CHAPTER 3

SYMBOLIC DYNAMICS

A symbolic space is a space of sequences of symbols, or a closed subspace of a (countable) product of finite discrete spaces. We say that (X, F) is a symbolic dynamical system, if X is a symbolic space. The importance of symbolic dynamics stems from the fact that any dynamical system has a **symbolic extension**. This means that points of the system can be encoded as symbolic sequences and dynamical properties of the system can be deduced from the combinatorial properties of these sequences. In many cases, a symbolic extension can be obtained as the **subshift of itineraries** of a cover. This happens whenever the cover is **generating**.

There are several ways of specifying a subshift. We can specify its set of forbidden words. The subshift consists of all points (sequences) which do not contain any forbidden word as a subword. We can also specify a subshift by its **language**, or its set of allowed words. The language of a subshift cannot be arbitrary. It must contain all subwords, and at least some prolongations (extensions) of any of its words. We say that the language of a subshift is **extendable**. Conversely, an extendable language specifies a subshift. A subshift can be also given as the orbit closure of a sequence.

Subshifts are very diversified dynamical systems and there are many subclasses with specific properties. A **subshift of finite type** (SFT) can be specified either by a finite set of **forbidden words**, or by a finite **transition graph**, or by a binary **adjacency matrix**. Many dynamical properties of subshifts of finite types such as transitivity and recurrence, depend on the structure of these graphs or matrices. In particular, the topological entropy of a subshift of finite type is the logarithm of the spectral radius of its matrix. This is a consequence of the Perron-Frobenius theory of nonnegative matrices. Subshifts of finite type have nice topological characterizations. They are exactly those subshifts which have the shadowing property, i.e., their ε-chains can be approximated by true trajectories. Another topological characterization is that of openness. Such subshifts map open sets to open sets.

With the subject of subshifts of finite type, topological dynamics touches computer science. The language of a subshift of finite type is **regular**. This is the lowest complexity class in the Chomsky hierarchy of formal languages. Subshifts with regular languages are called **sofic**. Sofic subshifts form a larger class, but they share certain properties of subshifts of finite type. They can be specified by finite graphs with labelled edges or by matrices whose entries are nonnegative integers. Once more these representations can be used to determine their dynamical properties or to compute their entropies.

3.1. Symbolic spaces

Symbolic power space is a space of infinite sequences of symbols. We call any finite set A with at least two elements an **alphabet**. We denote by A^n the set of words of A of length n, and by A^0 the set $\{\lambda\}$, where λ is the empty word. $A^{\mathbb{N}}$ is the set of infinite words, and

$$A^* = \bigcup_{n\geq 0} A^n, \quad A^+ = \bigcup_{n>0} A^n, \quad \overline{A^*} = A^* \cup A^{\mathbb{N}}.$$

The **concatenation** of words $u, v \in A^*$ is written as uv. We say that u is a **subword** of v and write $u \sqsubseteq v$ if there exist words x, y such that $xuy = v$. If $x = \lambda$ we say that u is a **prefix** of v and write $u \sqsubseteq_p v$. If $y = \lambda$, we say that u is a **suffix** of v and write $u \sqsubseteq_s v$. If $u \in A^+$, we denote by $u^\infty \in A^{\mathbb{N}}$ the infinite concatenation of u with itself. We denote by $\#A$ the number of elements of a finite set A.

We also consider symbolic spaces constructed as infinite products of possibly different alphabets. Let $\mathcal{A} = (A_i)_{i\geq 0}$ be an infinite sequence of alphabets. Set $\mathcal{A}^0 = \{\lambda\}$,

$$\mathcal{A}^n = \prod_{i=0}^{n-1} A_i, \quad \mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n, \quad \mathcal{A}^+ = \mathcal{A}^* \setminus \{\lambda\},$$

$$\mathcal{A}^{\mathbb{N}} = \prod_{i=0}^{\infty} A_i, \quad \overline{\mathcal{A}^*} = \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}.$$

The length of $u \in \mathcal{A}^n$ is $|u| = n$, the length of $u \in \mathcal{A}^{\mathbb{N}}$ is $|u| = \infty$. We denote by $u_i \in A_i$ the i-th letter of $u \in \overline{\mathcal{A}^*}$. If $i \geq |u|$, set $u_i = \lambda$. If $[j,k] \subseteq \mathbb{N}$ is an interval and $u \in \overline{\mathcal{A}^*}$, set $u_{[j,k]} = u_j \dots u_k \in \prod_{i=j}^{k} A_i$, and similarly for other types of intervals. If A is an alphabet and $u, v \in A^*$, then $u \sqsubseteq v$ iff there exist $0 \leq i \leq j < |v|$ such that $u = v_{[i,j)}$.

Definition 3.1. — We call **symbolic space** any closed subspace of $\overline{\mathcal{A}^*}$ with metric

$$d(u,v) = 2^{-n}, \text{ where } n = \min\{i \geq 0 : u_i \neq v_i\}.$$

A **symbolic product space** is any space $\mathcal{A}^{\mathbb{N}}$.

If $n < m$ and if $u \in \mathcal{A}^n$ is a prefix of $v \in \mathcal{A}^m$, then $u_n = \lambda \neq v_n$, so $d(u,v) = 2^{-n}$. Thus every finite word $u \in \mathcal{A}^*$ is isolated in $\overline{\mathcal{A}^*}$. For any $x \in \mathcal{A}^{\mathbb{N}}$ we have

$$x = \lim_{n\to\infty} x_{[0,n)},$$

so $\mathcal{A}^*$ is dense in $\overline{\mathcal{A}^*}$. The space $\overline{\mathbf{2}^*}$ can be visualized via the embedding $\psi : \overline{\mathbf{2}^*} \to \mathbb{I}$

$$\psi(u) = \sum_{i=1}^{|u|} 2u_{i-1}3^{-i} + \frac{3^{-|u|}}{2}$$

Then ψ maps $\mathbf{2}^{\mathbb{N}}$ to the Cantor middle third set and finite words of $\mathbf{2}^*$ to isolated points located in the middle of these middle intervals (Figure 1).

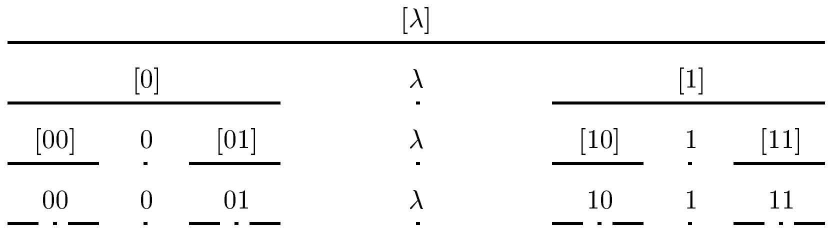


FIGURE 1. Binary symbolic space

The metric d on $\mathcal{A}^{\mathbb{N}}$ is equivalent to the product metric from Definition A.17. For $a, b \in A_i$ set $d_i(a,b) = 1$ iff $a \neq b$, so that (A_i, d_i) is a discrete metric space. For the product metric

$$d'(u,v) = \sum_{i=0}^{\infty} d_i(u_i, v_i) \cdot 2^{-i}$$

on $\mathcal{A}^{\mathbb{N}}$ we have $d(u,v) \leq d'(u,v) \leq 2 \cdot d(u,v)$. By Proposition A.19, any space $\mathcal{A}^{\mathbb{N}}$ is compact. A **cylinder** in a symbolic product space $X \subseteq \mathcal{A}^{\mathbb{N}}$ is any set

$$[u] = \{x \in X : x_{[0,n)} = u\},$$

where $u \in \mathcal{A}^*$. A nonempty cylinder set is a ball. If $x \in [u]$ then $[u] = B_{2^{-|u|+1}}(x)$, so $[u]$ is an open set. The complement of $[u]$ is a finite union of cylinders, so it is open too. Thus every cylinder is **clopen** (closed and open). For any $n > 0$ we have a **clopen partition** of X (Definition A.23).

$$\mathcal{P}_n = \{[u] : u \in \mathcal{A}^n\}.$$

Proposition 3.2. — *Any symbolic space is homeomorphic to a subspace of some symbolic product space.*

Proof. — Let $(A_i)_{i \geq 0}$ be a sequence of alphabets and let 0 be a symbol which does not belong to any A_i. Set $B_i = A_i \cup \{0\}$ and set

$$X = \{x \in \mathcal{B}^{\mathbb{N}} : \forall i \in \mathbb{N}, (x_i = 0 \;\Rightarrow\; x_{i+1} = 0)\}.$$

Then X is a closed subspace of $\mathcal{B}^n$ and there exists an isometry $\varphi : \overline{\mathcal{A}^*} \to \mathcal{B}^{\mathbb{N}}$ given by

$$\varphi(x) = \begin{cases} x & \text{if } x \in \mathcal{A}^{\mathbb{N}}. \\ x0^{\infty} & \text{if } x \in \mathcal{A}^*. \end{cases}$$

□

Proposition 3.3. — *Let $\mathcal{A} = (A_n)_{n \geq 0}$ be an infinite sequence of alphabets and let $(h_n : A_{n+1} \to A_n)_{n \geq 0}$ be a sequence of mappings. The* **inverse limit space**

$$X = \lim_{n \to \infty} (A_n, h_n) = \{x \in \mathcal{A}^{\mathbb{N}} : \forall n \geq 0, h_n(x_{n+1}) = x_n\}$$

is a symbolic space. If $\#h_n^{-1}(a) \geq 2$ for any $n \geq 0$ and any $a \in A_n$, then X is perfect, i.e., it does not have isolated points.

Proof. — If $x \in \mathcal{A}^{\mathbb{N}} \setminus X$, then for some n, $h_n(x_{n+1}) \neq x_n$ and $[x_{[0,n+1]}] \subseteq \mathcal{A}^{\mathbb{N}} \setminus X$, so X is a closed subspace of $\mathcal{A}^{\mathbb{N}}$ and is therefore a symbolic space. If for any n and any $a \in A_n$, $\#h_n^{-1}(a) \geq 2$, then for any $x \in X$ and any n there exists $y \in X$ with $x_{[0,n]} = y_{[0,n]}$, $x_{n+1} \neq y_{n+1}$ so $d(x,y) = 2^{-n-1}$ and X is perfect. □

The cylinder sets in the inverse limit space X have simpler form. For $a_n \in A_n$, set

$$[a_n] = \{x \in X : x_n = a_n\}.$$

Then $\{[a_n] : a_n \in A_n\}$ is a clopen partition.

Symbolic spaces can be characterized topologically. Recall that a compact space is a Cantor space (Definition A.34), if it is perfect, i.e., does not have isolated points, and totally disconnected, i.e., any two distinct points can be sparated by a clopen set.

Proposition 3.4. — *Any symbolic product space is a Cantor space. Any of its clopen sets is a finite union of cylinders.*

Proof. —

(1) The metric in $\mathcal{A}^{\mathbb{N}}$ is equivalent to the product metric, hence $\mathcal{A}^{\mathbb{N}}$ is compact by Proposition A.19.

(2) We show that $\mathcal{A}^{\mathbb{N}}$ is totally disconnected. If $x, y \in \mathcal{A}^{\mathbb{N}}$ and $x \neq y$, then there exists $n \geq 0$ with $x_n \neq y_n$. Thus $[x_{[0,n]}]$, $[y_{[0,n]}]$ are disjoint clopen sets containing x and y respectively.

(3) For $x \in \mathcal{A}^{\mathbb{N}}$ and $\varepsilon > 0$ we choose some $n \in \mathbb{N}$ with $2^{-n} < \varepsilon$. Since A_n contains at least two elements, there exists $y \in Y$ such that $y_i = x_i$ for $i < n$ and $x_n \neq y_n$. Thus $x \neq y$ and $d(x,y) < \varepsilon$, so X is perfect.

