CHAPTER 2

TOPOLOGICAL DYNAMICS

Bearing in mind the examples from Chapter 1, we will now develop dynamics in the setting of compact metric spaces. (The basic theory of compact metric spaces is reviewed in Appendix A.) We define a dynamical system to be a continuous map $F : X \to X$ on a compact metric space X and investigate its iterations and trajectories. To treat the basic concepts like transitivity, minimality or recurrence, we use dynamical relations considered by Akin [**3**]. Given a dynamical system $F : X \to X$, we ask whether a point $y \in X$ can be reached from a point $x \in X$. Depending on the exact meaning of "reached", we get several relations $R \subseteq X \times X$. It is desirable that these relations be transitive i.e., $R \circ R \subseteq R$, and topologically closed as subsets of $X \times X$. We start with the **orbit relation** $\mathfrak{O}$, which is the transitive closure of F. While $\mathfrak{O}$ is transitive, it is not topologically closed. We pass to the closure of its image sets $\mathfrak{O}(x)$ to obtain the **recurrence relation** $\mathfrak{R}$. Still larger is the **nonwandering relation** $\mathfrak{N}$, which is the topological closure of $\mathfrak{O}$ in $X \times X$. While $\mathfrak{N}$ is topologically closed, it is not necessarily transitive. The **chain relation** $\mathfrak{C}$ is both transitive and topologically closed.

Concepts such as minimality, transitivity or recurrence have now very natural form. A system is **minimal** if its recurrence relation $\mathfrak{R} = X \times X$ contains all pairs of points. A system is **transitive** if its nonwandering relation is $\mathfrak{N} = X \times X$. Recurrent points are points in the diagonal $|\mathfrak{R}|$. We extend this classification to chain-transitive systems ($\mathfrak{C} = X \times X$), nonwandering points $|\mathfrak{N}|$ and chain recurrent points $|\mathfrak{C}|$.

Next we generalize the concept of stability. We say that a point $x \in X$ is **equicontinuous**, if all neighbouring points have trajectories which stay close to the trajectory of x. We classify the dynamical systems according to the size of their equicontinuous set. This classification simplifies under the additional assumption of transitivity. **Sensitive** systems have no equicontinuous points at all while in **equicontinuous** systems all points are equicontinuous. Between these two extremes there are **almost equicontinuous** systems, whose equicontinuous set is topologically large (residual).

A particularly well understood class is of systems is that of minimal equicontinuous systems. All these systems are **rotations** on compact groups. The state space is equipped with the group structure and the group operations are continuous. The dynamics is of the form $F(x) = x + a$, where a is an element of X. Two examples are dyadic adding machine of Example 1.24 or the torus rotations of Example 1.31. Group rotations are conveniently described and classified by **spectral theory**. The spectrum of a dynamical system is the spectrum of the continuous linear operator which the system induces on the space $\mathbf{C}(X, \mathbb{C})$ of continuous complex-valued functions on X. Two group rotations are conjugate iff they have the same spectrum. A system which is not equicontinuous cannot be completely classified in this way, but is at least partially classified by its **maximal equicontinuous factor**.

In systems which are not transitive, the concept of **attractor** is of utmost importance. An attractor is a subsystem of the original system which attracts all trajectories starting in some neighbourhood. The largest neighbourhood with this property is called the **basin of attraction**. A dynamical system might possess several attractors with disjoint basins; likewise an attractor might include a smaller one. The structure of attractors can be read off from the chain relation $\mathfrak{C}$. This is based on the observation that chains never escape far from attractors.

The next topic is **topological entropy** which characterizes the mean amount of information which we gain when we observe a dynamical system through a finite cover. The future behaviour of systems with zero entropy can be to a large extent predicted in advance. In systems with positive entropy such a prediction is impossible. Another statistical characteristic is the limit frequency of visits to specified regions of the state space. In **strictly ergodic systems** this frequency is the same for all initial conditions. In other systems such a limit frequency may not exist, or it may depend on initial conditions. This leads to the concept of **invariant measure** on which ergodic theory is based.

2.1. Dynamical relations

Many important theorems of topological dynamics hold only under the assumption that the underlying state space is compact. We restrict therefore accordingly Definition 1.5 of dynamical system.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "G"] & Y \end{tikzcd}$$

Definition 2.1. —

(1) A **dynamical system** is a pair (X, F) where X is a compact metric space and $F : X \to X$ is a continuous mapping.

(2) A **morphism** $\pi : (X, F) \to (Y, G)$ between two dynamical systems is a continuous map $\pi : X \to Y$ satisfying $\pi \circ F = G \circ \pi$.

(3) If π is bijective, we say that π is a **conjugacy** and that (X, F) and (Y, G) are **conjugate**.

(4) If π is surjective, we say that π is a **factor map** and (Y, G) is a **factor** of (X, F) or that (X, F) is an **extension** of (Y, G).

(5) If π is injective, we say that (X, F) is a **subsystem** of (Y, G).

(6) A subset $Z \subseteq X$ of a dynamical set (X, F) is **invariant** if $F(Z) \subseteq Z$.

If $Z \subseteq Y$ is invariant, its closure $\overline{Z}$ is invariant too, since $G(\overline{Z}) \subseteq \overline{G(Z)} \subseteq \overline{Z}$. If $Z \subseteq Y$ is nonempty, closed and invariant, then it is compact and (Z, G) is a subsystem of (Y, G). Conversely, if $\pi : (X, F) \to (Y, G)$ is a subsystem then $Z = \pi(X) \subseteq Y$ is a closed invariant set. Thus subsystems of a dynamical system can be identified with its closed invariant subsets. Frequently, dynamical systems are constructed as **orbit closures**. The **orbit** of a point $x \in X$ is $\mathcal{O}(x) = \{F^n(x) : n > 0\}$. Clearly $\mathcal{O}(x)$ is an invariant set, so $\overline{\mathcal{O}(x)}$ is a closed and invariant set and therefore a subsystem of (X, F). For example, the Feigenbaum subshift of section 1.9 is the orbit closure of the Feigenbaum sequence. Other constructions are the product $(X \times Y, F \times G)$ and disjoint sum $(X \uplus Y, F \uplus G)$ of dynamical systems (X, F) and (Y, G).

$$(F \times G)(x, y) = (F(x), G(y)), \quad (F \uplus G)(z) = \begin{cases} F(z) & \text{if } z \in X \\ G(z) & \text{if } z \in Y \end{cases}$$

Definition 2.2. — Let (X, F) be a dynamical system. The **orbit relation** $\mathcal{O}$, the **recurrence relation** $\mathcal{R}$, the **nonwandering relation** $\mathcal{N}$, and the **chain relation** $\mathcal{C}$ are defined on X by

$$\begin{aligned}
(x, y) \in \mathcal{O} &\Leftrightarrow \exists n > 0, y = F^n(x) \\
(x, y) \in \mathcal{R} &\Leftrightarrow \forall \varepsilon > 0, \exists n > 0, d(y, F^n(x)) < \varepsilon \\
(x, y) \in \mathcal{N} &\Leftrightarrow \forall \varepsilon, \delta > 0, \exists n > 0, \exists z \in B_\delta(x), d(F^n(z), y) < \varepsilon \\
(x, y) \in \mathcal{C} &\Leftrightarrow \forall \varepsilon > 0, \exists n > 0, \exists x_0, \ldots, x_n, x = x_0, x_n = y, \ \& \\
&\qquad \forall i < n, d(F(x_i), x_{i+1}) < \varepsilon
\end{aligned}$$

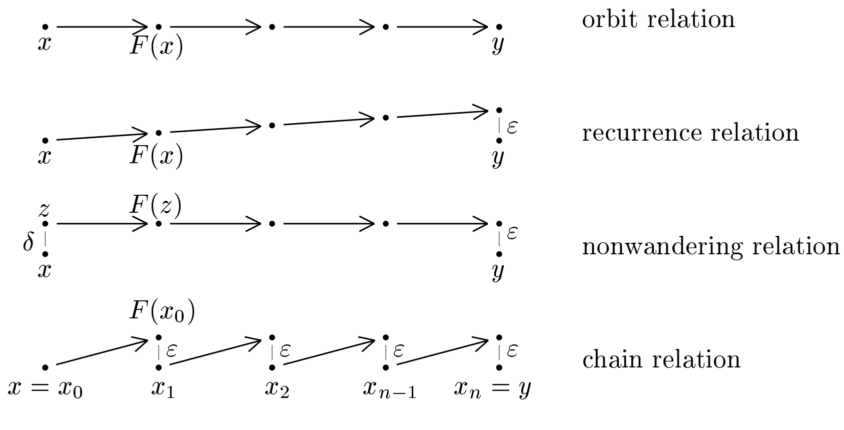

FIGURE 1. Dynamical relations

Thus a point $x \in X$ is periodic iff $x \in |\mathcal{O}|$. The **orbit** of a point $x \in X$ is the set $\mathcal{O}(x) = \{F(x), F^2(x), \ldots\}$. This coincides with our earlier definition. In contrast to a trajectory which is a sequence starting with the zeroth iteration, the orbit is a set and starts with the first iteration.

All these relations are topological concepts. This is clear for $\mathcal{O}$, as the metric does not enter the definition. For the other relations we get

$(x,y) \in \mathfrak{R}$ iff $y \in \overline{\mathfrak{O}(x)}$ iff for any neighbourhood U of y there exists $n > 0$ such that $F^n(x) \in U$

$(x,y) \in \mathfrak{N}$ iff $(x,y) \in \overline{\mathfrak{O}}$, i.e., iff for any neighborhood U of x and any neighbourhood V of y there exists $n > 0$ with $F^n(U) \cap V \neq \emptyset$ (or equivalently $U \cap F^{-n}(V) \neq \emptyset$).

$(x,y) \in \mathfrak{C}$ iff for any neighbourhood $U \subseteq X \times X$ of the diagonal there exists $n > 0$ and a sequence $x = x_0, \ldots, x_n = u$ such that $(F(x_i), x_{i+1}) \in U$.

The chain relation is defined in terms of pseudotrajectories or ε-chains. A (finite or infinite) sequence $(x_n)_{n\geq 0}$ is an ε-**chain**, if $d(F(x_n), x_{n+1}) < \varepsilon$ for all n. If a trajectory of a dynamical system is computed numerically with round-off errors non exceeding ε, we obtain rather an ε-chain than a trajectory.

We show an example where the dynamical relations differ from each other.

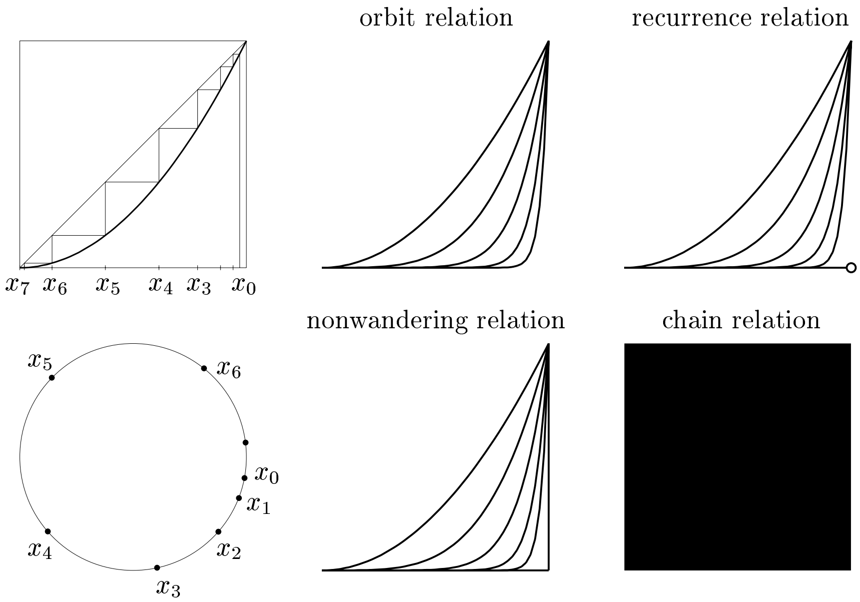

FIGURE 2. Quadratic map on the circle

Example 2.3 **(Quadratic system on the circle).** — $(\mathbb{T}, Q)$ where $Q(x) = x^2$.

The system has a unique fixed point 0 and for any $x \in \mathbb{T}$, $\lim_{n\to\infty} Q^n(x) = 0$. However, 0 is not stable. This shows that in Definition 1.7 of attracting fixed point, the assumption of stability cannot be left out. Dynamical relations of $(\mathbb{T}, Q)$ are in Figure 2.

$$
\begin{aligned}
\mathfrak{O} &= \{(x, Q^n(x)) : x \in \mathbb{T}, n > 0\} \\
\mathfrak{R} &= \mathfrak{O} \cup (\mathbb{T} \times \{0\}) \\
\mathfrak{N} &= \mathfrak{R} \cup (\{0\} \times \mathbb{T}) \\
\mathfrak{C} &= \mathbb{T} \times \mathbb{T}
\end{aligned}
$$

Since for any $x \in \mathbb{T}$, $\lim_{n\to\infty} Q^n(x) = 0$, $\mathcal{R} = \mathcal{O} \cup (\mathbb{T} \times \{0\})$. Since Q is a homeomorphism, negative iterations also exist and for any x, $\lim_{n\to\infty} Q^{-n}(x) = 0$. It follows $\mathcal{N} = \mathcal{R} \cup (\{0\} \times \mathbb{T})$. To see that $\mathcal{C} = \mathbb{T} \times \mathbb{T}$, let $x, y \in \mathbb{T}$ and $\varepsilon > 0$. There exists $n > 0$, such that $d(Q^n(x), 0) < \frac{\varepsilon}{2}$ and $d(Q^{-n}(y), 0) < \frac{\varepsilon}{2}$. Thus

$$x, Q(x), \ldots, Q^{n-1}(x), Q^{-n}(y), \ldots, Q^{-1}(y), y$$

is an ε-chain from x to y, so $(x, y) \in \mathcal{C}$. Note that the relations $\mathcal{N}$ and $\mathcal{C}$ are closed while $\mathcal{O}$ and $\mathcal{R}$ are not. Also, $\mathcal{O}$ and $\mathcal{C}$ are transitive, while $\mathcal{R}$ and $\mathcal{N}$ are not. This holds in general.

Proposition 2.4. — *Let (X, F) be a dynamical system.*
(1) $\mathcal{O} \subseteq \mathcal{R} \subseteq \mathcal{N} \subseteq \mathcal{C}$.
(2) $\mathcal{O}$ *is transitive.*
(3) $\mathcal{N} = \overline{\mathcal{O}}$ *is closed (in $X \times X$).*
(4) $\mathcal{C}$ *is transitive and closed.*

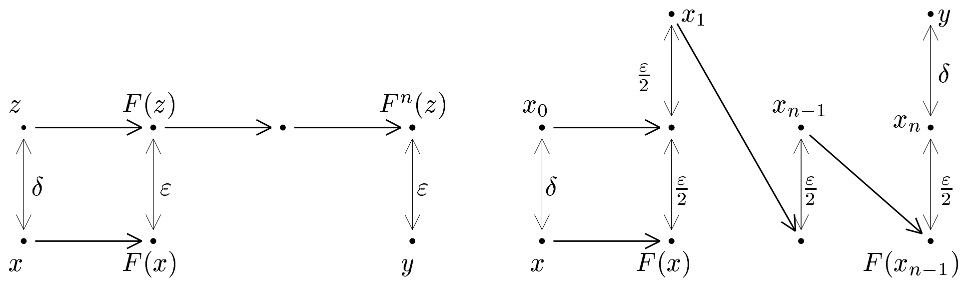

FIGURE 3. ε-chains

Proof. —

(1) $\mathcal{O} \subseteq \mathcal{R} \subseteq \mathcal{N}$ is clear. To prove $\mathcal{N} \subseteq \mathcal{C}$, suppose $(x, y) \in \mathcal{N}$, and let $\varepsilon > 0$. By Proposition A.29, F is uniformly continuous, so there exists δ such that for any $v, w \in X$,

$$d(v, w) < \delta \quad \Rightarrow \quad d(F(v), F(w)) < \varepsilon.$$

By the assumption there exists $z \in B_\delta(x)$ and $n > 0$, such that $d(F^n(z), y) < \varepsilon$ (Figure 3 left). Since $d(F(x), F(z)) < \varepsilon$, the sequence $x, F(z), \ldots, F^{n-1}(z), y$ is an ε-chain from x to y.

(2) The transitivity of $\mathcal{O}$ is clear.

(3) By definition,

$$(x, y) \in \mathcal{N} \;\Leftrightarrow\; \forall \varepsilon, \delta > 0, \exists z \in B_\delta(x), \exists w \in B_\varepsilon(y), (z, w) \in \mathcal{O} \;\Leftrightarrow\; (x, y) \in \overline{\mathcal{O}}$$

(4) The transitivity of $\mathcal{C}$ is clear. To show that $\mathcal{C}$ is closed, suppose that $(x, y) \in \overline{\mathcal{C}}$ and $\varepsilon > 0$. We construct an ε-chain from x to y (Figure 3 right). Since F is uniformly continuous, there exists $0 < \delta < \frac{\varepsilon}{2}$ such that for any $v, w \in X$,

$$d(v, w) < \delta \quad \Rightarrow \quad d(F(v), F(w)) < \frac{\varepsilon}{2}.$$

By the assumption there exists an $\frac{\varepsilon}{2}$-chain $x_0, x_1, \ldots, x_{n-1}, x_n$ from some point of $B_\delta(x)$ to some point of $B_\delta(y)$, so $d(x, x_0) < \delta$, $d(x_n, y) < \delta$ and $d(F(x_i), x_{i+1}) < \frac{\varepsilon}{2}$. It follows that $x, x_1, \ldots, x_{n-1}, y$ is an ε-chain from x to y. Indeed $d(F(x), x_1) \leq d(F(x), F(x_0)) + d(F(x_0), x_1) < \varepsilon$ and $d(F(x_{n-1}), y) \leq d(F(x_{n-1}), y_n) + d(y_n, y) < \varepsilon$. □

2.2. Minimal and transitive systems

We study systems for which one of the dynamical relations is whole of $X \times X$. For the orbit relation, the resulting concept would be trivial and would apply only to finite systems. The other relations yield the concepts of minimality and transitivity already encountered in Chapter 1 (Definitions 1.15 and 1.25). We obtain also the new concept of chain-transitivity.

Definition 2.5. — Let (X, F) be a dynamical system.
(1) (X, F) is **minimal**, if $\mathfrak{R} = X \times X$.
(2) (X, F) is **transitive**, if $\mathfrak{N} = X \times X$.
(3) (X, F) is **chain-transitive**, if $\mathfrak{C} = X \times X$.
(4) A point is transitive, if its orbit is dense. The set of **transitive points** is

$$\mathfrak{T} = \{x \in X : \overline{\mathfrak{O}(x)} = X\}$$

Proposition 2.6. —
(1) Any minimal system is transitive.
(2) Any transitive system is chain-transitive.
(3) Any chain-transitive system is surjective.

Proof. — Since $\mathfrak{R} \subseteq \mathfrak{N} \subseteq \mathfrak{C}$, (1) and (2) follow. If $F : X \to X$ is not surjective, then $X \setminus F(X)$ is a nonempty open set. If $x \in B_\varepsilon(x) \subseteq X \setminus F(X)$, then no ε-chain ends in x, so $(x, x) \notin \mathfrak{C}$. □

Proposition 2.7. — *The following conditions are equivalent.*
(1) (X, F) is a minimal dynamical system.
(2) Any point $x \in X$ is transitive, i.e., $\mathfrak{T} = X$.
(3) (X, F) does not have proper subsystems. If $Y \subseteq X$ is closed and invariant, then either $Y = \emptyset$ or $Y = X$.

Proof. —
(1) ⇔ (2): We have $(x, y) \in \mathfrak{R}$ iff $y \in \overline{\mathfrak{O}(x)}$.
(2) ⇒ (3): Suppose that (X, F) is minimal and $\emptyset \neq Y \neq X$ is its proper subsystem. Pick a point $x \in Y$. Since Y is invariant, $\mathfrak{O}(x) \subseteq Y$. Since Y is closed, $\overline{\mathfrak{O}(x)} \subseteq Y \neq X$ so $\mathfrak{O}(x)$ is not dense. This is a contradiction.
(3) ⇒ (2): Suppose that (X, F) does not have proper subsystems, and let $x \in X$. Then $\overline{\mathfrak{O}(x)}$ is a nonempty subsystem of (X, F), so $\overline{\mathfrak{O}(x)} = X$. □

The binary adding machine of Example 1.24 and irrational rotations (Proposition 1.38) are examples of minimal systems.

Example 2.8. — A finite system $(\mathbb{Z}(n), R_1)$ consisting of a single periodic orbit of length n is minimal. The singleton system $(\mathbb{Z}(1), R_1)$ is called **trivial**.

$$\mathbb{Z}(n) = \{0, \ldots, n-1\}, \quad R_1(i) = \mathrm{mod}_n(i+1).$$

Any finite minimal system (and any finite chain-transitive system) is conjugate to some $(\mathbb{Z}(n), R_1)$. We regard $\mathbb{Z}(n)$ as a **group of integers modulo** n with zero element 0 and addition modulo n. An infinite system which has periodic points cannot be minimal, since the orbit of any periodic point is its proper subsystem.

Theorem 2.9. — *The following conditions are equivalent.*

(1) (X, F) is a transitive dynamical system.

(2) For any nonempty open sets $U, V \subseteq X$ there exists $n > 0$ with $F^n(U) \cap V \neq \emptyset$, or equivalently, $U \cap F^{-n}(V) \neq \emptyset$.

(3) The set of transitive points $\mathfrak{T}$ is residual (Definition A.21).

(4) The set of transitive points $\mathfrak{T}$ is nonempty.

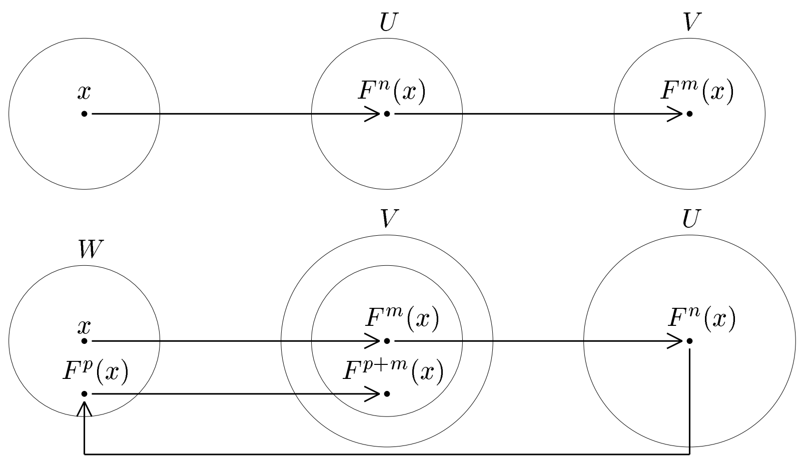

FIGURE 4. Transitive systems

Proof. —

(1) $\Rightarrow$ (2): Let $U, V \subseteq X$ be open and nonempty. Choose $x \in U$, $y \in V$ and $\delta, \varepsilon > 0$ with $B_\delta(x) \subseteq U$, $B_\varepsilon(y) \subseteq V$. There exists $z \in B_\delta(x)$ with $F^n(z) \in B_\varepsilon(y)$ for some $n > 0$. Thus $F^n(z) \in F^n(U) \cap V$ and $z \in U \cap F^{-n}(V)$.

(2) $\Rightarrow$ (3): By Proposition A.22, X has a countable base $(U_n)_{n \geq 0}$, whose unions are all open sets of X. For $n \geq 0$ set

$$V_n = \bigcup_{i>0} F^{-i}(U_n).$$

Since V_n is a union of open sets, it is open. If U is a nonempty open set, then by transitivity, there exists $i > 0$ such that $F^{-i}(U_n) \cap U \neq \emptyset$, so $V_n \cap U \neq \emptyset$. Thus V_n are open and dense and their intersection $V = \bigcap_{n \geq 0} V_n$ is residual. We show that V is exactly the set of points which have dense orbit. Indeed

$$x \in V \;\Leftrightarrow\; \forall n, \exists i > 0, F^i(x) \in U_n \;\Leftrightarrow\; \forall n, \mathfrak{O}(x) \cap U_n \neq \emptyset \;\Leftrightarrow\; \overline{\mathfrak{O}(x)} = X$$

(3) $\Rightarrow$ (4): By the Baire theorem A.25, any residual set is dense and therefore nonempty.

