$ is a **factor** of $(\mathbb{T}, D)$ or $(\mathbb{T}, D)$ is an **extension** of $(\mathbb{I}, Q_4)$.

The dynamics of the angle doubling system can be understood if we work with binary expansions. Binary expansion of a number $x \in [0, 1)$ is a sequence $(x_i)_{i\geq 0}$ of binary digits $x_i \in \{0, 1\}$ such that

$$x = 0.x_0x_1x_2\ldots = \frac{x_0}{2} + \frac{x_1}{4} + \frac{x_2}{8} + \cdots = \sum_{i=0}^{\infty} x_i \cdot 2^{-i-1}$$

The binary expansion of $D(x)$ is obtained from that of x if we forget the first digit and shift all other digits one place to the left

$$2x = x_0 + \frac{x_1}{2} + \frac{x_2}{4} + \cdots \qquad D(x) = \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} + \cdots = 0.x_1x_2x_3\ldots$$

Using this representation, we easily obtain periodic points. A point $x \in \mathbb{T}$ is periodic iff its binary expansion is periodic. For example $0.0101\ldots = \frac{1}{3}$ and $0.1010\ldots = \frac{2}{3}$ are periodic points with period 2, and $0.011011\ldots = \frac{3}{7}$ is a periodic point with period 3. If $x = 0.x_0x_1\ldots \in \mathbb{T}$ is an arbitrary point, then for any n there is a periodic point $y_n = 0.x_0\ldots x_{n-1}x_0\ldots x_{n-1}\ldots$ obtained form x by cyclic repetition of its first n digits. As n goes to infinity, y_n converge to x. Thus any point is a limit of a sequence of periodic points. We say that **periodic points are dense** in $(\mathbb{T}, D)$. Another important property of the angle doubling system is **transitivity**. Consider a point

$$x = 0.0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ldots$$

This is the **Champernowne sequence** which is obtained as concatenation of all finite binary words. First put words 0, 1 of length 1, then binary words 00, 01, 10, 11 of length 2, then all binary words of length 3, etc. Given an arbitrary point $y = 0.y_0y_1\ldots$, the iterations of x get arbitrarily close to y, as any binary word $y_0\ldots y_{n-1}$ occurs somewhere in x. Thus for some m, $D^m(x) = 0.y_0\ldots y_{n-1}x_{m+n}x_{m+n+1}\cdots$, and $d(D^m(x), y) < 2^{-n}$. We say that the point x has a **dense orbit**. In general, a set $Y \subseteq X$ of a metric space X is **dense** (Section A.3), if for any point $x \in X$ and any $\varepsilon > 0$ there exists $y \in Y$ with $d(y, x) < \varepsilon$. For example, the rational numbers are dense in $\mathbb{R}$.

Definition 1.15. — Let (X, F) be a dynamical system.

(1) The **orbit** of a point $x \in X$ is the set $\mathcal{O}(x) = \{F^n(x) : n > 0\}$.

(2) A dynamical system (X, F) is **transitive**, if at least one of its points has a dense orbit.
(3) A dynamical system is **chaotic**, if it is transitive, the set of periodic points is dense in X, and X is infinite.

The term "chaotic" has been used informally to describe dynamic systems which behave in an apparently irregular and unpredictable manner. They have been studied by many authors, for example Lorenz [**101**], Sharkovski [**143**], Hénon [**74**] or Li and Yorke [**96**]. Several formal definitions were suggested for them. We use a modified definition of Devaney [**48**].

If $x \in \mathbb{T}$ is periodic, then $H(x) \in \mathbb{I}$ is periodic too but its period might be smaller. For example $\frac{1}{3}$ has period 2 in $(\mathbb{T}, D)$ and $H(\frac{1}{3}) = \frac{3}{4}$ is a fixed point in $(\mathbb{I}, Q_4)$. Since periodic points are dense in $(\mathbb{T}, D)$, they are also dense in $(\mathbb{I}, Q_4)$. Moreover, the image of a dense orbit in $(\mathbb{T}, D)$ is a dense orbit in $(\mathbb{I}, Q_4)$. It follows that $(\mathbb{I}, Q_4)$ is chaotic too (Proposition 2.10).

In the quadratic family, only the last system $(\mathbb{I}, Q_4)$ is chaotic. However, there are many parameters r for which $(\mathbb{I}, Q_r)$ is chaotic on a subset consisting of several intervals. For example, for $r \doteq 3.67$ (Figure 7 bottom center), we have that $Q_r^2(\frac{1}{2}) = q_\lambda$ and the system $([q_\lambda, p_\lambda], Q_r^2)$ is chaotic.

1.6. Symbolic dynamics

The angle doubling system of Example 1.14 can be understood in terms of symbolic manipulation with digits. This approach is generalized in symbolic dynamics which studies dynamical systems in spaces of symbolic sequences. The method is due to Hadamard [**71**] and has been systematically developed by Morse and Hedlund [**111, 112**].

Consider the **binary alphabet** $\mathbf{2} = \{0, 1\}$ and the **binary power space** $\mathbf{2}^{\mathbb{N}}$ of all infinite binary sequences

$$\mathbf{2}^{\mathbb{N}} = \{x = x_0x_1x_2\ldots : x_i \in \mathbf{2}\}$$

Define the distance of two sequences $x, y \in \mathbf{2}^{\mathbb{N}}$ as

$$d(x, y) = 2^{-n} \text{ where } n = \min\{i \geq 0 : x_i \neq y_i\}.$$

For example $d(0011\ldots, 1001\ldots) = 2^{-0} = 1$ and $d(0011\ldots, 0110\ldots) = 2^{-1} = \frac{1}{2}$.

FIGURE 12. The Cantor set

The binary power space $\mathbf{2}^{\mathbb{N}}$ is in one-to-one correspondence with the **Cantor set**. This set is obtained from the unit interval $\mathbb{I} = [0,1]$ by deleting the open middle third interval $(\frac{1}{3}, \frac{2}{3})$ and repeating this successively with all intervals obtained (Figure 12).

$$C = [0,1] \setminus (\tfrac{1}{3}, \tfrac{2}{3}) \setminus (\tfrac{1}{9}, \tfrac{2}{9}) \setminus (\tfrac{7}{9}, \tfrac{8}{9}) \ldots$$

The interval $(\frac{1}{3}, \frac{2}{3})$ consists of numbers, which in their ternary (base three) expansion begin with digit 1. Indeed for $x_i \in \{0,1,2\}$ we have

$$\frac{x_0}{3} \leq \sum_{i=0}^{\infty} x_i \cdot 3^{-i-1} \leq \frac{x_0+1}{3}.$$

Likewise, $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ consists of numbers, whose second ternary digit is 1. Thus the Cantor set consists of real numbers, which have ternary expansions consisting only of zeros and twos. Some numbers have two ternary expansions. This happens exactly for rational numbers whose denominator is a power of 3. Such a number belongs to the Cantor set if at least one of its ternary expansions does not contain any ones. For example both $\frac{1}{3} = 0.1 = 0.0222\ldots$ and $\frac{2}{3} = 0.2 = 0.1222\ldots$ have two ternary expansions, one of which does not contain ones, so they belong to C. The correspondence between the binary space $\mathbf{2}^{\mathbb{N}}$ and the Cantor set is a **homeomorphism**: it is continuous and its inverse is continuous too.

Proposition 1.16. — *The binary power space $\mathbf{2}^{\mathbb{N}}$ is homeomorphic to the Cantor set.*

Proof. — Define a map $\varphi : \mathbf{2}^{\mathbb{N}} \to C$ by $\varphi(x) = \sum_{i=0}^{\infty} 2x_i 3^{-i-1}$. If $d(x,y) = 2^{-n}$, then $x_i = y_i$ for $i < n$, $x_n \neq y_n$, and

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &= \left|\sum_{i=n}^{\infty} 2(x_i - y_i)3^{-i-1}\right| \leq 2\sum_{i=n}^{\infty} 3^{-i-1} = \frac{2 \cdot 3^{-n-1}}{1 - \frac{1}{3}} = 3^{-n} \\
|\varphi(x) - \varphi(y)| &\geq 2 \cdot |x_n - y_n| \cdot 3^{-n-1} - 2\left|\sum_{i=n+1}^{\infty} (x_i - y_i)\right| \\
&\geq 2 \cdot 3^{-n-1} - 2\sum_{i=n+1}^{\infty} 3^{-i-1} = 3^{-n-1}
\end{aligned}$$

so we get

$$\begin{aligned}
d(x,y) \leq 2^{-n} &\Rightarrow |\varphi(x) - \varphi(y)| \leq 3^{-n} \\
|\varphi(x) - \varphi(y)| < 3^{-n-1} &\Rightarrow d(x,y) < 2^{-n}
\end{aligned}$$

This shows that both φ and φ^{-1} are continuous. □

More generally, for any finite alphabet A with at least two letters, we consider the power space $A^{\mathbb{N}}$ of all infinite sequence of letters of A. The metric is again

$$d(x,y) = 2^{-n}, \text{ where } n = \min\{i \geq 0 : x_i \neq y_i\}.$$

It turns out that all power spaces $A^{\mathbb{N}}$ are homeomorphic to each other (Section 3.1). For example, if $A = \{a,b,c\}$, we assign to letters of A binary words

$$\varphi_0(a) = 0, \quad \varphi_0(b) = 10, \quad \varphi_0(c) = 11$$

and define a bijective map $\varphi : A^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ by $\varphi(x_0x_1x_2\ldots) = \varphi_0(x_0)\varphi_0(x_1)\varphi_0(x_2)\ldots$. Both φ and φ^{-1} are continuous. Since a letter of A is coded by at least one binary digit and at most by two binary digits, we get

$$\frac{1}{2}d(x,y)^2 \leq d(\varphi(x),\varphi(y)) \leq d(x,y).$$

Besides infinite symbolic sequences we consider also finite symbolic sequences or **words** $u = u_0 \ldots u_{n-1}$ where $u_i \in A$. The length of u is denoted by $|u| = n$. There is also a word of length 0 which we denote by λ. The set of words of A of length n is denoted by A^n and the set of all words of A is A^*. For example the set of binary words is

$$\mathbf{2}^* = \bigcup_{n=0}^{\infty} \mathbf{2}^n = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \ldots\}$$

If $x \in A^{\mathbb{N}}$ is a word of A, denote by $x_{[i,j]} = x_i \ldots x_j$ the word which occurs in x between coordinates i and j. Similarly for open or semiopen intervals $x_{[i,j)} = x_i \ldots x_{j-1}$. If $u = u_0 \ldots u_{n-1} \in A^n$ is a word of length n, the **cylinder set** of u consists of all points $x \in A^{\mathbb{N}}$ whose **prefix** (initial part) is u.

$$[u] = \{x \in A^{\mathbb{N}} : x_{[0,n)} = u\}.$$

The cylinders are open balls in the power space. For any $x \in [u]$ the cylinder $[u]$ consists of all points whose distance from x is less than 2^{-n+1}.

$$x \in [u] \;\Rightarrow\; [u] = B_{2^{-n+1}}(x) = \{y \in \mathbf{2}^{\mathbb{N}} : d(y,x) < 2^{-n+1}\}$$

The symbolic version of the angle doubling system is the binary full shift.

***Example 1.17* (Full shift).** — $(A^{\mathbb{N}}, \sigma)$, where A is an alphabet and

$$\sigma(x_0x_1x_2\ldots) = x_1x_2\ldots, \quad \text{or} \quad \sigma(x)_i = x_{i+1}.$$

is a chaotic dynamical system.