(4) If U is a clopen set then for any $x \in U$ there exists a cylinder $x \in A_x \subseteq U$. Since U is compact, its open cover $\{A_x : x \in U\}$ has a finite subcover. Thus U is a finite union of cylinders. □

Proposition 3.5. — *Let X be a metric space which is compact and totally disconnected. Then the clopen sets form a base for its topology. For any $\varepsilon > 0$ there exists a finite clopen partition $\mathcal{V} = \{V_a : a \in A\}$ with* $\mathrm{diam}(\mathcal{V}) < \varepsilon$. *If X is also perfect then there exists $n_\varepsilon > 0$, such that $\#A$ can be any number larger than n_ε.*

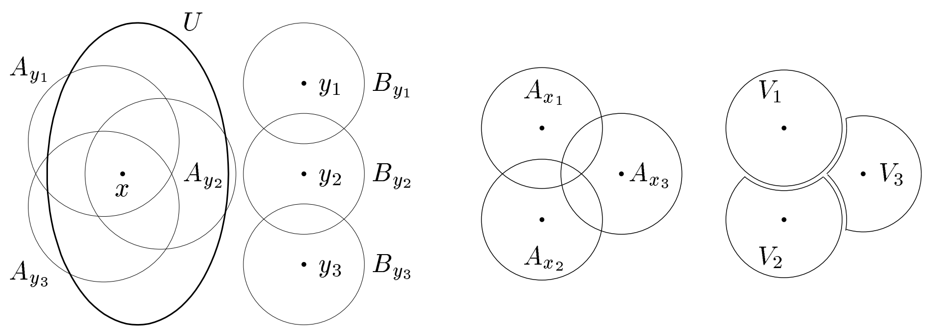


FIGURE 2. Clopen base

Proof. —

(1) Let $U \subseteq X$ be an open set and $x \in U$ (Figure 2 left). We construct a clopen set A with $x \in A \subseteq U$. If $U = X$, then U itself is clopen and we set $A = U$. Suppose $U \neq X$. For any $y \in X \setminus U$ there exist disjoint clopen sets A_y, B_y with $x \in A_y$ and $y \in B_y$. The open cover $\{B_y : y \in X \setminus U\}$ has a finite subcover $\{B_{y_1}, \dots, B_{y_n}\}$, and $A = A_{y_1} \cap \cdots \cap A_{y_n}$ is a clopen set with $x \in A \subseteq U$.

(2) Given $\varepsilon > 0$, for any $x \in X$ there exists a clopen set A_x containing x with $\operatorname{diam}(A_x) < \varepsilon$. Let $\{A_{x_1}, \dots, A_{x_n}\}$ be a minimal finite subcover of $\{A_x : x \in X\}$ (Figure 2 right) and set

$$V_1 = A_{x_1}, \quad V_2 = A_{x_2} \setminus V_1, \dots, \quad V_n = A_{x_n} \setminus (V_1 \cup \cdots \cup V_{n-1}),$$

so $\mathcal{V} = \{V_i : i \in [1, n]\}$ is a clopen partition of X with $\operatorname{diam}(\mathcal{V}) < \varepsilon$.

(3) If X is perfect, then any nonempty clopen set U contains at least two elements x, y which can be separated by clopen sets $V \ni x$, $X \setminus V \ni y$. Then $U \cap V$, $U \setminus V$ are nonempty disjoint clopen sets whose union is U. Thus if $\mathcal{V}$ is a clopen partition with $\operatorname{diam}(\mathcal{V}) < \varepsilon$, we can divide its elements to obtain a clopen partition with arbitrarily larger number of elements. □

Proposition 3.6. —

(1) Any compact, totally disconnected metric space is homeomorphic to a symbolic space.

(2) Any Cantor space is homeomorphic to $\mathbf{2}^{\mathbb{N}}$.

Proof. —

(1) Let X be compact and totally disconnected. We construct clopen partitions $\mathcal{V}_n = \{V_a : a \in A_n\}$ of X such that $\operatorname{diam}(\mathcal{V}_n) < 2^{-n}$ and $\mathcal{V}_{n+1} \succeq \mathcal{V}_n$ (Definition 2.78). By Proposition 3.5, there exists a partition $\mathcal{V}_0$ with $\operatorname{diam}(\mathcal{V}_0) < 1$. Given $\mathcal{V}_n$, we decompose any set $V \in \mathcal{V}_n$ to clopen sets with diameter less than 2^{-n-1}. Let $\mathcal{V}_{n+1} = \{V_a : a \in A_{n+1}\}$ consist of all these sets. For any $a \in A_{n+1}$ there exists a unique $h_n(a) \in A_n$ such that $V_a \subseteq V_{h_n(a)}$. This defines a mapping $h_n : A_{n+1} \to A_n$. Set

$$Y = \lim_{n \to \infty} (A_n, h_n) = \{x \in \mathcal{A}^{\mathbb{N}} : \forall n \geq 0, h_n(x_{n+1}) = x_n\}.$$

By Proposition 3.3, Y is a symbolic space. For any $u \in Y$, $V_{u_{n+1}} \subseteq V_{u_n}$, so $V_u = \bigcap_{n \geq 0} V_{u_n}$ is a nonempty closed set with diameter zero which contains exactly one element. Define a bijective mapping $\varphi : Y \to X$ by $\varphi(u) \in V_u$. Then

$$d(u, v) < 2^{-n} \Rightarrow \varphi(u), \varphi(v) \in V_{u_n} \Rightarrow d(\varphi(u), \varphi(v)) < 2^{-n},$$

so φ is continuous. Since X is compact, φ is a homeomorphism by Proposition A.20.

(2) Assume that X is perfect. There exists a clopen partition with diameter at most 1 whose number of elements is a power of 2, say 2^{k_0}. Index the sets of the partition by binary words of length k_0, so we get a clopen partition

$$\mathcal{V}_0 = \{V_u : u \in \mathbf{2}^{k_0}\}, \quad \operatorname{diam}(\mathcal{V}_0) < 2^0.$$

There exists $k_1 > k_0$ such that for any $u \in \mathbf{2}^{k_0}$, V_u has a clopen partition with number of elements $2^{k_1 - k_0}$ and diameter at most 2^{-1}. Let $\{V_{uv} : v \in \mathbf{2}^{k_1 - k_0}\}$ be this partition of V_u, so $\mathcal{V}_1 = \{V_w : w \in \mathbf{2}^{k_1}\}$ is a clopen partition of X. Continuing in this way we construct an increasing sequence $0 < k_0 < k_1 < \cdots$ and clopen partitions $\mathcal{V}_n = \{V_u : u \in \mathbf{2}^{k_n}\}$, such that $\mathrm{diam}(\mathcal{V}_n) < 2^{-n}$, and for $u \in \mathbf{2}^{k_n}$, $v \in \mathbf{2}^{k_{n+1}-k_n}$, $V_{uv} \subseteq V_u$, so $\mathcal{V}_{n+1} \succeq \mathcal{V}_n$. For $u \in \mathbf{2}^{\mathbb{N}}$, set $V_u = \bigcap_{n\geq 1} V_{u_{[0,k_n)}}$. Since X is compact, V_u is nonempty and its diameter is zero, so it contains exactly one element. Define a mapping $\varphi : \mathbf{2}^{\mathbb{N}} \to X$ by $\varphi(u) \in V_u$. Then φ is a homeomorphism. □

3.2. Symbolic extensions

The importance of symbolic spaces stems from the fact that any compact metric space can be constructed as a factor of a symbolic space. For example, the decadic representation of real numbers of the unit interval $\mathbb{I} = [0,1]$ is accomplished by a factor map $\varphi : A^{\mathbb{N}} \to \mathbb{I}$, where $A = \{0, 1, \ldots, 9\}$ and $\varphi(x) = \sum_{i=0}^{\infty} x_i \cdot 10^{-i-1}$. This representation is, however, not very convenient for symbolic computations, since continuous operations on $\mathbb{I}$ cannot be lifted to $A^{\mathbb{N}}$. For example if we add numbers $x = 0.033333\ldots$ and $y = 0.066666\ldots$ we are unable to determine the first digit of the sum unless we encounter position i, where $x_i + y_i \neq 9$. This means that a continuous addition (modulo 1) $G : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ cannot be lifted to any continuous function $F : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$, such that $\varphi \circ F = G \circ (\varphi \times \varphi)$. Such a lifting is possible in a **redundant positional system** with more digits than strictly necessary. For example we can use binary positional system with three digits $\mathbf{3} = \{0, 1, 2\}$ and construct a map $\varphi : \mathbf{3}^{\mathbb{N}} \to \mathbb{I}$ by

$$\varphi(x) = \sum_{i=0}^{\infty} x_i 2^{-i-2}.$$

We can then add (modulo 1) two numbers with symbolic representations $x, y \in \mathbf{3}^{\mathbb{N}}$ as follows. First add them componentwise $v_i = x_i + y_i$ to obtain $v_i \in \{0, 1, 2, 3, 4\}$. Then we perform a simultaneous carry to the left. For $i \geq -1$ set $w_i = \lfloor \frac{v_{i+1}}{2} \rfloor + \mathrm{mod}_2(v_i)$ (we assume here $v_{-1} = 0$). Then $w_i \in \{0, 1, 2, 3\}$ and w represents the same number as v. Indeed

$$\begin{aligned} \sum_{i=-1}^{\infty} w_i \cdot 2^{-i-2} &= \sum_{i=-1}^{\infty} \lfloor \tfrac{v_{i+1}}{2} \rfloor \cdot 2^{-i-2} + \sum_{i=0}^{\infty} \mathrm{mod}_2(v_i) \cdot 2^{-i-2} \\ &= \sum_{i=0}^{\infty} \left(2 \cdot \lfloor \tfrac{v_i}{2} \rfloor + \mathrm{mod}_2(v_i)\right) 2^{-i-2} = \sum_{i=0}^{\infty} v_i \cdot 2^{-i-2}. \end{aligned}$$

We perform the carry operation once more and get $z_i = \lfloor \frac{w_{i+1}}{2} \rfloor + \mathrm{mod}_2(w_i)$. Then $z \in \mathbf{3}^{\mathbb{N}}$ and z_i depends only on $x_{[i,i+2]}$, so the addition is continuous in $\mathbf{3}^{\mathbb{N}}$

$$\begin{array}{ll} x: & 0.002002222111\ldots \\ y: & 0.002101222210\ldots \\ \hline v: & 0.004103444321\ldots \\ w: & 0.02011322120\ldots \\ z: & 0.1001221020\ldots \end{array}$$

Definition 3.7. — We say that a factor map $\varphi : X \to Y$ has the **extension property**, if for any continuous $\psi : X \to Y$ there exists a map $F : X \to X$ with $\varphi \circ F = \psi$.

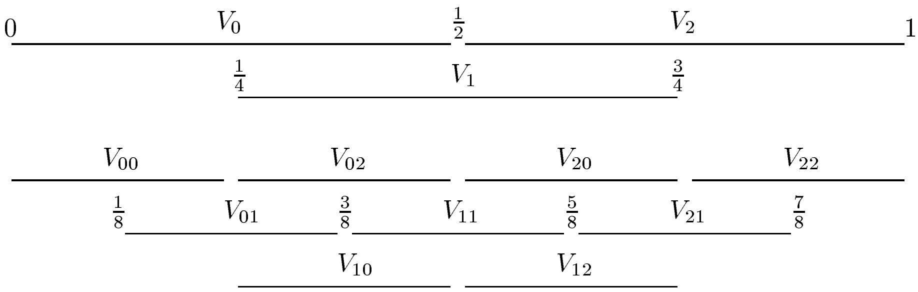


FIGURE 3. Redundant positional system

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "\psi"'] & X \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

We show that $\varphi : \mathbf{3}^{\mathbb{N}} \to \mathbb{I}$ has the extension property. For $u \in \mathbf{3}^*$ set $V_u = \varphi([u])^\circ = (\varphi(u0^\infty), \varphi(u2^\infty))$ (Figure 3). Then $\mathcal{V}_n = \{V_u : u \in \mathbf{3}^n\}$ is an open cover of $\mathbb{I}$ with $\operatorname{diam}(\mathcal{V}_n) = 2^{-n}$ and Lebesgue number $\lambda_n = 2^{-n-2}$. Moreover, for any $u \in \mathbf{3}^n$, $\{V_{ua} : a \in \mathbf{3}\}$ is an open cover of V_u with Lebesgue number 2^{-n-3}. Let $\psi : \mathbf{3}^{\mathbb{N}} \to \mathbb{I}$ be a continuous map. Given $n > 0$, there exists m_n such that

$$d(u,v) < 2^{-m_n} \ \Rightarrow\ d(\psi(u), \psi(v)) < \lambda_n = 2^{-n-2}.$$

Given $x \in \mathbf{3}^{\mathbb{N}}$, we construct $y = F(x)$. There exists $y_0 \in \mathbf{3}$ such that $\psi(B_{2^{-m_0}}(x)) \subseteq [y_0]$. Since $\{V_{y_0 a} : a \in \mathbf{3}\}$ is an open cover of V_{y_0} with diameter λ_1, there exists y_1 such that $\psi(B_{2^{-m_1}}(x)) \subseteq [y_0 y_1]$. We continue recursively to obtain all digits of y.