(4) $\Rightarrow$ (1): Suppose that there exists a point x with dense orbit, i.e., $\overline{\mathcal{O}(x)} = X$. Let $U, V \subseteq X$ be nonempty open sets. We show that for some $k > 0$, $F^k(U) \cap V \neq \emptyset$. Since $\mathcal{O}(x)$ is dense, there exist $n, m > 0$, such that $F^n(x) \in U$ and $F^m(x) \in V$. If $n < m$, then for $k = m - n$ we get $F^m(x) \in F^k(U) \cap V$ (Figure 4 top). If $m \leq n$ (Figure 4 bottom), there exists an open set W such that

$$x \in W, \quad \{F(x), \ldots, F^n(x)\} \cap W = \emptyset, \quad F^m(W) \subseteq V$$

Since $\mathcal{O}(x)$ is dense, there exists $p > 0$ for which $F^p(x) \in W$. It follows $p > n$, so for $k = p + m - n$ we get $F^{p+m}(x) \in F^k(U) \cap V$. □

A full shift $(A^{\mathbb{N}}, \sigma)$ has a dense orbit, so it is transitive. Since it also has periodic points, it is not minimal. Likewise, the angle doubling system $(\mathbb{T}, D)$ and the quadratic map $(\mathbb{I}, Q_4)$ are transitive but not minimal. The quadratic system on the circle from Example 2.3 is chain-transitive but not transitive.

Proposition 2.10. — *Let $\pi : (X, F) \to (Y, G)$ be a factor map.*

(1) If (X, F) is minimal then (Y, G) is minimal.

(2) If (X, F) is transitive then (Y, G) is transitive.

(3) If (X, F) is chain-transitive then (Y, G) is chain transitive.

(4) If (X, F) has a dense orbit, then (Y, F) has a dense orbit.

Proof. —

(1) If $\emptyset \neq Z \neq Y$ is a closed invariant set, then $\emptyset \neq \pi^{-1}(Z) \neq X$ is closed invariant set in (X, F).

(2) If $U, V \subseteq Y$ are nonempty open sets, then their preimages are nonempty and open, so $F^n(\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \emptyset$ for some $n > 0$ and $G^n(U) \cap V \neq \emptyset$.

(3) Let $y_0, y_1 \in Y$ and $\varepsilon > 0$. There exists $\delta > 0$ such that if $d(v, w) < \delta$ in X, then $d(\pi(v), \pi(w)) < \varepsilon$ in Y. Since π is surjective, there exist x_0, x_1 with $\pi(x_0) = y_0$, $\pi(x_1) = y_1$ and a δ-chain from x_0 to x_1. Its π-image is an ε-chain from y_0 to y_1.

(4) If $x \in X$ has dense orbit, then $\pi(x) \in Y$ has a dense orbit. □

2.3. Shadowing property

An ε-chain can be thought of as an approximation to a trajectory. In general, such an approximation works only in the short term. In some systems, however, the approximation may work also in the long term. An ε-chain may approximate the trajectory of its initial point, or of another point. In this case we say that such a point shadows the ε-chain in question. Shadowing properties have been studied in hyperbolic systems of differentiable dynamics, see e.g., Bowen [**28**], or Aoki and Hiraide [**5**].

Definition 2.11. — Let (X, F) be a dynamical system.

(1) A point $x \in X$ ε-**shadows** a finite sequence $x_0, \ldots, x_n$ (Figure 5), if for all $i \leq n$, $d(F^i(x), x_i) < \varepsilon$.

(2) (X, F) has the **shadowing property**, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any finite δ-chain is ε-shadowed by some point (Figure 5), i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_0, \ldots, x_n, \big(\forall i, d(F(x_i), x_{i+1}) < \delta \Rightarrow \exists x, \forall i, d(F^i(x), x_i) < \varepsilon\big)$$

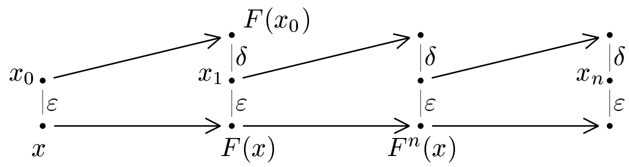

FIGURE 5. A chain and a trajectory

The approximation constant ε should be valid for δ-chains of arbitrary length. It follows that it is also valid for infinite δ-chains, except that strict inequality should be replaced by $\leq$. Suppose that $(x_n)_{n \geq 0}$ is an infinite δ-chain and set

$$X_n = \{x \in X : \forall i < n, d(F^i(x), x_i) \leq \varepsilon\}.$$

Then X_n are nonempty, closed and $X_{n+1} \subseteq X_n$. By compactness, there exists a point x contained in all the X_n and we have $d(F^i(x), x_i) \leq \varepsilon$ for all $i \in \mathbb{N}$.

Example 2.12. — The linear system $(\mathbb{I}, L_{\frac{1}{2}})$, where $L_{\frac{1}{2}}(x) = \frac{x}{2}$, has the shadowing property (Figure 6 left).

Proof. — For a given $\varepsilon > 0$ set $\delta = \frac{\varepsilon}{2}$. If $x_0, \ldots, x_n$ is a δ-chain, then we show by induction that $x = x_0$ ε-shadows $(x_i)_{0 \leq i \leq n}$. For $i = 0$ we have $|2^{-i}x - x_0| = 0 < \varepsilon$. If $|2^{-i}x - x_i| < \varepsilon$, then

$$|2^{-i-1}x - x_{i+1}| \leq \frac{|2^{-i}x - x_i|}{2} + \left|\frac{x_i}{2} - x_{i+1}\right| < \frac{\varepsilon}{2} + \delta = \varepsilon.$$

$\square$

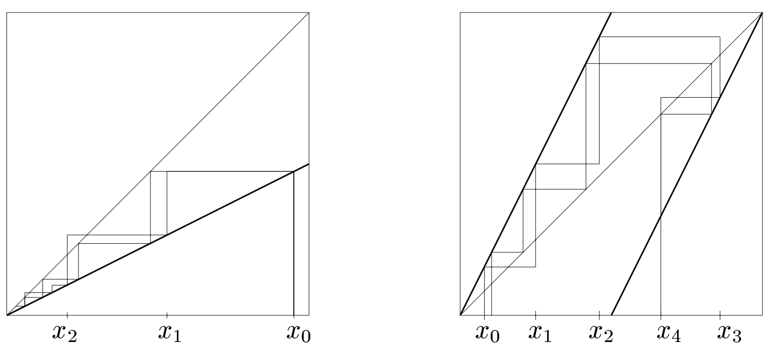

FIGURE 6. Shadowing property

Example 2.13. — The angle doubling system $(\mathbb{T}, D)$ of Example 1.14 has the shadowing property (Figure 6 right).

Proof. — For a given $\varepsilon < \frac{1}{4}$ set $\delta = \varepsilon$. Let $x_0, \ldots, x_n$ be a δ-chain. We construct a trajectory $y_i = f^i(y)$ backwards. Set $y_n = x_n$. Suppose that for some $i > 0$ we have constructed y_i with $d(y_i, x_i) < \varepsilon$. If $D(x_{i-1}) = 2x_{i-1}$, then set $y_{i-1} = \frac{y_i}{2}$. Since $d(2x_{i-1}, x_i) < \varepsilon$,

$$d(y_{i-1}, x_{i-1}) \leq d(\tfrac{y_i}{2}, \tfrac{x_i}{2}) + d(\tfrac{x_i}{2}, x_{i-1}) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

If $D(x_{i-1}) = 2x_{i-1} - 1$, set $y_{i-1} = \frac{y_i+1}{2}$. Since $d(2x_{i-1}, x_i + 1) < \varepsilon$,

$$d(y_{i-1}, x_{i-1}) \leq d(\tfrac{y_i+1}{2}, \tfrac{x_i+1}{2}) + d(\tfrac{x_i+1}{2}, x_{i-1}) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus $y = y_0$ ε-shadows the ε-chain $(x_i)_{0 \leq i \leq n}$. □

Example 2.14. — The identity $(A^{\mathbb{N}}, \mathrm{Id})$ has the shadowing property while the identity $(\mathbb{I}, \mathrm{Id})$ does not.

Proof. — If $\varepsilon = 2^{-n}$ and $(x_i \in A^{\mathbb{N}})_{0 \leq i \leq m}$ is an ε-chain then $d(x_i, x_{i+1}) < 2^{-n}$ so

$$(x_0)_{[0,n]} = (x_1)_{[0,n]} = \cdots = (x_m)_{[0,n]}$$

Thus x_0 ε-shadows $(x_i)_{0 \leq i \leq m}$. In $(\mathbb{I}, \mathrm{Id})$, for any $\varepsilon > 0$ there exists an ε-chain from 0 to 1 which for $\delta < 1$ is not δ-shadowed by any point. □

The quadratic system on the circle $(\mathbb{T}, Q)$ of Example 2.3 does not have the shadowing property. For any $\delta > 0$ there exists δ-chain from $\frac{1}{2}$ to itself and for $\varepsilon < \frac{1}{4}$, it is not ε-shadowed by any point.

Proposition 2.15. — *If (X, F) has the shadowing property, then $\mathcal{N} = \mathcal{C}$. In particular, if (X, F) is chain-transitive, then it is transitive.*

Proof. — Let $\varepsilon > 0$ and $(x, y) \in \mathcal{C}$. Let $\delta > 0$ be such that any δ-chain is ε-shadowed by some point. There exists a δ-chain $x = x_0, \ldots, x_n = y$. If w ε-shadows this δ-chain, then $d(w, x) < \varepsilon$ and $d(F^n(w), y) < \varepsilon$. Thus $(x, y) \in \mathcal{N}$. □

2.4. Recurrence

We recall that the diagonal of a relation $R \subseteq X \times X$ is $|R| = \{x \in X : (x, x) \in R\}$.

Definition 2.16. — Let (X, F) be a dynamical system. The points of $|\mathcal{O}|$, $|\mathcal{R}|$, $|\mathcal{N}|$, $|\mathcal{C}|$ are called **periodic**, **recurrent**, **nonwandering** and **chain-recurrent**, respectively.

Recurrent points have already been introduced in Definition 1.33. The quasi-periodic points of Definition 1.25 and the almost periodic points of Definition 1.33 are intermediate concepts between periodic and recurrent points. A quasi-periodic point returns periodically to any of its neighbourhood and the period depends on the size of this neighbourhood. An almost periodic point returns to any of its neighbourhood with bounded gaps. The period of a neighbourhood is the upper bound of these return times. A recurrent point returns to any of its neighbourhood infinitely often (Proposition 2.18) but with arbitrary gaps between these returns.

$$\begin{gathered}\text{periodic} \Rightarrow \text{quasi-periodic} \Rightarrow \text{almost periodic} \Rightarrow \text{recurrent} \Rightarrow \\ \text{nonwandering} \Rightarrow \text{chain-recurrent}\end{gathered}$$

Proposition 2.17. — *Let (X,F) be a dynamical system.*
(1) $|\mathfrak{O}| \subseteq |\mathfrak{R}| \subseteq |\mathfrak{N}| \subseteq |\mathfrak{C}|$
(2) $|\mathfrak{O}|$, $|\mathfrak{R}|$, $|\mathfrak{N}|$, $|\mathfrak{C}|$ *are invariant*
(3) $|\mathfrak{N}|$, $|\mathfrak{C}|$ *are closed.*

Proof. —]

(1) follows from Proposition 2.4.

(2) The invariance of $\mathfrak{O}$ is clear. For the invariance of $|\mathfrak{R}|$, $|\mathfrak{N}|$ and $|\mathfrak{C}|$ we need to use uniform continuity. Let $\varepsilon > 0$ be given and $\delta > 0$ be such that for any $v, w \in X$,

$$d(v,w) < \delta \;\Rightarrow\; d(F(v),F(w)) < \varepsilon.$$

If $x \in |\mathfrak{R}|$ then there exists $n > 0$ for which $d(F^n(x),x) < \delta$, so $d(F^{n+1}(x),F(x)) < \varepsilon$. Thus $F(x) \in |\mathfrak{R}|$ and $|\mathfrak{R}|$ is invariant.

If $x \in |\mathfrak{N}|$ then there exists $z \in B_\delta(x)$ and $n > 0$ for which $d(F^n(z),x) < \delta$. We get $d(F(z),F(x)) < \varepsilon$ and $d(F^{n+1}(z),F(x)) < \varepsilon$, so $F(x) \in |\mathfrak{N}|$ and $|\mathfrak{N}|$ is invariant.

If $x \in |\mathfrak{C}|$ then there exists $n > 0$ and a δ-chain $x = x_0, \dots, x_n = x$ from x to x. Then $F(x_0), \dots, F(x_n)$ is an ε-chain from $F(x)$ to $F(x)$, so $|\mathfrak{C}|$ is invariant.

(3) By Proposition 2.4, $\mathfrak{N}, \mathfrak{C}$ are closed in $X \times X$, so their intersection with diagonal $\Delta_X = \{(x,x) : x \in X\}$ is closed in Δ_X. Since Δ_X is homeomorphic to X, both $|\mathfrak{N}|$, $|\mathfrak{C}|$ are closed in X. □

In the quadratic system $(\mathbb{I}, Q_2)$, where $Q_2(x) = 2x(1-x)$, we have

$$|\mathfrak{O}| = |\mathfrak{R}| = |\mathfrak{N}| = |\mathfrak{C}| = \{0, \tfrac{1}{2}\}$$

In the binary full shift $(\mathbf{2}^{\mathbb{N}}, \sigma)$ we have

$$\emptyset \neq |\mathfrak{O}| \subset |\mathfrak{R}| \subset |\mathfrak{N}| = |\mathfrak{C}| = \mathbf{2}^{\mathbb{N}}, \quad \overline{|\mathfrak{O}|} = \overline{|\mathfrak{R}|} = \mathbf{2}^{\mathbb{N}}$$

Since $(\mathbf{2}^{\mathbb{N}}, \sigma)$ is transitive, $|\mathfrak{N}| = |\mathfrak{C}| = \mathbf{2}^{\mathbb{N}}$. The Champernowne sequence $x = 0100011011\cdots$, which concatenates all finite words, is recurrent but not periodic. A preperiodic point, e.g., $x = 1000\ldots$, is not recurrent but it is nonwandering. In fact all points are nonwandering since the system is transitive. In the quadratic system on the circle $(\mathbb{T}, Q)$ of Example 2.3,

$$\{0\} = |\mathfrak{O}| = |\mathfrak{R}| = |\mathfrak{N}| \subset |\mathfrak{C}| = \mathbb{T}$$

Since the system is chain-transitive, any its point is chain-recurrent. If $x \neq 0$, then x is wandering.

We have seen that there exist dynamic systems which do not have periodic points, for example irrational rotations of the circle. We show now that any dynamical system has at least one almost periodic point and therefore at least one recurrent point. Observe first that recurrent points visit their neighbourhoods infinitely often:

Proposition 2.18. — *Let $x \in X$ be a recurrent point and U a neighbourhood of x. Then the set $\{n \in \mathbb{N} : F^n(x) \in U\}$ is infinite.*

Proof. — If x is periodic with period n, then $F^{nk}(x) = x \in U$ for any k. Suppose that x is not periodic. Then its orbit is infinite and there exists $n_1 > 0$ for which $F^{n_1}(x) \in U = U_1$. The set $U_2 = U_1 \setminus \{F(x), \dots, F^{n_1}(x)\}$ is a neighbourhood of x, so there exists $n_2 > n_1$ for which $F^{n_2}(x) \in U_2$. We continue in this way constructing sets $U_{i+1} = U_i \setminus \{F(x), \dots, F^{n_i}(x)\}$ and $F^{n_{i+1}}(x) \in U_{i+1}$. □

While the return times to U form an infinite sequence, there might be arbitrarily large time gaps between consecutive returns. In contrast, almost periodic points return to their neighborhoods with bounded gaps. There is an important relation between almost periodicity and minimality.

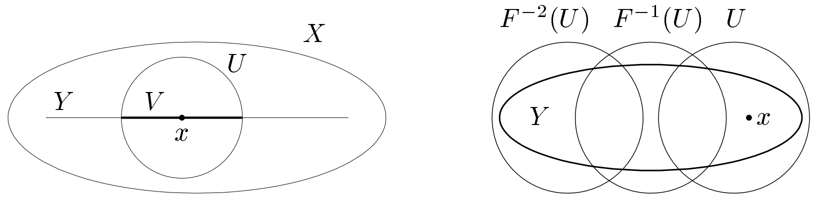

FIGURE 7. Minimal subsystem

***Theorem 2.19* (Birkhoff [18]).** — *A point x of a dynamical system (X, F) is almost periodic iff $\overline{\mathcal{O}(x)}$ is a minimal system.*

Proof. — Denote by $\overline{\mathcal{O}(x)}$ by Y. The point x is almost periodic in (X, F) iff it is almost periodic in (Y, F) (Figure 7 left). Indeed if $V \ni x$ is a neighbourhood of x in Y, then $V = U \cap Y$ for some neighbourhood $U \subseteq X$ of x in X, and $F^n(x) \in U$ iff $F^n(x) \in V$. Thus we can restrict ourselves to the system (Y, F) where $Y = \overline{\mathcal{O}(x)}$, and show that x is almost periodic iff (Y, F) is minimal.

Suppose that (Y, F) is minimal. Let $U \subseteq Y$ be a nonempty open set which contains x. For any $y \in Y$, $\mathcal{O}(y)$ is a dense set, so there exists $i > 0$ with $F^i(y) \in U$, or $y \in F^{-i}(U)$. This means that $\mathcal{U} = \{F^{-i}(U) : i \geq 0\}$ is an open cover of Y. By compactness, there exists $p > 0$, such that $\{U, F^{-1}(U), \dots, F^{-(p-1)}(U)\}$ is a finite subcover of $\mathcal{U}$ (Figure 7 right). Therefore, for any $n \in \mathbb{N}$ there exists $k < p$ with $F^n(x) \in F^{-k}(U)$, so $F^{n+k}(x) \in U$ and x is almost periodic.

Conversely, let x be almost periodic and suppose that there exists a proper subsystem $\emptyset \neq W \neq Y$ of Y (Figure 8). If the point x belongs to W, then $\overline{\mathcal{O}(x)} \subseteq W$ since W is closed and invariant and this contradicts $\overline{\mathcal{O}(x)} = Y$. Thus $x \notin W$, and there exists a closed neighbourhood V of x which does not intersect W, i.e., $x \in V^\circ$ and $V \cap W = \emptyset$. By the assumption, there exists $p > 0$ such that for any $n > 0$, there exists $k < p$ with $F^{n+k}(x) \in V$. Construct

$$V_p = V \cup F^{-1}(V) \cup \cdots \cup F^{-(p-1)}(V).$$

Since W is invariant, $W \cap V_p = \emptyset$. Thus $Y \setminus V_p$ is a nonempty open set, and since $\mathcal{O}(x)$ is dense, there exists $n > 0$, for which $F^n(x) \in Y \setminus V_p$. But this means that $F^{n+i}(x) \notin V$ for all $0 \leq i < p$. This is a contradiction. □

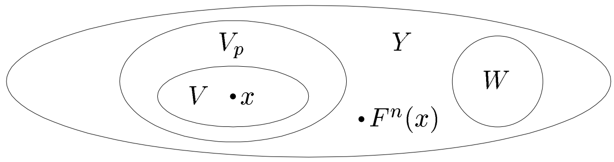

FIGURE 8. Almost periodic point

Proposition 2.20. — *Any dynamical system contains a minimal subsystem.*

Proof. — We use Zorn's lemma A.2. Consider the family $\mathcal{S}$ of all nonempty closed invariant sets of (X, F). This family is ordered by inclusion. Assume that $\mathcal{S}_0 \subseteq \mathcal{S}$ is a linearly ordered subfamily, i.e., if $Z, W \in \mathcal{S}_0$, then either $Z \subseteq W$ or $W \subseteq Z$. By compactness, the intersection $Y_0 = \bigcap \mathcal{S}_0$ is nonempty, and it is again closed and invariant. Thus any linearly ordered subfamily $\mathcal{S}_0 \subseteq \mathcal{S}$ has a lower bound Y_0. By Zorn's lemma there exists a minimal element in $\mathcal{S}$. □

Corollary 2.21 **(Birkhoff [18]).** — *Any dynamical system has an almost periodic point.*

Proof. — Any dynamical system contains a minimal subsystem, and any point of a minimal subsystem is almost periodic. □

If (X, F) is a dynamical system and $Y \subseteq X$ a closed invariant set, then $y \in Y$ is almost periodic in (Y, F) iff it is almost periodic in (X, F). We have used this fact in the proof of Birkhoff theorem 2.19. The same assertion holds for recurrent points. In fact, a set $V \subseteq Y$ is a neighbourhood of y in Y iff there exists a neighbourhood U of y in X and $V = U \cap Y$. Since Y is invariant,

$$F^n(y) \in V \quad \Leftrightarrow \quad F^n(y) \in U.$$

The analogous assertion for nonwandering and chain-recurrent points is false. A point $y \in Y$ may be nonwandering in (X, F) and wandering in (Y, F). Given the real function $f(x) = 1 - |2x - 1|$ and a sequence of real numbers $0 < a_1 < a_2 < \cdots < \frac{1}{2}$, we construct a sequence of dynamical systems

Example 2.22. — $(\mathbb{T}^n, F_n)$, where $n \geq 1$ and

$$\begin{aligned}
F_1(x) &= x + a_1 f(x) \\
F_2(x, y) &= (x + a_1 f(x), y + a_1 f(x) + a_2 f(y)) \\
F_3(x, y, z) &= (x + a_1 f(x), y + a_1 f(x) + a_2 f(y), z + a_1 f(x) + a_2 f(y) + a_3 f(z)) \\
&\vdots
\end{aligned}$$

Since F_1 is increasing on $[0, 1)$ and $F_1(0) = 0$ (Figure 9 left), $(\mathbb{T}, F_1)$ is conjugate to the quadratic system on the circle $(\mathbb{T}, Q)$ of Example 2.3. Its nonwandering set is $|\mathcal{N}| = \{0\}$. The nonwandering set of $(\mathbb{T}^2, F_2)$ is $|\mathcal{N}| = \{0\} \times \mathbb{T}$ (Figure 9 right). If $x \neq 0$ and y is arbitrary then the trajectory of (x, y) lies in a spiral which approaches $|\mathcal{N}|$. The system $(|\mathcal{N}|, F_2)$ is conjugate to $(\mathbb{T}, F_1)$ via an injective morphism φ :

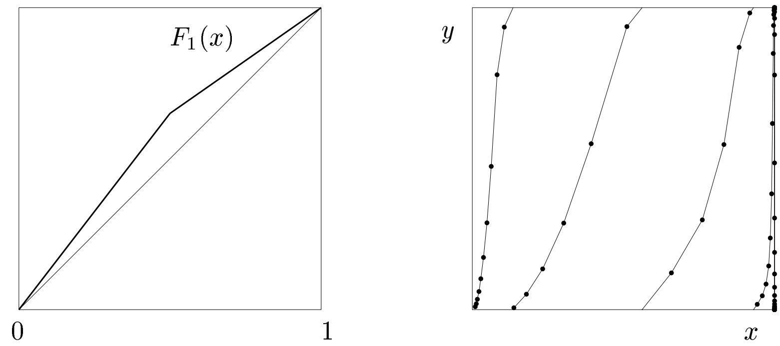

Figure 9. Nonwandering points

$(\mathbb{T}, F_1) \to (\mathbb{T}^2, F_2)$ given by $\varphi(x) = (0, x)$. The system $(\mathbb{T}^2, F_2)$ is in turn conjugate to the nonwandering set $|\mathfrak{N}| = \{0\} \times \mathbb{T}^2$ of the system $(\mathbb{T}^3, F_3)$.

Definition 2.23. — Let (X, F) be a dynamical system.

(1) The **n-th nonwandering set** $|\mathfrak{N}_n|$ of (X, F) is defined by induction. Let $|\mathfrak{N}_1| = |\mathfrak{N}|$ and let $|\mathfrak{N}_{n+1}|$ be the nonwandering set of the system $(|\mathfrak{N}_n|, F)$.
(2) The **center** of (X, F) is $\overline{|\mathfrak{R}|}$.

Proposition 2.24 **(Akin [3]).** — *Let (X, F) be a dynamical system.*

(1) For any $n \geq 1$, $\overline{|\mathfrak{R}|} \subseteq |\mathfrak{N}_n|$.
(2) If $|\mathfrak{N}| = X$ then $|\mathfrak{R}|$ is residual.

Proof. —
(1) Any system $(|\mathfrak{N}_n|, F)$ has the same set of recurrent points $|\mathfrak{R}|$. Since $|\mathfrak{N}_n|$ is closed, $\overline{|\mathfrak{R}|} \subseteq |\mathfrak{N}_n|$.

(2) Assume that every point is nonwandering. Let $\mathcal{U}_m$ be a finite open cover of X with $\mathrm{diam}(\mathcal{U}_m) < \frac{1}{m}$ and set

$$G_m = \bigcup \{F^{-n}(U) \cap U : n > 0, U \in \mathcal{U}_m\}.$$

G_m is a union of open sets, so it is open. We show that it is dense. Given $x \in X$ and $\delta > 0$, there exists $\varepsilon \leq \delta$ and $U \in \mathcal{U}_m$ such that $x \in B_\varepsilon(x) \subseteq U$. Since $|\mathfrak{N}| = X$, there exists $y \in B_\varepsilon(x)$ and $n > 0$ such that $F^n(y) \in B_\varepsilon(x)$. It follows that $y \in F^{-n}(U) \cap U \subseteq G_m$, so $B_\delta(x) \cap G_m \neq \emptyset$. Thus G_m is dense and $G = \bigcap_{m>0} G_m$ is residual. We show $|\mathfrak{R}| = G$.