Proof. — We show that σ is continuous. If $d(x,y) = 2^0 = 1$, then $d(\sigma(x),\sigma(y)) \leq 1$. If $d(x,y) = 2^{-n}$ and $n > 1$, then $d(\sigma(x),\sigma(y)) = 2^{-n+1}$. Thus for any $x, y \in \mathbf{2}^{\mathbb{N}}$,

$$d(\sigma(x),\sigma(y)) \leq 2 \cdot d(x,x)$$

Periodic points $x = (u_0 \ldots u_{n-1})^\infty = u_0 \ldots u_{n-1}u_0 \ldots u_{n-1} \ldots$ are dense in $A^{\mathbb{N}}$ and the sequence which is obtained as a concatenation of all finite words (the Champernowne sequence in the case of binary alphabet) has dense orbit in $A^{\mathbb{N}}$. □

Define a map $\psi : \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ by $\psi(x) = \mathrm{mod}_1\left(\sum_{i=0}^{\infty} x_i 2^{-i-1}\right)$. Then ψ is surjective and $\psi \circ \sigma = D \circ \psi$, so $\psi : (\mathbf{2}^{\mathbb{N}}, \sigma) \to (\mathbb{T}, D)$ is a factor map. Composing ψ with H we get $(\mathbb{I}, Q_4)$ as a factor of the binary shift $H \circ \psi : (\mathbf{2}^{\mathbb{N}}, \sigma) \to (\mathbb{I}, Q_4)$.

$$\begin{tikzcd}
\mathbf{2}^{\mathbb{N}} \arrow[r, "\sigma"] \arrow[d, "\psi"'] & \mathbf{2}^{\mathbb{N}} \arrow[d, "\psi"] \\
\mathbb{T} \arrow[r, "D"] & \mathbb{T}
\end{tikzcd}$$

1.7. Feigenbaum system

A third type of dynamics occurs in the quadratic family exactly at the limit of all period doubling bifurcations at the point $r_\infty \doteq 3.56994\ldots$ (Feigenbaum [**58**]). The system has no attracting points. Instead, there exists a **stable invariant subset** $X \subset \mathbb{I}$ which attracts all points with the exception of periodic and preperiodic ones. The concept of stable and attracting set generalizes the concepts of stable and attracting periodic points from Definition 1.13. For a set $Y \subseteq X$ and $x \in X$ denote by

$$\begin{aligned} d(x,Y) &= \inf\{d(x,y) : y \in Y\} \\ B_\delta(Y) &= \{x \in X : d(x,Y) < \delta\} \end{aligned}$$

the distance of x from Y (Figure 13 left) and the δ-neighbourhood of Y (Figure 13 right).

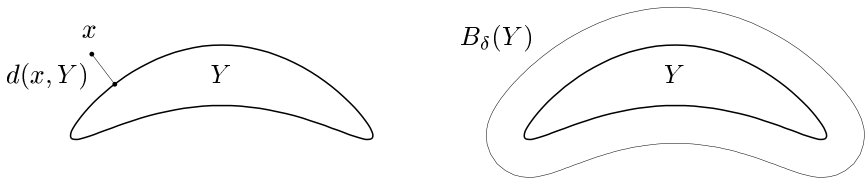

FIGURE 13. Neighbourhood of a set

Definition 1.18. — Let (X,F) be a dynamical system and $Y \subseteq X$.
(1) Y is **invariant** if $F(Y) \subseteq Y$.
(2) Y is **stable**, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B_\delta(Y), \forall n \geq 0, d(F^n(x), Y) < \varepsilon.$$

(3) Y is **attracting**, if it is stable and moreover

$$\exists \delta > 0, \forall x \in B_\delta(Y), \lim_{n\to\infty} d(F^n(x), Y) = 0.$$

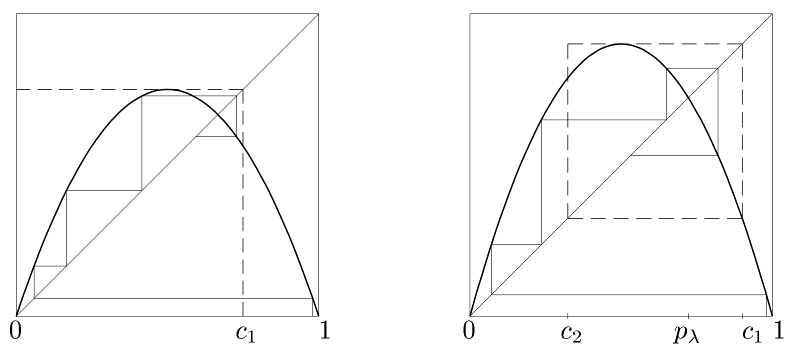

FIGURE 14. Attracting sets

In the quadratic system, attracting sets are obtained from the orbit of the **critical point** $c_0 = \frac{1}{2}$, at which the derivative vanishes. Set $c_i = Q_r^i(\frac{1}{2})$. If $r \geq 2$ then

$c_1 \geq \frac{1}{2}$ and $[0, c_1] = Q_r(\mathbb{I})$ is an attracting set (Figure 14 left). If moreover $c_2 \leq \frac{1}{2}$, then $[c_2, c_1]$ is an attracting set (Figure 14 right). Indeed, since $c_1 > p_\lambda$, we have $c_2 = Q_r(c_1) < c_1$. Since $c_2 < \frac{1}{2} < c_1$, and Q_r attains its maximum at $c_0 = \frac{1}{2}$, we have $Q_r([c_2, c_1]) = [Q_r(c_1), Q_r(c_0)] = [c_2, c_1]$. In the Feigenbaum system $(\mathbb{I}, Q_{r_\infty})$ there are many smaller attracting set, e.g., the union of intervals $[c_2, c_4] \cup [c_3, c_1]$.

It is, however, hard to analyze the Feigenbaum system directly. Instead, we shall consider a simpler system with similar behaviour considered by Preston [**131**]. This is a piecewise linear system $(\mathbb{I}, P)$ in Figure 15. We construct an increasing sequence $(a_i)_{i\geq 0}$ and a decreasing sequence $(b_i)_{i\geq 0}$ with common limit $\frac{1}{2}$. In any interval $[a_i, a_{i+1}]$ and $[b_{i+1}, b_i]$, the function P is linear, so it is determined by values $P(a_i) = P(b_i) = d_i$. The second iteration P^2 restricted to the interval $[a_1, b_1]$ has the same shape as P on $\mathbb{I}$, except inverted.

Example 1.19 **(Preston system).** — $(\mathbb{I}, P)$, where P is a piecewise linear map defined by $P(\frac{1}{2}) = \frac{4}{5}$, $P(a_i) = P(b_i) = d_i$. Here $d_0 = a_0 = 0$, $b_0 = 1$ and

$$a_i = \frac{1}{2}(1 - 3^{-i}), \quad b_i = \frac{1}{2}(1 + 3^{-i}), \quad d_i = \frac{4}{5}(1 - 6^{-i}), \quad i \geq 0$$

The gradient of P on the interval $[a_i, a_{i+1}]$ is

$$\frac{d_{i+1} - d_i}{a_{i+1} - a_i} = \frac{\frac{4}{5}(6^{-i} - 6^{-i-1})}{\frac{1}{2}(3^{-i} - 3^{-i-1})} = \frac{8}{5} \cdot \frac{5 \cdot 6^{-i-1}}{2 \cdot 3^{-i-1}} = 2^{-i+1}.$$

Similarly, the gradient of P on $[b_{i+1}, b_i]$ is -2^{-i+1}, so we have

$$x \in [a_i, a_{i+1}] \quad \Rightarrow \quad P(x) = d_i + 2^{1-i}(x - a_i),$$
$$x \in [b_{i+1}, b_i] \quad \Rightarrow \quad P(x) = d_i - 2^{1-i}(x - b_i).$$

In particular, $P(x) = 2x$ for $x \in [a_0, a_1]$ and $P(x) = 2(1 - x)$ for $x \in [b_1, b_0]$.

If $x \in [a_1, b_1]$ then $P(x) \in [b_1, b_0]$, where P has gradient -2. The gradients multiply under the composition, so P^2 has gradient -2 on $[a_1, a_2]$ and 2 on $[b_2, b_1]$. Similarly, P^2 has gradient -1 on $[a_2, a_3]$ and 1 on $[b_3, b_2]$. Thus the second iteration restricted to $[a_1, b_1]$ has the same shape as P on $\mathbb{I}$. It is smaller and inverted. Consider a linear map $\psi : [0, 1] \to [a_1, b_1]$ given by $\psi(0) = b_1$, $\psi(1) = a_1$, so

$$\psi(t) = \frac{2 - t}{3}, \quad \psi(a_i) = b_{i+1}, \quad \psi(b_i) = a_{i+1}.$$

We show that the diagram

$$\begin{tikzcd} {[a_0, b_0]} \arrow[r, "P"] \arrow[d, "\psi"'] & {[a_0, b_0]} \arrow[d, "\psi"] \\ {[a_1, b_1]} \arrow[r, "P^2"] & {[a_1, b_1]} \end{tikzcd}$$

commutes. Indeed

$$\psi(P(a_i)) = \psi(d_i) = \frac{2 + 8 \cdot 6^{-i-1}}{5} = 2(1 - d_{i+1}) = P(d_{i+1}) = P^2(b_{i+1}) = P^2(\psi(a_i))$$

and similarly $\psi(P(b_i)) = P^2(\psi(b_i))$. Since P and $P^2 \circ \psi$ are linear on both $[a_i, a_{i+1}]$ and $[b_{i+1}, b_i]$, we have $\psi \circ P = P^2 \circ \psi$.

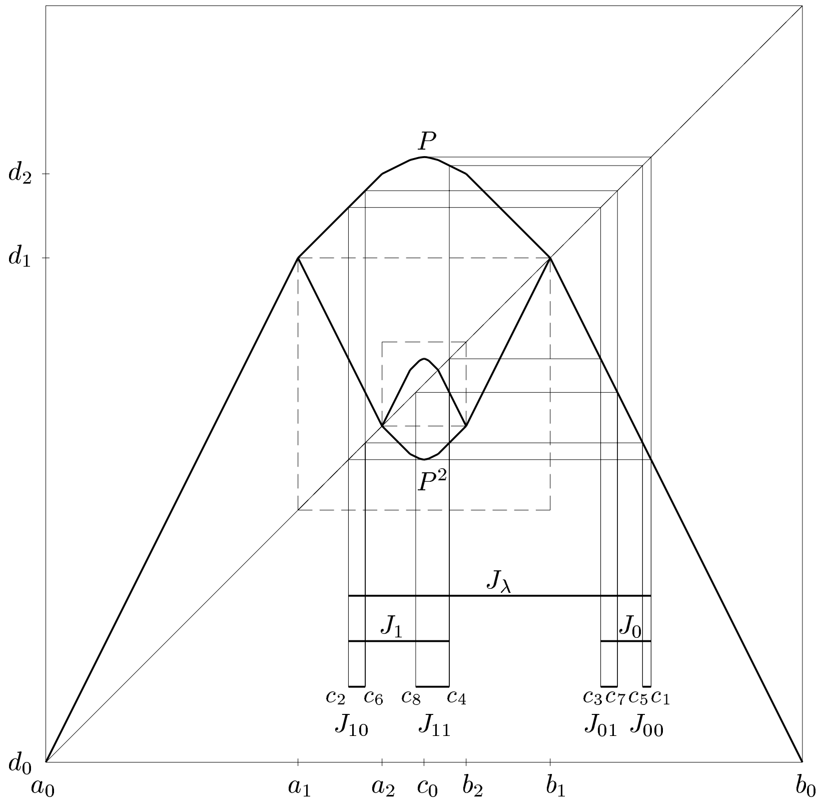

FIGURE 15. Preston system

The map $\psi : \mathbb{I} \to [a_1, b_1]$ is bijective. We say that $\psi : (\mathbb{I}, P) \to ([a_1, b_1], P^2)$ is a **conjugacy** and the systems $(\mathbb{I}, P)$ and $([a_1, b_1], P^2)$ are **conjugate**. This is a stronger property than just being a factor map. Conjugate systems have the same dynamical properties.

1.7.1. Critical point. — The dynamics of $(\mathbb{I}, P)$ can be understood by considering intervals whose end points are iterates of the critical point $c = c_0 = \frac{1}{2}$. Put $c_i = P^i(\frac{1}{2})$, so

$$c_0 = \tfrac{1}{2}, \quad c_1 = \tfrac{4}{5}, \quad c_2 = \tfrac{2}{5}, \quad c_3 = \tfrac{11}{15}, \quad c_4 = \tfrac{8}{15}, \dots$$

From $\psi \circ P = P^2 \circ \psi$ we get $\psi \circ P^n = P^{2n} \circ \psi$ for all $n > 0$. Since $\psi(c_0) = c_0$,

$$\psi(c_i) = \psi(P^i(c_0)) = P^{2i}(\psi(c_0)) = P^{2i}(c_0) = c_{2i}.$$

Using this formula we can verify by induction

$$c_{2^i} = \frac{1}{2} - \frac{(-3)^{1-i}}{10},$$

so c_{2^i} converge to c_0 as $i \to \infty$.