Theorem 3.8 **(Weihrauch[161]).** — *For any compact space Y there exists a factor map $\varphi : \mathbf{2}^{\mathbb{N}} \to Y$ with the extension property.*

Proof. — Let $\mathcal{V}_0 = \{V_a : a \in A_0\}$ be a finite open cover of Y with $\operatorname{diam}(\mathcal{V}_0) < 1$ and $\#A_0 \geq 2$. We allow some repetition of the V_a in $\mathcal{V}_0$, so we can have $V_a = V_b$ for $a \neq b$. We continue recursively. Let $\mathcal{V}_n = \{V_a : a \in A_n\}$ be a finite open cover of Y with $\operatorname{diam}(\mathcal{V}_n) < 2^{-n}$. For any $a \in A_n$, $\{B_{2^{-n-2}}(x) : x \in \overline{V_a}\}$ is an open cover of $\overline{V_a}$ and has a finite subcover $\{W_{(a,i)} : 0 \leq i < n_a\}$. Repeating some $W_{(a,i)}$, if necessary, we can assume $n_a \geq 2$. Set

$$A_{n+1} = \{(a,i) : a \in A_n, 0 \leq i < n_a\}.$$

Define $h_n : A_{n+1} \to A_n$ by $h_n(a,i) = a$ and let $V_{(a,i)} = W_{(a,i)} \cap V_a$. Then

$$\mathcal{V}_{n+1} = \{V_b : b \in A_{n+1}\}$$

is an open cover of Y with $\operatorname{diam}(\mathcal{V}_{n+1}) < 2^{-n-1}$ and $\mathcal{V}_{n+1} \succeq \mathcal{V}_n$. For any $b \in A_{n+1}$, $V_b \subseteq V_{h_n(b)}$. For any $a \in A_n$, $\#h_n^{-1}(a) \geq 2$ and $\bigcup\{V_b : b \in h_n^{-1}(a)\} = V_a$. We set

$$X = \lim_{n\to\infty}(A_n, h_n) = \left\{ x \in \prod_{n=0}^{\infty} : \forall n, h_n(x_{n+1}) = x_n \right\}$$

By Proposition 3.3, X is a Cantor space homeomorphic to $\mathbf{2}^{\mathbb{N}}$. Given $x \in X$, $\overline{V_{x_{n+1}}} \subseteq \overline{V_{x_n}}$ are nonempty closed sets so their intersection $V_x = \bigcap_{n>0} \overline{V_{x_n}}$ is nonempty. Since $\operatorname{diam}(V_{x_n}) < 2^{-n}$, $\operatorname{diam}(V_x) = 0$ and V_x contains a unique point $\varphi(x) \in V_x$. Clearly $\varphi : X \to Y$ is surjective. If $u, x \in X$, then

$$d(u,x) < 2^{-n} \Rightarrow u_n = x_n \Rightarrow \varphi(u), \varphi(x) \in \overline{V_{u_n}} \Rightarrow d(\varphi(u), \varphi(x)) < 2^{-n},$$

and φ is continuous. We show that φ has the extension property. Let λ_0 be a Lebesgue number of $\mathcal{V}_0$. For $n > 0$, let λ_{n+1} be the minimum of the Lebesgue numbers of all covers $\{W_{(a,i)} : 0 \le i < n_a\}$ of $\overline{V_a}$, with $a \in A_n$. Let $\psi : X \to Y$ be a continuous map. For a given $n \ge 0$, let m_n be such that for any $u, v \in X$,

$$d(u,v) \le 2^{-m_n} \Rightarrow d(\psi(u), \psi(v)) < \lambda_n.$$

We can assume that $m_{n+1} > m_n$. If $a \in A_{m_0}$ and $x \in [a]$ then $\psi([a]) \subseteq B_{\lambda_0}(\psi(x))$ and there exists $f_0(a) \in A_0$ such that $B_{\lambda_0}(\psi(x)) \subseteq V_{f_0(a)}$, so $\psi([a]) \subseteq V_{f_0(a)}$.

We continue recursively. Suppose that we have constructed $f_n : A_{m_n} \to A_n$ such that for any $b \in A_{m_n}$, $\psi([b]) \subseteq V_{f_n(b)}$. We have a map

$$h_{m_n} \circ h_{m_n+1} \circ \cdots \circ h_{m_{n+1}-1} = h_{[m_n, m_{n+1})} : A_{m_{n+1}} \to A_{m_n}$$

Given $a \in A_{m_{n+1}}$ then $b = h_{[m_n, m_{n+1})}(a) \in A_{m_n}$, so $[a] \subseteq [b]$ and $\psi([a]) \subseteq \psi([b]) \subseteq V_{f_n(b)}$. There exists $f_{n+1}(a) \in A_{n+1}$ such that for any $x \in [a]$,

$$\psi([a]) \subseteq B_{\lambda_{n+1}}(\psi(x)) \subseteq V_{f_{n+1}(a)} \subseteq V_{f_n(b)}$$

so $f_n \circ h_{[m_n, m_{n+1})}(a) = f_n(b) = h_n \circ f_{n+1}(a)$. Define $F : X \to \prod_{i \ge 0} A_i$ by $F(x)_n = f_n(x_{m_n}) \in A_n$. For any $x \in X$ and $n \ge 0$ we have

$$h_n(F(x)_{n+1}) = h_n \circ f_{n+1}(x_{m_{n+1}}) = f_n \circ h_{[m_n, m_{n+1})}(x_{m_{n+1}}) = f_n(x_{m_n}) = F(x)_n.$$

Thus $F(x) \in X$, so $F : X \to X$. The map F is continuous, since

$$d(u,v) < 2^{-m_n} \Leftrightarrow d(F(x), F(y)) < 2^{-n}$$

Moreover, $\psi([x_{m_n}]) \subseteq V_{f_n(x_{m_n})} = V_{F(x)_n}$, so $\psi(x) \in V_{F(x)}$ and $\varphi(F(x)) = \psi(x)$. $\square$

$$\begin{tikzcd}
A_{m_{n+1}} \arrow[r, "f_{n+1}"] \arrow[d, "h_{[m_n, m_{n+1})}"'] & A_{n+1} \arrow[d, "h_n"] \\
A_{m_n} \arrow[r, "f_n"] & A_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{2}^{\mathbb{N}} \arrow[r, "F"] \arrow[d, "\varphi"'] & \mathbf{2}^{\mathbb{N}} \arrow[d, "\varphi"] \\
Y \arrow[r, "G"] & Y
\end{tikzcd}$$

***Corollary 3.9* (Balcar and Simon [10]).** — *Any dynamical system is a factor of some dynamical system on the Cantor space.*

Proof. — Let (Y, G) be a dynamical system and $\varphi : \mathbf{2}^{\mathbb{N}} \to Y$ a map constructed in Theorem 3.8. For $\psi = G \circ \varphi : \mathbf{2}^{\mathbb{N}} \to Y$ there exists $F : \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ with $\varphi \circ F = G \circ \varphi$. Thus $\varphi : (\mathbf{2}^{\mathbb{N}}, F) \to (Y, G)$ is a factor map. $\square$

Example 3.10. — The linear system $(\mathbb{I}, L_{\frac{1}{2}})$ is a factor of $(\mathbf{2}^{\mathbb{N}}, F)$, where $F(x) = 0x$, with the factor map $\varphi(x) = \sum_{i \ge 0} x_i 2^{-i-1}$.

When we search for a symbolic extension of a given dynamical system, we usually first look into subshifts. This is, however, not always possible. A subshift has a finite topological entropy, so its factors have finite topological entropy too. Thus the system $(\mathbf{2}^{\mathbb{N}}, F)$ with $F(x)_i = x_{2i}$ (Example 2.87) is not a factor of any subshift. Boyle has constructed a dynamical system with finite entropy, which is not factor of any subshift. He defines residual entropy of a dynamical system as

$$h_{res}(X,F) = \inf\{h(\Sigma) - h(X,F) : \Sigma \text{ is a subshift extension of } (X,F)\}.$$

The residual entropy is infinite if there is no subshift which factors to (X, F). Downarowicz [**50**] constructs examples of dynamical systems with arbitrarily preset topological and residual entropies.

3.3. Subshifts

Subshifts have been introduced by Smale [**146**]. The simplest class of subshifts of finite type (SFT) were introduced by Parry [**121**]. Their generalization, sofic subshifts have been introduced by Weiss [**162**]. Many subshifts are constructed as orbit closures. For example, the Feigenbaum subshift of Example 1.30 is the orbit closure of the Feigenbaum sequence.

Definition 3.11. —

(1) A full shift is any dynamical system $(A^{\mathbb{N}}, \sigma)$, where A is an alphabet and σ is the shift map defined by $\sigma(x)_i = x_{i+1}$.
(2) A subshift is any subsystem of a full shift, i.e., a closed σ-invariant subset $\Sigma \subseteq A^{\mathbb{N}}$.

A subshift can be specified by its set of forbidden words which cannot occur in the points of the subshifts. Recall that $u \sqsubseteq v$, where $u \in A^*$ and $v \in \overline{A^*}$, means that u is a subword of v.

Proposition 3.12. — *Let $F \subseteq A^*$ be any set of words and set*

$$\Sigma_F = \{x \in A^{\mathbb{N}} : \forall u \in A^*, u \sqsubseteq x \;\Rightarrow\; u \notin F\}$$

If Σ_F is nonempty, then it is a subshift.

Proof. — If $x \in \Sigma_F$, then for any $0 \le i \le j$, $\sigma(x)_{[i,j]} = x_{[i+1,j+1]} \notin F$, so $\sigma(x) \in \Sigma_F$ and Σ_F is invariant. If $x \notin \Sigma_F$, then there exist $0 \le i \le j$ such that $x_{[i,j]} \in F$ and for any $y \in [x_{[0,j]}]$, $y \notin \Sigma_F$. Thus $[x_{[0,j]}] \subseteq A^{\mathbb{N}} \setminus \Sigma_F$ and Σ_F is closed. □

***Example 3.13* (golden mean subshift)**. — $\Sigma_{\{11\}} \subseteq \mathbf{2}^{\mathbb{N}}$

***Example 3.14* (even subshift)**. — (Σ_F, σ) where $F = \{01^{2n+1}0 : n \ge 0\}$

A subshift can be also finite. For example $\Sigma_{\{00,10\}} = \{01^\infty, 1^\infty\}$ contains only two elements.

3.3.1. Extendable languages. — By **language** we mean subset $L \subseteq A^*$, where A is an alphabet. A subshift can be characterized by the language of words which occur in its sequences. For example, the languages of the golden mean and even subshifts consist of words

$$\begin{aligned} \mathcal{L}(\Sigma_{\{11\}}) &= \{\lambda, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \ldots\}, \\ \mathcal{L}(\Sigma_F) &= \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 011, 100, 101, 110, 111, \ldots\}. \end{aligned}$$

Definition 3.15. — A nonempty language $L \subseteq A^*$ is **extendable**, if
(1) for any $u \in L$ and any $v \sqsubseteq u$, $v \in L$,
(2) for any $u \in L$ there exists $a \in A$ such that $ua \in L$.