Let $x \in G$ and $\varepsilon > 0$. If $\frac{1}{m} < \varepsilon$ then $x \in G_m$, so there exists $U \in \mathcal{U}_m$ and $n > 0$ with $x \in F^{-n}(U) \cap U$. Thus $d(F^n(x), x) < \varepsilon$ and therefore $x \in |\mathfrak{R}|$, so $G \subseteq |\mathfrak{R}|$.

Conversely, if $x \in |\mathfrak{R}|$ then for any m there exists $U \in \mathcal{U}_m$ with $x \in U$, and $F^n(x) \in U$ for some $n > 0$. Thus $x \in G_m$ and therefore $x \in G$ and $|\mathfrak{R}| \subseteq G$. Thus $|\mathfrak{R}| = G$ is residual. $\square$

2.5. Equicontinuity and sensitivity

In Chapter 1 we defined stable and attracting fixed points (Definition 1.7). These definitions can be generalized to arbitrary points. Stable points are also called equicontinuous since the family of iterations $(F^n)_{n\geq 0}$ is equicontinuous at them.

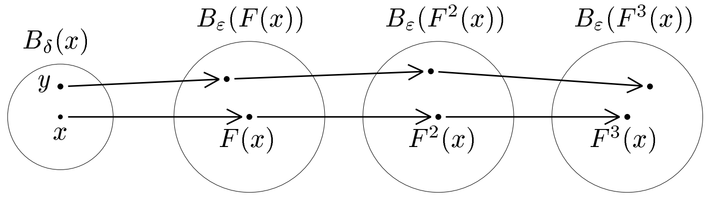

FIGURE 10. Equicontinuity

Definition 2.25. — Let (X, F) be a dynamical system.
(1) The set $\mathcal{E} \subseteq X$ of **equicontinuous points** of (X, F) is defined by

$$x \in \mathcal{E} \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall n \geq 0, d(F^n(y), F^n(x)) < \varepsilon$$

(2) We say that (X, F) is **equicontinuous**, if $\mathcal{E} = X$.
(3) We say that (X, F) is **almost equicontinuous**, if $\mathcal{E}$ is a residual set.

In particular, if F is an **isometry**, i.e., if $d(F(x), F(y)) = d(x, y)$, then $\mathcal{E} = X$ and (X, F) is equicontinuous. Thus binary adding machine and rotations of the circle are equicontinuous systems.

Example 2.26. — In the quadratic system $(\mathbb{I}, Q_r)$ we have

$$\begin{aligned} r \leq 1 \quad &\Rightarrow \quad \mathcal{E} = \mathbb{I} \\ 1 < r \leq 3 \quad &\Rightarrow \quad \mathcal{E} = (0,1) \\ 3 < r \leq \sqrt{6}+1 \quad &\Rightarrow \quad \mathcal{E} = \{x \in \mathbb{I} : \forall n, Q_r^n(x) \notin \{0, 1 - \tfrac{1}{r}\}\} \\ r = 4 \quad &\Rightarrow \quad \mathcal{E} = \emptyset \end{aligned}$$

We see that as r grows, the set of equicontinuous points shrinks. For $r \leq 1$, $(\mathbb{I}, Q_r)$ is equicontinuous and for $r \leq \sqrt{6}+1$, $(\mathbb{I}, Q_r)$ is almost equicontinuous. The opposite of equicontinuity is sensitivity (to initial conditions). In a sensititive system there exists a positive **sensitivity constant** ε, such that in any neighbourhood of a given point there is another point such that the trajectories of the two points are eventually at least ε apart.

Definition 2.27. — A dynamical system (X, F) is **sensitive** if

$$\exists \varepsilon > 0, \forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, d(F^n(y), F^n(x)) \geq \varepsilon.$$

For example, a full shift is sensitive with sensitivity constant 1: For any $x \in A^{\mathbb{N}}$ and any $\delta > 0$ pick some m with $2^{-m} < \delta$ and consider $y \in \mathbf{2}^{\mathbb{N}}$ with $y_{[0,m)} = x_{[0,m)}$ and $y_m \neq x_m$. Then $d(x, y) < \delta$ and $d(F^m(x), F^m(y)) = 1$. Similarly, the angle doubling system $(\mathbb{T}, D)$ and the quadratic system $(\mathbb{I}, Q_4)$ are sensitive.

A sensitive system has no equicontinuous points, since $\mathcal{E}$ is empty iff

$$\forall x \in X, \exists \varepsilon > 0, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, d(F^n(y), F^n(x)) \geq \varepsilon.$$

This differs from sensitivity only in the order of the first two quantifiers, so sensitivity implies $\mathcal{E} = \emptyset$. Nevertheless it is stronger. Consider a dynamical system (X, F), consisting of the unit disc in the xy plane together with a circle in the xz plane passing through the origin and whose dynamics is given by angle doubling map on any xy circle centered on origin and the angle doubling map on the xz circle. (Figure 11 left, the fixed points are marked by dots.)

Example 2.28. — The following system (X, F) has no equicontinuous points but is not sensitive:

$$\begin{aligned} X &= \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\} \cup \\ & \quad \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + z^2 = 1, y = 0\} \\ F(r\cos t, r\sin t, 0) &= (r\cos 2t, r\sin 2t, 0), \\ F(1-\cos t, 0, \sin t) &= (1-\cos 2t, 0, \sin 2t) \end{aligned}$$

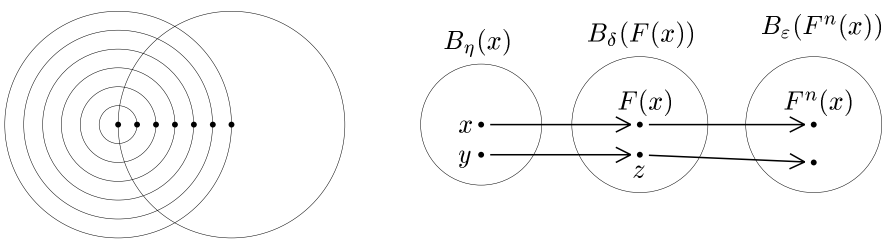

FIGURE 11. Equicontinuous set

We will show that this cannot happen in transitive systems.

Proposition 2.29. — *The equicontinuous set of a dynamical system (X, F) is* **inversely invariant**, *i.e., $F^{-1}(\mathcal{E}) \subseteq \mathcal{E}$, or $F(X \setminus \mathcal{E}) \subseteq X \setminus \mathcal{E}$.*

Proof. — Let $F(x) \in \mathcal{E}$, and $\varepsilon > 0$ (Figure 11 right). There exists $0 < \delta < \varepsilon$ such that $d(z, F(x)) < \delta$ implies $d(F^{n-1}(z), F^n(x)) < \varepsilon$ for all $n > 0$. There exists $0 < \eta < \varepsilon$ such that if $d(y, x) < \eta$, then $d(F(y), F(x)) < \delta$, and $d(F^n(y), F^n(x)) < \varepsilon$ for all n. Thus $x \in \mathcal{E}$. □

Proposition 2.30. — *Let (X, F) be a dynamical system, $\varepsilon > 0$ and*

$$\mathcal{E}_\varepsilon = \{x \in X : \exists \delta > 0, \forall y, z \in B_\delta(x), \forall n \geq 0, d(F^n(y), F^n(z)) < \varepsilon\}.$$

Then $\mathcal{E}_\varepsilon$ is inversely invariant, open, and $\mathcal{E} = \bigcap_{m>0} \mathcal{E}_{\frac{1}{m}}$.

Proof. —

(1) We show that $\mathcal{E}_\varepsilon$ is inversely invariant: Let $F(x) \in \mathcal{E}_\varepsilon$ and let $\delta < \frac{\varepsilon}{2}$ satisfy the condition from the definition of $\mathcal{E}_\varepsilon$ for $F(x)$. There exists a positive $\eta < \delta$, such that for any z, y,

$$d(z,y) < \eta \Rightarrow d(F(z), F(y)) < \delta.$$

If $y, z \in B_\eta(x)$, then $F(y), F(z) \in B_\delta(F(x))$, so $d(F^{n+1}(y), F^{n+1}(z)) < \varepsilon$ for any $n \geq 0$. Thus $x \in \mathcal{E}_\varepsilon$.

(2) If $x \in \mathcal{E}_\varepsilon$ and if δ satisfies the condition from the definition of $\mathcal{E}_\varepsilon$ for x, then $B_{\frac{\delta}{2}}(x) \subseteq \mathcal{E}_\varepsilon$. Indeed if $y \in B_{\frac{\delta}{2}}(x)$ and if $z, w \in B_{\frac{\delta}{2}}(y)$, then $z, w \in B_\delta(x)$, so $d(F^n(z), F^n(w)) < \varepsilon$ for all n. This shows that $\mathcal{E}_\varepsilon$ is open.

(3) If x belongs to all $\mathcal{E}_{\frac{1}{m}}$, then clearly $x \in \mathcal{E}$. Conversely, if $x \in \mathcal{E}$ and $m > 0$ then there exist $\delta > 0$ such that for all $y \in B_\delta(x)$, and all n, $d(F^n(x), F^n(y)) < \frac{1}{2m}$. Thus if $y, z \in B_\delta(x)$, then

$$d(F^n(y), F^n(z)) \leq d(F^n(y), F^n(x)) + d(F^n(x), F^n(z)) < \tfrac{1}{2m} + \tfrac{1}{2m} = \tfrac{1}{m}$$

□

***Theorem 2.31* (Akin,Auslander,Berg [4]).** — *A transitive system is either sensitive or almost equicontinuous.*

Proof. — Let (X, F) be a transitive system and $\varepsilon > 0$. Assume that $\mathcal{E}_\varepsilon$ is nonempty and nondense. Then $U = X \setminus \overline{\mathcal{E}_\varepsilon}$ is open and nonempty so,

$$\emptyset \neq U \cap F^{-n}(\mathcal{E}_\varepsilon) \subseteq U \cap \mathcal{E}_\varepsilon = \emptyset,$$

and this is a contradiction. Thus any $\mathcal{E}_\varepsilon$ is either empty or dense. If all $\mathcal{E}_\varepsilon$ are nonempty, then $\mathcal{E} = \bigcap_{m \geq 1} \mathcal{E}_{\frac{1}{m}}$ is a residual set, so (X, F) is almost equicontinuous. If $\mathcal{E}_\varepsilon = \emptyset$ for some $\varepsilon > 0$, then the system is sensitive, with sensitivity constant $\varepsilon/2$: For any $x \in X$ and for any $\delta > 0$ there exist $y, z \in B_\delta(x)$ and $n \geq 0$ with $d(F^n(y), F^n(z)) \geq \varepsilon$. It follows that either $d(F^n(y), F^n(x)) \geq \frac{\varepsilon}{2}$ or $d(F^n(z), F^n(x)) \geq \frac{\varepsilon}{2}$. □

2.5.1. Almost equicontinuous systems. — We have seen that there are two classes of transitive systems. The binary adding machine and irrational rotations of the circle are almost equicontinuous (even equicontinuous) while the binary shift, the angle doubling system and $(\mathbb{I}, Q_4)$ are sensitive. We will now study further properties of almost equicontinuous (transitive) systems.

Definition 2.32. — A dynamical system (X, F) is **uniformly rigid**, if

$$\forall \varepsilon > 0, \exists n > 0, \forall x \in X, d(x, F^n(x)) < \varepsilon.$$

The dyadic adding machine and any rotation of the circle are uniformly rigid.

***Theorem 2.33* (Glasner, Weiss [65], Akin et al.[4]).** — *Let (X, F) be a transitive, almost equicontinuous system. Then*

(1) $\mathcal{T} = \mathcal{E}$, a point is transitive iff it is equicontinuous.

(2) (X, F) is uniformly rigid.

(3) F is a homeomorphism.

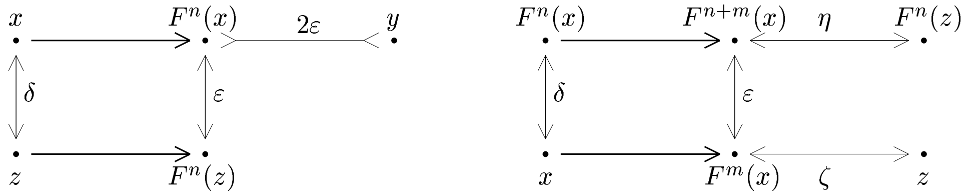

FIGURE 12. Uniformly rigid systems

Proof. — By Proposition 2.30, the set $\mathcal{E}_\varepsilon$ is dense, open and inversely invariant.

(1) If $x \in \mathcal{T}$, and $\varepsilon > 0$ then $F^n(x) \in \mathcal{E}_\varepsilon$ for some n, and since $\mathcal{E}_\varepsilon$ is inversely invariant, $x \in \mathcal{E}_\varepsilon$. Thus $x \in \mathcal{E}$ and $\mathcal{T} \subseteq \mathcal{E}$.

Conversely, let $x \in \mathcal{E}$, and suppose that $x \notin \mathcal{T}$ so there exists a nonempty open $U \subseteq X$, such that $F^n(x) \notin U$ for all $n > 0$. Pick $y \in U$ and $\varepsilon > 0$ such that $B_{2\varepsilon}(y) \subseteq U$, so $d(F^n(x), y) \geq 2\varepsilon$ for all n (Figure 12 left). Since $x \in \mathcal{E}$, there exists $\delta > 0$ such that for all $z \in B_\delta(x)$ and for all n, $d(F^n(z), F^n(x)) < \varepsilon$, so

$$d(y, F^n(z)) \geq d(y, F^n(x)) - d(F^n(x), F^n(z)) \geq \varepsilon.$$

Thus no $z \in B_\delta(x)$ is transitive and this is a contradiction, since the transitive points are dense by Theorem 2.9. We have proved that $\mathcal{E} \subseteq \mathcal{T}$.

(2) Let $\varepsilon > 0$ be given. Since (X, F) is transitive, there exists a point $x \in \mathcal{T}$ with dense orbit, so $F^n(x) \in \mathcal{E}_\varepsilon$ for some n. Since $\mathcal{E}_\varepsilon$ is inversely invariant, $x \in \mathcal{E}_\varepsilon$. Let $\delta > 0$ be such that

$$\forall y, z \in B_\delta(x), \forall m \geq 0, d(F^m(y), F^m(z)) < \varepsilon.$$

Since $x \in \mathcal{T}$ has dense orbit, $F^n(x) \in B_\delta(x)$ for some $n > 0$ (Figure 12 right), and for all m, $d(F^{n+m}(x), F^m(x)) < \varepsilon$. We show that for all $z \in X$, $d(F^n(z), z) \leq \varepsilon$. Since F^n is continuous, for any $\eta > 0$, there exists $\zeta > 0$ such that

$$\forall v, d(v, z) < \zeta \quad \Rightarrow \quad d(F^n(v), F^n(z)) < \eta.$$

Since $x \in \mathcal{T}$, there exists $m > 0$ with $d(z, F^m(x)) < \zeta$, so $d(F^n(z), F^{n+m}(x)) < \eta$ and

$$\begin{aligned} d(F^n(z), z) &\leq d(F^n(z), F^{n+m}(x)) + d(F^{n+m}(x), F^m(x)) + d(F^m(x), z) \\ &< \eta + \varepsilon + \zeta. \end{aligned}$$

Since η and ζ can be arbitrarily small, we get $d(F^n(z), z) \leq \varepsilon$.

(3) Since (X, F) is transitive, F is surjective by Proposition 2.6. Suppose that $F(x) = F(y)$ and $d(x, y) = \varepsilon > 0$. There exists $n > 0$ such that $d(F^n(x), x) < \frac{\varepsilon}{2}$ and $d(F^n(y), y) < \frac{\varepsilon}{2}$ so

$$0 = d(F^n(x), F^n(y)) \geq d(x, y) - d(x, F^n(x)) - d(y, F^n(y)) > 0,$$

and this is a contradiction. Thus F is bijective and therefore a homeomorphism. ☐

Since in transitive almost equicontinuous systems transitive points coincide with equicontinuous points, we get

Corollary 2.34. —

(1) A transitive equicontinuous system is minimal.

(2) A minimal almost equicontinuous system is equicontinuous.

Akin, Auslander and Berg [**4**] construct examples of dynamical systems which are transitive and almost equicontinuous but not minimal.

Theorem 2.35 **(Glasner and Weiss [65]).** — *A transitive system with a dense set of almost periodic points is either minimal or sensitive.*

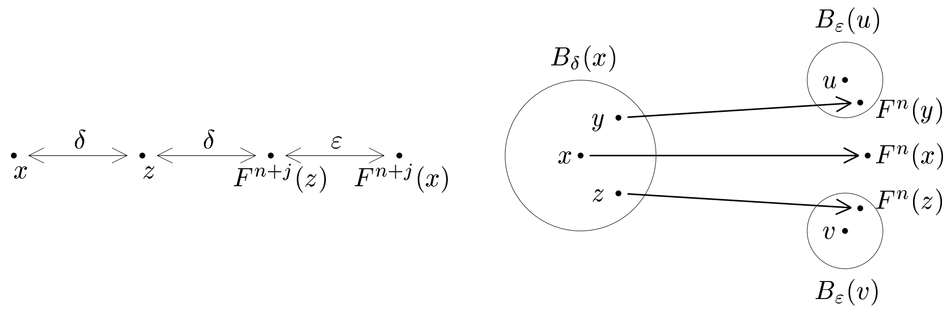

FIGURE 13. Sensitive systems

Proof. — Suppose that (X, F) is not sensitive, so it is almost equicontinuous. Let $x \in \mathfrak{E}$ be an equicontinuous point. We show that x is almost periodic. Let $U \ni x$ be a neighbourhood of x and choose $\varepsilon > 0$ such that $B_{3\varepsilon}(x) \subseteq U$. There exists $0 < \delta < \varepsilon$ such that if $d(x, y) < \delta$, then $d(F^n(x), F^n(y)) < \varepsilon$ for all n (Figure 13 left). Let $z \in B_\delta(x)$ be an almost periodic point; then there exists p such that for all n there exists $j < p$ with $d(z, F^{n+j}(z)) < \delta$. Then

$$d(x, F^{n+j}(x)) \leq d(x, z) + d(z, F^{n+j}(z)) + d(F^{n+j}(z), F^{n+j}(x)) \leq \delta + \delta + \varepsilon < 3\varepsilon$$

so $F^{n+j}(x) \in U$ and x is almost periodic. Since $\mathfrak{T} = \mathfrak{E}$, x is a transitive point, so $X = \overline{\mathfrak{O}(x)}$. Since x is almost periodic, X is a minimal system by Theorem 2.19. $\square$

2.5.2. Sensitive systems. — Transitive systems which are not almost equicontinuous are sensitive. They include chaotic systems, mixing systems and positively expansive systems. Recall that a dynamical system is chaotic (Definition 1.15), if it is infinite, transitive and has a dense set of periodic points.

Corollary 2.36. — *A chaotic system is sensitive.*

Proof. — If (X, F) is not sensitive, it is minimal by Theorem 2.35. Since it contains periodic points, it consists of a single periodic orbit. $\square$

Definition 2.37. — A dynamical system (X, F) is **mixing**, if for any nonempty open sets $U, V \subseteq X$

$$\exists m, \forall n \geq m, F^n(U) \cap V \neq \emptyset.$$

Clearly any mixing system is transitive and a factor of a mixing system is again mixing. The trivial system $(\mathbb{Z}(1), R_1)$ is mixing. If $n > 1$, the finite system $(\mathbb{Z}(n), R_1)$ of Example 2.8 is transitive but not mixing. If (X, F) is any transitive system and $n > 1$, then $(X \times \mathbb{Z}_n, F \times R_1)$ is transitive but not mixing.

Proposition 2.38. — *A mixing dynamical system with at least two points, is sensitive.*

Proof. — Let $u, v \in X$ be distinct points and set $\varepsilon = d(u, v)/4$. We show that ε is the sensitivity constant of (X, F) (Figure 13 right). For any $x \in X$ and any $\delta > 0$ there exists $n > 0$ such that $F^n(B_\delta(x)) \cap B_\varepsilon(u) \neq \emptyset$ and $F^n(B_\delta(x)) \cap B_\varepsilon(v) \neq \emptyset$. Thus there exist $y, z \in B_\delta(x)$ with $d(F^n(y), u) < \varepsilon$ and $d(F^n(z), v) < \varepsilon$ and

$$d(F^n(y), F^n(z)) \geq d(u, v) - d(F^n(y), u) - d(F^n(z), v) \geq 4\varepsilon - \varepsilon - \varepsilon = 2\varepsilon.$$

It follows that either $d(F^n(y), F^n(x)) \geq \varepsilon$ or $d(F^n(z), F^n(x)) \geq \varepsilon$. □

Definition 2.39. — A dynamical system (X, F) is **positively expansive**, if

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists n \geq 0, d(F^n(x), F^n(y)) \geq \varepsilon$$

The angle doubling system $(\mathbb{T}, D)$ and any full shift $(A^{\mathbb{N}}, \sigma)$ are positively expansive systems. The quadratic system $(\mathbb{I}, Q_4)$ is sensitive but not positively expansive. For any $\varepsilon > 0$ we have points $x = \frac{1}{2} - \frac{\varepsilon}{4}$, $y = \frac{1}{2} + \frac{\varepsilon}{4}$ whose trajectories are ε-close. Finite system $(\mathbb{Z}_n, R_1)$ with $n > 1$ is positively expansive but not sensitive. We recall that a metric space X is perfect , if it does not have isolated points.

Proposition 2.40. — *If (X, F) is positively expansive and if X is perfect, then (X, F) is sensitive.*

Proof. — Let (X, F) be positively expansive with expansivity constant $\varepsilon > 0$. If X is perfect and $x \in X$, then for any $\delta > 0$ there exists a point $y \neq x$ with $d(y, x) < \delta$, and for some $n \geq 0$, $d(F^n(y), F^n(x)) \geq \varepsilon$. □

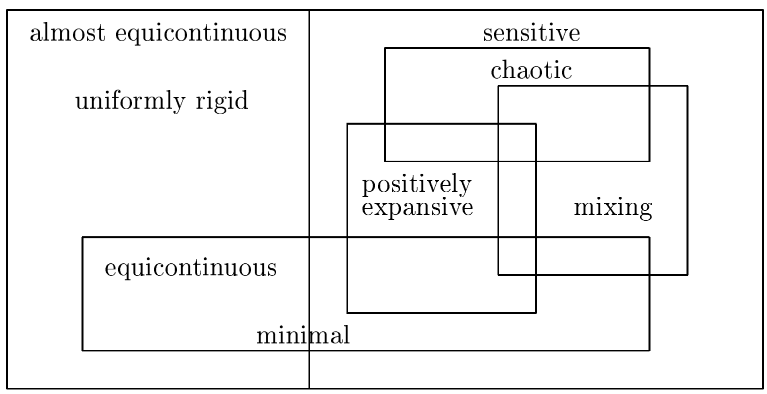

FIGURE 14. Classification of transitive systems

2.6. Maximal equicontinuous factor

Recall that a system is equicontinuous, if all its points are equicontinuous, i.e., if $\mathcal{E} = X$.

Proposition 2.41. — *Let (X,F) be an equicontinuous system. Then*

$$\forall\varepsilon > 0, \exists\delta > 0, \forall x, y \in X, (d(x,y) < \delta \;\Rightarrow\; \forall n \geq 0, d(F^n(x), F^n(y)) < \varepsilon)\,.$$

Moreover there exists an equivalent metric ϱ on X such that $\varrho(F(x), F(y)) \leq \varrho(x,y)$.

Proof. — Let $\varepsilon > 0$. By the assumption, for any z there exists δ_z, such that for any $y \in B_{\delta_z}(z) = U_z$, $d(F^n(z), F^n(y)) < \frac{\varepsilon}{2}$ for all n. The system $\{U_z : z \in X\}$ is an open cover of X. Let δ be its Lebesgue number, so that for any $x, y \in X$ with $d(x,y) < \delta$ there exists $z \in X$ such that $B_\delta(x) \subseteq U_z$ (Figure 15 left). Thus $x, y \in U_z$, and for all $n \geq 0$, $d(F^n(x), F^n(y)) \leq d(F^n(x), F^n(z)) + d(F^n(z), F^n(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

For $x, y \in X$ set

$$\varrho(x,y) = \sup\{d(F^n(x), F^n(y)) : n \geq 0\}.$$

For any dynamical system, ϱ is a metric and $\varrho(F(x), F(y)) \leq \varrho(x,y)$. Since $d(x,y) \leq \varrho(x,y)$, $\mathrm{Id} : (X,\varrho) \to (X,d)$ is continuous. Since (X,F) is equicontinuous, for any ε there exists δ such that if $d(x,y) < \delta$, then $\varrho(x,y) \leq \varepsilon$. This means that $\mathrm{Id} : (X,d) \to (X,\varrho)$ is continuous and therefore (X,d,F) and (X,ϱ,F) are conjugate. □

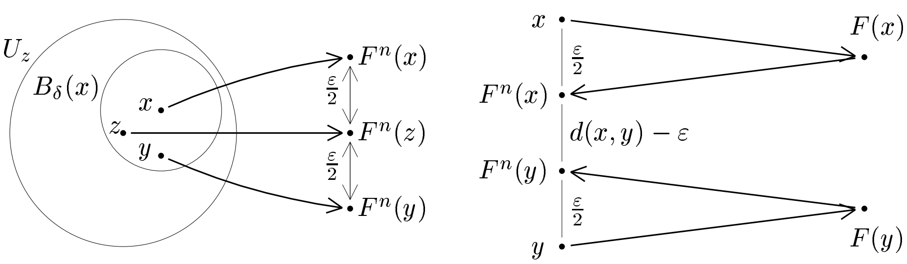

FIGURE 15. Equicontinuous systems

We have seen two examples of minimal equicontinuous systems: rotations of the circle and the adding machine. In both cases there is a group structure on the state space and the dynamics is addition of a specified element. This is also the case for finite minimal systems $\mathbb{Z}(n) = \{0, 1, \ldots, n-1\}$ which are groups with the addition modulo n. We show now that all minimal equicontinuous systems can be obtained as rotations of a compact Abelian (commutative) group. Recall that an Abelian group is a structure $(X, 0, +)$ with associative and commutative binary operation $+ : X \times X \to X$ and neutral element 0 in which there exist negative elements (see Section A.8). We say that X is a metric group if X is also a metric space and both operations $+$ and $-$ are continuous.