Proposition 1.20. — *The interval $[c_2, c_1]$ is an attracting set in $(\mathbb{I}, P)$.*

Proof. — Since c_2 is closer to $\frac{1}{2}$ than c_1, $P([c_2, c_1]) = [P(c_1), P(c_0)] = [c_2, c_1]$. If $0 < x < c_2$ then $d(P(x), [c_2, c_1]) < d(x, [c_2, c_1])$, so $[c_2, c_1]$ is stable. Moreover, for any $x \in (0, 1)$ there exists n with $P^n(x) \in [c_2, c_1]$, so $[c_2, c_1]$ is attracting. $\square$

We construct smaller attracting sets using the map ψ. Since $\psi(c_{2^i}) = c_{2^{i+1}}$, we get a sequence of conjugacies

$$\begin{tikzcd}
{[a_0, b_0]} \arrow[r, "P"] \arrow[d, "\psi"'] & {[a_0, b_0]} \arrow[d, "\psi"] \\
{[a_1, b_1]} \arrow[r, "P^2"] \arrow[d, "\psi"'] & {[a_1, b_1]} \arrow[d, "\psi"] \\
{[a_2, b_2]} \arrow[r, "P^4"] & {[a_2, b_2]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[c_2, c_1]} \arrow[r, "P"] \arrow[d, "\psi"'] & {[c_2, c_1]} \arrow[d, "\psi"] \\
{[c_2, c_4]} \arrow[r, "P^2"] \arrow[d, "\psi"'] & {[c_2, c_4]} \arrow[d, "\psi"] \\
{[c_8, c_4]} \arrow[r, "P^4"] & {[c_8, c_4]}
\end{tikzcd}$$

Since c_4 is closer to $\frac{1}{2}$ than c_2, $P([c_2, c_4]) = [P(c_2), P(c_0)] = [c_3, c_1]$. Since P is decreasing on $[c_3, c_1]$, we have $P([c_3, c_1]) = [P(c_1), P(c_3)] = [c_2, c_4]$. Set

$$J_\lambda = [c_2, c_1], \quad J_1 = [c_2, c_4], \quad J_0 = [c_3, c_1].$$

Thus $P(J_\lambda) = J_\lambda$, $P(J_1) = J_0$, $P(J_0) = J_1$ (Figure 16). The fixed point $b_1 \in J_\lambda$ is not stable and does not belong to $J_1 \cup J_0$. In next step set

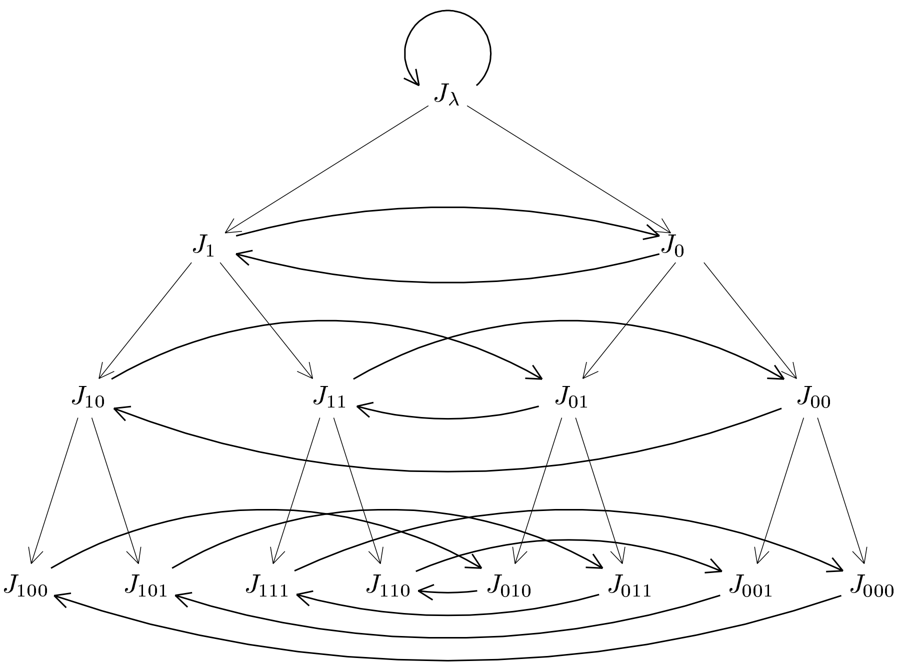

FIGURE 16. Invariant subsets

$$J_{11} = [c_8, c_4], \quad J_{00} = [c_5, c_1], \quad J_{10} = [c_2, c_6], \quad J_{01} = [c_3, c_7].$$
$$X_0 = J_\lambda, \quad X_1 = J_0 \cup J_1, \quad X_2 = J_{11} \cup J_{00} \cup J_{10} \cup J_{01}.$$

Then $P(J_{11}) = J_{00}$, $P(J_{00}) = J_{10}$, $P(J_{10}) = J_{01}$ and $P(J_{01}) = J_{11}$. The set X_1 is obtained from the interval X_0 by deleting its middle interval (c_4, c_3). Similarly X_2 is obtained from X_1 by deleting middles from its two intervals. This is similar to the construction of the Cantor set, except that the lengths of deleted intervals vary. The dynamics on these intervals is pictured in Figure 16. Vertical arrows symbolize inclusion and horizontal arrows symbolize dynamics. We see that for any binary word u, J_u has two subintervals J_{u0} and J_{u1}. We index these intervals by binary words. We define a mapping $f : \mathbf{2}^* \to \mathbf{2}^*$, by

$$f(1^n) = 0^n, \quad f(1^n 0u) = 0^n 1u$$

for all $n \geq 0$ and $u \in \mathbf{2}^*$. Thus f adds 1 to the zeroth bit of u with carry to the right. In particular, on words of length 3, f acts as

$$000 \mapsto 100 \mapsto 010 \mapsto 110 \mapsto 001 \mapsto 101 \mapsto 011 \mapsto 111 \mapsto 000$$

To simplify the notation, we disregard the order of endpoints of intervals and write

$$[a, b] = [b, a] = \{x \in \mathbb{R} : a \leq x \leq b \ \text{ or } \ b \leq x \leq a\}$$

Denote by $||J|| = |b - a|$ the length of an interval $J = [a, b]$. If $u \in \mathbf{2}^n$ is a binary word of length n, then for some $i < 2^n$, $u = f^i(0^n)$ and we set

$$u = f^i(0^n) \quad \Rightarrow \quad J_u = [c_{i+1}, c_{2^n+i+1}], \quad 0 \leq i < 2^n$$

In particular $J_{0^n} = [c_1, c_{2^n+1}]$. Since $1^n = f^{2^n-1}(0^n)$, $J_{1^n} = [c_{2^n}, c_{2^{n+1}}]$ contains c_0 in its interior. Let $Q : [0, d_1] \to [b_1, b_0]$ be the inverse map to the right branch of $P(x) = 2(1 - x)$ defined by $Q(x) = 1 - \frac{x}{2}$.

$$x \in [0, d_1] \quad \Rightarrow \quad P(Q(x)) = x,$$
$$x \in [b_1, b_0] \quad \Rightarrow \quad Q(P(x)) = x$$

Proposition 1.21. — *The system of closed intervals $\{J_u : u \in \mathbf{2}^*\}$ has the following properties.*

(1) $J_{1u} = \psi(J_u)$, $J_{0u} = Q(J_{1u})$.
(2) If $u \neq v$ and $|u| = |v|$, then $J_u \cap J_v = \emptyset$.
(3) $P(J_u) = J_{f(u)}$.
(4) J_{u0} and J_{u1} are disjoint subintervals of J_u each containing an endpoint of J_u.
(5) If $|u| = n$, then the length of J_u is at most $||J_u|| \leq ||J_{1^n}|| = 2 \cdot 3^{-n}/5$.

Proof. — The proposition holds for binary words of lengths 0 and 1. Assume that it holds for all binary words of length less than n and let us prove it for n.

(1) If $u = f^i(0^n) \in \mathbf{2}^n$, then $0u = f^{2i}(0^{n+1})$ and $1u = f^{2i+1}(0^{n+1})$. It follows

$$J_{1u} = [c_{2i+2}, c_{2i+2^{n+1}+2}] = [\psi(c_{i+1}), \psi(c_{i+2^n+1})] = \psi(J_u)$$

Since $c_{2k-1} \geq b_1$, $c_{2k-1} = Q(P(c_{2k-1})) = Q(c_{2k})$. It follows that

$$J_{0u} = [c_{2i+1}, c_{2i+2^{n+1}+1}] = Q(J_{1u}).$$

(2) By assumption, if $u, v \in \mathbf{2}^{n-1}$ and $u \neq v$, then $J_u \cap J_v = \emptyset$. It follows that

$$\begin{aligned} J_{1u} \cap J_{1v} &= \psi(J_u) \cap \psi(J_v) = \emptyset \\ J_{0u} \cap J_{0v} &= Q(J_{1u}) \cap Q(J_{1v}) = \emptyset \\ J_{0u} \cap J_{1v} &\subset J_0 \cap J_1 = \emptyset \end{aligned}$$

so $\{J_u : u \in \mathbf{2}^n\}$ are pairwise disjoint.

(3) Since $\{J_u : u \in \mathbf{2}^n\}$ are pairwise disjoint, J_{1^n} is the only interval which contains in its interior c_0. If $u \neq 1^n$ then P is monotone on J_u. Since P maps the endpoints of J_u to endpoints of $J_{f(u)}$, we have $P(J_u) = J_{f(u)}$. For $u = 1^n$ we get $P(J_{1^n}) = [P(c_0), P(c_{2^n})] = J_{0^n}$ since $c_{2^{n+1}}$ is closer to c_0 than c_{2^n}.

(4) Since J_1 and J_0 are disjoint subintervals of J_λ, each containing one of its endpoints, and since

$$J_{1^{n+1}} = \psi^n(J_1), \quad J_{1^n0} = \psi^n(J_0), \quad J_{1^n} = \psi^n(J_\lambda),$$

the statement holds for $u = 1^n$. If $u \in \mathbf{2}^n$, $u \neq 1^n$, then $f^k(u) = 1^n$ for some $k < 2^n$. Since P^k is monotone on P_u and since

$$P^k(J_{u0}) = J_{1^n0}, \quad P^k(J_{u1}) = J_{1^{n+1}}, \quad P^k(J_u) = J_{1^n}$$

the statement holds for u.

(5) Since P has gradient 2 on J_0, $||J_0|| = \frac{1}{2}||J_1|| \leq ||J_1||$. If $u \in \mathbf{2}^{n-1}$ then $\psi(J_u) = J_{1u}$ and

$$||J_{1u}|| = \tfrac{1}{3}||J_u|| \leq \tfrac{1}{3}||J_{1^{n-1}}|| = ||J_{1^n}||.$$

Since $J_{1u} = P(J_{0u})$ and P has gradient 2 on J_{0u},

$$||J_{0u}|| = \frac{1}{2}||J_{1u}|| \leq ||J_{1^n}||$$

and $||J_{1^n}|| = |c_{2^n} - c_{2^{n+1}}| = |3^{1-n} + 3^{-n}|/10 = 2 \cdot 3^{-n}/5$. □

Proposition 1.22. — *Set $X_0 = J_\lambda$, $X_1 = J_0 \cup J_1$,*

$$X_n = \bigcup\{J_u : u \in \mathbf{2}^n\}, \quad X = \bigcap_{n=0}^{\infty} X_n$$

Any X_n is invariant and attracting and X is nonempty, invariant and stable.