There is a one-to-one correspondence between subshifts and extendable languages.

Definition 3.16. —
(1) The **subshift of an extendable language** $L \subseteq A^*$ is

$$\mathcal{S}(L) = \{x \in A^{\mathbb{N}} : \forall n \geq 0, x_{[0,n)} \in L\}.$$

(2) The **language of a subshift** $\Sigma \subseteq A^{\mathbb{N}}$ is

$$\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}, \quad \mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n.$$

(3) The **language of a sequence** $x \in A^{\mathbb{N}}$ is

$$\mathcal{L}(x) = \{u \in A^* : u \sqsubseteq x\}, \quad \mathcal{L}^n(x) = \mathcal{L}(x) \cap A^n.$$

The language of a subshift $\Sigma \subseteq A^{\mathbb{N}}$ need not contain all letters of A. If $B = \mathcal{L}^1(\Sigma)$, then $\Sigma \subseteq B^{\mathbb{N}}$. The set B need not even be an alphabet, it may contain only one letter as is the case with the subshift $\{0^\infty\}$.

Proposition 3.17 **(Blanchard and Hansel [20]).** —
(1) If $L \subseteq A^*$ is an extendable language, then $\mathcal{S}(L)$ is a subshift and $\mathcal{L}(\mathcal{S}(L)) = L$.
(2) If $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift, then $\mathcal{L}(\Sigma)$ is an extendable language and $\mathcal{S}(\mathcal{L}(\Sigma)) = \Sigma$.
(3) If $x \in A^{\mathbb{N}}$ then $\mathcal{L}(x)$ is an extendable language and $\mathcal{S}(\mathcal{L}(x)) = \overline{\{x\} \cup \mathcal{O}(x)}$.
(4) If $x \in A^{\mathbb{N}}$, then $\overline{\mathcal{O}(x)} = \{y \in A^{\mathbb{N}} : \forall n \geq 0, \exists i > 0, y_{[0,n)} = x_{[i,i+n)}\}$.

Proof. —
(1) Let $L \subseteq A^*$ be an extendable language. For $n > 0$ set

$$X_n = \{x \in A^{\mathbb{N}} : x_{[0,n)} \in L\}, \text{ so } \mathcal{S}(L) = \bigcap_{n>0} X_n.$$

Since L contains words of any length, X_n is nonempty. Since X_n is a finite union of cylinders, it is closed. Since $X_{n+1} \subseteq X_n$, their intersection $\mathcal{S}(L)$ is nonempty and closed. Clearly, $\mathcal{S}(L)$ is invariant, so it is a subshift. If $u \in L$, $|u| = n$, then there exists $u_n \in A$ such that $u_{[0,n]} \in L$. Repeating this infinitely many times we extend u to a point $x \in A^{\mathbb{N}}$ such that for any m, $x_{[0,m)} \in L$. Thus $x \in \mathcal{S}(L)$ and $u \in \mathcal{L}(\mathcal{S}(L))$, so $L \subseteq \mathcal{L}(\mathcal{S}(L))$. If $u \in \mathcal{L}(\mathcal{S}(L))$, then there exists $x \in \mathcal{S}(L)$ with $u = x_{[i,j)}$ for some $i < j$. Since $x_{[0,j)} \in L$ and u is its subword, $u \in L$. Thus $\mathcal{L}(\mathcal{S}(L)) \subseteq L$.

(2) If $u \sqsubseteq x \in \Sigma$ and $v \sqsubseteq u$, then $v \sqsubseteq x$. If $u = x_{[i,i+|u|)}$, then $ux_{i+|u|} \in \mathcal{L}(x)$, so $\mathcal{L}(\Sigma)$ is extendable. If $x \in \Sigma$, then for any n, $x_{[0,n)} \in \mathcal{L}(\Sigma)$, so $x \in \mathcal{S}(\mathcal{L}(\Sigma))$. Thus $\Sigma \subseteq \mathcal{S}(\mathcal{L}(\Sigma))$. Suppose that $x \in \mathcal{S}(\mathcal{L}(\Sigma))$ and $x \notin \Sigma$. Since $A^{\mathbb{N}} \setminus \Sigma$ is open, there

exists n such that $[x_{[0,n)}] \subseteq A^{\mathbb{N}} \setminus \Sigma$. Since $x \in \mathcal{S}(\mathcal{L}(\Sigma))$, $x_{[0,n)} \in \mathcal{L}(\Sigma)$ and there exists $y \in \Sigma$ such that $y_{[j,j+n)} = x_{[0,n)}$. Thus $\sigma^j(y) \in [x_{[0,n)}]$ and this is a contradiction. Thus $\mathcal{S}(\mathcal{L}(\Sigma)) \subseteq \Sigma$.

(3),(4) Clearly $\mathcal{L}(x)$ is an extendable language. For $y \in A^{\mathbb{N}}$

$$\begin{aligned}
y \notin \mathcal{S}(\mathcal{L}(x)) &\Leftrightarrow \exists n > 0, y_{[0,n)} \notin \mathcal{L}(x) \Leftrightarrow \exists n > 0, \forall j \geq 0, y_{[0,n)} \neq x_{[j,j+n)} \\
&\Leftrightarrow \exists n > 0, d(y, \{x\} \cup \mathcal{O}(x)) \geq 2^{-n} \Leftrightarrow y \notin \overline{\{x\} \cup \mathcal{O}(x)}, \\
y \in \overline{\mathcal{O}(x)} &\Leftrightarrow \forall n \geq 0, d(y, \mathcal{O}(x)) \leq 2^{-n} \\
&\Leftrightarrow \forall n \geq 0, \exists i > 0, d(y, \sigma^i(x)) \leq 2^{-n} \\
&\Leftrightarrow \forall n \geq 0, \exists i > 0, y_{[0,n)} = x_{[i,i+n)}.
\end{aligned}$$

□

In particular, if $x \in A^{\mathbb{N}}$ is a recurrent point, then $\mathcal{S}(\mathcal{L}(x)) = \overline{\mathcal{O}(x)}$ is the closure of its orbit. For example, the Champernowne sequence

$$x = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111 \ldots$$

is recurrent and $\overline{\mathcal{O}(x)} = \mathbf{2}^{\mathbb{N}}$. If a point $x \in A^{\mathbb{N}}$ is almost periodic, then $\overline{\mathcal{O}(x)}$ is a minimal subshift. For example the Feigenbaum subshift is minimal since it is the orbit closure of an almost periodic sequence.

3.3.2. Covers and Partitions. — We have seen that any dynamical system is a factor of a symbolic dynamical system. In many cases, this symbolic system can be constructed as the subshift of itineraries of some finite cover. A **finite cover** of a space X is a system $\mathcal{V} = \{V_a : a \in A\}$ indexed by an alphabet A which satisfies $\bigcup_{a \in A} V_a = X$. If $V_a \cap V_b = \emptyset$ for $a \neq b$ we say that $\mathcal{V}$ is a **partition**. (There is a slight disaccord with Definition A.23, where a cover is thought of as a set of subsets of X.) In Section 1.9 we have used finite closed covers, but in general we do not make any assumptions about the topological properties of these covers. Howevewr, we must change the definition of $\Sigma_{\mathcal{V}}(X,F)$. We form the n-th power $\mathcal{V}^n$ of $\mathcal{V}$, which contains all nonempty intersections $V_{u_0} \cap F^{-1}(V_{u_1}) \cap \cdots \cap F^{-(n-1)}(V_{u_{n-1}})$.

Definition 3.18. — Let (X,F) be a dynamical system and $\mathcal{V} = \{V_a : a \in A\}$ a finite cover of X.

(1) The set of points with itinerary $v \in A^* \cup A^{\mathbb{N}}$ is
$$V_v = \{x \in X : \forall i < |v| : F^i(x) \in V_{v_i}\} = \bigcap_{i<|v|} F^{-i}(V_{v_i})$$
(2) The n-th power of $\mathcal{V}$ (Definition 2.80) is
$$\mathcal{V}^n = \{V_v : v \in A^n, V_v \neq \emptyset\} = \mathcal{V} \vee F^{-1}(\mathcal{V}) \vee \cdots \vee F^{-(n-1)}(\mathcal{V})$$
(3) The language and subshift of $\mathcal{V}$ are
$$\mathcal{L}_{\mathcal{V}}(X,F) = \{v \in A^* : V_v \neq \emptyset\}, \quad \Sigma_{\mathcal{V}}(X,F) = \{z \in A^{\mathbb{N}} : \forall n > 0, V_{z_{[0,n)}} \neq \emptyset\}$$
(4) $\mathcal{V}$ is generating if $\operatorname{diam}(\mathcal{V}^n) \to 0$ as $n \to \infty$.
(5) If $\mathcal{V}$ is a partition, the itinerary map $\mathcal{I} : X \to \Sigma_{\mathcal{V}}(X,F)$ is defined by
$$\mathcal{I}(x)_n = a \Leftrightarrow F^n(x) \in V_a.$$

It is immediate that $\Sigma_{\mathcal{V}}(X,F)$ is a subshift with language $\mathcal{L}_{\mathcal{V}}(X,F)$. If $\mathcal{V}$ is a partition, then the itinerary map satisfies $\mathcal{I}\circ F=\sigma\circ\mathcal{I}$, but $\mathcal{I}$ is not continuous unless $\mathcal{V}$ is a clopen partition.

Proposition 3.19. — *Let (X,F) be a dynamical system and $\mathcal{V}=\{V_a : a\in A\}$ a finite cover of X.*

(1) If $\mathcal{V}$ is generating, then there is a factor map $\varphi:(\Sigma_{\mathcal{V}}(X,F),\sigma)\to(X,F)$ which satisfies

$$z\in\Sigma_{\mathcal{V}}(X,F)\ \Rightarrow\ \varphi(z)\in\bigcap_{n>0}\overline{V_{z_{[0,n)}}},\quad u\in\mathcal{L}_{\mathcal{V}}(X,F)\ \Rightarrow\ \varphi([u])=\overline{V_u}$$

(2) If $\mathcal{V}$ is a clopen partition, then $\mathcal{I}:(X,F)\to(\Sigma_{\mathcal{V}}(X,F),\sigma)$ is a factor map.

(3) If $\mathcal{V}$ is a clopen generating partition, then $\varphi=\mathcal{I}^{-1}$ is a conjugacy.

$$\begin{tikzcd}
\Sigma_{\mathcal{V}}(X,F) \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma_{\mathcal{V}}(X,F) \arrow[d, "\varphi"] \\
X \arrow[r, "F"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\mathcal{I}"'] & X \arrow[d, "\mathcal{I}"] \\
\Sigma_{\mathcal{V}}(X,F) \arrow[r, "\sigma"] & \Sigma_{\mathcal{V}}(X,F)
\end{tikzcd}$$

Proof. —

(1) If $z\in\Sigma_{\mathcal{V}}(X,F)$, then $\lim_{n\to\infty}\operatorname{diam}(\overline{V_{z_{[0,n)}}})=\lim_{n\to\infty}\operatorname{diam}(V_{z_{[0,n)}})=0$. The intersection $\bigcap_{n>0}\overline{V_{z_{[0,n)}}}$ is nonempty by compactness and has diameter 0, so it contains a unique point $\varphi(z)$. We show that φ is continuous. Given $\varepsilon>0$ there exists n such that $\operatorname{diam}(\mathcal{V}^n)<\varepsilon$. If $w,z\in\Sigma_{\mathcal{V}}(X,F)$ and $d(w,z)\le 2^{-n}$, then $w_{[0,n)}=z_{[0,n)}=u\in\mathcal{L}_{\mathcal{V}}(X,F)$ and $\varphi(w),\varphi(z)\in\overline{V_u}$. Since $\operatorname{diam}(V_u)<\varepsilon$, it follows that $d(\varphi(w),\varphi(z))\le\varepsilon$. We show that $\varphi([u])=\overline{V_u}$. If $w\in[u]$, then $\varphi(w)\in\overline{V_u}$, so $\varphi([u])\subseteq\overline{V_u}$. If $x\in V_u$, then there exists an itinerary w of x with $w_{[0,|u|)}=u$. For $i>|u|$ simply pick any w_i for which $F^i(x)\in V_{v_i}$. Then $w\in\Sigma_{\mathcal{V}}(X,F)$ and $x=\varphi(w)$, so $x\in\varphi([u])$. We have shown $V_u\subseteq\varphi([u])$. Since φ is continuous, $\overline{V_u}\subseteq\varphi([u])$, so $\varphi([u])=\overline{V_u}$. We now show that $\varphi:(\Sigma_{\mathcal{V}}(X,F),\sigma)\to(X,F)$ is a factor map. If $z\in\Sigma_{\mathcal{V}}(X,F)$, then for any n, $\varphi(z)\in\overline{V_{z_{[0,n)}}}$, so

$$F(\varphi(z))\in F(\overline{V_{z_{[0,n)}}})\subseteq\overline{F(V_{z_{[0,n)}})}=\overline{V_{\sigma(z)_{[0,n)}}}$$

and $F(\varphi(z))=\varphi(\sigma(z))$.