Theorem 2.42. — *Let (X,F) be a minimal equicontinuous system. Then*

(1) (X,F) is conjugate to an isometry.

(2) There exists an Abelian group structure on X with continuous group operations and a point $a \in X$ such that $F(x) = x + a$.

Proof. —

(1) By Proposition 2.41, there exists an equivalent metric d such that for any $x, y \in X$, $d(F(x), F(y)) \leq d(x, y)$. Assume that there exist x, y with $\varepsilon = d(x, y) - d(F(x), F(y)) > 0$. By Theorem 2.33 (X, F) is uniformly rigid, so there exists $n > 0$ such that $d(F^n(x), x) < \frac{\varepsilon}{2}$ and $d(F^n(y), y)) < \frac{\varepsilon}{2}$ (Figure 15 right). We get

$$d(x, y) \leq d(x, F^n(x)) + d(F^n(x), F^n(y)) + d(F^n(y), y) < d(F(x), F(y)) + \varepsilon$$

and this is a contradiction. Thus F is an isometry.

(2) Since F is bijective there exist negative iterations $F^{-n} = (F^{-1})^n$. Choose a point from X and denote it by 0. Define a group structure on $Y = \{F^n(0) : n \in \mathbb{Z}\}$ by $F^n(0) + F^m(0) = F^{n+m}(0)$. We show that this addition is continuous on Y.

$$\begin{aligned} & d(F^{n_1}(0), F^{n_2}(0)) < \varepsilon \;\; \& \;\; d(F^{m_1}(0), F^{m_2}(0)) < \varepsilon \\ & \qquad \Rightarrow \;\; d(F^{n_1 - n_2}(0), 0) < \varepsilon \;\; \& \;\; d(0, F^{m_2 - m_1}(0)) < \varepsilon \\ & \qquad \Rightarrow \;\; d(F^{n_1 - n_2}(0), F^{m_2 - m_1}(0)) < 2\varepsilon \\ & \qquad \Rightarrow \;\; d(F^{n_1 + m_1}(0), F^{m_2 + n_2}(0)) < 2\varepsilon \end{aligned}$$

Since Y is dense in X, the addition can be continuously extended to X. Put $a = F(0)$. If $x = F^n(0) \in Y$, then $F(x) = F^{n+1}(0) = x + a$. Since Y is dense, $F(x) = x + a$ for all $x \in X$. □

If X is a compact Abelian group and $a \in X$, we denote by (X, R_a) the rotation defined by $R_a(x) = x + a$. Any rotation is equicontinuous, but not necessarily minimal.

Proposition 2.43. —

(1) A subsystem of an equicontinuous system is equicontinuous.

(2) The product of equicontinuous systems is equicontinuous.

The proof follows from the definition. Using the product construction we obtain many equicontinuous systems. However, the product of minimal systems need not be minimal. For example the product of finite systems $(\mathbb{Z}(n), R_1)$ and $(\mathbb{Z}(m), R_1)$ is minimal iff n, m are mutually prime. In this case the product is conjugate to $(\mathbb{Z}(nm), R_1)$. The product of two irrational rotations of the circle $(\mathbb{T}, R_\alpha)$ and $(\mathbb{T}, R_\beta)$ is minimal iff α/β is irrational (Proposition 1.38). If $(\alpha_i)_{i \geq 0}$ is a sequence of irrational numbers such that all ratios α_i/α_j are irrational, then the infinite product $(\mathbb{T}^{\mathbb{N}}, \prod_{i \geq 0} R_{\alpha_i})$ is minimal. We have also minimal systems $(\mathbb{T} \times \mathbb{Z}(n), R_\alpha \times R_1)$ for α irrational.

The Feigenbaum subshift of Example 1.30 is not equicontinuous, but it has as an equicontinuous factor - namely the dyadic adding machine $(\mathbf{2}^{\mathbb{N}}, R_1)$ of Example 1.24. The Feigenbaum subshift has many other equicontinuous factors, e.g., the trivial system on $\mathbb{Z}(1) = \{0\}$ and finite systems $(\mathbb{Z}(2^n), R_1)$. It turns out, however, that any equicontinuous factor of the Feigenbaum subshift is also a factor of the dyadic adding machine. We say that the dyadic adding machine is the **maximal equicontinuous factor** of the Feigenbaum subshift.

In fact any dynamical system possesses its maximal equicontinuous factor. We obtain it when we identify points whose trajectories approach one another. To carry out this construction, we have to step out of the category of compact metric spaces into larger categories of compact Hausdorff topological spaces (Section A.6) and compact uniform spaces (Section A.7). Given a dynamical system (X, F), any factor map $\varphi : (X, F) \to (Z, H)$ yields a closed equivalence relation

$$R_\varphi = \{(x_0, x_1) \in X \times X : \varphi(x_0) = \varphi(x_1)\}$$

which is F-invariant, i.e., $(F \times F)(R_\varphi) \subseteq R_\varphi$. Using the theory of compact Hausdorff spaces, and particularly Proposition A.54, we can reconstruct the factort map from the relation. If $R \subseteq X \times X$ is a closed F-invariant equivalence relation, then there exists a compact metric space X/R and a factor map $\varphi_R : (X, F) \to (X/R, F/R)$ with $R_\varphi = R$.

The theory of uniform spaces studies uniformly continuous maps in an abstract setting. Any metric space yields a uniformity

$$\mathcal{U} = \{R \subseteq X \times X : \exists \varepsilon > 0, \Delta_X^\varepsilon \subseteq R\},$$

where $\Delta_X^\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$. Given a uniform space $(X, \mathcal{U})$, a map $F : X \to X$ is uniformly continuous, if for any $S \in \mathcal{U}$ there exists $R \in \mathcal{U}$ such that $(F \times F)(R) \subseteq S$. It is equicontinuous, if for any $S \in \mathcal{U}$ there exists $R \in \mathcal{U}$ such that for any $n \geq 0$, $(F^n \times F^n)(R) \subseteq S$. As in the case of metric space, we find that a subsystem of an equicontinuous system is equicontinuous and a product (finite or infinite) of equicontinuous systems is equicontinuous. We say that $\pi : (X, F) \to (Y, G)$ is an equicontinuous factor, if π is a factor map and (Y, G) is equicontinuous.

Theorem 2.44. — *Any dynamical system (X, F) possesses a* **maximal equicontinuous factor** *$\pi : (X, F) \to (Y, G)$, such that for any equicontinuous factor $\varphi : (X, F) \to (Z, H)$, there is a unique factor map $\psi : (Y, G) \to (Z, H)$ with $\psi \circ \pi = \varphi$. The system (Y, G) is unique up to conjugacy.*

$$\begin{tikzcd}
(X,F) \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & (Y,G) \arrow[d, "\psi"] \\
 & (Z,H)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,F) \arrow[r, "\xi"] \arrow[dr, "\varphi_S"'] & (W,K) \arrow[d, "\pi_S"] \\
 & (X/S,F/S)
\end{tikzcd}$$

Proof. — Let $\mathcal{R}$ be the family of all closed F-invariant relations on X which give rise to equicontinuous factors. The family $\mathcal{R}$ contains $X \times X$ so it is nonempty. The intersection $R = \bigcap \mathcal{R}$ is a nonempty closed F-invariant equivalence relation, so $Y = X/R$ is a compact metric space. There exists a unique continuous map $G : Y \to Y$ which commutes with these projection $\pi : X \to Y$. If $\varphi : (X, F) \to (Z, H)$ is a factor map with (Z, H) equicontinuous, then $R \subseteq R_\varphi$, so there exists a unique factor $\psi : (Y, G) \to (Z, H)$ with $\psi \circ \pi = \varphi$. To show that (Y, G) itself is equicontinuous, we consider infinite product $W = \prod_{S \in \mathcal{R}} X/S$ with projections $\pi_S : W \to X/S$. Here W a uniform space and it is compact by the Tychonoff theorem A.52. There exists a unique continuous $K : W \to W$ which commutes with the projections. The system (W, K) is equicontinuous and we have a morphism $\xi : (X, F) \to (W, K)$, such that

$\pi_S \circ \xi = \varphi_S : X \to X/S$ is the factor corresponding to S. Thus (Y, G) is conjugate to the subsystem $(\xi(X), K)$ of (W, K), and therefore (Y, G) is also equicontinuous. □

Recall that a dynamical system (X, F) is mixing (Definition 2.37), if for any nonempty open sets $U, V \subseteq X$ there exists $n > 0$, such that $F^m(U) \cap V \neq \emptyset$ for all $m > n$.

Proposition 2.45. — *The maximal equicontinuous factor of a mixing system is trivial.*

Proof. — A factor of a mixing system is mixing, so the maximal equicontinuous factor is mixing too. If it were not trivial, then it would be sensitive by Proposition 2.38 and this is impossible. □

If (X, F) is a minimal system, then its maximal equicontinuous factor is a rotation of a compact group. We now give two results which help to identify maximal equicontinuous factors.

Definition 2.46. — Let (X, F) be a dynamical system. We say that $z, w \in X$ are **regionally proximal**, if there exists an increasing integer sequence $(n_i)_{i \geq 0}$ and convergent sequences such that

$$\lim_{i\to\infty} z_i = z, \quad \lim_{i\to\infty} w_i = w, \quad \lim_{i\to\infty} d(F^{n_i}(z_i), F^{n_i}(w_i)) = 0.$$

Proposition 2.47. — *Let (X, F) be a minimal system and let $\pi : (X, F) \to (Y, G)$ be the maximal equicontinuous factor. If $z, w \in X$ are regionally proximal, then $\pi(z) = \pi(w)$.*

Proof. — By Proposition 2.41 we can assume that (Y, G) is an isometry. If $\pi(z) \neq \pi(w)$, then

$$0 = \lim_{i\to\infty} d(\pi(F^{n_i}(z_i)), \pi(F^{n_i}(w_i))) = \lim_{i\to\infty} d(\pi(z_i), \pi(w_i)) = d(\pi(z), \pi(w)) > 0$$

and this is a contradiction. □

In the full shift $(\mathbf{2}^{\mathbb{N}}, \sigma)$ any two points are regionally proximal, so the maximal equicontinuous factor is trivial. (This follows also from Proposition 2.45.) In the cubic root system $([-1, 1], \sqrt[3]{x})$, we have two attracting fixed points -1 and 1 and one unstable fixed point 0. Any point $x \in [-1, 1]$ is regionally proximal to 0, so the maximal equicontinuous factor is again trivial.

The regionally proximal relation is reflexive and symmetric but not necessarily transitive. Proposition 2.47 says that the regionally proximal relation is contained in the equivalence of the maximal equicontinuous factor. Auslander [**9**] shows that the maximal equicontinuous equivalence is the smallest closed equivalence which contains the regionally proximal relation. If the system is minimal, then these two relations coincide.

Definition 2.48. — We say that $\varphi : (X, F) \to (Y, G)$ is an **almost one-to-one extension**, if it is an extension and there exists $y \in Y$ such that $\varphi^{-1}(y)$ is a singleton.

***Theorem 2.49* (Paul [123]).** — *Let (X,F) be minimal and let $\varphi:(X,F)\to(Z,H)$ be an almost one-to-one extension of a compact group Z, where $H(z)=c+z$ for some $c\in Z$. Then (Z,H) is a maximal equicontinuous factor of (X,F).*

Proof. — Let $\pi:(X,F)\to(Y,G)$ be the maximal equicontinuous factor of (X,F) and $G(y)=b+y$. By Theorem 2.44, there exists a factor map $\psi:(Y,G)\to(Z,H)$. Define $\psi_0:Y\to Z$ by $\psi_0(y)=\psi(y)-\psi(0)$. For any $n\in\mathbb{Z}$ we have

$$\psi_0(nb)=\psi(G^n(0))-\psi(0)=H^n(\psi(0))-\psi(0)=nc$$

so ψ_0 is a group homomorphism on the orbit $\mathcal{O}(0)=\{nb:n\in\mathbb{Z}\}$. Since this orbit is dense in Y, $\psi_0:Y\to Z$ is a group homomorphism. Given $z_0,z_1\in Z$, pick $y_{01}\in Y$ with $\psi_0(y_{01})=z_1-z_0$. Then $\xi:\psi_0^{-1}(z_0)\to\psi_0^{-1}(z_1)$ defined by $\xi(y)=y+y_{01}$ is bijective, so all fibers $\psi_0^{-1}(z)$ have the same cardinality and therefore also all fibers $\psi^{-1}(z)$ have the same cardinality. It follows that all $\psi^{-1}(z)$ are singletons, so ψ is a conjugacy. □

Let Σ be the Feigenbaum subshift from Example 1.30, and let $\chi:(\Sigma,\sigma)\to(\mathbf{2}^{\mathbb{N}},R_1)$ be the factor map onto the dyadic adding machine. If $\mathcal{K}$ is the kneading sequence, then $\chi^{-1}(\chi((\mathcal{K}))=\{\mathcal{K}\}$ is a sigleton, so $(\mathbf{2}^{\mathbb{N}},R_1)$ is the maximal equicontinuous factor of (Σ,σ).

2.7. Spectral theory

Let $\mathbb{C}$ be the space of complex numbers. For a given dynamical system (X,F) we consider the space $\mathbf{C}(X,\mathbb{C})$ of continuous complex-valued functions $f:X\to\mathbb{C}$ (Section A.5) with metric

$$d(f,g)=\sup\{|f(x)-g(x)|:x\in X\}$$

By Proposition A.38, the space $\mathbf{C}(X,\mathbb{C})$ has a countable dense set. It is also an (infinite dimensional) vector space over $\mathbb{C}$. The map $F:X\to X$ induces a continuous linear operator $\mathcal{F}:\mathbf{C}(X,\mathbb{C})\to\mathbf{C}(X,\mathbb{C})$ given by $\mathcal{F}(f)=f\circ F$. As in the case of a linear operator on a finite-dimensional vector space, the operator $\mathcal{F}$ has **eigenvalues** and eigenvectors, which are called **eigenfunctions**. Recall that a set $\mathcal{A}\subseteq\mathbf{C}(X,\mathbb{C})$ is **linarly independent**, if for any distinct functions $f_1,\dots,f_n\in\mathcal{A}$ and any nonzero complex numbers $c_1,\dots,c_n\in\mathbb{C}$, the linear combination $c_1f_1+\cdots+c_nf_n$ is nonzero. A linearly independent set $\mathcal{A}\subseteq\mathbf{C}(X,\mathbb{C})$ is a **basis**, if its **linear span** $\mathbf{L}(\mathcal{A})$, which consists of all linear combinations of elements of $\mathcal{A}$, is dense in $\mathbf{C}(X,\mathbb{C})$.

Definition 2.50. — Let (X,F) be a dynamical system.

(1) A continuous nonzero function $f:X\to\mathbb{C}$ is an **eigenfunction** of (X,F) with an **eigenvalue** $\lambda\in\mathbb{C}$, if $f\circ F=\lambda\cdot f$.
(2) The set $\mathcal{S}(X,F)$ of all eigenvalues of (X,F) is called its **spectrum**.
(3) We say that (X,F) has a **discrete spectrum**, if the set of its eigenfunctions is the basis of the vector space $\mathbf{C}(X,\mathbb{C})$.
(4) We say that (X,F) has **trivial spectrum**, if $\mathcal{S}(X,F)=\{1\}$.

Any dynamical system has eigenvalue 1 with eigenfunction $\mathbf{e}(x)=1$.

Example 2.51. — The finite system $(\mathbb{Z}(n), R_1)$ has discrete spectrum

$$\mathcal{S}(\mathbb{Z}(n), R_1) = \{e^{2\pi i k/n} : k = 0, 1, \ldots, n-1\}$$

Proof. — The space $\mathbf{C}(\mathbb{Z}(n), \mathbb{C})$ is simply the n-dimensional complex vector space $\mathbb{C}^n$. Set $\alpha = e^{2\pi i/n}$. If $f : \mathbb{Z}(n) \to \mathbb{C}$ is an eigenfunction with eigenvalue λ, then $f(x) = f(R_1^n(x)) = \lambda^n f(x)$. Since $f(x) \neq 0$, for some x, we get $\lambda^n = 1$, and $\lambda = \alpha^k$ for some $k < n$. The corresponding eigenfunction is $f_k(j) = \alpha^{kj}$. Indeed

$$f_k(R_1(j)) = \alpha^{k(j+1)} = \alpha^k \cdot f_k(j).$$

We verify easily that the matrix

$$(f_k(j))_{j,k<n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \alpha & \cdots & \alpha^{n-1} \\ & & \vdots & \\ 1 & \alpha^{n-1} & & \alpha^{(n-1)^2} \end{bmatrix}$$

is regular, so the eigenfunctions form a basis of $\mathbb{C}^n$. □

Example 2.52. — A rotation of the circle $(\mathbb{T}, R_a)$ has spectrum and eigenfunctions

$$\mathcal{S}(\mathcal{T}, R_a) = \{\alpha^n : n \in \mathbb{Z}\}, \quad f_n(x) = e^{2\pi i n x}$$

where $\alpha = e^{2\pi i a}$. The system has discrete spectrum iff a is irrational.

Proof. — We have $f_n(R_a(x)) = e^{2\pi i n(x+a)} = \alpha_n \cdot f_n(x)$, so f_n is an eigenfunction with eigenvalue α^n. If $a = \frac{p}{q}$ is rational, with p, q relatively prime, then the spectrum

$$\mathcal{S}(\mathbb{T}, R_a) = \{\alpha^n : n = 0, 1, \ldots, q-1\}$$

is finite and the eigenfunctions $\{f_n : n = 0, \ldots, q-1\}$ do not span the infinite-dimensional space $\mathbf{C}(\mathbb{T}, \mathbb{C})$. If a is irrational, the spectrum $\mathcal{S}(\mathbb{T}, R_a) = \{\alpha^n : n \in \mathbb{Z}\}$ is infinite and the functions f_n span $\mathbf{C}(\mathbb{T}, \mathbb{C})$. This is simply the classical theorem of Fourier analysis. A function $f \in \mathbf{C}(\mathbb{T}, \mathbb{C})$ can be regarded as a periodic continuous function $f : \mathbb{R} \to \mathbb{C}$ with period 1, or $f(x+1) = f(x)$. Any such function can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n f_n(x), \quad a_n = \int_0^1 f(x) \cdot \overline{f_n(x)}\, dx.$$

□

Denote by $\mathbb{K} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle in the complex plane. With standard multiplication, $(\mathbb{K}, \cdot, 1)$ is a compact group isomorphic to $(\mathbb{T}, +, 0)$.

Proposition 2.53. — *Let (X, F) be a transitive system.*

(1) If $f \in \mathbf{C}(X, \mathbb{C})$ is an eigenfunction with eigenvalue 1, then f is a constant function.

(2) If f is an eigenfunction with eigenvalue λ then $|\lambda| = 1$ and $|f|$ is a constant function.

(3) If f, g are both eigenfunctions corresponding to the same eigenvalue λ, then there exists $c \in \mathbb{C}$ such that $f = cg$.

(4) A finite set of eigenfunctions corresponding to distinct eigenvalues is linearly independent in $\mathbf{C}(X, \mathbb{C})$.

(5) The spectrum $\mathcal{S}(X, F)$ is a countable subgroup of $\mathbb{K}$.

Proof. —

(1) Let $x \in X$ be a transitive point. Then f is constant on $\mathcal{O}(x)$, so it is constant on $X = \overline{\mathcal{O}(x)}$.

(2) Since F is surjective and $|f(F(x))| = |\lambda| \cdot |f(x)|$, we have

$$\sup\{|f(x)| : x \in X\} = \sup\{f(F(x)) : x \in X\} = |\lambda| \cdot \sup\{|f(x)| : x \in X\}$$

Since f is nonzero, $|\lambda| = 1$ and $|f|$ is constant by 1.

(3) If f, g are eigenfunctions with eigenvalue λ, then $g(x) \neq 0$ by 2, and f/g is an eigenfunction with eigenvalue 1, so it is constant by 1.

(4) Let $f_i(F(x)) = \lambda_i f_i(x)$ be eigenfunctions for $i = 1, \ldots, k$ with $\lambda_1, \ldots, \lambda_k$ distinct. Assume that there exist complex nonzero vector $(a_1, \ldots, a_k)$ such that

$$a_1 f_1(x) + \cdots + a_k f_k(x) = 0.$$

If we apply F^i, we get $a_1 \lambda_1^i f_1(x) + \cdots + a_k \lambda_k^i f_k(x)$, or, in matrix form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ & & \vdots & \\ \lambda_1^{k-1} & \lambda_2^{k-1} & & \lambda_k^{k-1} \end{bmatrix} \cdot \begin{bmatrix} a_1 f_1(x) \\ a_2 f_2(x) \\ \vdots \\ a_k f_k(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the eigenvalues are distinct, the matrix is regular and $a_i f_i(x) = 0$. Since $f_i(x) \neq 0$, we get $a_i = 0$, so $f_1, \ldots, f_k$ are linearly independent.

(5) It is clear that the eigenvalues form a subgroup of $\mathbb{K}$. If f, g are eigenfunctions with eigenvalues α, β, then $f \cdot g$, f/g are eigenfunctions with eigenvalues $\alpha \cdot \beta$ and α/β. Assume that f is an eigenfunction with eigenvalue $\alpha \neq 1$. After normalization $|f| = 1$. We choose an arbitrary $x_0 \in X$. There exists $k > 0$ such that the real part of $\alpha^k f(x_0)$ is negative.

$$d(f, \mathbf{e}) = \sup_{x \in X} d(f(x), 1) \geq d(f(F^k(x_0)), 1) = d(\alpha^k f(x_0), 1) > \tfrac{1}{4}$$

If f, g are eigenvectors corresponding to distinct eigenvalues, then $d(f/g, \mathbf{e}) > \frac{1}{4}$. If $|f| = |g| = 1$, then $d(f, g) > \frac{1}{4}$. By Theorem A.38, the space $\mathbf{C}(X, \mathbb{C})$ has a countable dense set, so there exists at most a countable number of eigenvalues. □

A **character** of a metric group X is a continuous group homomorphism $f : X \to \mathbb{K}$ (Definition A.66). The product of characters is a character too, so the characters form a group $\widehat{X}$ which is called the **dual group** of X. By the Pontrjagin duality theorem A.69, the dual $\widehat{\widehat{X}}$ of $\widehat{X}$ is isomorphic to X. For example the characters of a finite group $\mathbb{Z}(n)$ are functions $f_j(k) = e^{2\pi kj/n}$, $j = 0, \ldots, n-1$, so $\widehat{\mathbb{Z}(n)} \approx \mathbb{Z}(n)$ is self-dual. The characters of $(\mathbb{T}, +, 0)$ are exponential functions $f_n(x) = e^{2\pi inx}$, $n \in \mathbb{Z}$, so $\widehat{\mathbb{T}} \approx \mathbb{Z}$. In the multiplicative group $(\mathbb{K}, \cdot, 1)$ which is isomorphic to $(\mathbb{T}, +, 0)$, the characters are just the powers $f_n(z) = z^n$, for $n \in \mathbb{Z}$. The group $\mathbb{Z}$ has characters $f_z(n) = z^n$ indexed by $z \in \mathbb{K}$, so $\widehat{\mathbb{Z}} \approx \mathbb{K}$, as predicted by the Pontrjagin theorem.

We use also another consequence of the Pontrjagin duality theorem which says that characters separate points (Proposition A.70). If X is a compact group, and $x, y \in X$ are distinct, then there exists a character $f : X \to \mathbb{K}$ such that $f(x) \neq f(y)$. Consider the linear span $\mathcal{A} = \mathbf{L}(\widehat{X})$ of all linear combinations of characters. Then $\mathcal{A}$ separates points as well and the Stone-Weierstrass theorem A.37 can be applied.

If (X, R_a) is a rotation of a compact group and if $f : X \to K$ is a character, then $f(R_a(x)) = f(a) \cdot f(x)$, so f is an eigenfunction with eigenvalue $f(a)$. If (X, R_a) is minimal, these are the only eigenvalues.