Proof. — We have already proved that all X_n are nonempty and invariant, and X_0 is attracting. Suppose that X_{n-1} is attracting, so for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, X_{n-1}) < \delta \Rightarrow \forall i, d(P^i(x), X_{n-1}) < \varepsilon, \quad \lim_{i \to \infty} d(P^i(x), X_{n-1}) = 0$$

We have $X_n = \psi(X_{n-1}) \cup Q \circ \psi(X_{n-1})$ and

$$\begin{aligned}
d(x, \psi(X_{n-1})) < \tfrac{\delta}{3} \quad &\Rightarrow \quad d(\psi^{-1}(x), X_{n-1}) < \delta \\
&\Rightarrow \quad \forall i, d(P^i(\psi^{-1}(x)), X_{n-1}) < \varepsilon \\
&\Rightarrow \quad \forall i, d(P^{2i}(x), \psi(X_{n-1})) < \tfrac{\varepsilon}{3} \ \& \\
&\qquad\qquad d(P^{2i-1}(x), Q(\psi(X_{n-1}))) < \tfrac{\varepsilon}{6} \\
&\Rightarrow \quad \forall i, d(P^i(x), X_n) < \tfrac{\varepsilon}{3} \\
d(x, Q(\psi(X_{n-1}))) < \tfrac{\delta}{6} \quad &\Rightarrow \quad d(P(x), \psi(X_{n-1})) < \tfrac{\delta}{3} \\
&\Rightarrow \quad \forall i, d(P^i(x), X_n) < \tfrac{\varepsilon}{3} \\
d(x, X_n) < \tfrac{\delta}{6} \quad &\Rightarrow \quad \forall i, d(P^i(x), X_n) < \tfrac{\varepsilon}{3}
\end{aligned}$$

Similarly we prove $\lim_{i\to\infty} d(P^i(x), X_n) = 0$. Since $X_{n+1} \subset X_n$ and all X_n are nonempty and closed, their intersection X is nonempty and closed too. We show that it is stable. Given $\varepsilon > 0$ there exists $n > 0$ such that for any x,

$$d(x, X_n) < \tfrac{\varepsilon}{2} \ \Rightarrow \ d(x, X) < \varepsilon.$$

Since X_n is stable, there exists $\delta > 0$ such that for any x,

$$d(x, X_n) < \delta \quad \Rightarrow \quad \forall i, d(P^n(x), X_n) < \tfrac{\varepsilon}{2}.$$

If $d(x, X) < \delta$, then $d(x, X_n) < \delta$, so for all i, $d(P^i(x), X_n) < \frac{\varepsilon}{2}$ and $d(P^i(x), X) < \varepsilon$. Thus X is stable. □

The limit set X is, however, not attracting. Any interval J_u contains in its interior a periodic point, so there are periodic points arbitrarily close to X and their orbits do not converge to X. Periodic points can also be described symbolically. We denote by $p_\lambda = b_1 \in J_\lambda$ the fixed point of P other than 0. Then $p_1 = a_2 = \psi(p_\lambda) \in J_1$ is a periodic point with period 2 and $p_0 = P(p_1) = Q(p_1) \in J_0$. By induction we can prove

Proposition 1.23. — *There exists a system of periodic points $\{p_u : u \in \mathbf{2}^*\}$ with*

(1) $p_u \in J_u \setminus (J_{u1} \cup J_{u0})$.

(2) $p_{1u} = \psi(p_u)$, $p_{0u} = Q(p_{1u})$.

(3) $P(p_u) = p_{f(u)}$.

1.8. Dyadic adding machine

The set X from Proposition 1.22 has been constructed in a similar way to the Cantor set by successively deleting the middles of intervals, although the lengths of the deleted intervals vary. For this reason, X is homeomorphic to the Cantor set, and also, to the binary symbolic space $\mathbf{2}^{\mathbb{N}}$. The dynamics of P on X can be understood in terms of a symbolic dynamical system called the **adding machine**. Sequences $u \in \mathbf{2}^{\mathbb{N}}$ with only finitely many ones represent binary expansions of nonnegative integers written backwards, the least significant digits first, e.g., $0 = 0000\ldots$, $1 = 1000\ldots$,

$2 = 0100\ldots$, $3 = 1100\ldots$, etc. Negative integers are represented by binary sequences with only finitely many zeros. This representation is a map $\psi : \mathbb{Z} \to \mathbf{2}^{\mathbb{N}}$ given by

$$\psi(n)_i = \mathrm{mod}_2(n_i), \quad \text{where } n_0 = n, \quad \text{and } n_{i+1} = \left\lfloor \frac{n_i}{2} \right\rfloor.$$

Here $\mathrm{mod}_2 : \mathbb{Z} \to \{0,1\}$ is the modulo 2 function (see page 18) defined by

$$\mathrm{mod}_2(m) = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

The integer part $\lfloor z \rfloor$ of z is the unique integer satisfying $\lfloor z \rfloor \le z < \lfloor z \rfloor + 1$. For example

n	n_1	n_2	n_3		$\psi(n)$
-4	-2	-1	-1	$\ldots$	$001111\ldots$
-3	-2	-1	-1	$\ldots$	$101111\ldots$
-2	-1	-1	-1	$\ldots$	$011111\ldots$
-1	-1	-1	-1	$\ldots$	$111111\ldots$
0	0	0	0	$\ldots$	$000000\ldots$
1	0	0	0	$\ldots$	$100000\ldots$

In this representation, integers are added with carry to the right. Given $u, v \in \mathbf{2}^{\mathbb{N}}$, denote by c_i the carry from the i-th position to the $(i+1)$-st position. We have $c_{-1} = 0$ and $c_0 = \lfloor (u_0 + v_0)/2 \rfloor$. Then $(u+v)_1 = \mathrm{mod}_2(u_1 + v_1 + c_0)$ and in general,

$$(u+v)_i = \mathrm{mod}_2(u_i + v_i + c_{i-1}), \quad c_i = \lfloor \tfrac{u_i + v_i + c_{i-1}}{2} \rfloor$$

$$\begin{array}{rl} -2: & 01111\ldots \\ 2: & 01000\ldots \\ \hline 0: & 00000\ldots \end{array} \qquad \begin{array}{rl} -2: & 01111\ldots \\ 3: & 11000\ldots \\ \hline 1: & 10000\ldots \end{array} \qquad \begin{array}{rl} -2: & 01111\ldots \\ -3: & 10111\ldots \\ \hline -5: & 11011\ldots \end{array}$$

The space $\mathbf{2}^{\mathbb{N}}$ with this addition is a group (Section A.8). The zero element is $0000\ldots$, the inverse element to $x \in \mathbf{2}^{\mathbb{N}}$ is $(-x)_i = 1 - x_i$. The map $\psi : \mathbb{Z} \to \mathbf{2}^{\mathbb{N}}$ has a partial inverse

$$\psi^{-1}(u) = \sum_{i=0}^{\infty} 2^i u_i, \quad \psi^{-1}(u) = 1 - \sum_{i=0}^{\infty} 2^i (1 - u_i)$$

The map ψ^{-1} is defined for those $u \in \mathbf{2}^{\mathbb{N}}$, for which one of these sums is finite. This happens if u contains only finitely many ones, or only finitely many zeros. The dyadic adding machine is simply adding 1, $R_1(x) = x + 1$. It is an **isometry** (Definition A.11), i.e., it preserves the distance $d(R_1(x), R_1(y)) = d(x, y)$.

Example 1.24 **(Dyadic adding machine)**. — $(\mathbf{2}^{\mathbb{N}}, R_1)$ where

$$R_1(u)_n = \begin{cases} \mathrm{mod}_2(u_n + 1) & \text{if } u_0 = u_1 = \cdots = u_{n-1} = 1 \\ u_n & \text{otherwise} \end{cases}$$

is an isometry and it is conjugate to (X, P) of Proposition 1.22.

$$\begin{tikzcd} \mathbf{2}^{\mathbb{N}} \arrow[r, "R_1"] \arrow[d, "\pi"'] & \mathbf{2}^{\mathbb{N}} \arrow[d, "\pi"] \\ X \arrow[r, "P"] & X \end{tikzcd}$$

Proof. —

(1) Let $d(u,v) = 2^{-n}$, so $u_{[0,n)} = v_{[0,n)}$ and $u_n \neq v_n$. Then $R_1(u)_{[0,n)} = R_1(v)_{[0,n)}$ and $R_1(u)_n \neq R_1(v)_n$, so $d(R_1(u), R_1(v)) = 2^{-n} = d(u,v)$. Thus R_1 is an isometry.

(2) We construct a conjugacy $\pi : (\mathbf{2}^{\mathbb{N}}, R_1) \to (X, P)$. For $u \in \mathbf{2}^{\mathbb{N}}$ we set

$$J_u = \bigcap_{n=1}^{\infty} J_{u_{[0,n)}}$$

Since for any n, $J_{u_{[0,n+1)}} \subseteq J_{u_{[0,n)}}$ are nonempty closed intervals, their intersection J_u is nonempty too. Since $||J_{u_{[0,n)}}|| \leq 2 \cdot 3^{-n}/5$, the length of J_u is zero, so it contains exactly one point. If $\pi(u) \in J_u$ is the unique point contained in J_u, then $\pi : \mathbf{2}^{\mathbb{N}} \to X$ is a bijective mapping.

(3) For any $u \in \mathbf{2}^*$ we have $R_1([u]) = [f(u)]$, where $[u] = \{x \in \mathbf{2}^{\mathbb{N}} : x_{[0,|u|)} = u\}$ is the cylinder set and $f : \mathbf{2}^* \to \mathbf{2}^*$ is the adding machine acting on finite words. It follows that for $x \in \mathbf{2}^{\mathbb{N}}$, $P(J_x) = J_{R_1(x)}$ and therefore $\pi \circ R_1 = P \circ \pi$.

(4) If $u, v \in \mathbf{2}^{\mathbb{N}}$ and $d(u,v) = 2^{-n}$, then $u_{[0,n)} = v_{[0,n)}$ and both $\pi(u)$ and $\pi(v)$ belong to $J_{u_{[0,n)}}$. Since the length of this interval is at most $2 \cdot 3^{-n}/5$, we get $|\pi(u) - \pi(v)| \leq 2 \cdot 3^{-n}/5$. Thus π is continuous. □

In chaotic systems some points have dense orbits but not all. (For example periodic points do not.) In the adding machine, all points have dense orbits. Such systems are called **minimal** (Birkhoff [**18**]). Moreover, the points of the adding machine are **quasi-periodic:** they return periodically to any neighborhood of the initial point. For example, the point $0 = 0000\ldots$ visits the cylinder $[0]$ every even step and the cylinder $[00]$ every fourth step.

Definition 1.25. —

(1) A dynamical system (X, F) is **minimal**, if any its orbit is dense.
(2) A point $x \in X$ of a dynamical system is **quasi-periodic**, if for any $\varepsilon > 0$ there exits a positive integer p called period, such that for any $n \geq 0$, $d(F^{np}(x), x) < \varepsilon$.

Proposition 1.26. — *The dyadic adding machine is minimal. All of its points are quasi-periodic but not periodic.*

Proof. — For $x, y \in \mathbf{2}^{\mathbb{N}}$ we set $n = \mathrm{mod}_{2^n}(y_{[0,n)} - x_{[0,n)})$ (finite binary words $u \in \mathbf{2}^*$ are identified with $u0^\infty \in \mathbf{2}^{\mathbb{N}}$). Then $d(R_1^n(x), y) \leq 2^{-n}$, so the orbit of x is dense and $(\mathbf{2}^{\mathbb{N}}, R_1)$ is minimal. For any $x \in X$ and $n, m > 0$ we have $d(R_1^{m2^n}(x), x) \leq 2^{-n}$, so x is quasi-periodic. □

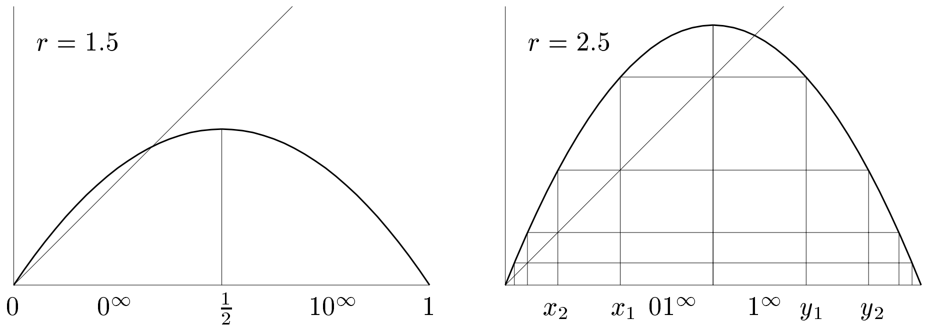

FIGURE 17. Itineraries of quadratic system

1.9. Covers and itineraries

Besides the dyadic adding machine there is another symbolic system which describes the Feigenbaum or Preston system. It is based on the idea of an itinerary. Consider two intervals $V_0 = [0, \frac{1}{2}]$, $V_1 = [\frac{1}{2}, 1]$ which intersect in the critical point $\frac{1}{2}$. We say that $\mathcal{V} = \{V_0, V_1\}$ is an **interval cover** of the space $\mathbb{I}$ and $\frac{1}{2}$ is its **cut point**. Given a quadratic system $(\mathbb{I}, Q_r)$ and a point $x \in \mathbb{I}$, we record at any time n, the element of the cover $\mathcal{V}$, to which $Q_r^n(x)$ belongs. We say that a binary sequence $v \in \mathbf{2}^{\mathbb{N}}$ is an **itinerary** of x, if

$$x \in V_{v_0}, \quad Q_r(x) \in V_{v_1}, \quad Q_r^2(x) \in V_{v_2}, \ldots$$

If $Q_r^n(x) \neq \frac{1}{2}$ for any n, then the itinerary of a point $x \in \mathbb{I}$ is unique. In general a point may have two or more itineraries. For example for $r = 1.5$, any point $x \in [0, \frac{1}{2}]$ has itinerary $000\ldots$ and any point in $[\frac{1}{2}, 1]$ has itinerary $1000\ldots$. Any point except $\frac{1}{2}$ has a unique itinerary (Figure 17 left).