(2) If $\mathcal{V}$ is a clopen partition, then the function $\alpha:X\to A$ defined by $\alpha^{-1}(a)=V_a$ is continuous. The continuous maps $\alpha\circ F^n:X\to A$ yield a unique continuous map $\mathcal{I}:X\to A^{\mathbb{N}}$ with $\pi_n\circ\mathcal{I}=\alpha\circ F^n$. Then $\Sigma_{\mathcal{V}}(X,F)=\mathcal{I}(X)$ and $\mathcal{I}:(X,F)\to(\Sigma_{\mathcal{V}}(X,F),\sigma)$ is a factor map. □

Let Σ be a subshift over $A=\mathcal{L}^1(\Sigma)$. Recall that the **cylinder** of a word $u\in\mathcal{L}(\Sigma)$ is

$$[u]=\{w\in\Sigma : w_{[0,|u|)}=u\}.$$

The **canonical clopen partition** of Σ consists of cylinders of length 1. Its n-th power consists of cylinders of length n.

$$\mathcal{P}=\{[a]:a\in A\},\quad \mathcal{P}^n=\{[u]:u\in\mathcal{L}(\Sigma)\}.$$

We have $\operatorname{diam}(\mathcal{P}^n) = 2^{-n}$, so $\mathcal{P}$ is generating. The language of Σ is simply its language with respect to $\mathcal{P}$

$$\mathcal{L}(\Sigma) = \mathcal{L}_{\mathcal{P}}(\Sigma, \sigma) = \{u \in A^* : [u] \neq \emptyset\}.$$

A dynamical system (X, F) is **positively expansive** (Definition 2.39), if

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists n \geq 0, d(F^n(x), F^n(y)) \geq \varepsilon.$$

The number ε is called the **expansivity constant**. Any subshift $\Sigma \subseteq A^{\mathbb{N}}$ is positively expansive with the expansivity constant $\varepsilon = 1$. If $x \neq y$, then for some n, $x_n \neq y_n$, so $\sigma^n(x)_0 \neq \sigma^n(y)_0$ and $d(\sigma^n(x), \sigma^n(y)) = 1$. Conversely, any positively expansive system is a factor of a subshift. Positively expansive systems on symbolic spaces are even conjugate to subshifts.

Proposition 3.20. — *Any positively expansive system has a generating cover.*

Proof. — Let $\varepsilon > 0$ be an expansivity constant. The open cover $\{B_{\frac{\varepsilon}{2}}(x) : x \in X\}$ has a finite subcover $\{B_{\frac{\varepsilon}{2}}(x_0), \ldots, B_{\frac{\varepsilon}{2}}(x_{n-1})\}$. Set $V_i = B_{\frac{\varepsilon}{2}}(x_i)$ and $A = \{0, \ldots, n-1\}$. Then $\mathcal{V} = \{V_i : i \in A\}$ is an open cover with diameter $\operatorname{diam}(\mathcal{V}) < \varepsilon$. Suppose that $\mathcal{V}$ is not generating. Since $\operatorname{diam}(\mathcal{V}^{n+1}) \leq \operatorname{diam}(\mathcal{V}^n)$, there exists a limit $a = \lim_{n\to\infty} \operatorname{diam}(\mathcal{V}^n) > 0$. By compactness there exists some $u \in A^{\mathbb{N}}$, such that V_u contains two distinct $x, y \in X$. Then for any $n \geq 0$ there exists i with $F^n(x), F^n(y) \in V_i$, so $d(F^n(x), F^n(y)) < \varepsilon$. This is a contradiction. ☐

Proposition 3.21. — *A dynamical system is conjugate to a subshift iff it is positively expansive and its state space is totally disconnected.*

Proof. — If Σ is a subshift then it is totally disconnected. If $x, y \in \Sigma$ and $x \neq y$, then $x_n \neq y_n$ for some $n \geq 0$ and $d(\sigma^n(x), \sigma^n(y)) = 1$, so (Σ, σ) is positively expansive. Conversely let (X, F) be positively expansive and let X be totally disconnected. Since clopen sets form a base of the space, there exists a generating clopen partition $\mathcal{V}$ of X. By Proposition 3.19, $\mathcal{I} : (X, F) \to (\Sigma_{\mathcal{V}}(X, F), \sigma)$ is a conjugacy. ☐

3.4. Topological entropy

Topological entropy of a subshift can be computed from its complexity function which gives for any $n \in \mathbb{N}$ the number of words of length n.

Definition 3.22. — The **complexity function** $P : \mathbb{N} \to \mathbb{N}$ of a subshift $\Sigma \subseteq A^{\mathbb{N}}$ is given by

$$P(n) = \#\mathcal{L}^n(\Sigma) = \#\{u \in A^n : \exists x \in \Sigma, u \sqsubseteq x\}.$$

In a finite subshift $\Sigma_{\{00,10\}} = \{01^\infty, 1^\infty\}$ we have $P(n) = 2$.

Proposition 3.23. — *If $\Sigma \subseteq A^{\mathbb{N}}$ is an infinite subshift, then its complexity function satisfies $P(n) \geq n + 1$.*

Proof. — Since Σ is infinite, its alphabet has at least two letters and $P(1) \geq 2$. Assume that for some $n > 0$, $P(n+1) = P(n)$. For any $u \in \mathcal{L}^n(\Sigma)$ there exists $a \in A$ such that $ua \in \mathcal{L}(\Sigma)$, and since $P(n+1) = P(n)$, this a is unique. If $m \geq n$ and $u \in \mathcal{L}^m(\Sigma)$, then its suffix $u_{[m-n,m-1]}$ has a unique extension, so u has a unique extension. It follows that for any $u \in \mathcal{L}^n(\Sigma)$ there exists a unique $x \in \Sigma$ with prefix u, so Σ has $P(n)$ elements and this is a contradiction. Thus $P(n+1) > P(n)$ for any n. Since $P(1) \geq 2$, we get $P(n) \geq n+1$. □

Proposition 3.24. — *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift with complexity function $P(n) = \#\mathcal{L}^n(\Sigma)$. Then*

$$h(\Sigma, \sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{n}.$$

Proof. — The canonical clopen partition $\mathcal{P} = \{[a] : a \in A\}$ is generating since $\mathcal{P}^n = \mathcal{P}_n = \{[u] : u \in \mathcal{L}^n(\Sigma)\}$ and $\operatorname{diam}(\mathcal{P}^n) = 2^{-n}$. Since $|\mathcal{P}^n| = P(n)$, we get the result. □

The complexity function of the full shift $(A^{\mathbb{N}}, \sigma)$ is $P(n) = (\#A)^n$, so $h(A^{\mathbb{N}}, \sigma) = \ln \#A$. For any subshift $\Sigma \subseteq A^{\mathbb{N}}$ we have $h(\Sigma, \sigma) \leq \ln \#A$.

In the examples which we have studied, minimal systems have zero topological entropy while chaotic systems have positive topological entropy. This is not, however, a general rule. In Theorem 4.77 we construct a minimal system with arbitrarily large (finite) topological entropy. Conversely, we have the following example.

***Example 3.25* (Glasner and Weiss [65]).** — There exists a chaotic system with zero topological entropy (and infinite polynomial entropy - see Section 2.11).

Proof. — Construct sets $A_i \subseteq \mathbf{2}^{3^i}$ such that

$$A_0 = \mathbf{2}, \quad A_{i+1} = \{uuv : u, v \in A_i\}.$$

The general form of words in the sets A_i is

$i = 1$: aab
$i = 2$: aabaabccd
$i = 3$: aabaabccdaabaabccdeefeefggh

where $a, b, \ldots \in \mathbf{2}$. The language $L = \{u \in \mathbf{2}^* : \exists i, \exists v \in A_i, u \sqsubseteq v\}$ is extendable and determines a subshift $\Sigma = \mathcal{S}(L)$.

If $U, V \subseteq \Sigma$ are nonempty open sets, then there exist words $u, v \in \mathbf{2}^*$ such that $[u] \subseteq U$, $[v] \subseteq V$. There exist i and $w, y \in A_i$, such that $u \sqsubseteq w$ and $v \sqsubseteq y$. Since $wwy \in A_{i+1}$, there exists $x \in \Sigma$ which has prefix u and contains v, so Σ is transitive. Since $w^\infty \in \Sigma$, there exists a periodic point in U, so Σ has a dense set of periodic points. Since Σ is infinite, it is chaotic.

We evaluate the topological entropy of Σ. We show $\#A_i = 2^{2^i}$. Indeed $\#A_0 = 2$, and by induction,

$$\#A_{i+1} = (\#A_i)^2 = 2^{2^i} \cdot 2^{2^i} = 2^{2^{i+1}}.$$

If $3^{i-1} < n \leq 3^i$ and $|u| = n$, then there exist $v, w \in A_i$ and $k \leq 3^i$ such that $u = (vw)_{[k,k+|u|)}$. We have 2^{2^i} possibilities for u and v and 3^i possibilities for k, so $P(n) \leq 3^i \cdot 2^{2^i}$. It follows

$$h(\Sigma, \sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{n} \leq \lim_{i\to\infty} \frac{i \cdot \ln 3 + 2^i \cdot \ln 2}{3^{i-1}} = 0.$$

Since $P(3^i) \geq \#A_i = 2^{2^i}$, we have

$$h_p(\Sigma, \sigma) \geq \liminf_{i\to\infty} \frac{\ln P(3^i)}{\ln 3^i} \geq \lim_{i\to\infty} \frac{2^i \cdot \ln 2}{i \cdot \ln 3} = \infty.$$

□

3.4.1. Monotone characteristics. — The monotone characteristics introduced in Section 2.11 also apply to subshifts. Recall that if (X, F) is a dynamical system and $U, V \subseteq X$, then the transition time (Definition 2.94) from U to V is

$$\tau(U, V) = \min\{n > 0 : F^n(U) \cap V \neq \emptyset\}.$$

Proposition 3.26. — *Let (X, F) be a dynamical system, let $\mathcal{V}$ be a partition of X and $\Sigma_{\mathcal{V}}(X, F)$ its subshift. Then for $u, v \in \mathcal{L}_{\mathcal{V}}(X, F)$, $\tau([u],[v]) = \tau(V_u, V_v)$.*

Proof. — Assume that $z \in [u] \cap \sigma^{-n}([v])$ and set $m = \max\{|u|, |v| + n\}$. For $w = z_{[0,m)}$ there exists $x \in V_w$, so $x \in V_u \cap F^{-n}(V_v)$. Conversely, if $x \in V_u \cap F^{-n}(V_v)$, then there exists $w \in \Sigma_{\mathcal{V}}(X, F)$ with $x \in V_w$, $w_{[0,|u|)} = u$ and $w_{[n,n+|v|)} = v$, so $[u] \cap \sigma^{-1}([v]) \neq \emptyset$. We have proved that $[u] \cap \sigma^{-n}([v]) = \emptyset$ iff $V_u \cap F^{-n}(V_v) = \emptyset$, so $\tau_\sigma([u],[v]) = \tau_F(V_u, V_v)$. □

Definition 2.96 yields the following integer functions.