Proposition 2.54. — *If (X, R_a) is a minimal rotation on a compact Abelian group, then any eigenfunction of (X, R_a) is a constant multiple of a character f of X and the corresponding eigenvalue is $\alpha = f(a)$.*

Proof. — Assume that α is an eigenvalue. The corresponding eigenfunctions are constant multiples of each other, so we can assume $f(0) = 1$. Since $f(x+a) = \alpha \cdot f(x)$, $f(na) = \alpha^n$, so $f(x + y) = f(x)f(y)$ for $x, y \in \{na : n \in \mathbb{Z}\}$. Since this is a dense subset of X, f is a character. Moreover, $f(a) = \alpha \cdot f(0) = \alpha$. □

***Theorem 2.55* (Halmos and von Neumann).** — *For a dynamical system (X, F) the following conditions are equivalent.*

(1) (X, F) is transitive and has a discrete spectrum.

(2) (X, F) is conjugate to a minimal rotation on a compact Abelian group.

Proof. —

(1) $\Rightarrow$ (2) : Assume that (X, F) is transitive and there exist eigenvalues $(\lambda_n)_{n\geq 0}$ with eigenfunctions $(f_n)_{n\geq 0}$ which form a basis of $\mathbf{C}(X, \mathbb{C})$. Then for any distinct $x, y \in X$ there exists an eigenfunction $f : X \to \mathbb{K}$ with $f(x) \neq f(y)$, so we may define the metric

$$\varrho(x, y) = \sum_{i=0}^{\infty} |f_i(x) - f_i(y)| \cdot 2^{-i}$$

Since $|\lambda_i| = 1$, we get

$$\varrho(F(x), F(y)) = \sum_{i=0}^{\infty} |\lambda_i f_i(x) - \lambda_i f_i(y)| \cdot 2^{-i} = \varrho(x, y)$$

so F is an isometry with respect to ϱ. We show that the identity $\mathrm{Id} : (X, d) \to (X, \varrho)$ is continuous. Given $\varepsilon > 0$, choose n such that $\sum_{i=n}^{\infty} 2^{1-i} < \frac{\varepsilon}{2}$. There exists $\delta > 0$ such that if $d(x, y) < \delta$, then $|f_i(x) - f_i(y)| < \frac{\varepsilon}{4}$ for $0 \leq i < n$. Thus

$$d(x, y) < \delta \;\Rightarrow\; \varrho(x, y) \leq \sum_{i=0}^{n-1} \varepsilon 2^{-i-2} + \tfrac{\varepsilon}{2} = \varepsilon$$

By Proposition A.20, the continuous image (X, ϱ) of a compact space (X, d) is compact and $\mathrm{Id} : (X, d) \to (X, \varrho)$ is a homeomorphism. Thus (X, F) is equicontinuous. By Corollary 2.34 it is minimal and by Theorem 2.42 it is conjugate to a minimal rotation on a compact Abelian group.

(2) $\Rightarrow$ (1): If (X, F) is a minimal rotation, then each character is an eigenfunction. Let $\mathcal{A} = \mathbf{L}(\widehat{X})$ be the linear span of characters. By Proposition A.70, $\mathcal{A}$ is a subalgebra, contains constants, and is closed with respect to the complex conjugation. By the Stone-Weierstrass theorem A.37, $\mathcal{A}$ is dense in $\mathbf{C}(X, \mathbb{C})$, so (X, F) has discrete spectrum. □

Theorem 2.56. —

(1) Two minimal equicontinuous systems have the same spectrum iff they are conjugate.

(2) Two dynamical systems have the same spectrum iff their maximal equicontinuous factors are conjugate.

Proof. —

(1) We can assume that the two systems are rotations (X, R_a) and (Y, R_b). If they have the same spectrum, then

$$\{f(a) : f \in \widehat{X}\} = \{g(b) : g \in \widehat{Y}\}$$

Define a map $\psi : \widehat{Y} \to \widehat{X}$ by $(\psi(g))(a) = g(b)$. Then ψ is a bijective group homomorphism. By the Pontrjagin duality theorem A.69 there exists a bijective group homomorphism $\varphi : X \to Y$ with $\widehat{\varphi} = \psi$, or $\psi(g) = g \circ \varphi$. For any $g \in \widehat{Y}$ we have that

$$g(\varphi(a)) = (\psi(g))(a) = g(b).$$

Since $\widehat{Y}$ separates points, $\varphi(a) = b$, so $\varphi : (X, R_a) \to (Y, R_b)$ is a conjugacy.

(2) If $\pi : (X, F) \to (Y, G)$ is a factor map, and if $g \in \mathbf{C}(Y, \mathbb{C})$ is an eigenfunction of (Y, G) with eigenvalue α, then $f \circ \pi \in \mathbf{C}(X, \mathbb{C})$ is an eigenfunction of (X, F) with eigenvalue α, so $\mathcal{S}(Y, G) \subseteq \mathcal{S}(X, F)$. Assume now that (Y, G) is a maximal equicontinuous factor of (X, F). If $f \in \mathbf{C}(X, \mathbb{C})$ is an eigenfunction of (X, F) with eigenvalue α, then $f : (X, F) \to (\mathbb{K}, R_\alpha)$ is a factor map, so there exists a factor map $g : (Y, G) \to (\mathbb{K}, R_\alpha)$ with $g \circ \pi = f$. This means that $g \in \mathbf{C}(Y, \mathcal{C})$ is an eigenfunction of (Y, G) with eigenvalue α and $\mathcal{S}(X, F) \subseteq \mathcal{S}(Y, G)$. Thus a dynamical system has the same spectrum as its maximal equicontinuous factor. □

The spectrum of an irrational circle rotation is $\mathcal{S}(\mathbb{T}, R_\alpha) = \{e^{2\pi i n \alpha} : n \in \mathbb{Z}\}$. Two irrational rotations $(\mathbb{T}, R_\alpha)$ and $(\mathbb{T}, R_\beta)$ are therefore conjugate iff either $\alpha = \beta$ or $\alpha + \beta = 1$.

2.8. Attractors

The concept of an attractor generalizes the concept of an attracting fixed point or an attracting periodic orbit. A subsystem of a dynamical system is an attractor, if the trajectories of all neighbouring points do not escape and further converge on the subsystem. First, we define omega limit of a set. The omega limit of a point consists of all limit points of the point's trajectory.

Definition 2.57. — Let (X, F) be a dynamical system. The **omega limit** of a set $Y \subseteq X$ is

$$\omega(Y) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{m>n} F^m(Y)}.$$

If $Y \subseteq X$ is a closed invariant set, the definition simplifies to

$$\omega(Y) = \bigcap_{n=0}^{\infty} F^n(Y).$$

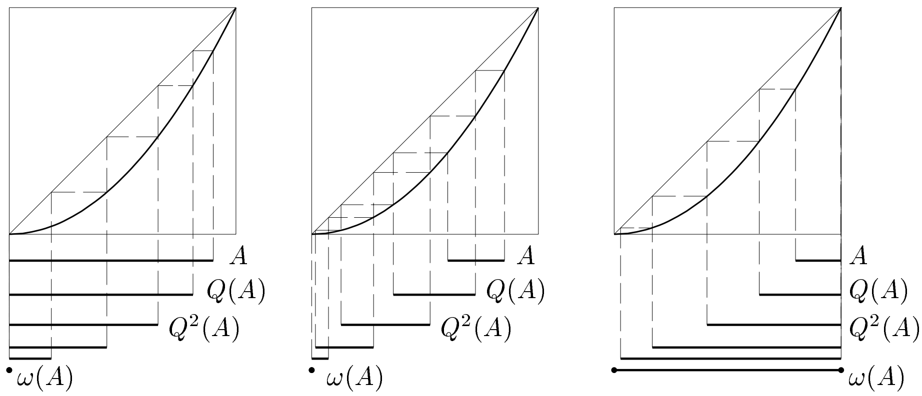

FIGURE 16. Omega limit of a set

If $y \in X$ and $\lim_{n\to\infty} F^n(y) = x$, then x is a fixed point and $\omega(y) = \{x\}$. Similarly, if the trajectory of a point y converges to a periodic orbit of period p, i.e., if $\lim_{n\to\infty} F^{np}(y) = x$, then $\omega(y) = \mathcal{O}(x)$. In general we have $\overline{\mathcal{O}(x)} = \mathcal{O}(x) \cup \omega(x)$. If (X, F) is a transitive system, then $\omega(U) = X$ for any nonempty open set U. Clearly, if $Y \subseteq Z$, then $\omega(Y) \subseteq \omega(Z)$.

In a quadratic system $(\mathbb{I}, Q)$, where $Q(x) = x^2$, we have $\omega(a, b) = \{0\}$ for $0 \leq a < b < 1$ (Figure 16 left and center) and $\omega(a, 1) = [0, 1]$ (Figure 16 right).

Proposition 2.58. — *A point $y \in X$ belongs to $\omega(Y)$ iff there exists a sequence of points $y_k \in Y$ and an increasing sequence $(n_k)_{k\geq 0}$, such that $\lim\limits_{k\to\infty} F^{n_k}(y_k) = y$.*

Proof. — Let $y \in \omega(Y)$. We construct sequences of y_k and n_k by induction. Since Y is nonempty, we may pick arbitrary $y_0 \in Y$ and $n_0 = 0$. Suppose y_{k-1} and n_{k-1} have been constructed. Since $y \in \overline{\bigcup\{F^m(Y) : m > n_{k-1}\}}$, the ball $B_{\frac{1}{k}}(y)$ has nonempty intersection with $\bigcup\{F^m(Y) : m > n_{k-1}\}$, so there exists $y_k \in Y$ and $n_k > n_{k-1}$ with $d(F^{n_k}(y_k), y) < \frac{1}{k}$. Thus $\lim_{k\to\infty} F^{n_k}(y_k) = y$. Conversely, if the condition is satisfied, then for any $n > 0$ and any neighbourhood U of y there exists $y_k \in Y$ and $n_k > n$, such that $F^{n_k}(y_k) \in U$, so $y \in \overline{\bigcup\{F^m(Y) : m > n\}}$. Thus $y \in \omega(Y)$. □

Proposition 2.59. — *Let (X, F) be a dynamical system and $Y \subseteq X$ a nonempty set. Then $\omega(Y)$ is nonempty and closed, and*

$$\omega(Y) = F(\omega(Y)) = \omega(\omega(Y)) = \omega(\overline{Y}).$$

Proof. —

(1) Since X is compact and $\omega(Y)$ is an intersection of a decreasing sequence of nonempty closed sets, it is nonempty and closed.

(2) Let $y \in \omega(Y)$. By Proposition 2.58, there exist n_k and $y_k \in Y$ with $F^{n_k}(y_k) \to y$ as $k \to \infty$, so $F^{n_k+1}(y_k) \to F(y)$ as $k \to \infty$. Thus $F(y) \in \omega(Y)$, and $F(\omega(Y)) \subseteq \omega(Y)$.

(3) On the other hand, since X is compact, the sequence $F^{n_k-1}(y_k)$ contains a converging subsequence $F^{n_{k_i}-1}(y_{k_i}) \to z$ as $i \to \infty$, so $z \in \omega(Y)$. Then $F(z) = y$, so $\omega(Y) \subseteq F(\omega(Y))$.

(4) From $F(\omega(Y)) = \omega(Y)$ it is immediate that $\omega(\omega(Y)) = \omega(Y)$.

(5) Since $Y \subseteq \overline{Y}$, $\omega(Y) \subseteq \omega(\overline{Y})$. Since $F^n(\overline{Y}) \subseteq \overline{F^n(Y)} \subseteq \overline{\bigcup\{F^m(Y) : m \geq n\}}$, we get $\omega(\overline{Y}) \subseteq \omega(Y)$. □

We have already introduced stable and attracting sets in Definition 1.18. The concept of an attractor is even stronger. An attractor is an attracting set which is closed and **strongly invariant** i.e., $F(Y) = Y$.

Definition 2.60. — Let (X, F) be a dynamical system and $Y \subseteq X$.

(1) Y is an **attractor** if it is nonempty, closed, $F(Y) = Y$ and if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for any $x \in X$

$$d(x, Y) < \delta \quad \Rightarrow \quad \forall n \geq 0, d(F^n(x), Y) < \varepsilon \ \& \ \lim_{n\to\infty} d(F^n(x), Y) = 0.$$

(2) Y is a **minimal attractor**, if no set $Z \subset Y$ is an attractor.

(3) Y is a **quasi-attractor**, if it is the intersection of a countable number of attractors but is not an attractor.

(4) The **basin of an attractor** Y is

$$\mathfrak{B}(Y) = \{x \in X : \lim_{n\to\infty} d(F^n(x), Y) = 0\}.$$

The cubic system $([-1,1], F)$, where $F(x) = x^3$, has attractors $\{0\}$, $[-1,0]$, $[0,1]$ and $[-1,1]$ with basins

$$\mathfrak{B}(\{0\}) = (-1,1), \quad \mathfrak{B}([-1,0]) = [-1,1), \quad \mathfrak{B}([0,1]) = (-1,1], \quad \mathfrak{B}([-1,1]) = [-1,1]$$

The minimal attractor is $\{0\}$ (Figure 17 top left). The cubic root system $([-1,1], F^{-1})$, where $F^{-1}(x) = \sqrt[3]{x}$, has attractors $\{-1\}$, $\{1\}$ and $[-1,1]$ with basins

$$\mathfrak{B}(-1) = [-1,0), \quad \mathfrak{B}(1) = (0,1], \quad \mathfrak{B}([-1,1]) = [-1,1]).$$

The minimal attractors are $\{-1\}$ and $\{1\}$ (Figure 17 top right). There exist also systems with infinite numbers of attractors.

Example 2.61. — $(\mathbb{I}, F)$ where $F(x) = x + \frac{x}{a+1}|\sin(a \ln x)|$, $a > 0$.

There are fixed points (Figure 17 bottom left)

$$p_0 = 1, \quad p_1 = e^{-\pi/a}, \ldots, \quad p_n = e^{-\pi n/a}, \ldots \quad p_\infty = 0.$$

In any interval (p_{n+1}, p_n), F has positive derivation

$$F'(x) = 1 \pm \frac{1}{a+1}(\sin(a \ln x) + a \cos(a \ln x)) > 0,$$

so F is an increasing function with $F(x) \geq x$. It follows that p_0 is an attracting fixed point, so $\{p_0\}$ is a minimal attractor and any interval $[p_n, p_0]$, $n \geq 0$ is an attractor. Its basin is $\mathfrak{B}([p_n, p_0]) = (p_{n+1}, p_0]$. The largest attractor is $[p_\infty, p_0] = \mathbb{I}$.

Example 2.62. — $(\mathbb{I}, G)$ where $G(x) = x - \frac{x}{a+1}|\sin(a \ln x)|$ and $a > 0$, has a quasi-attractor $\{0\}$ (Figure 17 bottom right).

Now G is an increasing function with $G(x) \leq x$ and positive derivative. Once more G has fixed points $p_n = e^{-\pi n/a}$, $p_\infty = 0$. The attractors are the intervals $[0, p_n]$ whose basins are $\mathfrak{B}([0, p_{n+1}]) = [0, p_n)$. Note that $\{0\}$ is not an attractor, since for any $x > 0$, $F^k(x)$ tends to some $p_n \neq 0$ as $k \to \infty$. Thus $\{0\}$ is a quasi-attractor.

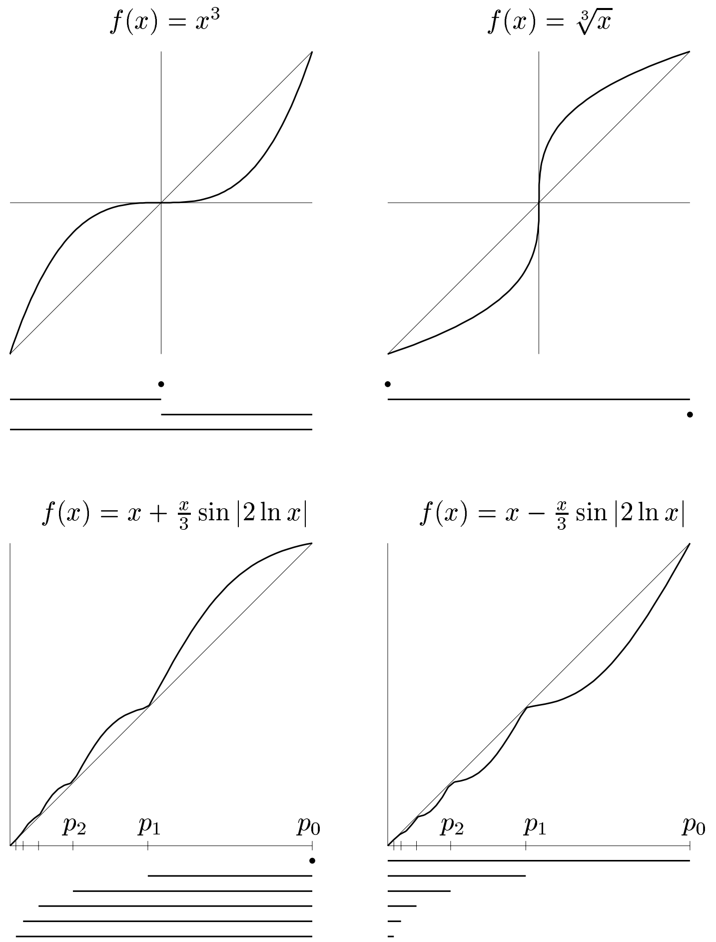

FIGURE 17. Attractors

In the Preston system of Example 1.19, all X_n are attractors and their intersection X is a quasi-attractor.

2.8.1. Inward sets. — We now characterize attractors as omega limits of inward sets. This is a stronger condition than invariance. The whole boundary of an inward set is required to map into its interior.

Definition 2.63. — Let (X, F) be a dynamical system. A set $V \subseteq X$ is **inward**, if $F(\overline{V}) \subseteq V^{\circ}$.

For the quadratic system $(\mathbb{I}, Q)$, with $Q(x) = x^2$, for any $0 < a < 1$, $[0, a)$ and $[0, a]$ are inward sets, while $[0, 1)$ is not. In any system (X, F), X is an inward set since $\overline{X} = X^{\circ} = X$. Clearly if V is an inward set then $\overline{V}$ is also an inward set since $F(\overline{V}) \subseteq V^{\circ} \subseteq \overline{V}^{\circ}$.

Proposition 2.64. — *Let (X, F) be a dynamical system, and $V \subseteq X$ a nonempty inward set. Then $Y = \omega(V)$ is an attractor.*

Proof. —

(1) By Proposition 2.59, Y is nonempty, closed and $F(Y) = Y$.

(2) Suppose that Y is not stable. Then there exists $\varepsilon > 0$, such that for any $\delta = \frac{1}{k}$ there exists $x_k \in X$ such that $d(x_k, Y) < \frac{1}{k}$, and $d(F^{n_k}(x_k), Y) \geq \varepsilon$ for some n_k (Figure 18 left). Since $Y \subseteq F(V) \subseteq V^\circ$, by Proposition A.28 there exists k_0 such that $B_{1/k_0}(Y) \subseteq V$, so for all $k > k_0$, $x_k \in V$. The sequence $F^{n_k}(x_k)$ has a converging subsequence $F^{n_{k_i}}(x_{k_i}) \to x$ as $i \to \infty$, so $x \in \omega(V) = Y$. This is contradicts the fact that $d(F^{n_k}(x_k), Y) \geq \varepsilon$.

(3) Since $Y \subseteq V^\circ$ there exists $\delta > 0$ such that $B_\delta(Y) \subseteq V^\circ$. Suppose that $d(x, Y) < \delta$ and that $d(F^n(x), Y)$ does not converge to zero. Then there exists $\varepsilon > 0$, such that for infinitely many different n, $d(F^n(x), Y) \geq \varepsilon$. Hence, there exists a converging sequence $F^{n_i}(x) \to y$ as $i \to \infty$, and $d(F^{n_i}(x), Y) \geq \varepsilon$. Since $x \in V$, $y \in \omega(V) = Y$ and $d(y, Y) \geq \varepsilon$. This is a contradiction. ☐

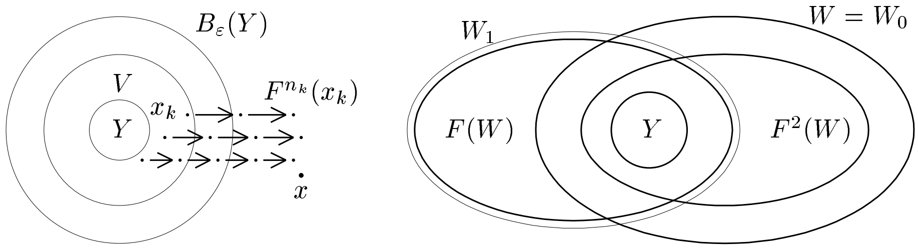

FIGURE 18. Inward sets

Proposition 2.65. — *Let $Y \subseteq X$ be an attractor. Then for any $\varepsilon > 0$ there exists a closed inward set $V \subseteq B_\varepsilon(Y)$ such that $Y = \omega(V)$.*

Proof. — For a given $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that for all $x \in X$,

$$d(x, Y) < \delta \quad \Rightarrow \quad \forall n, d(F^n(x), Y) < \varepsilon, \text{ and } \lim_{n\to\infty} d(F^n(x), Y) = 0.$$

Choose $0 < \varepsilon_0 < \delta$ and $0 < \delta_0 < \varepsilon_0$ such that for all $x \in X$,

$$d(x, Y) < \delta_0 \quad \Rightarrow \quad \forall n, d(F^n(x), Y) < \varepsilon_0.$$

Finally choose a closed set W with $B_{\varepsilon_0}(Y) \subseteq W^\circ \subseteq B_\delta(Y)$, so we have inclusions

$$B_{\delta_0}(Y) \subseteq B_{\varepsilon_0}(Y) \subseteq W^\circ \subseteq \overline{W} \subseteq B_\delta(Y) \subseteq B_\varepsilon(Y).$$

If $x \in W$, then $d(F^n(x), Y) \to 0$ as $n \to \infty$, so there exists $n_x > 0$ with $F^{n_x}(x) \in B_{\delta_0}(Y)$. By continuity of F, there exists an open $W_x \subseteq B_\delta(x)$, such that $x \in W_x$, $F^{n_x}(W_x) \subseteq B_{\delta_0}(Y)$, so for all $n \geq n_x$, $F^n(W_x) \subseteq B_{\varepsilon_0}(Y)$. Then $\{W_x : x \in W\}$ is an open cover of W, so it has a finite subcover $\{W_{x_1}, \ldots, W_{x_m}\}$. Set $n = \max\{n_{x_1}, \ldots, n_{x_m}\}$. Then

$$F^n(W) \subseteq B_{\varepsilon_0}(Y) \subseteq W^\circ.$$

We have obtained inside $B_\varepsilon(Y)$ a closed set W, which is inward for F^n. To obtain a set inward for F, we consider iterates of W (Figure 18 right). Set $W_0 = W$. Since $F(F^{n-1}(W)) \subseteq W_0^\circ \subseteq B_\varepsilon(Y)$ and $d(F^n(W), X \setminus W^\circ) > 0$, there exists a closed set

$W_1 \subseteq B_\varepsilon(Y)$, for which $F^{n-1}(W) \subseteq W_1^\circ$, $F(W_1) \subseteq W_0^\circ$. Iterationg we obtain closed sets $W_0, \dots, W_{n-1}$, such that

$$F^{n-k}(W) \subseteq W_k^\circ, \quad F(W_k) \subseteq W_{k-1}^\circ,$$

and finally $F(W_0) \subseteq W_{n-1}^\circ$. For $V = W_0 \cup \cdots \cup W_{n-1}$ we get

$$F(V) \subseteq W_{n-1}^\circ \cup W_0^\circ \cup \cdots \cup W_{n-2}^\circ \subseteq V^\circ.$$

Since for any k, $F^k(W) \subseteq B_\varepsilon(Y)$, we get $V \subseteq B_\varepsilon(Y)$ and $\omega(V) = Y$. □

Proposition 2.66. — *Let (X, F) be a dynamical system.*
(1) There exists a maximal attractor $\omega(X)$.
(2) If Y_0, Y_1 are attractors then $Y_0 \cup Y_1$ is an attractor.
(3) If Y_0, Y_1 are attractors with nonempty intersection, then $\omega(Y_0 \cap Y_1)$ is the largest attractor contained in both of them.
(4) If Y is an attractor and $\mathcal{B}$ is a countable base, then there exists an inward set $V = V_1 \cup \cdots \cup V_n$, such that $V_i \in \mathcal{B}$ and $\omega(V) = Y$.
(5) The set of attractors of (X, F) is at most countable.