The quadratic system $(\mathbb{I}, Q_{2.5})$ (Figure 17 right) has an attracting fixed point $p_\lambda > \frac{1}{2}$ with itinerary 1^∞. Let x_n, y_n be the preimages of $\frac{1}{2}$ defined by

$$x_0 = y_0 = \frac{1}{2}, \quad x_{n+1} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{x_n}{r}}, \quad y_{n+1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{x_n}{r}},$$

so $Q_r(x_{n+1}) = Q_r(y_{n+1}) = x_n$. These are the only points whose itineraries are not unique. The itinerary of $x_0 = \frac{1}{2}$ is $\mathcal{I}(x_0) = *1^\infty$, where the star stands for either 0 or 1. Similarly $\mathcal{I}(x_1) = 0{*}1^\infty$, $\mathcal{I}(y_1) = 1{*}1^\infty$ and in general

$$\mathcal{I}(x_{n+1}) = 0^{n+1}{*}1^\infty, \quad \mathcal{I}(y_{n+1}) = 10^n{*}1^\infty$$

Between consecutive preimages of $\frac{1}{2}$, the points have unique itineraries.

$$x \in (x_{n+1}, x_n) \quad \Rightarrow \quad \mathcal{I}(x) = 0^{n+1}1^\infty$$

$$x \in (y_n, y_{n+1}) \quad \Rightarrow \quad \mathcal{I}(x) = 10^n1^\infty$$

Given a finite or infinite binary sequence $v \in \mathbf{2}^* \cup \mathbf{2}^{\mathbb{N}}$, we denote by

$$V_v = \{x \in X : \forall i < |v|, F^i(x) \in V_{v_i}\} = \bigcap_{i<|v|} F^{-i}(V_{v_i})$$

the set of points with itinerary v. In the quadratic system, this set is always an interval, which may be also empty or a singleton (one point set). For example for

$r = 1.5$, $V_{0^n} = [0, \frac{1}{2}]$ and $V_{10^n} = [\frac{1}{2}, 1]$ for any n, even for $n = \infty$. If $r = 2.5$ we have that for infinite binary words

$$V_{0^\infty} = \{0\}, \quad V_{10^\infty} = \{1\}, \quad V_{0^{n+1}1^\infty} = [x_{n+1}, x_n], \quad V_{10^n1^\infty} = [y_n, y_{n+1}].$$

We call the sets

$$\mathcal{L}_\mathcal{V}(\mathbb{I}, Q_r) = \{v \in \mathbf{2}^* : V_v \neq \emptyset\}, \quad \Sigma_\mathcal{V}(\mathbb{I}, Q_r) = \{v \in \mathbf{2}^\mathbb{N} : V_v \neq \emptyset\}$$

the **language** and **subshift** of $(\mathbb{I}, Q)$ on the cover $\mathcal{V}$. For example

$$\begin{aligned} \Sigma_\mathcal{V}(\mathbb{I}, Q_{1.5}) &= \{0^\infty, 10^\infty\} \\ \Sigma_\mathcal{V}(\mathbb{I}, Q_{2.5}) &= \{0^n1^\infty, 10^n1^\infty : 0 \leq n \leq \infty\} \end{aligned}$$

The first subshift contains only two points while the second is infinite. The subshift $\Sigma = \Sigma_\mathcal{V}(\mathbb{I}, Q_r)$ is invariant with respect to σ. If $V_v \neq \emptyset$, and $x \in V_v$, then $Q_r(x) \in V_{\sigma(v)}$, so $\sigma(v) \in \Sigma$ and $\sigma(\Sigma) \subseteq \Sigma$. Thus a subshift $(\Sigma_\mathcal{V}(\mathbb{I}, Q_r), \sigma)$ is regarded as a dynamical system.

1.9.1. Generating covers. — In some cases, $(\mathbb{I}, Q_r)$ can be reconstructed from its subshift. This happens whenever for any $v \in \Sigma_\mathcal{V}(\mathbb{I}, Q_r)$, V_v is a sigleton. In this case we say that $\mathcal{V}$ is **generating** for $(\mathbb{I}, Q_r)$. We then define a map $\varphi : \Sigma_\mathcal{V}(\mathbb{I}, Q_r) \to \mathbb{I}$ by $\varphi(v) \in V_v$.

Proposition 1.27. — *If $\mathcal{V}$ is generating for $(\mathbb{I}, Q_r)$, then $\varphi : (\Sigma_\mathcal{V}(\mathbb{I}, Q_r), \sigma) \to (\mathbb{I}, Q_r)$ is a factor map.*

Proof. — For any $v \in \Sigma_\mathcal{V}(\mathbb{I}, Q_r)$ we have

$$V_v = \{\varphi(v)\} = \bigcap_{n>0} V_{v_{[0,n)}}$$

where $v_{[0,n)}$ is the prefix (initial subword) of v of length n. Given $\varepsilon > 0$ there exists $n > 0$ such that the length $||V_{v_{[0,n)}}|| < \varepsilon$ and it follows that for any $u \in \Sigma_\mathcal{V}(\mathbb{I}, Q_r)$,

$$d(u, v) < 2^{-n} \Rightarrow d(\varphi(u), \varphi(v)) < \varepsilon.$$

Thus φ is continuous. Since $\varphi(v) \in V_v$, $F(\varphi(v)) \in F(V_v) \subseteq V_{\sigma(v)}$, so $F(\varphi(v)) = \varphi(\sigma(v))$ and $\varphi \circ \sigma = Q_r \circ \varphi$. □

$$\begin{tikzcd} \Sigma_\mathcal{V}(\mathbb{I}, Q_r) \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma_\mathcal{V}(\mathbb{I}, Q_r) \arrow[d, "\varphi"] \\ \mathbb{I} \arrow[r, "Q_r"] & \mathbb{I} \end{tikzcd}$$

Example 1.28. — The binary interval cover $\mathcal{V} = \{V_0, V_1\}$ is generating for $(\mathbb{I}, Q_4)$ and $\Sigma_\mathcal{V}(\mathbb{I}, Q_4) = \mathbf{2}^\mathbb{N}$.

Proof. — We use the factor map $H : (\mathbb{T}, D) \to (\mathbb{I}, Q_4)$ constructed in Section 1.5. Suppose that $x, y \in \mathbb{I}$ are different points with the same itinerary. There exists different $u, v \in \mathbb{T}$ with $H(u) = x$, $H(v) = y$ and both u and v belong to one of the intervals

$$J_0 = [0, \tfrac{1}{4}], \quad J_1 = [\tfrac{1}{4}, \tfrac{1}{2}], \quad J_2 = [\tfrac{1}{2}, \tfrac{3}{4}], \quad J_4 = [\tfrac{3}{4}, 1) \cup \{0\}.$$

We continue by induction. If both $D^n(u)$ and $D^n(v)$ belong to one of the intervals J_i, then both $D^{n+1}(u)$ and $D^{n+1}(v)$ belong to one of the intervals $D(J_i) \cap H^{-1}(\frac{k}{2}, \frac{k+1}{2}]$ where $k = 0$ or $k = 1$. This intersection is again one of the intervals J_j. Thus for all $n \geq 0$, $d(D^n(u), D^n(v)) \leq \frac{1}{4}$ and this is impossible. We show that $\Sigma_{\mathcal{V}}(\mathbb{I}, Q_4) = \mathbf{2}^{\mathbb{N}}$. If $u \in \mathcal{L}_{\mathcal{V}}(\mathbb{I}, Q_4)$, and $x \in V_u$, the x has a preimage in both V_0 and V_1, so both $0u$ and $1u$ belong to the language. This shows that $\mathcal{L}_{\mathcal{V}}(\mathbb{I}, Q_4) = \mathbf{2}^*$. If $v \in \mathbf{2}^{\mathbb{N}}$, then for any n, $V_{v_{[0,n)}}$ is nonempty, and their intersection V_v is nonempty too, so $\Sigma_{\mathcal{V}}(\mathbb{I}, Q_4) = \mathbf{2}^{\mathbb{N}}$. □

1.10. Feigenbaum subshift

Proposition 1.29. — *The binary interval cover $\mathcal{V} = \{V_0, V_1\}$ is generating for the Preston system $(\mathbb{I}, P)$ of Example 1.19.*

Proof. — Suppose that there exists $u \in \mathbf{2}^{\mathbb{N}}$ such that V_u contains two distinct points $x < y$. Then $[x, y] \subseteq V_u$, so V_u is an interval. Take u such that V_u has maximal length. We show first that V_u does not contain any a_n or b_n. We have $\mathcal{I}(0) = 0^\infty$, $\mathcal{I}(1) = 10^\infty$. Since 0 is not stable, for any $x \in (0, 1)$ there exists $i > 0$ with $P^i(x) > \frac{1}{2}$, so $\mathcal{I}(x)$ differs from both $\mathcal{I}(0)$ and $\mathcal{I}(1)$. It follows that $0, 1 \notin V_u$. We repeat this argument for P^2. We have $\mathcal{I}(b_1) = 1^\infty$, $\mathcal{I}(a_1) = 01^\infty$. Since $([a_1, b_1], P^2)$ is conjugate to $(\mathbb{I}, P)$, the points inside (a_1, b_1) have distinct itineraries from either a_1 or b_1. If $0 < x < a_1$ or $b_1 < x < 1$, then for some n, $P^n(x) \in [a_1, b_1]$, so x has a different itinerary from either a_1 or b_1. Thus V_u does not contain any a_n or b_n and there exists n, such that $V_u \subseteq (a_n, a_{n+1})$ or $V_u \subseteq (b_{n+1}, b_n)$. On these intervals, P^{2^n} has gradient 2, so

$$||P^{2^n}(V_u)|| = ||V_{\sigma^{2^n}(u)}|| = 2||V_u||$$

is twice longer than V_u and this is a contradiction. □

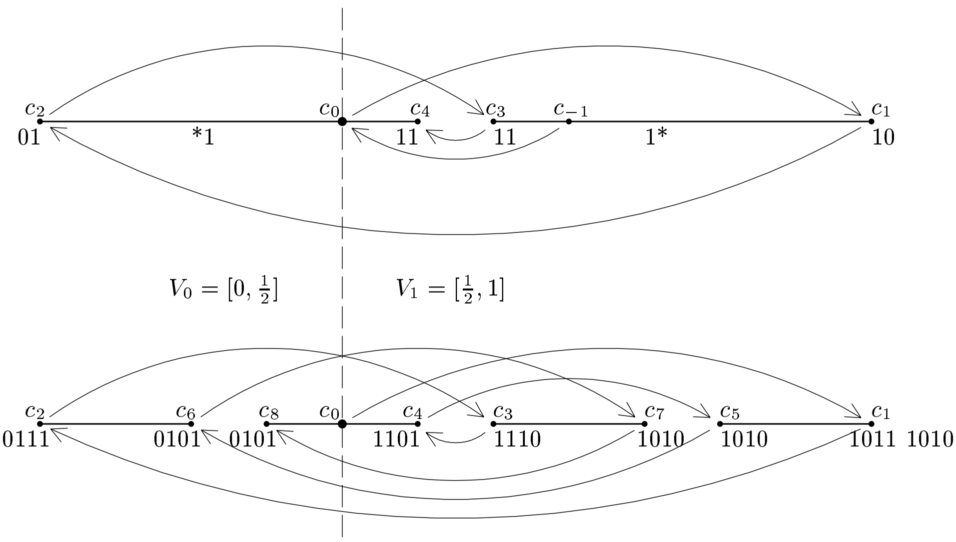

FIGURE 18. Kneading sequence

1.10.1. Kneading sequence. — The key role in the subshift $\Sigma_{\mathcal{V}}(\mathbb{I}, P)$ is played by its **kneading sequence**, which is the itinerary $\mathcal{K} = \mathcal{I}(c_1)$ of $c_1 = P(c_0) = \frac{4}{5}$ (see Figures 15 and 18). Since the trajectory of c_1 never visits c_0, the kneading sequence is determined uniquely.