Definition 3.27. — Let $\Sigma \subseteq A^{\mathbb{N}}$ and $\mathcal{P} = \{[a] : a \in A\}$ be the canonical clopen partition. Set

$$\begin{array}{rcll} Q(n) &=& Q(\mathcal{P}^n) = \min\{\tau([u]) : u \in \mathcal{L}^n(\Sigma)\} & \text{return time} \\ R(n) &=& R(\mathcal{P}^n) = \max\{\tau([u]) : u \in \mathcal{L}^n(\Sigma)\} & \text{recurrence time} \\ S(n) &=& S(\mathcal{P}^n) = \max\{\tau([u],[v]) : u, v \in \mathcal{L}^n(\Sigma)\} & \text{transition time} \\ T(n) &=& T(\mathcal{P}^n) = \max\{\tau(x,[u]) : x \in \Sigma, u \in \mathcal{L}^n(\Sigma)\} & \text{running time} \end{array}$$

By Proposition 2.97 all these functions are nondecreasing and

$$Q(n) \leq R(n) \leq S(n) \leq T(n), \quad Q(n) \leq P(n) \leq T(n).$$

These characteristics may be distinct from each other. For example in the Feigenbaum subshift we have

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P	2	3	5	6	8	10	11	12	14	16	18	20	21	22	23	24	26
Q	1	1	2	2	2	4	4	4	4	4	4	8	8	8	8	8	8
R	2	2	4	4	8	8	8	8	16	16	16	16	16	16	16	16	32
S	2	2	4	5	8	9	10	11	16	17	18	19	20	21	22	23	32
T	4	7	14	14	28	28	28	28	56	56	56	56	56	56	56	56	112

Proposition 3.28. — *If (Σ, σ) is surjective, then $T(n) = T_0(n)$, where*

$$T_0(n) = \min\{m > 0 : \forall v \in \mathcal{L}^{m+n-1}(\Sigma), \forall u \in \mathcal{L}^n(\Sigma), u \sqsubseteq v\}$$

Proof. — If $m < T(n)$, then there exists $x \in \Sigma$ and $u \in \mathcal{L}^n(\Sigma)$, such that $\tau(x,[u]) > m$, so the word $v = x_{[1,m+n-1]}$ of length $m+n-1$ does not contain u as a subword, and therefore $m < T_0(n)$. Conversely if $m < T_0(n)$, then there exists $v \in \mathcal{L}^{m+n-1}(\Sigma)$ and $u \in \mathcal{L}^n(\Sigma)$, such that $u \not\sqsubseteq v$. Since (Σ,σ) is surjective, there exists $x \in \Sigma$ with $\sigma(x) \in [v]$. This means $\tau(x,[u]) > m$, so $m < T(n)$. We have proved that for any m, $m < T(n)$ iff $m < T_0(m)$, so $T(n) = T_0(n)$. □

Proposition 3.29. — *If Σ is infinite, then $T(Q(n)) > n$ and $Q(T(n)) > n$.*

Proof. — In infinite subshifts, $P(n) > n$ by Proposition 3.23. Assume first that $Q(n) = m < n$. Then there exists $u \in \mathcal{L}^n(\Sigma)$ such that $\tau(u) = m$ and $w \in \mathcal{L}^{n+m}(\Sigma)$ with $w_{[0,n)} = w_{[m,m+n)} = u$. If $T(m) \le n$, then $w_{[0,m+n-1)}$ contains all words of length m. Since $w_{[i,i+m)} = w_{[i+m,i+2m)}$ for $0 \le i \le n-m$, we get $P(m) \le m$ and this is a contradiction, so $T(Q(n)) = T(m) > n$. If $Q(n) \ge n$, then $T(Q(n)) \ge T(n) \ge P(n) > n$, so in both cases $T(Q(n)) > n$. Assume that $Q(T(n)) \le n$. Then $T(Q(T(n))) \le T(n)$ and this contradicts the preceding result for $m = T(n)$. □

Proposition 3.30. — *If Σ is an infinite minimal subshift, then $R(n) > n$ for an infinite number of n.*

Proof. — Assume that there exists n_0 such that for all $n \ge n_0$, $R(n) \le n$. We construct a periodic point in Σ. Pick a word $u^{(0)} \in \mathcal{L}^{n_0}(\Sigma)$. We proceed recursively. Given a word $u^{(i)} \in \mathcal{L}^{n_i}(\Sigma)$, set $n_{i+1} = n_i + \tau(u^{(i)}) < 2n_i$, $a_i = n_i - \tau(u^{(i)}) \ge 0$. There exists a word $u^{(i+1)} \in \mathcal{L}^{n_{i+1}}(\Sigma)$, such that $u^{(i)}$ is both its prefix and suffix and these two occurrences of $u^{(i)}$ in $u^{(i+1)}$ overlap by a_i. Now $u^{(i+2)}$ has four occurrences of $u^{(i)}$ and the second and third overlap by a_{i+1}. It follows $a_{i+1} \le a_i$, so the sequence (a_i) is eventually constant. There exists $j \ge 0$ such that for any $i \ge j$, $a_i = a_j$. Since $u^{(i)}$ is a prefix of $u^{(i+1)}$, there exists a limit $x = \lim_{i\to\infty} u^{(i)}$ and x is then a periodic point with period $\tau(u^{(j)})$. □

3.5. Subshifts of finite type

Definition 3.31. — A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is a **subshift of finite type** (**SFT**) if there exists a finite set $F \subseteq A^*$ of forbidden words such that

$$\Sigma = \Sigma_F = \{x \in A^{\mathbb{N}} : \forall u \sqsubseteq x, u \notin F\}.$$

An SFT Σ has **order** $p \ge 2$, if there exists $F \subseteq A^p$ such that $\Sigma = \Sigma_F$. Subshifts of order 2 are called **Markov subshifts**.

Thus the golden mean subshift $\Sigma_{\{11\}}$ of Example 3.13 is a Markov subshift. Further examples of Markov subshifts are

$$\begin{aligned}
\Sigma_{\{10\}} &= \{0^n1^\infty : n \geq 0\} \cup \{0^\infty\} \\
\Sigma_{\{01,10\}} &= \{0^\infty, 1^\infty\} \\
\Sigma_{\{00,10\}} &= \{01^\infty, 1^\infty\} \\
\Sigma_{\{00,11\}} &= \{(01)^\infty, (10)^\infty\} \\
\Sigma_{\{01,10,11\}} &= \{0^\infty\}.
\end{aligned}$$

To forbid a word $u \in A^*$ is equivalent to forbidding words ua for all $a \in A$. Thus any SFT has an order. For example $\Sigma_{\{00,111\}} = \Sigma_{\{000,001,111\}}$ is an SFT of order 3.

Proposition 3.32. — *Any SFT has an order. If $\Sigma \subseteq A^{\mathbb{N}}$ is an SFT of order p, then*

$$\begin{aligned}
x \in \Sigma &\Leftrightarrow \forall n \geq 0, x_{[n,n+p)} \in \mathcal{L}(\Sigma) \\
u \in \mathcal{L}(\Sigma) &\Leftrightarrow \forall n \leq |u| - p, u_{[n,n+p)} \in \mathcal{L}(\Sigma)
\end{aligned}$$

The even subshift of Example 3.14 is not an SFT. If it were, and if its order were $p > 1$, then $x = 10^{2p+1}1^\infty$ would belong to it, because all subwords of x of length p belong to the language.

3.5.1. Shadowing property. — There are two distinct topological characterization of SFT. Recall that a dynamical system (X, F) has the shadowing property (Definition 2.11), if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-chain is ε-shadowed by some point. If $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift, a sequence $(x_i \in \Sigma)_{i \geq 0}$ is a 2^{-n}-chain, if $d(x_{i+1}, \sigma(x_i)) < 2^{-n}$, or $(x_{i+1})_{[0,n]} = \sigma(x_i)_{[0,n]} = (x_i)_{[1,n+1]}$. In Figure 4, we see 2^{-2}-chains in the golden mean subshift of Example 3.13 (left) and the even subshift of Example 3.14 (right). We obtain them by applying the shift σ and changing the digits arbitrarily behind the bars. We see that the 2^{-2}-chain in the golden mean subshift is 1-shadowed (and also 2^{-3}-shadowed) by the point $00101\ldots$ which appears in the first column. In contrast, the 2^{-2}-chain in the even subshift is not 1-shadowed by any point, since the sequence $01110\ldots$ of the first column does not belong to the subshift.

$$\begin{array}{r|l}
001 & 0000\ldots \\
010 & 1000\ldots \\
101 & 0000\ldots \\
010 & 0000\ldots \\
100 & 1000\ldots
\end{array}
\qquad\qquad
\begin{array}{r|l}
011 & 1100\ldots \\
111 & 0000\ldots \\
110 & 0000\ldots \\
100 & 0000\ldots \\
000 & 1000\ldots
\end{array}$$

FIGURE 4. ε-chains in golden mean and even subshifts

Theorem 3.33 **(Walters [158]).** — *A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is an SFT iff (Σ, σ) has the shadowing property.*

Proof. —

(1) Let Σ be an SFT of order $p > 1$. Let $\varepsilon > 0$ and pick $m \geq p$ with $2^{-m} < \varepsilon$. We show that any 2^{-m+2}-chain is ε-shadowed by some point. Let $(x_n \in \Sigma)_{n\geq 0}$ be a 2^{-m+2}-chain, so $d(\sigma(x_n), x_{n+1}) < 2^{-(m-2)}$ and

$$(x_{n+1})_{[0,m-2]} = \sigma(x_n)_{[0,m-2]} = (x_n)_{[1,m-1]}.$$

We define $y \in A^{\mathbb{N}}$ by $y_n = (x_n)_0$. For $0 \leq i < m$ we have

$$y_{n+i} = (x_{n+i})_0 = (x_{n+i-1})_1 = \cdots = (x_n)_i,$$

so $y_{[n,n+m-1]} = (x_n)_{[0,m-1]} \in \mathcal{L}(\Sigma)$. Thus $y \in \Sigma$ and $d(\sigma^n(y), x_n) \leq 2^{-m} < \varepsilon$, so y ε-shadows $(x_n)_{n\geq 0}$.

(2) Suppose that Σ is a subshift with the shadowing property. For $\varepsilon = 1$ there exists $\delta > 0$, such that any δ-chain is 1-shadowed by some point. Let $p > 1$ be such that $2^{-p+1} < \delta$. We show that Σ is an SFT of order p. Let $y \in A^{\mathbb{N}}$, and suppose that for any $n \in \mathbb{N}$, $y_{[n,n+p)} \in \mathcal{L}(\Sigma)$. There exist points $x_n \in \Sigma$, such that $(x_n)_{[0,p)} = y_{[n,n+p)}$. Since

$$(x_{n+1})_{[0,p-2]} = y_{[n+1,n+p-1]} = (x_n)_{[1,p-1]} = \sigma(x_n)_{[0,p-2]},$$

$d(\sigma(x_n), x_{n+1}) \leq 2^{-(p-1)} < \delta$. Thus $(x_n)_{n\geq 0}$ is a δ-chain, so it is 1-shadowed by some point $z \in \Sigma$. Since $d(\sigma^n(z), x_n) < 1$, $z_n = (x_n)_0 = y_n$, so $z = y$ and $y \in \Sigma$. ☐

Definition 3.34. — A dynamical system (X, F) is **open**, if $F(U)$ is open for any open $U \subseteq X$.

In the golden mean subshift, both of the sets

$$\sigma([0]) = \Sigma, \quad \sigma([1]) = [0]$$

are open. In contrast, in the even subshift, the set $\sigma([0])$ is not open. We have $1^\infty \in \sigma([0])$ but for any $n \geq 0$, $1^{2n+1}0^\infty \notin \sigma([0])$.