Proof. —
(1) Since X is an inward set, $\omega(X)$ is an attractor.
(2) Let V_0 and V_1 be inward sets and $Y_i = \omega(V_i)$. Since

$$F(\overline{V_0 \cup V_1}) = F(\overline{V_0} \cup \overline{V_1}) \subseteq V_0^\circ \cup V_1^\circ \subseteq (V_0 \cup V_1)^\circ,$$

$\omega(V_0 \cup V_1)$ is an attractor. Clearly $Y_0 \cup Y_1 \subseteq \omega(V_0 \cup V_1)$. If $y \in \omega(V_0 \cup V_1)$, then there exist $y_k \in V_0 \cup V_1$ with $F^{n_k}(y_k) \to y$ as $k \to \infty$. At least one of the sets $\{k : y_k \in V_0\}$ and $\{k : y_k \in V_1\}$ is infinite, so either $y \in \omega(V_0)$ or $y \in \omega(V_1)$. Thus $\omega(V_0 \cup V_1) = Y_0 \cup Y_1$ is an attractor.
(3) If $Y_0 \cap Y_1 \neq \emptyset$ then $V_0 \cap V_1 \neq \emptyset$ and

$$F(\overline{V_0 \cap V_1}) \subseteq F(\overline{V_0} \cap \overline{V_1}) \subseteq V_0^\circ \cap V_1^\circ = (V_0 \cap V_1)^\circ,$$

so $\omega(V_0 \cap V_1)$ is an attractor. Since $\omega(Y_0 \cap Y_1) \subseteq \omega(V_0 \cap V_1) \subseteq Y_0 \cap Y_1$, we get that $\omega(Y_0 \cap Y_1) = \omega(V_0 \cap V_1)$ is an attractor.
(4) Let W be an inward set and $\omega(W) = Y$. For any $x \in \overline{W}$ there exists a set $x \in V_x \in \mathcal{B}$ such that $F(\overline{V_x}) \subseteq W^\circ$. The open cover $\{V_x : x \in \overline{W}\}$ of $\overline{W}$ has a finite subcover $\{V_{x_1}, \cdots, V_{x_n}\}$ and $V = V_{x_1} \cup \cdots \cup V_{x_n}$ is an inward set with $\omega(V) = Y$.
(5) The number of finite unions of sets of $\mathcal{B}$ is countable, hence the same is true of the number of attractors. □

2.9. Chain-invariant sets

The structure of attractors can be understood in terms of the chain relation $\mathfrak{C}$. Since an attractor is omega limit of an inward set, ε-chains cannot escape too far from it. For example, in the quadratic system $(\mathbb{I}, Q)$, an ε-chain starting from 0 satisfies $x_{n+1} < x_n^2 + \varepsilon$. For small ε, function $f_\varepsilon(x) = x^2 + \varepsilon$ has a unique fixed point x_ε, which converges to zero as $\varepsilon \to 0$. Thus ε-chains cannot escape from 0 farther than $x_\varepsilon + \varepsilon$. On the other hand, any point in an attractor can be reached by ε-chains from a chain recurrent point.

Let (X, F) be a dynamical system. We say that a set $Y \subseteq X$ is **chain-invariant**, or $\mathfrak{C}$-invariant, if $\mathfrak{C}(Y) \subseteq Y$, i.e., if $x \in Y$ and $(x, y) \in \mathfrak{C}$ implies $y \in Y$. For a set $Y \subseteq X$ and $\delta > 0$, let

$$\mathfrak{C}_\delta(Y) = \{x \in X : \exists y_0, \ldots, y_n, y_0 \in Y, y_n = x, \forall i < n, d(F(y_i), y_{i+1}) < \delta\}$$

be the set of points which can be reached from Y by δ-chains.

Proposition 2.67. — *Let (X, F) be a dynamical system and $Y \subseteq X$. Then*
(1) If $\delta < \varepsilon$, then $\overline{\mathfrak{C}_\delta(Y)} \subseteq \mathfrak{C}_\varepsilon(Y)$.
(2) $\mathfrak{C}_\delta(Y)$ is inward and open.
(3) Any inward set is chain-invariant.
(4) Any neighbourhood of a chain-invariant set contains an inward set.

Proof. —
(1) If $x \in \overline{\mathfrak{C}_\delta(Y)}$ then there exists $y \in \mathfrak{C}_\delta(Y)$ with $d(y, x) < \varepsilon - \delta$. If $y_0, \ldots, y_{n-1}, x$ is a δ-chain with $y_0 \in Y$, then $y_0, \ldots, y_{n-1}, y$ is an ε-chain, so $x \in \mathfrak{C}_\varepsilon(Y)$.

(2) Clearly $\mathfrak{C}_\delta(Y)$ is open. Since F is uniformly continuous, there exists $\eta > 0$, such that $d(u, v) < \eta$ implies $d(F(u), F(v)) < \delta$. Assume $x \in \overline{\mathfrak{C}_\delta(Y)}$ (Figure 19 left). There exists $y \in \mathfrak{C}_\delta(Y)$ with $d(y, x) < \eta$, so $d(F(y), F(x)) < \delta$. There exists a δ-chain $y_0, \ldots y_{n-1}, y$ with $y_0 \in Y$, so $y_0, \ldots, y_{n-1}, y, F(x)$ is a δ-chain and $F(x) \in \mathfrak{C}_\delta(Y)$. Thus $\mathfrak{C}_\delta(Y)$ is an inward set.

(3) Let V be an inward set. If $V = X$, the claim is trivial. If $V \neq X$, $\varepsilon = d(F(V), X \setminus V^\circ) > 0$ by Proposition A.28, and any ε-chain which starts in V rests in V forever (Figure 19 right). Thus V is chain-invariant.

(4) Let Y be chain-invariant. For any $\delta > 0$, $V_\delta = \overline{\mathfrak{C}_\delta(Y)}$ is chain-invariant and $Y = \bigcap_{\delta > 0} V_\delta$. By Proposition A.28, for any $\varepsilon > 0$ there exists $\delta > 0$ with $V_\delta \subseteq B_\varepsilon(Y)$. $\square$

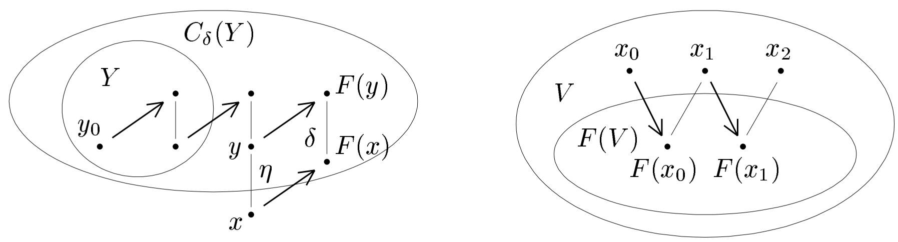

FIGURE 19. Chain-invariant sets

***Theorem 2.68* (Akin [3]).** — *Let $Y \subseteq X$ be an attractor of (X, F). Then Y is chain-invariant and $Y = \mathfrak{C}(Y \cap |\mathfrak{C}|)$.*

Proof. — By Propositions 2.65 and 2.67, for any $\varepsilon > 0$ there exists a chain invariant set $Y \subseteq V \subseteq B_\varepsilon(Y)$, so Y is chain invariant and it follows $\mathfrak{C}(Y \cap |\mathfrak{C}|) \subseteq Y$. Assume $y \in Y$. Since $F(Y) = Y$, y has an infinite number of preimages in Y, i.e., there exists a sequence $(y_n \in Y)_{n \geq 0}$ with $y_0 = y$ and $F(y_{n+1}) = y_n$. There exists a converging subsequence $y_{n_i} \to x \in Y$ as $i \to \infty$, so $(x, y) \in \mathfrak{N} \subseteq \mathfrak{C}$ and $x \in |\mathfrak{N}| \subseteq |\mathfrak{C}|$. We get $y \in \mathfrak{C}(Y \cap |\mathfrak{C}|)$. □

Theorem 2.69. — *An attractor is a minimal attractor iff it is chain-transitive. A dynamical system (X, F) has a unique attractor iff $\omega(X)$ is chain-transitive.*

Proof. — Suppose that an attractor Y in (X, F) contains a smaller attractor $Y_0 \subset Y$. If $y_0 \in Y_0$ and $y \in Y \setminus Y_0$, then $(y_0, y) \notin \mathfrak{C}$, so Y is not chain transitive.

Conversely, suppose that $y_0, y \in Y$, and $(y_0, y) \notin \mathfrak{C}$. Then for some $\delta_0 > 0$, there exists no δ_0-chain from y_0 to y. Choose δ such that $0 < \delta < \delta_0$. By Proposition 2.67, $V = \overline{\mathfrak{C}_\delta(y_0)} \subseteq \mathfrak{C}_{\delta_0}(y_0)$ is an inward set and $y \notin V$. It follows that $\omega(V) \subset Y$ is a proper subattractor of Y. □

2.9.1. Chain components. — Since $\mathfrak{C}$ is a transitive relation, the restriction of $\mathfrak{C} \cap \mathfrak{C}^{-1}$ to $|\mathfrak{C}|$ is an equivalence. Note that $\mathfrak{C} \cap \mathfrak{C}^{-1}$ need not be an equivalence on X since it is not necessarily reflexive. Call the equivalence classes of $\mathfrak{C} \cap \mathfrak{C}^{-1}$ on $|\mathfrak{C}|$ **chain components**. Theorem 2.69 says that an attractor is a minimal attractor iff it is a chain component. Subsets of $|\mathfrak{C}|$ which are invariant for $\mathfrak{C}$ restricted to $|\mathfrak{C}|$ are called **chain-terminal**.

Definition 2.70. — Let (X, F) be a dynamical system.

(1) A set $C \subseteq |\mathfrak{C}|$ is a **chain component** if

$$\forall x \in C, \forall y \in |\mathfrak{C}|, \Big((x, y) \in \mathfrak{C} \cap \mathfrak{C}^{-1} \;\Leftrightarrow\; y \in C\Big).$$

(2) A set $C \subseteq |\mathfrak{C}|$ is a **chain-terminal set** if

$$\forall x \in C, \forall y \in |\mathfrak{C}|, ((x, y) \in \mathfrak{C} \;\Rightarrow\; y \in C).$$

(3) The **graph of chain components** is an oriented graph whose vertices are chain components and whose oriented edges are

$$C_1 \to C_2 \quad \Leftrightarrow \quad \exists y_1 \in C_1, \exists y_2 \in C_2, (y_1, y_2) \in \mathfrak{C}.$$

If B is a chain-terminal set and if C is a chain component, then either $C \subseteq B$ or $B \cap C = \emptyset$. Thus any chain-terminal set is a union of some chain components. In the graph of chain components there are no loops and chain-terminal sets are in one-to-one correspondence with subgraphs from which no edges lead outside. From Theorem 2.68 we get

Corollary 2.71. — *Distinct attractors contain distinct sets of chain-recurrent points. If Y is an attractor, then $Y \cap |\mathfrak{C}|$ is a chain terminal set.*

In some systems attractors are in one-to-one correspondence with chain-terminal sets. In general, however, not every chain-terminal set yields an attractor.

In the cubic system on page 77, the graph of chain components is

$$\{-1\} \to \{0\} \leftarrow \{1\}.$$

The attractors are in one-to-one correspondence with chain-terminal sets.

$$[-1, 1] \cap |\mathfrak{C}| = \{-1, 0, 1\}, \qquad [-1, 0] \cap |\mathfrak{C}| = \{-1, 0\}$$
$$[0, 1] \cap |\mathfrak{C}| = \{0, 1\}, \qquad \{0\} \cap |\mathfrak{C}| = \{0\}$$

In the cubic root system on page 77, the graph of chain components is

$$\{-1\} \leftarrow \{0\} \rightarrow \{1\}$$

The attractors are in one-to-one correspondence with chain-terminal sets.

$$[-1,1] \cap |\mathfrak{C}| = \{-1,0,1\}, \quad \{-1\} \cap |\mathfrak{C}| = \{-1\}, \quad \{1\} \cap |\mathfrak{C}| = \{1\}$$

In Example 2.61, the chain components are singletons $\{p_i\}$ and there is an edge $\{p_i\} \rightarrow \{p_j\}$ iff $i > j$.

$$\{p_\infty\} \ \cdots \rightarrow \{p_2\} \rightarrow \{p_1\} \rightarrow \{p_0\}$$

The chain components of attractors are

$$[p_\infty, p_0] \cap |\mathfrak{C}| = \{p_\infty, \ldots, p_1, p_0\}, \quad [p_n, p_0] \cap |\mathfrak{C}| = \{p_n, p_{n-1}, \ldots, p_0\}$$

In Example 2.62, the chain components are $\{p_i\}$ and there is an edge $\{p_i\} \rightarrow \{p_j\}$ iff $i < j$. The chain components of attractors are

$$[p_\infty, p_n] \cap |\mathfrak{C}| = \{p_n, p_{n+1}, \ldots, p_\infty\}.$$

There exists no attractor containing only the chain component $\{p_\infty\}$, so there exists no minimal attractor. Attractors are therefore not in one-to-one correspondence with chain-terminal sets. The set $\{p_\infty\}$ is a quasi-attractor since it is the intersection of all attractors.

Similar situation is in the Preston system of Example 1.19. By Proposition 1.23, for any $m > 0$ there exists a periodic orbit $\{p_u : u \in \mathbf{2}^m\}$ of period 2^m. Any periodic orbit is a chain component. The only other chain component is $X = \bigcap_n X_n$. The graph of chain components is

$$\{0\} \rightarrow \{p_\lambda\} \rightarrow \{p_1, p_0\} \rightarrow \{p_{00}, p_{01}, p_{10}, p_{11}\} \rightarrow \cdots \ X$$

and corresponding attractors are $[0, c_1]$, X_0, X_1, X_2, etc. There is no attractor containing only X, which is a quasi-attractor. An abstract example with similar properties is the identity on the Cantor space. However the identity on a connected space possesses a unique attractor.

Example 2.72. — The system $(\mathbb{I}, \mathrm{Id})$ has a unique attractor. In $(A^{\mathbb{N}}, \mathrm{Id})$, any clopen set is an attractor and any point is a quasi-attractor.

Proposition 2.73. — *The basin of an attractor is open.*

Proof. — Let $\delta > 0$ be such that for any x with $d(x,Y) < \delta$, $d(F^n(x), Y) \rightarrow 0$ as $n \rightarrow \infty$. If $x \in \mathfrak{B}(Y)$, then for some n, $d(F^n(x), Y) < \delta$, and for some neighbourhood $U \ni x$, $F^n(U) \subseteq B_\delta(Y)$, so $U \subseteq \mathfrak{B}(Y)$. □

Proposition 2.74. — *If Y is an attractor in (X,F) and if $Y \subseteq \mathfrak{E}$, then $\mathfrak{B}(Y) \subseteq \mathfrak{E}$. In particular the basin of an attracting periodic orbit consists of equicontinuous points.*

Proof. — Let $\varepsilon > 0$. For any $y \in Y$ let δ_y be such that for any w,

$$d(w,y) < \delta_y \quad \Rightarrow \quad \forall n \geq 0, d(F^n(w), F^n(y)) < \tfrac{\varepsilon}{2}.$$

Let ξ be the Lebesgue number of the cover $\{B_{\delta_y}(y) : y \in Y\}$ of Y. Let $x \in \mathfrak{B}(Y)$ and $n > 0$ be such that there exists $y \in Y$ with $d(F^n(x), y) < \xi$ (Figure 20 left). There exists $\eta > 0$, such that $F^n(B_\eta(x)) \subseteq B_\xi(y)$ and for all $i < n$, $F^i(B_\eta(x)) \subseteq B_\varepsilon(F^i(x))$. If $d(z,x) < \eta$, then both $F^n(x), F^n(z) \in B_\xi(y)$, so for any $m \geq 0$,

$$d(F^{n+m}(z), F^{n+m}(x)) \leq d(F^{n+m}(z), F^m(y)) + d(F^m(y), F^{n+m}(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

□

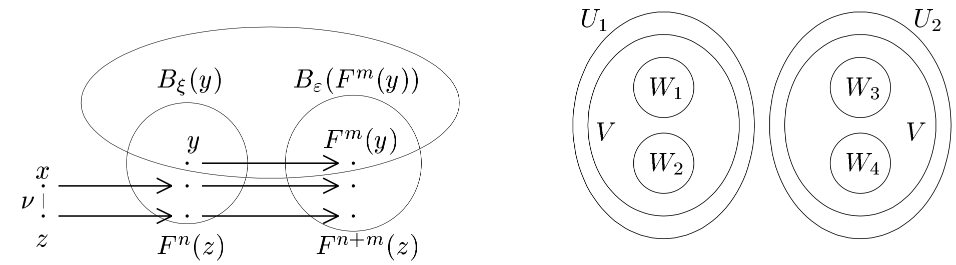

FIGURE 20. Equicontinuous basin

2.9.2. Maximal totally disconnected factor. — Recall that a metric space is connected (Section A.4), if its only clopen (closed and open) sets are $\emptyset$ and X. A connected component of X is a maximal connected subset of X. Connected components are the equivalence classes of the inseparability relation (Definition A.30) $\mathfrak{M} \subseteq X \times X$, so $\mathfrak{M}(x)$ is the unique connected component which contains $x \in X$. The inseparability relation is closed and $X/\mathfrak{M}$ is a totally disconnected compact metric space by Proposition A.54. It is called the **space of connected components** of X. If $\pi : X \to X/\mathfrak{M}$ is the factor map and if $\varphi : X \to Y$ is a surjective map to a totally disconnected space Y, then there exists a unique continuous map $\psi : X/\mathfrak{M} \to Y$ with $\psi \circ \pi = \varphi$.

Proposition 2.75. — *Any dynamical system (X, F) possesses the* **maximal totally disconnected factor** *$\pi : (X, F) \to (Y, G)$ such that for any factor map $\varphi : (X, F) \to (Z, H)$ with totally disconnected Z there is a factor map $\psi : (Y, G) \to (Z, H)$ with $\psi \circ \pi = \varphi$.*

Proof. — Let $Y = X/\mathfrak{M}$ be the space of connected components with the projection $\pi : X \to Y$. If $V \subseteq X$ is a connected component of X, then $F(V) \subseteq X$ is connected and therefore is a part of some connected component of X. This means $(F \times F)(\mathfrak{M}) \subseteq \mathfrak{M}$, so there exists a map $G = F/\mathfrak{M}$ on Y with $G \circ \pi = \pi \circ F$. □

Recall that a space X is **locally connected** if any neighbourhood of any point contains a connected neighbourhood (Definition A.33).

***Theorem 2.76* (Buescu and Stewart [33]).** — *Let (X, F) be a dynamical system on a locally connected space X. Let (Y, F) be a transitive subsystem of X and assume that $Y \subseteq X$ is stable (Definition 1.18). Then the maximal totally disconnected factor of (Y, F) is equicontinuous.*

Proof. — Let $\varphi : (Y, F) \to (Z, G)$ be the maximal totally disconnected factor. Given $\varepsilon > 0$, there exists a clopen partition $\mathcal{C} = \{C_1, \dots, C_m\}$ of Z with $\mathrm{diam}(\mathcal{C}) < \varepsilon$. Since $\varphi^{-1}(C_j)$ are disjoint and closed, they are also closed as subsets of X and there exist disjoint open $U_j \subseteq X$ with $\varphi^{-1}(C_j) \subseteq U_j$, such that $\{U_1 \cap Y, \dots, U_m \cap Y\}$ is a clopen partition of Y. The set $U = U_1 \cup \cdots \cup U_m$ is an open neighbourhood of Y. Since Y is stable, there exists an open set $Y \subseteq V \subseteq U$ such that $F^k(\overline{V}) \subseteq U$ for any $k \geq 0$. Since X is locally connected, for any $y \in Y$ there exists an open set $y \in V_y \subseteq V$ such that $\overline{V_y}$ is connected. The open cover $\{V_y \cap Y : y \in Y\}$ has a finite subcover $\{V_y \cap Y : y \in K\}$. The union $\bigcup_{y \in K} \overline{V_y}$ has a finite number of components and each such component is a union of some $\overline{V_y}$. Thus we may write K as a disjoint union $K = K_1 \cup \cdots \cup K_n$, such that $W_i = \bigcup_{y \in K_i} V_y$ are open sets, and the $\overline{W_i}$ are connected and disjoint (Figure 20 right). Since $F^k(\overline{W_i}) \subseteq U$ is connected, there exists j (which depends on k and i), such that $F^k(\overline{W_i}) \subseteq U_j$. Since $\{\overline{W_1} \cap Y, \dots, \overline{W_n} \cap Y\}$ is a clopen partition of Y, $\{\varphi(\overline{W_1} \cap Y), \dots, \varphi(\overline{W_n} \cap Y)\}$ is a clopen partition of Z. Let $\delta > 0$ be its Lebesgue number and assume that $z_0, z_1 \in Z$ and $d(z_0, z_1) < \delta$. There exists i such that $z_0, z_1 \in \varphi(\overline{W_i})$, so there exist $y_0, y_1 \in \overline{W_i} \cap Y$ with $\varphi(y_0) = z_0$, $\varphi(y_1) = z_1$. For any k there exists j with $F^k(\overline{W_i}) \subseteq U_j$, so $F^k(y_0) \in U_j \cap Y = \varphi^{-1}(C_j)$. Thus $G^k(z_0) = \varphi(F^k(y_0)) \in C_j$ and similarly $G^k(z_1) \in C_j$, so $d(G^k(z_0), G^k(z_1)) < \varepsilon$. We have shown that (Z, G) is equicontinuous. □

In Section 4.1 we shall generalize the dyadic adding machine of Example 1.24 to adding machines on arbitrary sequences of alphabets.

Corollary 2.77. — *Under the assumptions of Theorem 2.76, the maximal totally disconnected factor is conjugate to an adding machine.*

Proof. — Since (Y, F) is transitive, its factor (Z, G) is transitive. Since it is also equicontinuous, it is minimal by Corollary 2.34. By Theorem 4.4 any minimal equicontinuous system on a totally disconnected space is conjugate to an adding machine. □

The state space of the Preston system $(\mathbb{I}, P)$ from Example 1.19 is locally connected and $X \subseteq I$ is its stable transitive set. Since X is itself totally disconnected, we get that (X, P) is minimal and conjugate to an adding machine; in fact, to the dyadic adding machine.

2.10. Topological entropy

Suppose that we observe a dynamical system (X, F) equipped with a finite cover $\mathcal{V} = \{V_a : a \in A\}$. As in Section 1.9) we may considerthe itinerary which enumerates the sets containing the points of a trajectory. Sometimes, the next letter is uniquely determined by the previous letters. For example in quadratic system $(\mathbb{I}, Q_{1.5})$ of Figure 17 on page 33, the only itineraries are 0^∞ and 10^∞, so any letter except the first can be predicted. In contrast, an itinerary in $(\mathbb{I}, Q_4)$ can be an arbitrary sequence in $\mathbf{2}^{\mathbb{N}}$ (Example 1.28), so it is completely unpredictable. At any step we obtain one bit of information. The mean amount of information obtained per a step is the entropy of the cover. The topological entropy of (X, F) is the maximal entropy of any cover of X. The concept of topological entropy is due to Adler, Konheim and McAndrew [**1**].

The number of pieces of information needed to determine an element in a set of size n is $\log_2 n$. If $n = 2^m$, then we need m binary questions (with answers yes or no) to determine an unknown element of the set. In topological dynamics, we use natural logarithm instead and define the information of a finite set A to be $\ln(\#A)$. The entropy of a dynamical system (X, F) with respect to a cover $\mathcal{V} = \{V_a : a \in A\}$ is

$$H(X, F, \mathcal{V}) = \lim_{n\to\infty} \frac{\ln \#\{v \in A^n : V_v \neq \emptyset\}}{n}$$

For the quadratic system $(\mathbb{I}, Q_{1.5})$ (Figure 17 left), we have for any n exactly two possible itineraries 0^n and 10^{n-1}, so

$$H(\mathbb{I}, Q_1, \mathcal{V}) = \lim_{n\to\infty} \frac{\ln 2}{n} = 0$$

In $(\mathbb{I}, Q_4)$, any itinerary is possible (Example 1.28), so

$$H(\mathbb{T}, D, \mathcal{V}) = \lim_{n\to\infty} \frac{\ln 2^n}{n} = \ln 2$$

To obtain a simpler theory of topological entropy, we consider arbitrary (infinite) open covers of the state space (Definition A.23). The set of itineraries of a cover is obtained using the operation of inverse image and join.

Definition 2.78. — Let $\mathcal{U}$, $\mathcal{V}$ be open covers of a compact space X.

(1) We say that $\mathcal{V}$ is **finer** than $\mathcal{U}$, if any set of $\mathcal{V}$ is included in some set of $\mathcal{U}$

$$\mathcal{U} \succeq \mathcal{V} \quad \Leftrightarrow \quad \forall U \in \mathcal{U}, \exists V \in \mathcal{V}, U \subseteq V.$$

(2) The **join** of $\mathcal{U}$ and $\mathcal{V}$ is $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\} \setminus \{\emptyset\}$.

(3) The **size** of an open cover is $|\mathcal{U}| = \min\{\#\mathcal{U}_0 : \mathcal{U}_0 \text{ is a finite subcover of } \mathcal{U}\}$.

(4) The **diameter** of an open cover is $\operatorname{diam}(\mathcal{U}) = \sup\{\operatorname{diam}(U) : U \in \mathcal{U}\}$.