$$\mathcal{K}_i = 0 \quad \Leftrightarrow \quad c_{i+1} = P^{i+1}(\tfrac{1}{2}) < \tfrac{1}{2}.$$

Since $[a_1, b_1]$ is P^2-invariant and $c_0 \in [a_1, b_1]$, we get $c_{2i} \in [a_1, b_1]$ for any $i \geq 0$. Since $P(x) > \frac{1}{2}$ on $x \in [a_1, b_1]$, we get $c_{2i+1} > \frac{1}{2}$ and $\mathcal{K}_{2i} = 1$. For odd entries of $\mathcal{K}$ we use the conjugacy $\psi : (I, P) \to ([a_1, b_1], P^2)$ defined by $\psi(t) = \frac{2-t}{3}$. Since ψ is decreasing and $\psi(c_i) = c_{2i}$, we get

$$\begin{aligned} \mathcal{K}_i = 0 \;\Leftrightarrow\; c_{i+1} < \tfrac{1}{2} \quad &\Leftrightarrow \quad c_{2i+2} > \tfrac{1}{2} \;\Leftrightarrow\; \mathcal{K}_{2i+1} = 1 \\ \mathcal{K}_i = 0 \quad &\Rightarrow \quad \mathcal{K}_{[2i,2i+1]} = 11 \\ \mathcal{K}_i = 1 \quad &\Rightarrow \quad \mathcal{K}_{[2i,2i+1]} = 10 \end{aligned}$$

This is a recursive formula which together with $\mathcal{K}_0 = 1$ determines $\mathcal{K}$ completely. Define a map $\vartheta : \mathbf{2}^* \to \mathbf{2}^*$ by $|\vartheta(u)| = 2|u|$ and

$$\begin{aligned} u_i = 0 \quad &\Rightarrow \quad \vartheta(u)_{[2i,2i+1]} = 11 \\ u_i = 1 \quad &\Rightarrow \quad \vartheta(u)_{[2i,2i+1]} = 10 \end{aligned}$$

We have $\vartheta^{-1}(\lambda) = \lambda$ and $\vartheta(uv) = \vartheta(u)\vartheta(v)$. A map with these two properties is called a **substitution**. Minimal subshifts constructed from substitutions will be studied in Chapter 4. Substitutions can be iterated. The iterates of 0 and 1 are

$$\vartheta = \left\{ \begin{array}{l} 0 \;\mapsto\; 11 \;\mapsto\; 1010 \;\mapsto\; 1011\;1011 \;\mapsto\; 1011\;1010\;1011\;1010\ldots \\ 1 \;\mapsto\; 10 \;\mapsto\; 1011 \;\mapsto\; 1011\;1010 \;\mapsto\; 1011\;1010\;1011\;1011\ldots \end{array} \right.$$

Set $W_k = \vartheta^k(1)$. Since $\mathcal{K}_0 = 1$, $\mathcal{K}_{[0,2^k)} = W_k$, the second row yields prefixes of $\mathcal{K}$.

$$W_0 = 1, \quad W_1 = 10, \quad W_2 = 1011, \quad W_3 = 1011\;1010, \ldots$$

Note that

$$W_{k+1} = W_k \widehat{W_k} = \vartheta^{k+1}(1), \quad W_k W_k = \vartheta^{k+1}(0)$$

where $\widehat{u}$ is obtained from u by changing the last bit. This is another recursive formula which determines $\mathcal{K}$.

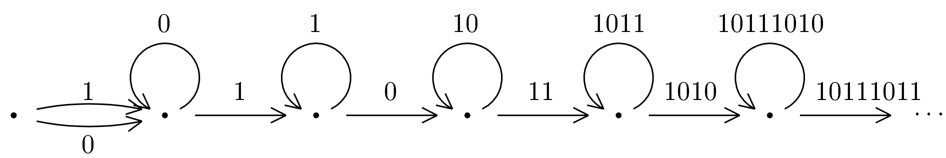

FIGURE 19. The subshift of the Preston system

It can be shown that the subshift $\Sigma_{\mathcal{V}}(X, P)$ consists of all labels of all paths in an infinite graph in Figure 19. The loops in this graph correspond to the itineraries of periodic points (Proposition 1.23)

$$\mathcal{I}(0) = 0^\infty, \quad \mathcal{I}(p_\lambda) = 1^\infty, \quad \mathcal{I}(p_0) = (10)^\infty, \quad \mathcal{I}(p_{00}) = (1011)^\infty, \ldots$$

Like the Preston system, $\Sigma_{\mathcal{V}}(\mathbb{I}, P)$ is neither chaotic nor minimal. It has, however, a minimal subsystem which corresponds to the minimal subsystem (X, P) of the Preston system.

Example 1.30 **(Feigenbaum subshift).** — The sequence $\mathcal{K}$ is quasi-periodic and the subshift

$$\Sigma = \{x \in \mathbf{2}^{\mathbb{N}} : \forall m > 0, \exists i, x_{[0,m]} = \mathcal{K}_{[i,i+m]}\}$$

is minimal. Moreover, $\varphi(\Sigma) = X$, where φ is constructed in Proposition 1.27.

Proof. — Since $\vartheta(0)$ and $\vartheta(1)$ have common prefix 1 and $\mathcal{K}_{[2i,2i+1]} = \vartheta(\mathcal{K}_i)$, $\mathcal{K}_{2i} = 1$ for any i. Similarly, $\mathcal{K}_{[4i,4i+3]} = \vartheta(\mathcal{K}_{[2i,2i+1]}) = \vartheta^2(\mathcal{K}_i)$. Since $\vartheta^2(0)$ and $\vartheta^2(1)$ have common prefix 101, we get $\mathcal{K}_{[4i,4i+2]} = 101$. In the next step we get $\mathcal{K}_{[8i,8i+6]} = 1011101$, the common prefix of $\vartheta^3(0)$ and $\vartheta^3(1)$. In general W_n and $\widehat{W_n}$ have a common prefix of length $2^n - 1$, so $d(\sigma^{2^n}(\mathcal{K}), \mathcal{K}) = 2^{-2^n}$ and $\mathcal{K}$ is quasi-periodic.

Let $x, y \in \Sigma$, and $m > 0$. We show that for some $k > 0$, $d(\sigma^k(x), y) < 2^{-m}$. There exist i, j such that $x_{[0,m]} = \mathcal{K}_{[i,i+m]}$ and $y_{[0,m]} = \mathcal{K}_{[j,j+m]}$. Since $\mathcal{K}$ is quasi-periodic, we may assume $i > j$, so $d(\sigma^{i-j}(x), y) < 2^{-m}$. Thus the trajectory of x is dense, and (Σ, σ) is minimal. Since c_0 is contained in all intervals J_{1^n}, $c_0 \in X$ and therefore $\varphi(\mathcal{K}) = c_1 \in X$. Since $\mathcal{O}(\mathcal{K})$ is dense in Σ, $\mathcal{O}(c_1)$ is dense in X and $\varphi(\Sigma) = X$. $\square$

In the factor map $\varphi : (\Sigma, \sigma) \to (X, P)$ a point $x \in X$ has unique preimage iff it has a unique itinerary and this happens iff its trajectory never visits the critical point c_0. Since P is bijective on X, there exists unique $c_{-1} \in X$ with $P(c_{-1}) = c_0$, and in general a unique $c_{-n} \in X$ with $P^n(c_{-n}) = c_0$. These are the only points with two preimages

$$\varphi^{-1}(c_0) = \{0\mathcal{K}, 1\mathcal{K}\}, \quad \varphi^{-1}(c_{-1}) = \{10\mathcal{K}, 11\mathcal{K}\}, \quad \varphi^{-1}(c_{-2}) = \{010\mathcal{K}, 011\mathcal{K}\}, \ldots$$

Since (X, P) is conjugate to $(\mathbf{2}^{\mathbb{N}}, R_1)$, we have a factor map $\chi : (\Sigma, \sigma) \to (\mathbf{2}^{\mathbb{N}}, R_1)$ with $\pi \circ \chi = \varphi$.

1.11. Rotations

Minimal systems occur also in theories of planetary motions. Assume for simplicity that every planet moves around the sun in a circle with constant velocity. Its position can be represented by a point of $\mathbb{T} = [0, 1)$. The vector of positions of n planets is a point in the n-dimensional torus $\mathbb{T}^n$. The two-dimensional torus $\mathbb{T}^2$ is a surface which can be embedded into $\mathbb{R}^3$. It can be constructed from a square. Identify first the lower and upper sides of the square to obtain a cylinder (Figure 20 center) and then identify the left and right boundary circles (Figure 20 right). The distance in $\mathbb{T}^n$ is defined by

$$d(x, y) = \max\{d_i(x_i, y_i) : 1 \leq i \leq n\}.$$

Suppose that the i-th planet moves with angular velocity $a_i = 1/T_i$, where T_i is its orbit time. If the position of the i-th planet at time 0 is $x_i \in \mathbb{T}$, its position at time t is $\mathrm{mod}_1(x_i + a_i t)$. Thus we get a flow $f : \mathbb{T}^n \times \mathbb{R} \to \mathbb{T}^n$ defined by

$$f(x, t)_i = \mathrm{mod}_1(x_i + a_i t).$$

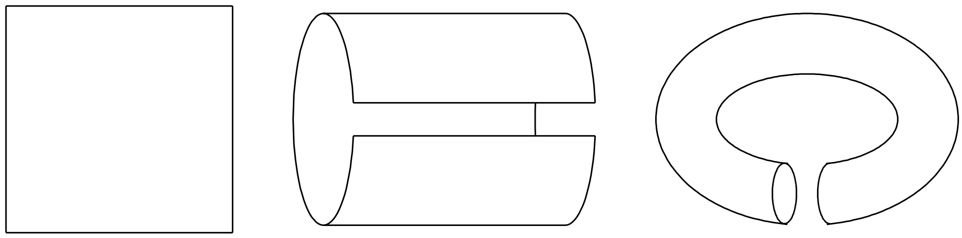

FIGURE 20. Torus

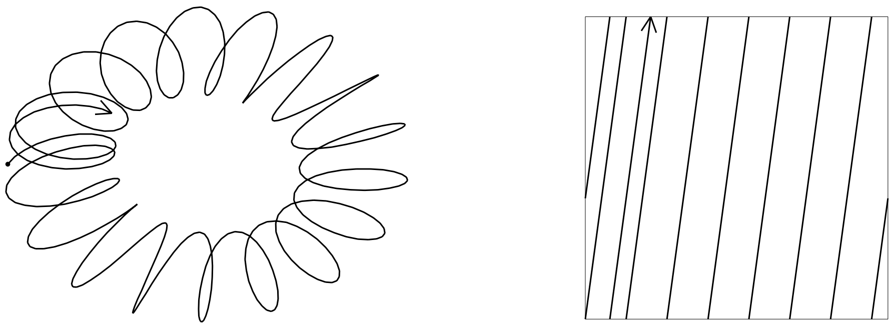

FIGURE 21. Spirals on the torus

Trajectories of f are spirals on the surface of the torus (Figure 21). If we record the positions only at discrete time instants, say every midnight, we get a dynamical system

Example 1.31 **(Rotation)**. — $(\mathbb{T}^n, R_a)$ where $a = (a_1, \ldots a_n) \in \mathbb{T}^n$ and

$$R_a(x)_i = \mathrm{mod}_1(x_i + a_i).$$

Like the adding machine, rotations are isometries defined on a group. The addition in $\mathbb{T}^n$ is defined by

$$(x + y)_i = \mathrm{mod}_1(x_i + y_i)$$

For any $a, b \in \mathbb{T}^n$, $R_a \circ R_b = R_b \circ R_a$, so $R_b : (\mathbb{T}^n, R_a) \to (\mathbb{T}^n, R_a)$ is a conjugacy. For arbitrary points x, y there exists $b = y - x$ such that $y = R_b(x)$, so any two points have the same dynamical properties. In particular to show that $(\mathbb{T}^n, \mathbb{R}_a)$ is minimal, it suffices to show that one of its orbits is dense. We study first one-dimensional rotations $(\mathbb{T}, R_a)$ where $a \in \mathbb{T}$ and $R_a(x) = \mathrm{mod}_1(x + a)$ (Figure 22).