***Theorem 3.35* (Parry [122]).** — *A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is an SFT iff (Σ, σ) is open.*

Proof. —

(1) Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift of order p. Assume that $n \geq p$, $u \in \mathcal{L}^n(\Sigma)$ and $x \in [u]$. We show that $\sigma(x)$ is an inner point of $\sigma([u])$ (Figure 5 left). Let $y \in \Sigma$, $d(y, \sigma(x)) < 2^{-n+1}$, so $y_{[0,n)} = \sigma(x)_{[0,n)} = x_{[1,n]}$. Define $w \in A^{\mathbb{N}}$ by $w_0 = u_0$, $w_{i+1} = y_i$ for $i \geq 0$. Then

$$w_{[0,p)} = x_{[0,p)} \in \mathcal{L}(\Sigma), \quad w_{[i+1,i+p]} = y_{[i,i+p)} \in \mathcal{L}(\Sigma), \quad i > 0,$$

so $w \in [u]$ and $y = \sigma(w) \in \sigma([u])$. Thus $B_{2^{-n+1}}(\sigma(x)) \subseteq \sigma([u])$ and $\sigma([u])$ is open.

(2) Suppose that (Σ, σ) is open. Then for any $a \in A$, $\sigma([a])$ is clopen and therefore, for some $p > 1$, $\sigma([a])$ is a finite union of p-cylinders. This means that for any $a \in A$ there exists $B_a \subseteq A^p$ such that

$$\sigma([a]) = \bigcup\{[u] : u \in B_a\}.$$

We show that Σ is an SFT of order $p+1$, i.e., if $|u| \geq p+1$, then

$$u \in \mathcal{L}(\Sigma) \quad \Leftrightarrow \quad \forall n < |u| - p, u_{[n,n+p]} \in \mathcal{L}^{p+1}(\Sigma)$$

The condition holds trivially for $|u| = p+1$. Suppose that it holds for any word of length $m \geq p+1$ and let $u \in A^{m+1}$ be such that all its subwords of length $p+1$

belong to $\mathcal{L}(\Sigma)$ (Figure 5 right). By the induction hypothesis, $u_{[1,m]} \in \mathcal{L}(\Sigma)$, so there exists $y \in [u_{[1,m]}]$. Since $u_{[0,p]} \in \mathcal{L}(\Sigma)$, $\sigma([u_0]) \cap ([u_{[1,p]}]) \neq \emptyset$. Thus $u_{[1,p]} \in B_{u_0}$ and therefore $[u_{[1,p]}] \subseteq \sigma([u_0])$. It follows that $y \in \sigma([u_0])$, so there exists $x \in \Sigma$ with $x_0 = u_0$ and $\sigma(x) = y$. Thus $x \in \Sigma$ and $u \in \mathcal{L}(\Sigma)$. $\square$

$x:$	u_0	u_1		u_{n-1}	x_n	x_{n+1}
$y:$		u_1		u_{n-1}	x_n	y_n
$w:$	u_0	u_1		u_{n-1}	x_n	y_n

$x:$	u_0	u_1		u_m	
$y:$		u_1		u_m	y_m

FIGURE 5. Open subshifts

Proposition 3.36. — *Any SFT is conjugate to a Markov subshift, i.e., to an SFT of order 2.*

Proof. — Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift of order $p > 2$. Consider the alphabet $B = \mathcal{L}^{p-1}(\Sigma)$. We write points of $B^{\mathbb{N}}$ as

$$x = \begin{bmatrix} (x_0)_0 \\ (x_0)_1 \\ \vdots \\ (x_0)_{p-2} \end{bmatrix} \begin{bmatrix} (x_1)_0 \\ (x_1)_1 \\ \vdots \\ (x_1)_{p-2} \end{bmatrix} \begin{bmatrix} (x_2)_0 \\ (x_2)_1 \\ \vdots \\ (x_2)_{p-2} \end{bmatrix} \cdots$$

We define a subshift $\Theta \subseteq B^{\mathbb{N}}$ by $x \in \Theta$ iff

$$\forall n \geq 0, (x_n)_{[1,p-2]} = (x_{n+1})_{[0,p-3]} \;\&\; (x_n)_0 (x_n)_1 \ldots (x_n)_{p-2} (x_{n+1})_{p-2} \in \mathcal{L}(\Sigma).$$

Then Θ is a Markov subshift and the map $\varphi : \Sigma \to \Theta$ given by

$$\varphi(x) = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p-1} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix} \cdots$$

is a conjugacy. $\square$

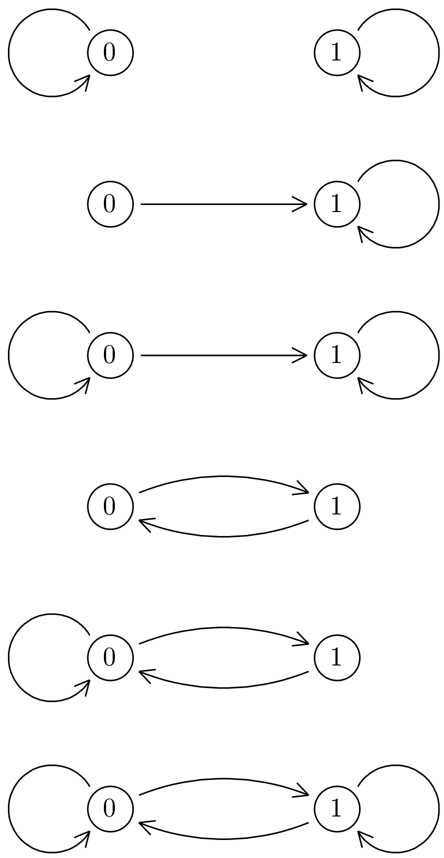
Example 3.37. — $\Sigma = \Sigma_{\{00,010,111\}} = \{(011)^\infty, (110)^\infty, (101)^\infty\}$

This is a finite subshift of order 3 and $\mathcal{L}^3(\Sigma) = \{011, 110, 101\}$. If we denote these words a, b, c, we see that Σ is conjugate to a Markov subshift

$$\Sigma_{\{aa,ac,ba,bb,cb,cc\}} = \{(abc)^\infty, (bca)^\infty, (cab)^\infty\}.$$

3.6. Transition graphs and their matrices

A Markov subshift $\Sigma \subseteq A^{\mathbb{N}}$ can be described by an oriented graph. The vertices of the graph are letters of A and there is an arrow from a to b if $ab \in \mathcal{L}^2(\Sigma)$. If Σ is a subshift with alphabet $\mathcal{L}^1(\Sigma) = A$, then for any $a \in A$ there exists at least one $b \in A$ with $ab \in \mathcal{L}^2(\Sigma)$. Otherwise, no word $u \in A^{\mathbb{N}}$ containing the letter a would belong to Σ and Σ would be a subshift in the alphabet $A \setminus \{a\}$.



$\Sigma = \{0^\infty, 1^\infty\}$

$\Sigma = \{01^\infty, 1^\infty\}$

$\Sigma = \{0^k 1^\infty, k \geq 0\} \cup \{0^\infty\}$

$\Sigma = \{(01)^\infty, (10)^\infty\}$

golden mean subshift

full shift

FIGURE 6. Binary Markov subshifts

Definition 3.38. — A **transition graph** is a pair (A, E), where A is a nonempty finite set and $E \subseteq A \times A$ is a set of edges satisfying

$$\forall a \in A, \exists b \in A, (a, b) \in E.$$

The subshift $\Sigma_E \subseteq A^{\mathbb{N}}$ of a transition graph (A, E) is defined by

$$u \in \Sigma_E \text{ iff } \forall i \geq 0, (u_i, u_{i+1}) \in E.$$

A subshift of a transition graph is a Markov subshift and any SFT is conjugate to a subshift of some transition graph. Figure 6 shows graphs of some binary Markov subshifts (subsystems of $\mathbf{2}^{\mathbb{N}}$). In Figure 7 there are transition graphs of subshifts conjugate to $\Sigma_{\{00,111\}}$ (top) and $\Sigma_{\{000,111\}}$ (bottom) of order 3. We can characterize transitive and mixing SFTs in terms of their graphs. A Markov subshift is transitive iff for any $a, b \in A$ there is a path leading from a to b. If there is such a path of arbitrary length (sufficiently large), then the subshift is mixing.

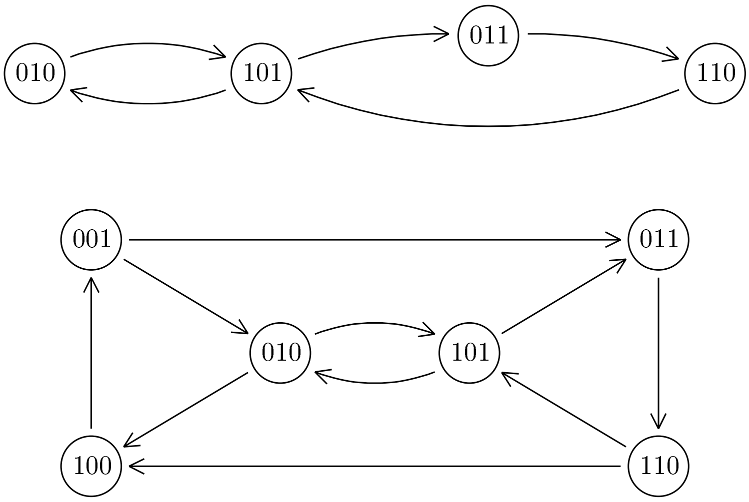


FIGURE 7. Subshifts $\Sigma_{\{00,111\}}$ and $\Sigma_{\{000,111\}}$

Proposition 3.39. — *Let Σ be a Markov subshift with $\mathcal{L}^1(\Sigma) = A$.*
(1) (Σ, σ) is transitive iff

$$\forall a, b \in A, \exists n > 1, \exists u \in \mathcal{L}^n(\Sigma), u_0 = a, u_{n-1} = b.$$

(2) (Σ, σ) is mixing iff

$$\exists m > 0, \forall a, b \in A, \forall n \geq m, \exists u \in \mathcal{L}^n(\Sigma), u_0 = a, u_{n-1} = b.$$

(3) Any transitive SFT has a dense set of periodic points.

Proof. —
(1) Suppose that the condition is satisfied and let $U, V \subseteq \Sigma$ be nonempty and open. There exists $k > 0$ and $u, v \in \mathcal{L}^k(\Sigma)$, such that the cylinders $[u]$ and $[v]$ are contained in U and V respectively. For $a = u_{k-1}$ and $b = v_0$ there exists $m > 0$ and $w \in \mathcal{L}^m(\Sigma)$ for which $w_0 = u_{k-1}$ and $w_{m-1} = v_0$. If

$$x \in [u_0 \dots u_{k-2} w_0 \dots w_{m-2} v_0 \dots v_{k-1}],$$

then $x \in U \cap \sigma^{-k-m+1}(V)$. Thus (Σ, σ) is transitive. Conversely suppose that (Σ, σ) is transitive and $a, b \in A$. The cylinders $[a]$ and $[b]$ are nonempty and open, so there exists $k > 0$ and $x \in \Sigma$ such that $x \in [a] \cap \sigma^{-k}[v] \cap \Sigma$. Then $u = x_{[0,k]} \in \mathcal{L}(\Sigma)$, $u_0 = a$ and $u_k = b$.
(2) The proof is similar to that given for the part 1 of the proposition.
(3) Let $U \subseteq \Sigma$ be nonempty and open. There exists $k > 0$ and $u \in \mathcal{L}^k(\Sigma)$, such that $[u] \subseteq U$. For $a = u_{k-1}$ and $b = u_0$ there exists $m > 0$ and $w \in \mathcal{L}^m(\Sigma)$ for which $w_0 = u_{k-1}$ and $w_{m-1} = v_0$. Then $(u_0 \dots u_{k-1} w_1 \dots w_{m-2})^\infty \in [u]$ is a periodic point. $\square$

The first three subshifts in Figure 6 are not transitive. The subshifts

$$\Sigma_{\{00,11\}} = \{(01)^\infty, (10)^\infty\}, \quad \Sigma_{\{00,010,111\}} = \{(011)^\infty, (110)^\infty, (101)^\infty\}$$

are transitive but not mixing. The first subshift has period 2, there is no path of odd length from 0 to 0. The second subshift has period 3. The golden mean subshift and the full shift are both transitive and mixing. The subshifts of Figure 7 are also mixing.