(5) Denote by $\mathfrak{U}_X$ the family of open covers of X.

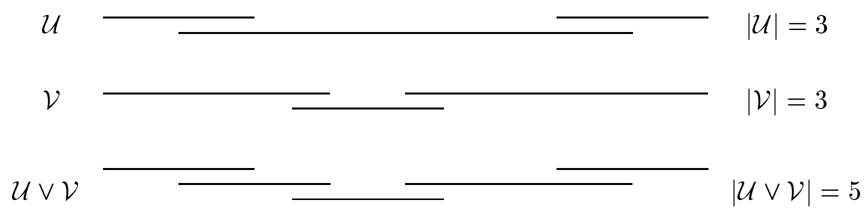

FIGURE 21. Join of covers

Proposition 2.79. — *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a compact space X. Then*

(1) $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$

(2) $|\mathcal{U} \vee \mathcal{V}| \leq |\mathcal{U}| \cdot |\mathcal{V}|$

(3) $\mathcal{U} \succeq \mathcal{V} \Rightarrow |\mathcal{U}| \geq |\mathcal{V}|$

Proof. —

(1) If $U \cap V \in \mathcal{U} \vee \mathcal{V}$, then $U \cap V \subseteq U \in \mathcal{U}$, so $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$.

(2) If $\mathcal{U}_0$, $\mathcal{V}_0$ are finite subcovers of $\mathcal{U}$ and $\mathcal{V}$, each having minimal number of elements, then $\mathcal{U}_0 \vee \mathcal{V}_0$ is a finite subcover of $\mathcal{U} \vee \mathcal{V}$ and

$$|\mathcal{U} \vee \mathcal{V}| \leq \#(\mathcal{U}_0 \vee \mathcal{V}_0) \leq \#\mathcal{U}_0 \cdot \#\mathcal{V}_0 = |\mathcal{U}| \cdot |\mathcal{V}|.$$

(3) Let $\mathcal{U}_0$ be a finite subcover of $\mathcal{U}$ having a minimal number of elements. For any $U \in \mathcal{U}_0$ there exists $V_U \in \mathcal{V}$ with $U \subseteq V_U$. Thus, $\mathcal{V}_0 = \{V_U : U \in \mathcal{U}_0\}$ is a finite subcover of $\mathcal{V}$ and $|\mathcal{V}| \leq \#\mathcal{V}_0 = \#\mathcal{U}_0 = |\mathcal{U}|$. □

Definition 2.80. —

(1) If $F : X \to Y$ is a continuous map, then the inverse image of a cover $\mathcal{V}$ of Y is a cover $F^{-1}(\mathcal{V}) = \{F^{-1}(V) : V \in \mathcal{V}\}$ of X.
(2) For a dynamical system (X, F) and a cover $\mathcal{U}$ of X define its n-th power by

$$\mathcal{U}^n \quad = \quad \mathcal{U} \vee F^{-1}(\mathcal{U}) \vee \cdots \vee F^{-(n-1)}(\mathcal{U}).$$

(3) A cover $\mathcal{U}$ is **generating** for (X, F), if $\lim_{n\to\infty} \operatorname{diam}(\mathcal{U}^n) = 0$.

The elements of $\mathcal{U}^n$ are nonempty sets $U_0 \cap F^{-1}(U_1) \cap \cdots \cap F^{-(n-1)}(U_{n-1})$, where $U_i \in \mathcal{U}$. These are sets of points with a given itinerary of length n. If $(A^{\mathbb{N}}, \sigma)$ is a full shift and $\mathcal{P} = \{[a] : a \in A\}$ is the canonical clopen partition, then $[a] \cap \sigma^{-1}[b] = [ab]$, so $\mathcal{P}^2 = \{[u] : u \in A^2\}$ and $\mathcal{P}^n = \{[u] : u \in A^n\}$. Since $\operatorname{diam}(\mathcal{P}^n) = 2^{-n}$, $\mathcal{P}$ is generating.

Proposition 2.81. — *Let $F : X \to Y$ be continuous and $\mathcal{U}$, $\mathcal{V}$ open covers of Y.*
(1) $F^{-1}(\mathcal{U} \vee \mathcal{V}) = F^{-1}(\mathcal{U}) \vee F^{-1}(\mathcal{V})$.
(2) If $\mathcal{U} \succeq \mathcal{V}$ then $F^{-1}(\mathcal{U}) \succeq F^{-1}(\mathcal{V})$.
(3) $|F^{-1}(\mathcal{U})| \leq |\mathcal{U}|$.
(4) If F is surjective, then $|F^{-1}(\mathcal{U})| = |\mathcal{U}|$.

Proof. —

(1) $F^{-1}(\mathcal{U} \vee \mathcal{V}) = \{F^{-1}(U \cap V) : U \in \mathcal{U}, V \in \mathcal{V}\} = F^{-1}(\mathcal{U}) \vee F^{-1}(\mathcal{V})$.

(2) If $U \in \mathcal{U}$, $V \in \mathcal{V}$ and $U \subseteq V$, then $F^{-1}(U) \subseteq F^{-1}(V)$.

(3) If $\mathcal{U}_0$ is a finite subcover of $\mathcal{U}$ having a minimal number of elements, then $F^{-1}(\mathcal{U}_0)$ is a finite subcover of $F^{-1}(\mathcal{U})$ so $|F^{-1}(\mathcal{U})| \leq \#\mathcal{U}_0 = |\mathcal{U}|$.

(4) Let $\mathcal{V}_0$ be a finite subcover of $F^{-1}(\mathcal{U})$ having a minimal number of elements. For any $V \in \mathcal{V}_0$ there exists $U_V \in \mathcal{U}$ with $V = F^{-1}(U_V)$. Since F is surjective, $\mathcal{U}_0 = \{U_V : V \in \mathcal{V}_0\}$ is a subcover of $\mathcal{U}$ and $|\mathcal{U}| \leq \#\mathcal{U}_0 = \#\mathcal{V}_0 = |F^{-1}(\mathcal{U})|$. □

Proposition 2.82. — *Let (X, F) be a dynamical system and $\mathcal{U}$, $\mathcal{V}$ open covers.*
(1) $(\mathcal{U}^n)^m = \mathcal{U}^{n+m-1}$.
(2) $|\mathcal{U}^{n+m}| \leq |\mathcal{U}^n| \cdot |\mathcal{U}^m|$
(3) $\mathcal{U} \succeq \mathcal{V} \;\Rightarrow\; \mathcal{U}^n \succeq \mathcal{V}^n$.

Proof. —

$$\begin{aligned}
(\mathcal{U}^n)^m &= \left(\mathcal{U} \vee \cdots \vee F^{-(n-1)}(\mathcal{U})\right) \vee \left(F^{-1}(\mathcal{U}) \vee \cdots \vee F^{-n}(\mathcal{U})\right) \\
&\quad \vee \cdots \vee \left(F^{-(m-1)}(\mathcal{U}) \vee \cdots \vee F^{-(m+n-2)}(\mathcal{U})\right) = \mathcal{U}^{n+m-1} \\
|\mathcal{U}^{n+m}| &= \left|\left(\mathcal{U} \vee \cdots \vee F^{-(n-1)}(\mathcal{U})\right) \vee F^{-n}\left(\mathcal{U} \vee \cdots \vee F^{-(m-1)}(\mathcal{U})\right)\right| \\
&\leq |\mathcal{U} \vee \cdots \vee F^{-(n-1)}(\mathcal{U})| \cdot |F^{-n}(\mathcal{U} \vee \cdots \vee F^{-(m-1)}(\mathcal{U}))| \\
&= |\mathcal{U}^n| \cdot |\mathcal{U}^m|
\end{aligned}$$

We have $F^{-i}(\mathcal{U}) \succeq F^{-i}(\mathcal{V})$, so $\mathcal{U}^n \succeq \mathcal{V}^n$. □

Proposition 2.83. — *Let (X,F) be a dynamical system and $\mathcal{U}$ an open cover of X. There exists a limit*

$$H(X,F,\mathcal{U}) = \lim_{n\to\infty} \frac{\ln|\mathcal{U}^n|}{n}.$$

If $\mathcal{U} \succeq \mathcal{V}$, then $H(X,F,\mathcal{U}) \geq H(X,F,\mathcal{V})$.

Proof. — Put $a_n = \ln|\mathcal{U}^n| \geq 0$. By Proposition 2.82, $a_{n+m} \leq a_n + a_m$. For a fixed $m > 0$ and $n > 0$ set $q_n = \lfloor \frac{n}{m} \rfloor$, so $(q_n - 1)\cdot m < n \leq q_n \cdot m$ and $a_n \leq q_n \cdot a_m$. We get

$$\begin{aligned}
\frac{a_n}{n} &\leq \frac{q_n}{q_n - 1} \cdot \frac{a_m}{m} \\
\limsup_{n\to\infty} \frac{a_n}{n} &\leq \limsup_{n\to\infty} \frac{q_n}{q_n-1} \cdot \frac{a_m}{m} = \frac{a_m}{m} \\
\limsup_{n\to\infty} \frac{a_n}{n} &\leq \liminf_{m\to\infty} \frac{a_m}{m}
\end{aligned}$$

so $\lim_{n\to\infty} \frac{a_n}{n}$ exists. If $\mathcal{U} \succeq \mathcal{V}$, then $\mathcal{U}^n \succeq \mathcal{V}^n$, so $H(X,F,\mathcal{U}) \geq H(X,F,\mathcal{V})$. □

Definition 2.84. — The topological entropy of a dynamical system (X,F) is

$$h(X,F) = \sup\{H(X,F,\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\}.$$

Here $\mathfrak{U}_X$ is the family of open covers of X. To evaluate topological entropy using all open covers may be a difficult task. In next proposition we show that we can restrict ourselves to one conveniently chosen sequence of open covers.

Proposition 2.85. — *Let (X,F) be a dynamical system and $(\mathcal{U}_n)_{n\geq 0}$ a sequence of open covers of X, such that $\lim_{n\to\infty} \mathrm{diam}(\mathcal{U}_n) = 0$. Then*

$$h(X,F) = \lim_{n\to\infty} H(X,F,\mathcal{U}_n).$$

Proof. — Let $\mathcal{V}$ be an open cover of X and λ its Lebesgue number. There exists n_0, such that for any $n \geq n_0$, $\mathrm{diam}(\mathcal{U}_n) < \lambda$. Given $U \in \mathcal{U}_n$ pick some $x \in U$. There exists $V \in \mathcal{V}$, such that $U \subseteq B_\lambda(x) \subseteq V$. This means $\mathcal{U}_n \succeq \mathcal{V}$, so $H(X,F,\mathcal{U}_n) \geq H(X,F,\mathcal{V})$, and $\liminf_{n\to\infty} H(X,F,\mathcal{U}_n) \geq H(X,F,\mathcal{V})$. Thus

$$\liminf_{n\to\infty} H(X,F,\mathcal{U}_n) \geq \sup\{H(X,F,\mathcal{V}) : \mathcal{V} \in \mathfrak{U}_X\} \geq \limsup_{n\to\infty} H(X,F,\mathcal{U}_n),$$

and the statement follows. □

Proposition 2.86. — *Let (X,F) be a dynamical system and let $\mathcal{V}$ be its open generating cover. Then $h(X,F) = H(X,F,\mathcal{V})$.*

Proof. — Since $\mathrm{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$, we apply Proposition 2.85 to $(\mathcal{V}^n)_{n\geq 0}$. Since

$$H(X,F,\mathcal{V}^n) = \lim_{m\to\infty} \frac{\ln |(\mathcal{V}^n)^m|}{m} = \lim_{m\to\infty} \frac{\ln |\mathcal{V}^{n+m-1}|}{m} = H(X,F,\mathcal{V}),$$

we get $h(X,F) = H(X,F,\mathcal{V})$. □

We show that the topological entropy of a rotation $(\mathbb{T}, R_\alpha)$ of the circle is zero. For an integer $n > 0$ consider an open cover $\mathcal{U}_n = \{(a,b) \subseteq \mathbb{T} : b - a \leq 2^{-n}\}$ all of whose intervals have length at most 2^{-n}. Clearly $|\mathcal{U}_n| = 2^n + 1$. Since $R^{-1}(c,d)$ is an interval of the same length as (c,d), we get $\mathcal{U}_n \vee R_a^{-1}(\mathcal{U}_n) = \mathcal{U}_n$ and

$$H(\mathbb{T}, R_a, \mathcal{U}_n) = \lim_{m\to\infty} \frac{\ln(2^n+1)}{m} = 0.$$

We show that the topological entropy of the angle doubling system $(\mathbb{T}, D)$ is $\ln 2$. For $n > 0$ consider an open cover $\mathcal{V}_n$ of intervals of length at most 2^{-n}. The intersection

$$(a,b) \cap D^{-1}(c,d) = (a,b) \cap \left[\left(\frac{c}{2},\frac{d}{2}\right) \cup \left(\frac{c+1}{2},\frac{d+1}{2}\right)\right]$$

consists of a single interval of length at most $(d-c)/2$, so $\mathcal{U}_n \vee D^{-1}(\mathcal{U}_n) = \mathcal{U}_{n+1}$ and, more generally, $(\mathcal{U}_n)^m = \mathcal{U}_{n+m}$. Since $|\mathcal{U}_n| = 2^n + 1$,

$$H(\mathbb{T}, D, \mathcal{U}_n) = \lim_{m\to\infty} \frac{\ln(2^{n+m}+1)}{m} = \lim_{m\to\infty} \frac{(n+m)\ln 2}{m} = \ln 2, \quad h(\mathbb{T}, D) = \ln 2.$$

We now show that the topological entropy of the dyadic adding machine $(\mathbf{2}^{\mathbb{N}}, R_1)$ is zero. Consider a clopen partition $\mathcal{P}_n = \{[u] : u \in \mathbf{2}^n\}$. For any $u \in \mathbf{2}^n$, $R_1^{-1}([u]) = [f^{-1}(u)]$ is again a cylinder of length n, so $R_1^{-1}(\mathcal{P}_n) = \mathcal{P}_n$, $(\mathcal{P}^n)^m = \mathcal{P}^n$ and $H(X,F,\mathcal{P}_n) = 0$. Topological entropy may be also infinite.

Example 2.87. — The dynamical system $(\mathbf{2}^{\mathbb{N}}, F)$ where $F(x)_i = x_{2i}$ has infinite topological entropy. The system consists of two disjoint conjugate chaotic systems.

Proof. — The map F is continuous, since

$$\begin{aligned} x_{[0,2n]} = y_{[0,2n]} &\quad\Rightarrow\quad F(x)_{[0,n]} = F(y)_{[0,n]}, \\ d(x,y) < 2^{-2n} &\quad\Rightarrow\quad d(F(x),F(y)) < 2^{-n}. \end{aligned}$$

To evaluate topological entropy, we need to use generalized cylinders. Let $K \subseteq \mathbb{N}$ be a finite set and $u : K \to \mathbf{2}$ a map. The cylinder of u is the set

$$[u] = \{x \in \mathbf{2}^{\mathbb{N}} : \forall i \in K, x_i = u_i\}.$$

Then $\mathcal{P}(K) = \{[u] : u \in \mathbf{2}^K\}$ is a clopen partition of $\mathbf{2}^{\mathbb{N}}$ and $|\mathcal{P}(K)| = 2^{\#K}$. For a given $n > 0$ consider the cover $\mathcal{V}_n = \mathcal{P}([0,2n-1])$. If $u,v : [0,2n-1] \to A$, then $[u] \cap F^{-1}[v]$ is a generalized cylinder which specifies coordinates in interval $[0,2n-1]$ together with coordinates $0,2,\ldots,2(2n-1)$. From these coordinates, n are not included in $[0,2n-1]$. We get

$$\begin{aligned} (\mathcal{V}_n)^2 &= \mathcal{P}([0,2n-1] \cup \{2n, 2(n+1), \ldots, 2(2n-1)\}), \\ (\mathcal{V}_n)^3 &= \mathcal{P}([0,2n-1] \cup \{2n, \ldots, 2(2n-1)\} \cup \{4n, \ldots, 4(2n-1)\}. \end{aligned}$$

For each successive higher power we get n new coordinates, so $|(\mathcal{V}_n)^k| = n(k+1)$. Thus

$$H(\mathbf{2}^{\mathbb{N}}, F, \mathcal{V}_n) = \lim_{k\to\infty} \frac{(k+1)n\ln 2}{k} = n\ln 2,$$

and $h(X,F) = \infty$. Since $F(x)_0 = x_0$, both $X_0 = [0]$ and $X_1 = [1]$ are invariant. If $u, v \in \mathbf{2}^n$ and $u_0 = u_1 = 0$, then there exists $x \in [u]$, such that for any $1 \le i \le n$, $x_{2^m i} = v_i$, so $F^m(x) \in [v]$ and (X_0, F) is transitive. We construct a periodic point $y \in [u]$. Set $y_{[0,n)} = u$. For $i = n, n+1, n+2, \ldots$ we define recursively $y_i = y_j$ if $i = 2^m j$ and $y_i = 0$ if i is not divisible by 2^m. Then $F^m(y) = y$, so (X_0, F) is chaotic. Similarly, (X_1, F) is chaotic. □

Proposition 2.88. —
(1) If (X,F) is a subsystem of (Y,G), then $h(X,F) \le h(Y,G)$.
(2) If (Y,G) is a factor of (X,F), then $h(X,F) \ge h(Y,G)$.

Proof. — Let $\varphi : (X,F) \to (Y,G)$ be a morphism of dynamical systems.

(1) Let φ be injective, and let $\mathcal{U}$ be an open cover of X (Figure 22 left). For any $U \in \mathcal{U}$ there exists an open set $V_U \subseteq Y$ such that $\varphi(U) = V_U \cap \varphi(X)$. Set

$$\mathcal{V} = \{V_U : U \in \mathcal{U}\} \cup \{Y \setminus \varphi(X)\}.$$

Then $\mathcal{V}$ is an open cover of Y and $\varphi^{-1}(\mathcal{V}) = \mathcal{U}$. We have

$$\varphi^{-1}(G^{-1}(\mathcal{V})) = F^{-1}(\varphi^{-1}(\mathcal{V})) = F^{-1}(\mathcal{U}),$$

so $\varphi^{-1}(\mathcal{V}^n) = \mathcal{U}^n$ and $|\mathcal{U}^n| \le |\mathcal{V}^n|$. It follows $H(X,F,\mathcal{U}) \le H(Y,G,\mathcal{V}) \le h(Y,G)$ and $h(X,F) \le h(Y,G)$.

(2) Let $\mathcal{V}$ be an open cover of Y. Then $\mathcal{U} = \varphi^{-1}(\mathcal{V})$ is an open cover of X and $\mathcal{U}^n = \varphi^{-1}(\mathcal{V}^n)$ (Figure 22 right). Since φ is onto, $|\mathcal{U}^n| = |\mathcal{V}^n|$ and

$$H(Y,G,\mathcal{V}) = H(X,F,\mathcal{U}) \le h(X,F),$$

so $h(Y,G) \le h(X,F)$. □

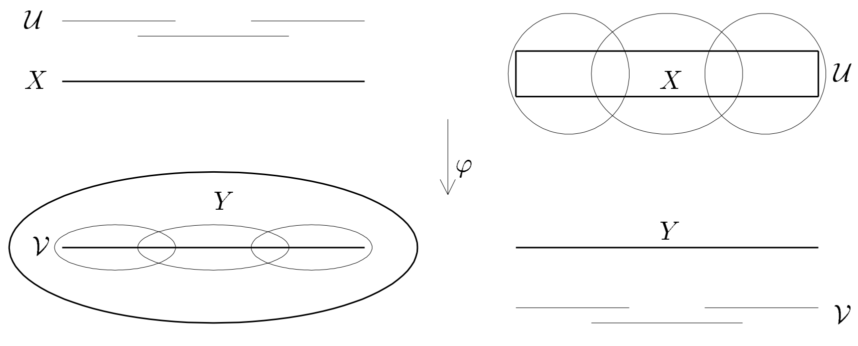

FIGURE 22. Entropy of a subsystem and a factor

Recall that the disjoint union $(X \uplus Y, F \uplus G)$ of dynamical systems (X,F), (Y,G) contains the two systems as disjoint subsystems. The product $(X \times Y, F \times G)$ is their least common extension (see Section 2.1).

Proposition 2.89. — *Let (X,F) and (Y,G) be dynamical systems. Then*

$$\begin{aligned} h(X \uplus Y, F \uplus G) &= \max\{h(X,F), h(Y,G)\}, \\ h(X \times Y, F \times G) &= h(X,F) + h(Y,G). \end{aligned}$$

Proof. —

(1) Let $(\mathcal{U}_n)_{n\geq 0}$, $(\mathcal{V}_n)_{n\geq 0}$ be sequences of open covers of X and Y whose diameters tend to zero. Since X, Y are disjoint closed subsets of $X \uplus Y$, if n is large enough, if $U \in \mathcal{U}_n$ and $V \in \mathcal{V}_n$, then $U \cap V = \emptyset$. For the open cover $\mathcal{W}_n = \mathcal{U}_n \cup \mathcal{V}_n$ of $X \uplus Y$ we get $(\mathcal{W}_n)^m = (\mathcal{U}_n)^m \cup (\mathcal{V}_n)^m$. Assume $h(X,F) \leq h(Y,G)$. For any $\varepsilon > 0$ there exists n_0 such that for any $n \geq n_0$,

$$H(X,F,\mathcal{U}_n) < H(Y,G,\mathcal{V}_n) + \varepsilon$$

For a fixed $n \geq n_0$ there exists m_0 such that for all $m \geq m_0$,

$$\begin{aligned} \frac{\ln|(\mathcal{U}_n)^m|}{m} &< \frac{\ln|(\mathcal{V}_n)^m|}{m} + \varepsilon \\ |(\mathcal{W}_n)^m| &= |(\mathcal{U}_n)^m| + |(\mathcal{V}_n)^m| < |(\mathcal{V}_n)^m| \cdot (e^{m\varepsilon} + 1) \\ H(X \uplus Y, F \uplus G, \mathcal{W}_n) &\leq \lim_{m\to\infty} \frac{\ln|(\mathcal{V}_n)^m|}{m} + \lim_{m\to\infty} \frac{\ln(e^{m\varepsilon}+1)}{m} = H(Y,G,\mathcal{V}_n) + \varepsilon \end{aligned}$$

Thus $h(X \uplus Y, F \uplus G) \leq h(Y,G) + \varepsilon$ for any $\varepsilon > 0$ and hence $h(X \uplus Y, F \uplus G) = h(Y,G) = \max\{h(X,F), h(Y,G)\}$.