Proposition 1.32. — *Let $(\mathbb{T}, R_a)$ be a rotation of the circle.*

(1) If a is rational, then each point is periodic and no trajectory is dense.

(2) If a is irrational, then no point is periodic and every trajectory is dense, so the system is minimal.

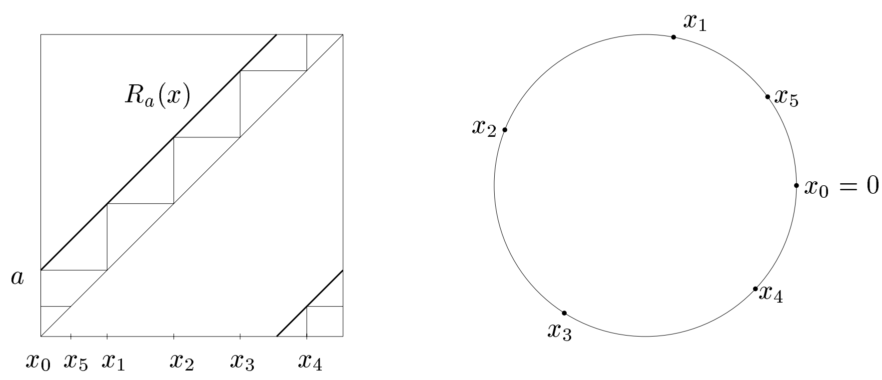

FIGURE 22. Rotation of the circle

Proof. — Let $a = \frac{p}{q}$ be rational with p, q relatively prime. Then for any point x, $R_a^q(x) = \mathrm{mod}_1(x + p) = x$, so any point is periodic with the same period q, but no point has a dense orbit.

If a is irrational, then for any integer q, $\mathrm{mod}_1(qa) \neq 0$, so no point is periodic. We show that 0 has dense orbit. For a given $\varepsilon > 0$ pick an integer n with $\frac{1}{n} < \varepsilon$, and consider the points $0, R_a(0), \dots, R_a^{n-1}(0)$. Since a is irrational, all these points are distinct. Since the length of the circle is 1, at least two of them are at distance at most $\frac{1}{n}$. There exist $0 \leq i < j < n$ such that $d(R_a^i(0), R_a^j(0)) < \frac{1}{n}$ and, since R_a is an isometry,

$$d(0, R_a^{j-i}(0)) = d(R_a^i(0), R_a^j(0)) \leq \frac{1}{n} < \varepsilon.$$

Now $R_a^{j-i} = R_b$ is also a rotation of the circle and either $0 < b < \frac{1}{n}$ or $1 - \frac{1}{n} < b < 1$. In either case, for any $x \in \mathbb{T}$ there exists $k > 0$ such that $d(R_a^{k(j-i)}(0), x) < \varepsilon$. This shows that the orbit of 0 is dense. It follows that any point has a dense orbit. □

1.11.1. Recurrent and almost periodic points. — If a is irrational, the points of $(\mathbb{T}, R_a)$ are not quasi-periodic. A point $x \in \mathbb{T}$ does return to all of its neighbourhoods, but not periodically. We say that x **recurrent**. Moreover, the times between successive returns form a bounded sequence. We say that x is **almost periodic**.

Definition 1.33. — Let (X, F) be a dynamical system and $x \in X$.

(1) The point x is **almost periodic**, if

$$\forall \varepsilon > 0, \exists p > 0, \forall n \geq 0, \exists k < p, d(F^{n+k}(x), x) < \varepsilon$$

(2) The point x is **recurrent**, if

$$\forall \varepsilon > 0, \exists n > 0, d(F^n(x), x) < \varepsilon$$

We have implications

$$\text{periodic} \Rightarrow \text{quasi-periodic} \Rightarrow \text{almost periodic} \Rightarrow \text{recurrent}$$

If x is recurrent, it returns to all of its neighbourhoods but we have no information on the gaps between return times. If x is almost periodic, any of its neighbourhoods $U \ni x$ has a period p, such that the difference between consecutive return times of x to U is less than p. If x is quasi-periodic, the point returns precisely at the p-th step.

Finally if it is periodic, the period of return is the same for all neighbourhoods. In the binary full shift, there exist points which are recurrent but not almost periodic.

Example 1.34. — The Champernowne sequence $x = 0100011011\ldots$ which concatenates all finite binary words, is recurrent but not almost periodic.

Proof. — Suppose that x were almost periodic, and let p be its period of return into the cylinder set $[0]$. Since x contains all finite words, for some n, $x_{[n,n+p)} = 1^p$, so for any $0 \le k < p$, $\sigma^{n+k}(x) \notin [0]$. Thus x is not almost periodic. We show that x is recurrent. Let $\varepsilon > 0$ and $2^{-m} < \varepsilon$. There exists $n > 0$ such that $x_{[n,n+m)} = x_{[0,m)}$, so $d(\sigma^n(x), x) \le 2^{-m} < \varepsilon$. □

Almost periodic point which are not quasi-periodic occur in irrational rotations.

Proposition 1.35. — *Any point of an irrational rotation of the circle is almost periodic but not quasi-periodic.*

Proof. —

(1) Let a be irrational and $\varepsilon > 0$, By Proposition 1.32, there exists $i > 0$ such that $b = d(0, R_a^i(0)) < \varepsilon$. Let j be the unique integer with $\frac{1}{j} < d(0, R_a^i(0)) < \frac{1}{j-1}$ and set $p = ji$. Given $n > 0$, consider the sequence

$$R^n(0), R^{n+i}(0), R^{n+2i}(0) \ldots R^{n+ji}(0)$$

The distance of any two consecutive points is less than ε. Since $jb > 1$, there exists $k < j$, such that $d(R_a^{n+ki}(0), 0) < \varepsilon$. We have $ki < p$, so 0 is almost periodic. It follows that any point of $\mathbb{T}$ is almost periodic.

(2) For any $p > 0$, $(\mathbb{T}, R_a^p)$ is an irrational rotation of the circle, so its orbits are dense and not confined to neighbourhoods of the initial points. □

1.11.2. Dirichlet theorem.

— To analyze the properties of multidimensional rotations we first prove the Dirichlet theorem.

***Theorem 1.36* (Dirichlet).** — *For real numbers $a_1, \ldots, a_n$ and an integer $t > 0$ there exists an integer $q \in [1, t^n]$ and integers $p_1, \ldots, p_n$, such that for any $1 \le i \le n$, $|qa_i - p_i| < \frac{1}{t}$.*

Proof. — For $m = 0, \ldots, t^n$ and $i = 1, \ldots, n$ put $b_{m,i} = \mathrm{mod}_1(ma_i)$ and consider vectors

$$(\lfloor tb_{m,1} \rfloor, \ldots, \lfloor tb_{m,n} \rfloor) \in \{0, 1, \ldots, t-1\}^n$$

Since there are only t^n different vectors in $\{0, \ldots, t-1\}^n$, there exist $0 \le m < k \le t^n$ such that

$$(\lfloor tb_{m,1} \rfloor, \ldots, \lfloor tb_{m,n} \rfloor) = (\lfloor tb_{k,1} \rfloor, \ldots, \lfloor tb_{k,n} \rfloor)$$

and therefore $|tb_{k,i} - tb_{m,i}| < 1$. There exist integers $c_1, \ldots, c_n, d_1, \ldots, d_n$, such that $b_{m,i} = ma_i - c_i$ and $b_{k,i} = ka_i - d_i$, so

$$|(ka_i - d_i) - (ma_i - c_i)| < \frac{1}{t}, \quad i = 1, \ldots, n$$

For $q = k - m$ and $p_i = d_i - c_i$ we get $|qa_i - p_i| < \frac{1}{t}$ for any $1 \le i \le n$. □

The Dirichlet theorem says that in any rotation $(\mathbb{T}^n, R_a)$, the point $0 = (0, \dots, 0)$ (and hence any point) is recurrent. (If all the a_i are rational, it is even periodic.) Minimality of rotations depends not only on the rationality of its parameters but also on the rationality of their ratios.

Example 1.37. — Let $(\mathbb{T}^2, R_a)$ be a rotation with a_1, a_2 irrational and a_1/a_2 rational. If p_i are integers with $p_1 a_1 = p_2 a_2$, then

$$\mathcal{O}(0) \subseteq \{x \in \mathbb{T}^2 : \mathrm{mod}_1(p_1 x_1) = \mathrm{mod}_1(p_2 x_2)\}$$

The orbit of 0 is contained in a closed spiral $x(t) = \mathrm{mod}_1(at)$ on the surface of the torus and the system is not minimal (Figure 23 left). If a_1/a_2 is irrational, the spiral $x(t) = \mathrm{mod}_1(at)$ fills densely all the torus and the system is minimal. We show this in general.

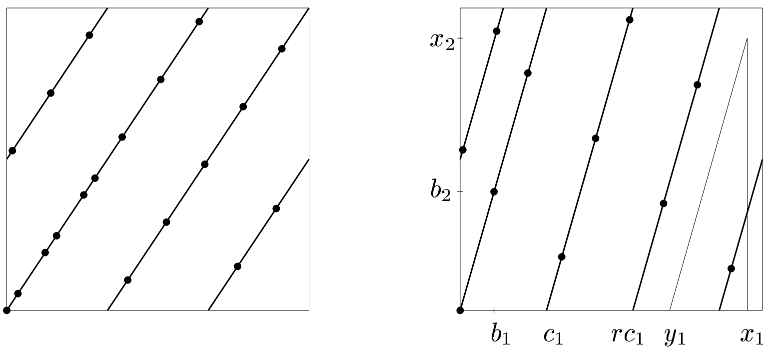

FIGURE 23. Two-dimensional rotations

Proposition 1.38. — *Let $a \in \mathbb{T}^n$ be such that all a_i and all their ratios a_i/a_j, $i \neq j$ are irrational. Then $(\mathbb{T}^n, R_a)$ is minimal and all its points are almost periodic.*

Proof. — For $n = 1$, the statement is proved in Propositions 1.32 and 1.35. Assume that the statement holds for $n-1$ and let us prove it for n. Let $x \in \mathbb{T}^n$ and $\varepsilon > 0$ (Figure 23 right). By the Dirichlet theorem 1.36, there exists $s > 0$ such that for $b = R_a^s(0)$, $d(b, 0) < \frac{\varepsilon}{2}$. Without loss of generality we can assume that b_n is the largest component of b and set $c = (\frac{b_1}{b_n}, \dots, \frac{b_{n-1}}{b_n}) \in \mathbb{T}^{n-1}$. By the induction hypothesis, $(\mathbb{T}^{n-1}, R_c)$ is minimal, so for the vector

$$y = x - c x_n = (x_1 - c_1 x_n, \dots, x_{n-1} - c_{n-1} x_n)$$

there exists an integer $r > 0$ such that $d(R_c^r(0), y) < \frac{\varepsilon}{2}$. Thus there exist integers $p_1, \dots, p_{n-1}$, such that

$$|rc_i - x_i + c_i x_n - p_i| < \frac{\varepsilon}{2}, \quad 1 \le i \le n-1.$$

Since $d(b_n, 0) < \frac{\varepsilon}{2}$, there exists an integer $q > 0$ such that $|qb_n - r - x_n| < \frac{\varepsilon}{2}$. Multiplying this inequality by $c_i = b_i/b_n$ we get

$$|qb_i - rc_i - c_i x_n| < \frac{\varepsilon c_i}{2} < \frac{\varepsilon}{2}, \quad 1 \le i \le n-1$$

We add the last two inequalities and get $|qb_i - p_i - x_i| < \varepsilon$, so $d(R_b^q(0), x) < \varepsilon$ and $d(R_a^{sq}(0), x) < \varepsilon$. This means that the orbit of 0 is dense and $(\mathbb{T}^n, R_a)$ is minimal. Similarly we can prove that 0 is almost periodic in $(\mathbb{R}^n, T_a)$. □

1.12. Spiral phyllotaxis

The theory of irrational rotations has a nice application in botany. Many higher plants position their leaves around the axis of their stem in a strikingly regular pattern. The leaves form a system of spirals winding in the same direction, and another system of spirals winding in the opposite direction (see e.g., D'Arcy Thompson [**150**]). The numbers of spirals in these two systems are usually two consecutive Fibonacci numbers. This phenomenon can be explained if we suppose that consecutive leaves have constant angular difference $2\pi\alpha$, α being the golden number, and constant vertical difference $c > 0$. The leaves then lie on **genetic spiral** on the surface of a cylinder with equation

$$x(t) = \cos(2\pi\alpha t), \quad y(t) = \sin(2\pi\alpha t), \quad z(t) = ct$$

The n-th leaf has coordinates $(x(n), y(n), z(n))$. If we parametrize the circle by $\mathbb{T} = [0, 1)$, we get the angle and height of the n-th leaf (Figure 24) as

$$r_n = \mathrm{mod}_1(n\alpha), \quad z_n = cn$$

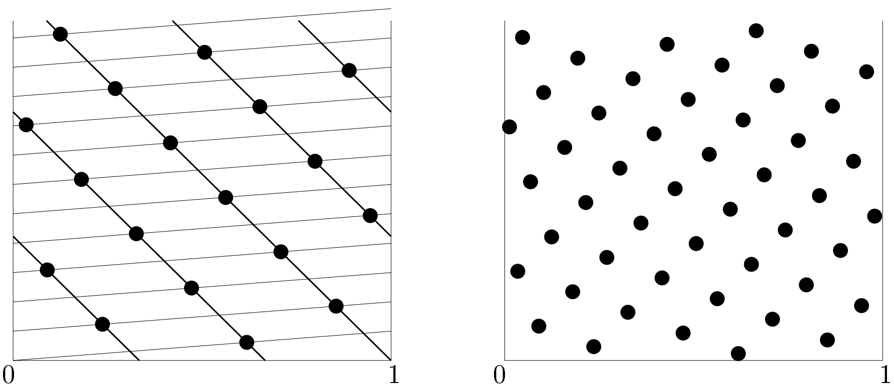

Figure 24. Spiral phyllotaxis

Besides the genetic spiral, many other spirals can be found in the pattern. For any $m > 1$ there exists a spiral linking leaves number $m, 2m, 3m, \ldots$, along with parallel spirals $k, k + m, k + 2m, \ldots$ for $k < m$. The equations of these systems of spirals are

$$x(t) = \cos 2\pi(tr_m + k\alpha), \quad y(t) = \sin 2\pi(tr_m + k\alpha), \quad z(t) = c(m + k)t$$

However, if the m-th leaf is far from 0, this system of spirals is not easily discernible. The most conspicuous systems of spirals are obtained for leaves whose angles are close to 0, i.e., for those m for which (r_m, z_m) is close to zero.

1.12.1. Golden number. — The numbers m are obtained when we write the golden number $\alpha = \frac{\sqrt{5}-1}{2}$ as a **continued fraction** (see Section A.10). From $\frac{1}{\alpha} = \alpha + 1$ we get

$$\alpha = \frac{1}{1+\alpha} = \frac{1}{1+\dfrac{1}{1+\alpha}} = \cdots = \frac{1}{1+\dfrac{1}{1+\cdots}}$$

A finite continued fraction in which only 1's appear can be written in the following way:

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}} = \frac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}} = \frac{1}{1+\dfrac{2}{3}} = \frac{3}{5}$$

so the n-th convergent of α is F_{n-1}/F_n, where F_n is the n-th Fibonacci number.

Example 1.39. — The Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}$$

have equation

$$F_n = \frac{\alpha^{-n} - (-\alpha)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Proof. — We search the solution of the equation $F_{n+1} = F_n + F_{n-1}$ of the form $F_n = x^n$. We get $x^2 = x+1$ and there are two solutions $-\alpha$ and $1/\alpha = \frac{\sqrt{5}+1}{2} \doteq 1.618$. Any linear combination $x_n = a(\alpha)^{-n} + b(-\alpha)^n$ then satisfies $x_{n+1} = x_n + x_{n-1}$. The initial condition $F_0 = 0$, $F_1 = 1$ yields the formula. □

The sequence $r_n = \mathrm{mod}_1(n\alpha)$ comes closer to 0 at Fibonacci times. Indeed,

$$F_n\alpha - F_{n-1} = \frac{(-\alpha)^{n+1} + (-\alpha)^{n-1}}{\sqrt{5}} = \frac{(-\alpha)^{n-1}(\alpha^2+1)}{\sqrt{5}} = -(-\alpha)^n$$

so the distance $d(r_{F_n}, 0) = \alpha^n$ is a geometric sequence. The square of the distance of the F_n-th leaf from the starting point is

$$(r_{F_n})^2 + (z_{F_n})^2 = \alpha^{2n} + c^2(F_n)^2 = \frac{c^2+5}{5}\alpha^{2n} + \frac{c^2}{5}\alpha^{-2n} + \frac{2c^2}{5}(-1)^n$$

If we evaluate the derivative with respect to n, we find that this expression attains minimum at

$$n = \frac{\ln(1+5c^{-2})}{4\ln\alpha^{-1}} \doteq 0.52 \cdot \ln\left(1 + \frac{5}{c^2}\right)$$

We compute c from this equation, and get

$$c_n = \sqrt{\frac{5}{\alpha^{-4n} - 1}} \doteq \sqrt{5} \cdot \alpha^{2n}$$

For example for $n = 4$, $c_4 = 0.048$, $F_4 = 3$, so the most conspicuous system contains three parallel spirals winding to the left (Figure 24 left). We can also see two parallel right-winding spirals with $F_3 = 2$ and also five parallel right-winding spirals with $F_5 = 5$. For $n = 5$ we get $c_5 = 0.0182$, $F_5 = 5$ and the most conspicuous system contains five spirals winding to the right (Figure 24 right).

1.12.2. Sunflower head. — Another example of spiral phyllotaxis can be seen in a sunflower head. Here we see systems of spirals which wind from the center to periphery. As in the preceding case, there are many systems of parallel spirals. Which system is most conspicuous depends on the distance from the center. Closer to the center we see rather systems of spirals with low Fibonacci numbers, and as we move to the perimeter we see systems with ever higher Fibonacci numbers. This pattern can be explained if we assume that the florets are positioned on a planar genetic spiral in which the golden angle separates any two consecutive florets. However, the shape of the spiral causes some problems. The classical cases of geometric spiral $r = a \cdot \exp\varphi$ or arithmetic spiral $r = a \cdot \varphi$ are not appropriate since the density of florets would decrease with increasing radius. To ensure constant density of florets independent of the radius we need to use the **cyclometric spiral** whose equation is $r = a \cdot \sqrt{\varphi}$ (Vogel [**156**], Figure 25 left). The position of the n-th floret is

$$x_n = a\sqrt{n}\cos 2\pi n\alpha, \quad y_n = a\sqrt{n}\sin 2\pi n\alpha.$$

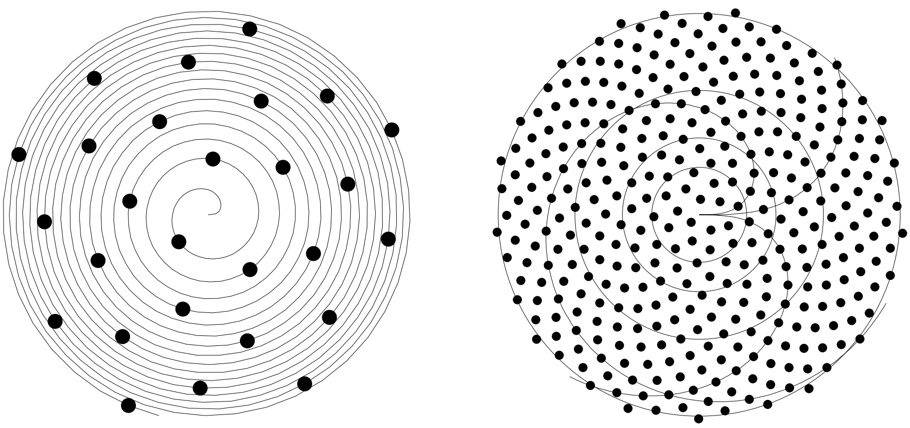

Figure 25. Cyclometric spirals

The disk of radius r then contains r^2/a^2 florets and this is proportional to its area. The m-th floret is at radius $r_m = a\sqrt{m}$. The ratio of the difference of consecutive radii to the length of the circle, is

$$c = \frac{r_{m+1} - r_m}{2\pi r_m} \doteq \frac{a}{2\sqrt{m}\cdot 2\pi a\sqrt{m}} = \frac{1}{4\pi m}$$

The system of F_n spirals is therefore best visible at the floret number

$$M(n) = \frac{1}{4\pi c_n} = \frac{\sqrt{\alpha^{-4n} - 1}}{4\pi\sqrt{5}} \doteq \frac{\alpha^{-2n}}{4\pi\sqrt{5}} \doteq 0.036\cdot\alpha^{-2n}$$

This floret is positioned at radius $R(n) = a\sqrt{M(n)} \doteq 0.189\cdot a\cdot\alpha^{-n}$. The spirals in the n-th system are given by the equations

$$r_{nk}(t) = a\sqrt{tF_n + k}, \quad \varphi_{nk}(t) = 2\pi t(F_n\alpha - F_{n-1}) + k\alpha, \quad k = 0, 1, \dots, F_n - 1$$

In Figure 25 right we see three spirals with $F_7 = 13$, $F_8 = 21$ and $F_9 = 35$. The circles at radii $R(6.5)$, $R(7.5)$, $R(8.5)$ separate regions where a particular system of spirals is best visible. In Figure 26 we see a model of the sunflower head with $R(10.5) = 871$ florets where the systems of spirals up to $F_{10} = 55$ can be observed.

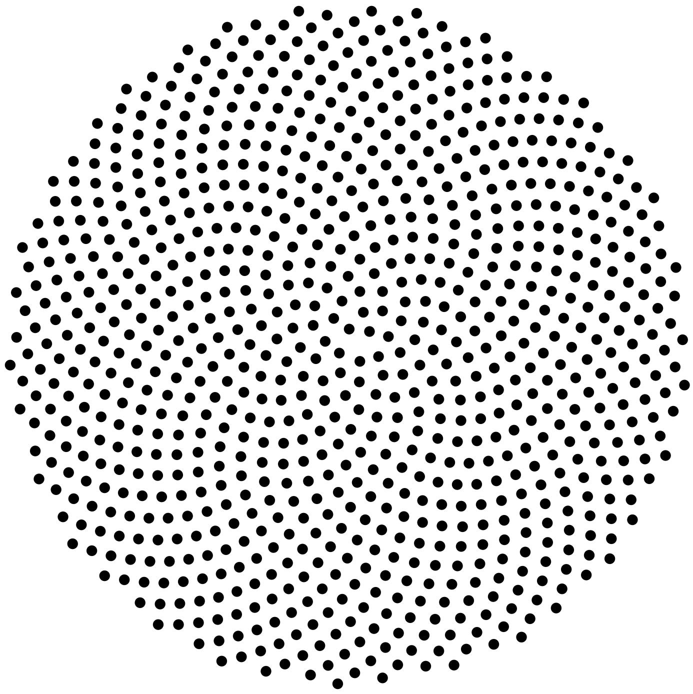

FIGURE 26. Sunflower head

Exercise 1.1. — Find fixed points and their stabilities in the following dynamical systems

(1) $([-1,1], x^3)$.
(2) $([-1,1], \sqrt[3]{x})$.
(3) $(\mathbb{I}, F_a)$, where $F_a(x) = x + \frac{x}{a+1}\sin(a \ln x), a > 0$.
(4) $(\mathbb{T}, x + a \sin 2\pi x)$, $a > 0$
(5) $(\mathbb{I}, F_{r,a})$, where $0 < a < \frac{1}{2}$, $F_{r,a}(x) = 0$ for $x \in [0,a] \cup [1-a,1]$, and $F_{r,a}(x) = r(x-a)(1-a-x)$ for $x \in [a, 1-a]$.

Show that the following systems are chaotic

(6) $(\mathbb{I}, F)$, where $F(x) = 1 - 2|x - \frac{1}{2}|$
(7) $(\mathbb{I}, F)$, where $F(x) = \frac{x}{1-x}$ for $x \leq \frac{1}{2}$ and $F(x) = \frac{1-x}{x}$ for $x \geq \frac{1}{2}$.
(8) $([-1,1], F)$ where $F(x) = 3x - 4x^3$ (use conjugacy with $(\mathbb{T}, \mathrm{mod}_1(3x))$.
(9) Find invariant sets of $(\mathbb{R}, 5x(1-x))$.