(2) Let $(\mathcal{U}_n)_{n\geq 0}$, $(\mathcal{V}_n)_{n\geq 0}$ be sequences of open covers of X and Y whose diameters tend to zero. Then

$$\mathcal{W}_n = \{U \times V : U \in \mathcal{U}_n, V \in \mathcal{V}_n\}$$

is a sequence of open covers of $X \times Y$ whose diameters tend to zero. For any $m > 0$,

$$(\mathcal{W}_n)^m = \{U \times V : U \in (\mathcal{U}_n)^m, V \in (\mathcal{V}_n)^m\}$$

It follows $|(\mathcal{W}_n)^m| = |(\mathcal{U}_n)^m| \cdot |(\mathcal{V}_n)^m|$ and

$$H(X \times Y, F \times G, \mathcal{W}_n) = H(X,F,\mathcal{U}_n) + H(Y,G,\mathcal{V}_n)$$

so $h(X \times Y, F \times G) = h(X,F) + h(Y,G)$. □

***Theorem 2.90* (Bowen [28]).** — *The topological entropy of a dynamical system is concentrated on its nonwandering set.*

$$h(X,F) = h(|\mathbf{N}|, F)$$

Proof. — Let $\mathcal{U}$ be an open cover of X and denote by $\mathcal{U}_0 = \{U \cap |\mathbf{N}| : U \in \mathcal{U}\}$ the open cover of $|\mathbf{N}|$ obtained from $\mathcal{U}$ by restriction. We show first

$$H(X,F,\mathcal{U}) \leq \ln|\mathcal{U}_0|.$$

Let $|\mathcal{U}_0| = k$, and let $U_1, \ldots, U_k \in \mathcal{U}$ be such that $|\mathbf{N}| \subseteq U_1 \cup \cdots \cup U_k = U$. For any $x \in X \setminus U$ choose some set $x \in A_x \in \mathcal{U}$, and an open set $x \in B_x \subseteq A_x$ such that $F^n(B_x) \cap B_x = \emptyset$ for all $n > 0$. Thus $\{B_x : x \in X \setminus U\}$ is an open cover of $X \setminus U$ and it has some finite subcover $\{U_{k+1}, \ldots, U_{k+m}\}$ (Figure 23). We get an open cover of X

$$\mathcal{V} = \{U_1, \ldots, U_k, U_{k+1}, \ldots, U_{k+m}\} \succeq \mathcal{U}.$$

Then $F^{-p}(U_{k+j}) \cap U_{k+j} = \emptyset$ for $j, p \geq 1$. If $U_{i_0} \cap F^{-1}(U_{i_1}) \cap \cdots \cap F^{-(n-1)}(U_{i_{n-1}}) \neq \emptyset$ is a nonempty set, where $U_{i_j} \in \mathcal{V}$, then the sequence $i = (i_0, \ldots, i_{n-1})$ contains at most one index $k+1$, at most one index $k+2$, etc. The number of sequences $i = (i_0, \ldots, i_{n-1})$ in which no $k+j$ occurs twice is bounded by

$$\#\{i \in \{1, \ldots, k+m\}^n : i_j = i_l > k \Rightarrow j \neq l\} \leq n^m \cdot m! \cdot k^n$$

There is at most n^m possibilities to choose indices j with $i_j > k$ and at most $m!$ possibilities for their values. For the other indices there is at most k^n possibilities. It follows

$$H(X, F, \mathcal{U}) \leq H(X, F, \mathcal{V}) \leq \lim_{n \to \infty} \frac{m \ln n + \ln m! + n \ln k}{n} = \ln k = \ln |\mathcal{U}_0|.$$

For $p > 0$ we set $\mathcal{W} = \mathcal{U}^p$, $\mathcal{W}_0 = (\mathcal{U}_0)^p = \{W \cap |\mathbf{N}| : W \in \mathcal{W}\}$. Since the nonwandering set of F^p is included in the nonwandering set of F, we can apply the preceding result to $\mathcal{W}$, $\mathcal{W}_0$ and F^p. We get

$$\begin{aligned} H(X, F, \mathcal{U}) &= \lim_{n \to \infty} \frac{\ln |\mathcal{U} \vee \cdots \vee F^{-(np-1)}(\mathcal{U})|}{np} \\ &= \lim_{n \to \infty} \frac{\ln |\mathcal{W} \vee F^{-p}(\mathcal{W}) \vee \cdots \vee F^{-p(n-1)}(\mathcal{W})|}{np} \\ &= \frac{H(X, F^p, \mathcal{W})}{p} \leq \frac{\ln |\mathcal{W}_0|}{p} = \frac{\ln |(\mathcal{U}_0)^p|}{p} \end{aligned}$$

so $H(X, F, \mathcal{U}) \leq H(|\mathbf{N}|, F, \mathcal{U}_0) \leq h(|\mathbf{N}|, F)$. It follows $h(X, F) \leq h(|\mathbf{N}|, F)$ and by Proposition 2.88, $h(X, F) = h(|\mathbf{N}|, F)$. ☐

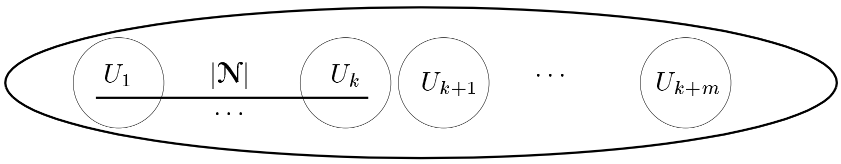

FIGURE 23. Bowen theorem

2.11. Characteristics of recurrence and transition

If the entropy $H(X, F, \mathcal{U}) = a$ of an open cover $\mathcal{U}$ is positive, then $|\mathcal{U}^n| \approx e^{an}$ grows exponentially. If the topological entropy is zero, this growth may be polynomial, for example $|\mathcal{U}^n| \approx n^a$ for some $a > 0$. The exponent a can then be obtained as $a = \lim_{n \to \infty} \ln |\mathcal{U}^n| / \ln n$. If $a = 1$, the growth is linear $|\mathcal{U}^n| \approx bn$, and the coefficient b can be obtained as $b = \lim_{n \to \infty} |\mathcal{U}^n|/n$. In general these limits need not exist. We define therefore the upper and lower **polynomial entropies** and upper and lower **linear entropies** as

$$\begin{aligned} H^p(X,F,\mathcal{U}) &= \limsup_{n\to\infty} \frac{\ln|\mathcal{U}^n|}{\ln n}, & H_p(X,F,\mathcal{U}) &= \liminf_{n\to\infty} \frac{\ln|\mathcal{U}^n|}{\ln n}, \\ H^l(X,F,\mathcal{U}) &= \limsup_{n\to\infty} \frac{|\mathcal{U}^n|}{n}, & H_l(X,F,\mathcal{U}) &= \liminf_{n\to\infty} \frac{|\mathcal{U}^n|}{n}. \end{aligned}$$

All these functions are monotone. If $\mathcal{U} \succeq \mathcal{V}$ then $\mathcal{U}^n \succeq \mathcal{V}^n$ and $H^p(X,F,\mathcal{U}) \geq H^p(X,F,\mathcal{V})$. Many other monotone characteristics of open covers can be constructed. Besides characteristics based on $|\mathcal{U}^n|$ there exist characteristics based on recurrence and transitive properties of open covers. Recall that $\mathfrak{U}_X$ is the family of open covers of X.

Definition 2.91. — Let X be a compact metric space. A **monotone characteristic** is a mapping $K : \mathfrak{U}_X \to [0,\infty]$ satisfying

$$\mathcal{U} \succeq \mathcal{V} \quad \Rightarrow \quad K(\mathcal{U}) \geq K(\mathcal{V}).$$

Proposition 2.92. — *Let X be a compact metric space, $K : \mathfrak{U}_X \to [0,\infty]$ a monotone characteristic and $(\mathcal{U}_n)_{n\geq 0}$ a sequence of open covers with $\lim\limits_{n\to\infty} \operatorname{diam}(\mathcal{U}_n) = 0$. Then*

$$\lim_{n\to\infty} K(\mathcal{U}_n) = \sup\{K(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\}$$

The proof is a straightforward generalization of the proof of Proposition 2.85.

For $K(\mathcal{U}) = |\mathcal{U}|$ we get $\sup\{|\mathcal{U}| : \mathcal{U} \in \mathfrak{U}_X\} = \infty$ so we use rather dynamically defined characteristics $H^p(X,F,\mathcal{U})$ or $H_p(X,F,\mathcal{U})$. This construction will be carried out in general. The **dynamical characteristics** of a monotone characteristic $K : \mathfrak{U}_X \to [0,\infty]$ are defined by

$$\begin{aligned} K_e(\mathcal{U}) &= \liminf_{n\to\infty} \frac{\ln K(\mathcal{U}^n)}{n}, & K^e(\mathcal{U}) &= \limsup_{n\to\infty} \frac{\ln K(\mathcal{U}^n)}{n}, \\ K_p(\mathcal{U}) &= \liminf_{n\to\infty} \frac{\ln K(\mathcal{U}^n)}{\ln n}, & K^p(\mathcal{U}) &= \limsup_{n\to\infty} \frac{\ln K(\mathcal{U}^n)}{\ln n} \\ K_l(\mathcal{U}) &= \liminf_{n\to\infty} \frac{K(\mathcal{U}^n)}{n}, & K^l(\mathcal{U}) &= \limsup_{n\to\infty} \frac{K(\mathcal{U}^n)}{n} \end{aligned}$$

Clearly all these functions are monotone characteristics and

$$K_e(\mathcal{U}) \leq K_p(\mathcal{U}), \quad K^e(\mathcal{U}) \leq K^p(\mathcal{U}).$$

Proposition 2.93. — *Let (X,F) be a dynamical system, $K : \mathfrak{U}_X \to [0,\infty]$ a monotone characteristic and $\widetilde{K} : \mathfrak{U}_X \to [0,\infty]$ one of the dynamical characteristics K^e, K_e, K^p, K_p, K^l or K_l. If $\mathcal{V} \in \mathfrak{U}_X$ is a generating, then*

$$\sup\{\widetilde{K}(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} = \widetilde{K}(\mathcal{V}).$$

Proof. — We have

$$\begin{aligned} K_e(\mathcal{V}^m) &= \liminf_{n\to\infty} \frac{\ln K(\mathcal{V}^{n+m-1})}{n} = \liminf_{n\to\infty} \frac{\ln K(\mathcal{V}^n)}{n} = K_e(\mathcal{V}) \\ K_p(\mathcal{V}^m) &= \liminf_{n\to\infty} \frac{\ln K(\mathcal{V}^{n+m-1})}{\ln n} = \liminf_{n\to\infty} \frac{\ln K(\mathcal{V}^n)}{\ln n} = K_e(\mathcal{V}) \end{aligned}$$

and similarly for the other cases. □

If (X,F) is a dynamical system and $K : \mathfrak{U}_X \to [0,\infty]$ a monotone characteristic, its **dynamical rates** are defined by

$$\begin{aligned}
k_e(X,F) &= \sup\{K_e(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{lower exponential rate}\\
k^e(X,F) &= \sup\{K^e(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{upper exponential rate}\\
k_p(X,F) &= \sup\{K_p(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{lower polynomial rate}\\
k^p(X,F) &= \sup\{K^p(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{upper polynomial rate}\\
k_l(X,F) &= \sup\{K_l(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{lower linear rate}\\
k^l(X,F) &= \sup\{K^l(\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\} && \text{upper linear rate}
\end{aligned}$$

In particular, lower and upper polynomial entropies are defined by

$$\begin{aligned}
h_p(X,F) &= \sup\{H_p(X,F,\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\}\\
h^p(X,F) &= \sup\{H^p(X,F,\mathcal{U}) : \mathcal{U} \in \mathfrak{U}_X\}
\end{aligned}$$

In a transitive system (X,F), for any nonempty open set $U \subseteq X$ there exists $n > 0$ with $F^n(U) \cap U \neq \emptyset$. The smallest n with this property is called the **recurrence time** of U. If U shrinks, its recurrence time grows. Dimension like characteristics based on recurrence time have been studied by Afraimovich [**2**], Penné et al. [**124**] or Kůrka et al. [**94**] in the context of general Pesin theory of dimension [**126**].

Definition 2.94. — Let (X,F) be a dynamical system.

(1) The **transition time** from a set $U \subseteq X$ to a set $V \subseteq X$ is

$$\tau(U,V) = \min\{n > 0 : F^n(U) \cap V \neq \emptyset\}.$$

(2) The **recurrence time** of a set $U \subseteq X$ is $\tau(U) = \tau(U,U)$.

We write $\tau(x,U)$ instead of $\tau(\{x\},U)$. In transitive systems, $\tau(U,V)$ is finite for any pair of nonempty open sets. Moreover, it is monotone

$$U_0 \subseteq U_1 \;\;\&\;\; V_0 \subseteq V_1 \to \tau(U_0,V_0) \geq \tau(U_1,V_1)$$

Proposition 2.95. — *Let (X,F) be a minimal system and $\mathcal{U}$ a finite open cover of X. Then $\max\{\tau(x,U) : x \in X, U \in \mathcal{U}\}$ is finite.*

Proof. — Suppose that there exists a sequence of points x_n and a set $U \in \mathcal{U}$ such that $\lim_{n\to\infty} \tau(x_n,U) = \infty$. By compactness, we can assume that there exists a convergent sequence $x_n \to x$ with $\tau(x_n,U) \to \infty$ as $n \to \infty$. But $\tau(x,U)$ is finite and there exists $\varepsilon > 0$ such that for any $y \in B_\varepsilon(x)$, $\tau(y,U) \leq \tau(x,U)$. This is a contradiction. $\square$

The return time of an open cover may be calculated as the minimal recurrence time of its sets. Whilst the minimum gives rise to a monotone function, the maximum in general not. We can, however, consider finite subcovers, and evaluate maximal recurrence time of their elements. The resulting characteristic is monotone.

Definition 2.96. — Let (X, F) be a dynamical system, $\mathcal{U}$ and open cover of X and denote by $\mathfrak{U}^f(\mathcal{U})$ the family of finite subcovers of $\mathcal{U}$. We set

$$\begin{array}{rcll} Q(\mathcal{U}) & = & \inf\{\tau(U) : U \in \mathcal{U}\} & \textbf{return time} \\ R(\mathcal{U}) & = & \min\{\max\{\tau(U) : U \in \mathcal{U}_0\} : \mathcal{U}_0 \in \mathfrak{U}^f(\mathcal{U})\} & \textbf{recurrence time} \\ S(\mathcal{U}) & = & \min\{\max\{\tau(U,V) : U, V \in \mathcal{U}_0\} : \mathcal{U}_0 \in \mathfrak{U}^f(\mathcal{U})\} & \textbf{transition time} \\ T(\mathcal{U}) & = & \min\{\max\{\tau(x,U) : x \in X, U \in \mathcal{U}_0\} : \mathcal{U}_0 \in \mathfrak{U}^f(\mathcal{U})\} & \textbf{running time} \end{array}$$

For a full shift $(A^{\mathbb{N}}, \sigma)$ and the canonical clopen partition $\mathcal{P} = \{[a] : a \in A\}$ we have

$$Q(\mathcal{P}^n) = 1, \quad R(\mathcal{P}^n) = S(\mathcal{P}^n) = n, \quad T(\mathcal{P}^n) = \infty.$$

Proposition 2.97. — *The functions $Q, R, S, T : \mathfrak{U}_X \to [0, \infty]$ are monotone characteristics and satisfy*

$$Q(\mathcal{U}) \leq R(\mathcal{U}) \leq S(\mathcal{U}) \leq T(\mathcal{U}), \quad Q(\mathcal{U}) \leq |\mathcal{U}|.$$

If $\mathcal{U}$ is a finite partition, then $|\mathcal{U}| \leq T(\mathcal{U})$.

Proof. —

(1) Let $\mathcal{U} \succ \mathcal{V}$ be open covers of X. For any $U \in \mathcal{U}$ there exists $V_U \in \mathcal{V}$ with $U \subseteq V_U$ and $\tau(U) \geq \tau(V_U) \geq Q(\mathcal{V})$, so $Q(\mathcal{U}) \geq Q(\mathcal{V})$.

(2) Let $\mathcal{U}_0 \in \mathfrak{U}^f(\mathcal{U})$ be such that $R(\mathcal{U}) = \max\{\tau(U) : U \in \mathcal{U}_0\}$. For any $U \in \mathcal{U}_0$ pick some $V_U \in \mathcal{V}$ with $U \subseteq V_U$ and set $\mathcal{V}_0 = \{V_U : U \in \mathcal{U}_0\}$. Then $\mathcal{V}_0$ is a finite subcover of $\mathcal{V}$ and

$$R(\mathcal{V}) \leq \max\{\tau(V_U) : U \in \mathcal{U}_0\} \leq \max\{\tau(U) : U \in \mathcal{U}_0\} = R(\mathcal{U}).$$

(3),(4) By an analogous argument for the transition and running times we get

$$S(\mathcal{V}) \leq \max\{\tau(V_U, V_W) : U, W \in \mathcal{U}_0\} \leq \max\{\tau(U, W) : U, W \in \mathcal{U}_0\} = S(\mathcal{U})$$

$$T(\mathcal{V}) \leq \max\{\tau(x, V_U) : x \in X, U \in \mathcal{U}_0\} \leq \max\{\tau(x, U) : x \in X, U \in \mathcal{U}_0\} = T(\mathcal{U})$$

(5) It is clear that $Q(\mathcal{U}) \leq R(\mathcal{U}) \leq S(\mathcal{U})$. Let $m = T(\mathcal{U})$ and let $\mathcal{U}_0 \subseteq \mathcal{U}$ be a subcover such that $m = \max\{\tau(x, U) : x \in X, U \in \mathcal{U}_0\}$. Given $U, V \in \mathcal{U}_0$, pick $x \in U \setminus V$. There exist $0 \leq i \leq m$ with $F^i(x) \in V$, so $\tau(U, V) \leq m$ and $S(\mathcal{U}) \leq T(\mathcal{U})$.

(6) Let $\mathcal{U}_0 \subseteq \mathcal{U}$ be a subcover with $|\mathcal{U}_0| = |\mathcal{U}|$. For a point $x \in X$ there exist $0 \leq i < j \leq |\mathcal{U}_0|$ and $U \in \mathcal{U}_0$ such that $F^i(x) \in U$ and $F^j(x) \in U$, so $\tau(U) \leq |\mathcal{U}_0|$ and $Q(\mathcal{U}) \leq |\mathcal{U}|$.

(7) Assume that $\mathcal{U}$ is a finite open partition and $T(\mathcal{U}) = m$. There exists $x \in X$ and $V \in \mathcal{U}$ such that $\tau(x, V) = m$. For any $U \in \mathcal{U}$ there exists i with $1 \leq i \leq m$ and $F^i(x) \in U$, Since $\mathcal{U}$ is a partition, $|\mathcal{U}| \leq m$. □

2.12. Strictly ergodic systems

In minimal systems, the trajectory of any point pays repeated visits to any nonempty open set. If the frequency of these visits has a limit which is independent of the initial point, we say that the system is **strictly ergodic**.

Definition 2.98. — A dynamical system (X,F) is **strictly ergodic**, if for any nonempty open set $U \subseteq X$ there exists a positive number $\mu(U)$, such that for any $x \in X$,

$$\lim_{n\to\infty} \frac{\#\{i < n : F^i(x) \in U\}}{n} = \mu(U).$$

A strictly ergodic system is minimal, since every its point has a dense trajectory. In Theorem 4.78 we shall see that there exist minimal systems which are not strictly ergodic. In one-dimensional spaces $\mathbb{I}$ or $\mathbb{T}$, any open set is a union of a finite or infinite number of open intervals. It sufficestherefore to check the condition only for open intervals. Similarly, in symbolic spaces, any open set is a finite or infinite union of cylinders, so it will be enough to check the condition only for cylinders.

Example 2.99. — The dyadic adding machine $(\mathbf{2}^{\mathbb{N}}, R_1)$ is strictly ergodic.

Proof. — Any point is quasi-periodic and visits periodically any cylinder of length m, with period 2^m. If $u \in \mathbf{2}^m$, then for any $x \in \mathbf{2}^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\#\{i < n : F^i(x) \in [u]\}}{n} = 2^{-m}.$$

□

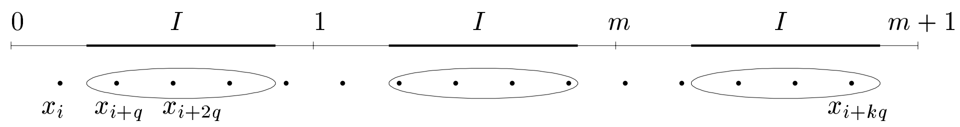

Figure 24. Strict ergodicity of rotations

Example 2.100. — An irrational rotation of the circle is strictly ergodic. If $I \subseteq \mathbb{T}$ is an interval, then $\mu(I) = ||I||$ is its length.

Proof. — Let $0 < \varepsilon < ||I||/2$. There exists $q > 0$ such that $0 < R^q_\alpha(0) = \beta < \varepsilon$. Let $(x_i = R^i_\alpha(x))_{i\geq 0}$ be the trajectory of a point $x = x_0$ and let $k > 0$. Set

$$p = \left\lfloor \frac{||I||}{\beta} \right\rfloor, \quad m = \lfloor k\beta \rfloor.$$

For $0 \leq i < q$ consider the trajectory $x_i,\ x_{i+q}, \dots\ x_{i+kq}$ of x_i in $(\mathbb{T}, R_\beta)$. This trajectory visits the interval I in several consecutive blocks (Figure 24). The length of each block is either p or $p+1$, the number of these blocks is either m or $m+1$.

$$(\tfrac{||I||}{\beta} - 1)(k\beta - 1) \leq pm \leq \#\{j \leq k : x_{i+jq} \in I\} \leq (p+1)(m+1) \leq (\tfrac{||I||}{\beta} + 1)(k\beta + 1)$$

Summing for $i = 0, 1, \dots, q-1$ we get

$$(k||I|| - k\beta - \tfrac{||I||}{\beta} + 1)q \leq \#\{j < (k+1)q : x_j \in I\} \leq (k||I|| + k\beta + \tfrac{||I||}{\beta} + 1)q.$$

Given $n > 0$, let k satisfy $kq < n \leq (k+1)q$. Letting n go to infinity, k goes to infinity too and we get

$$||I|| - \beta \leq \liminf_{n\to\infty} \frac{\#\{j < n : x_j \in I\}}{n} \leq \limsup_{n\to\infty} \frac{\#\{j < n : x_j \in I\}}{n} \leq ||I|| + \beta.$$

Since this holds for arbitrarily small β, the limit exists and is equal to the length of the interval. □

2.12.1. Invariant measures. — For strictly ergodic systems we can obtain the frequency of visits not only for open sets but also for closed sets as $\mu(V) = 1-\mu(X\setminus V)$. If $(V_i)_{i\geq 0}$ is a sequence of closed sets then their union $V = \bigcup_{i\geq 0} V_i$ need not be either closed or open. Countable unions of closed sets are called F_σ-sets and dually, countable intersections of open sets are called G_δ-sets. A measure can be extended to these two types of sets. If we iterate this construction, we obtain the system of Borel sets $\mathcal{B}(X)$. This is the smallest system which contains all open sets and is closed with respect to the operations of complement and countable union. For example the sets $\mathfrak{T}$ of transitive points and the set $\mathcal{E}$ of equicontinuous points are both F_δ-sets so they are Borel sets.

Definition 2.101. — A Borel probability measure on a compact metric space X is a map $\mu : \mathcal{B}(X) \to [0,1]$ such that
(1) $\mu(X) = 1$,
(2) $\mu(X \setminus U) = 1 - \mu(U)$ for any $U \in \mathcal{B}(X)$,
(3) $\mu(\bigcup_{i\geq 0} U_i) = \sum_{i\geq 0} \mu(U_i)$ for any pairwise disjoint sets $U_i \in \mathcal{B}(X)$.

We interpret $\mu(U)$ as the probability that the system is in a state $x \in U$. If μ represents the probability distribution at time zero, then the probability that the system is in a set U at time 1 is $\mu\{x \in X : F(x) \in U\} = \mu(F^{-1}(U))$. In this way a dynamical system acts on measures of X. If μ is a Borel measure, then $\nu = F(\mu)$ defined by $\nu(U) = \mu(F^{-1}(U))$ is also a Borel measure. We say that μ is an invariant measure if $F(\mu) = \mu$. If (X, F) is strictly ergodic, then there exists a unique invariant measure defined by the limit frequency of visits to open sets. The Krylov-Bogoljubov theorem states that any dynamical system possesses at least one invariant measure (see Denker et al. [**47**] or Akin [**3**] for a proof). A dynamical system may possess, however, many distinct invariant measures. For example, if x is a periodic point with period $p \geq 1$, then we have an invariant measure

$$\mu(U) = \frac{\#\{i < n : F^i(x) \in U\}}{p}$$

which is concentrated on the orbit $\mathcal{O}(x) = \{x, \ldots, F^{p-1}(x)\}$. Ergodic theory studies properties of a dynamical system equipped with an invariant measure. If the system (X, F) is not uniquely ergodic, then the limit frequency of visits of a point x in an open set U need not exist or it may be different for different points. However, if we fix an invariant measure μ, then for a given Borel set $U \subseteq X$ there exists a set $X_U \subseteq X$ with $\mu(X_U) = 1$, such that

$$x \in X_U \;\Rightarrow\; \lim_{n\to\infty} \frac{\#\{i < n : F^i(x) \in U\}}{n} = \mu(U)$$

This is a special case of the Birkhoff Ergodic theorem (see e.g., Walters [**158**]). Many topological properties of dynamical systems have their counterparts in ergodic theory. For example transitive systems correspond to ergodic systems. A dynamical system (X, F, μ) is μ-ergodic if for any Borel invariant set $Y \subseteq X$ either $\mu(Y) = 1$ or $\mu(Y) = 0$.

Exercise 2.1. — Find dynamical relations, their diagonals, equicontinuity points and attractors in following dynamical systems

(1) $(\mathbb{I}, \mathrm{Id})$ (identity on the unit real interval)
(2) $(\mathbb{I}, F)$ where $F(x) = x + x \cdot \ln(\sin x)$.
(3) $(\mathbb{I}, F)$ where $F(x) = x + \frac{1}{k\pi}\sin(k\pi x)$, k is a positive integer.
(4) $(\mathbb{I}, F)$ where $F(x) = x + \max\{0, \frac{1}{k\pi}\sin(k\pi x)\}$
(5) $(\overline{\mathbb{C}}, z+1)$, $(\overline{\mathbb{C}}, 2z)$, $(\overline{\mathbb{C}}, z^2)$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the complex sphere.
(6) $(\overline{\mathbb{C}}, F)$ where $F(z) = z \cdot e^{i|z|}$
(7) $(\mathbb{I}^2, F)$ where $F(x, y) = (x^2, \frac{y}{2})$
(8) (X, F) where $X = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x+y \leq 0\}$ and $F(x, y) = (x^2+y^2, 2xy)$.
(9) Find a transitive dynamical system whose subsystem is not transitive.
(10) Find transitive dynamical systems whose product is not transitive.
(11) Find a minimal dynamical system whose product is not minimal.
(12) Show that if (X, F) is mixing, then $(X \times X, F \times F)$ is transitive.
(13) Determine $h(\mathbb{I}, Q_r)$ for $r \leq r_\infty$.
(14) Show that the entropy of the Preston system of Example 1.19 is 0.
(15) Evaluate the entropy of systems $(\overline{\mathbb{C}}, z+1)$, $(\overline{\mathbb{C}}, 2z)$, $(\overline{\mathbb{C}}, z^2)$ on the complex sphere.