3.6.1. Attractors. — Since any SFT has the shadowing property, the nonwandering and chain relations coincide, i.e., $\mathcal{N} = \mathcal{C}$ (Proposition 2.15). The structure of the chain relation is determined by the path relation. In particular, chain components and chain-terminal sets (Definition 2.70) are determined by **path components** and **path-terminal sets**.

Definition 3.40. — Let Σ be a Markov subshift and $A = \mathcal{L}^1(\Sigma)$.

(1) The **path relation** $\mathcal{P} \subseteq A \times A$ is defined by

$$(a, b) \in \mathcal{P} \text{ iff } \exists n > 1, \exists u \in \mathcal{L}^n(\Sigma), u_0 = a, u_{n-1} = b.$$

(2) A set $B \subseteq |\mathcal{P}|$ is **path terminal**, if

$$\forall b \in B, \forall a \in |\mathcal{P}|, ((b, a) \in \mathcal{P} \Rightarrow a \in B).$$

(3) A set $B \subseteq |\mathcal{P}|$ is a **path component**, if

$$\forall b \in B, \forall a \in |\mathcal{P}|, \left(a \in B \Leftrightarrow (a, b) \in \mathcal{P} \cap \mathcal{P}^{-1}\right).$$

(4) The **graph of path components** is an oriented graph whose vertices are path components and whose oriented edges are

$$B \to C \quad \Leftrightarrow \quad \exists b \in B, \exists c \in C, (b, c) \in \mathcal{P}.$$

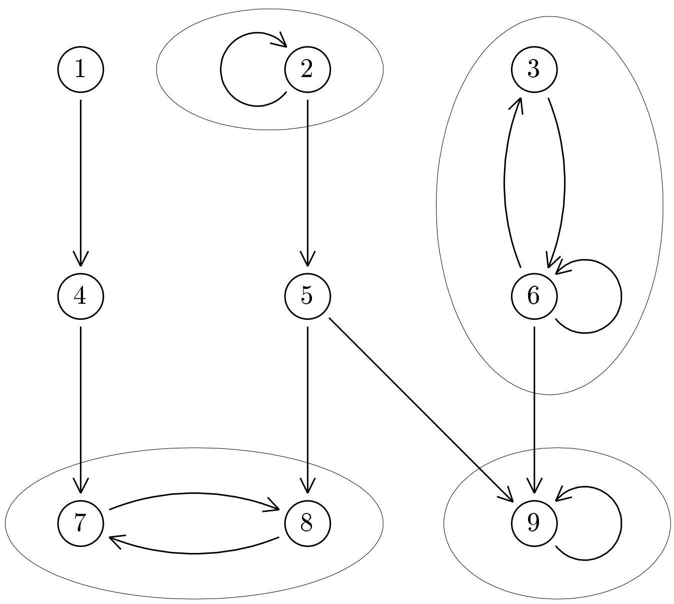


FIGURE 8. Graph of path components

Since $\mathcal{P}$ is transitive, $\mathcal{P}\cap\mathcal{P}^{-1}$ is an equivalence on the set $|\mathcal{P}|$ and path components are its equivalence classes. Any path terminal set is a union of some set of path components. The graph of path components has no loops. An SFT Σ with $\mathcal{L}^1(\Sigma) = A$ is transitive iff its only path component is A, i.e., if $\mathcal{P} = A \times A$. In the graph in Figure 8, $|\mathcal{P}| = \{2,3,6,7,8,9\}$ and the graph of path components is

$$\{7,8\} \longleftarrow \{2\} \longrightarrow \{9\} \longleftarrow \{3,6\}$$

Path terminal sets are $\{9\}$, $\{7,8\}$, $\{2,7,8,9\}$ and $\{3,6,9\}$. We shall see that any path component gives rise to a chain component and any path terminal set gives rise to an attractor. In Figure 8, the chain components are

$$\{2^\infty\}, \quad \{3,6\}^{\mathbb{N}} \cap \Sigma, \quad \{7,8\}^{\mathbb{N}} \cap \Sigma, \quad \{9^\infty\}.$$

We have attractors

$$\{9^\infty\}, \quad \{7,8\}^{\mathbb{N}} \cap \Sigma, \quad \{2,5,7,8,9\}^{\mathbb{N}} \cap \Sigma, \quad \{3,6,9\}^{\mathbb{N}} \cap \Sigma$$

and minimal attractors are $\{9^\infty\}$ and $\{(78)^\infty, (87)^\infty\}$.

Proposition 3.41. — *Let Σ be a Markov subshift with $\mathcal{L}^1(\Sigma) = A$ and $x, y \in \Sigma$.*
(1) If $\forall i \geq 0, (x_i, y_0) \in \mathcal{P}$, then $(x,y) \in \mathcal{N}$.
(2) If $x \in |\mathcal{N}|$ and $(x,y) \in \mathcal{N}$, then $\forall i \geq 0, (x_i, y_0) \in \mathcal{P}$.

Proof. —
(1) Given $\varepsilon > 0$, pick some i with $2^{-i} < \varepsilon$. Since $(x_i, y_0) \in \mathcal{P}$, there exist $n > i$ and $z \in \Sigma$ with $z_{[0,i]} = x_{[0,i]}$, $z_{[n,n+i]} = y_{[0,i]}$, so $d(z,x) < 2^{-i}$ and $d(F^n(z), y) < 2^{-i}$. Thus $(x,y) \in \mathcal{N}$.

(2) Let $x \in |\mathcal{N}|$ and $(x,y) \in \mathcal{N}$. We show first $\forall i \geq 0, (x_i, x_0) \in \mathcal{P}$. This is clear if x is periodic. If not, it is not eventually periodic and there exists an open set $x \in U \subseteq B_1(x)$, such that $U, \sigma(U), \dots, \sigma^i(U)$ are pairwise disjoint. Since $x \in |\mathcal{N}|$, there exists $n > 0$ with $U \cap \sigma^n(U) \neq \emptyset$ and therefore $n > i$. If $z \in \sigma^n(U) \cap U$, then $z_0 = x_0 = z_n$, so $(x_i, x_0) \in \mathcal{P}$. Since $(x,y) \in \mathcal{N}$, there exists $m > 0$ and $w \in \Sigma$, such that $d(w,x) < 1$, $d(F^m(w), y) < 1$, and hence $w_0 = x_0$, $w_m = y_0$. Thus $(x_0, y_0) \in \mathcal{P}$ and therefore, $(x_i, y_0) \in \mathcal{P}$. ☐

Proposition 3.42. — *Let Σ be a Markov subshift with $\mathcal{L}^1(\Sigma) = A$.*
(1) If $B \subseteq |\mathcal{P}|$ is a path component then $B^{\mathbb{N}} \cap \Sigma$ is a chain component. Any chain component of Σ is of this form.
(2) If $B_1, \dots, B_m$ are path components and $B = B_1 \cup \cdots \cup B_m$ is a path terminal set, then $(B_1^{\mathbb{N}} \cup \cdots \cup B_m^{\mathbb{N}}) \cap \Sigma$ is a chain terminal set. Any chain terminal set is of this form.
(3) If B is a chain terminal set and $C = \{c \in A : \exists b \in B, (b,c) \in \mathcal{P}\}$, then $C^{\mathbb{N}} \cap \Sigma$ is an attractor. Any attractor is of this form.

Proof. — Since Σ has the shadowing property, $\mathcal{N} = \mathcal{C}$.
(1) If $x \in B^{\mathbb{N}} \cap \Sigma$, then for any $i \geq 0$ we have $(x_i, x_0) \in \mathcal{P}$, so $x \in |\mathcal{N}|$ and $B^{\mathbb{N}} \cap \Sigma \subseteq |\mathcal{N}|$. Assume that $x \in B^{\mathbb{N}} \cap \Sigma$, $y \in |\mathcal{N}|$. If $(x,y) \in \mathcal{N} \cap \mathcal{N}^{-1}$ then $(x_0, y_0) \in \mathcal{P}$, so $y_0 \in B$ and $y \in B^{\mathbb{N}} \cap \Sigma$. If $y \in B^{\mathbb{N}} \cap \Sigma$, then for any $i \geq 0$, $(x_i, y_0) \in \mathcal{P}$ and $(y_i, x_0) \in \mathcal{P}$, so $(x,y) \in \mathcal{N} \cap \mathcal{N}^{-1}$. Thus $B^{\mathbb{N}} \cap \Sigma$ is a chain component.

If $Y \subseteq \Sigma$ is a chain component, set

$$B = \{b \in A : \exists y \in Y, y_0 = b\}.$$

If $b \in B$, there exists $y \in Y$ such that $y_0 = b$. Since $y \in |\mathcal{N}|$, there exists $i > 0$ and $z \in \Sigma$ such that $z_i = z_0 = b$, so $b \in |\mathcal{P}|$. Thus $B \subseteq |\mathcal{P}|$. Assume that $b \in B$, $a \in A$. If $(b, a) \in \mathcal{P} \cap \mathcal{P}^{-1}$, there exists $y \in Y$ such that $y_0 = b$ and a periodic point $x \in \Sigma$ such that $x_0 = a$, $x_n = b = y_0$ for some $n > 0$. Then $x, y \in |\mathcal{N}|$ and for any $i \geq 0$,

$$(x_i, y_0) \in \mathcal{P}, \quad (y_i, y_0) \in \mathcal{P}, \quad (y_0, x_0) \in \mathcal{P},$$

so $(y, x) \in \mathcal{N} \cap \mathcal{N}^{-1}$. Since Y is a chain component, $x \in Y$ and $a \in B$. If $a, b \in B$, there exist $x, y \in Y$ with $x_0 = a$, $y_0 = b$. Since Y is a chain component, $(x, y) \in \mathcal{N} \cap \mathcal{N}^{-1}$, so $(b, a) \in \mathcal{P} \cap \mathcal{P}^{-1}$. Thus B is a path component and $Y \subseteq B^{\mathbb{N}} \cap \Sigma$. If $y \in Y$, $x \in B^{\mathbb{N}} \cap \Sigma$, then for any $i \geq 0$, $(y_i, x_0) \in \mathcal{P}$, so $(y, x) \in \mathcal{N}$ and $x \in Y$. Thus $Y = B^{\mathbb{N}} \cap \Sigma$.

(2) If $x \in (B_1^{\mathbb{N}} \cup \cdots \cup B_m^{\mathbb{N}}) \cap \Sigma$, $y \in |\mathcal{N}|$ and $(x, y) \in \mathcal{N}$, then for any $i \geq 0$ we have $(x_i, y_0) \in \mathcal{P}$, so $y_0 \in B$. Since $y \in |\mathcal{N}|$, there exists $j \leq m$ such that $y \in B_j^{\mathbb{N}} \cap \Sigma$. Thus $(B_1^{\mathbb{N}} \cup \cdots \cup M_m^{\mathbb{N}}) \cap \Sigma$ is a chain-terminal set.

If $Y \subseteq \Sigma$ is a chain-terminal set then it is a union of chain components, so there exist path components $B_1, \ldots, B_m$ such that $Y = (B_1^{\mathbb{N}} \cup \cdots \cup B_m^{\mathbb{N}}) \cap \Sigma$. Assume that $B = B_1 \cup \cdots \cup B_m$ is not a path-terminal set. Then there exist $a \in B$, $b \in |\mathcal{P}| \setminus B$ with $(a, b) \in \mathcal{P}$. There exist $n > 0$ and $x \in \Sigma$ with $x_0 = a$, $x_n = b$, $\sigma^n(x) \in |\mathcal{N}|$ and $(x, \sigma^n(x)) \in \mathcal{N}$. Thus $\sigma^n(x) \in Y$ and this is a contradiction.

(3) The set $Y = \{x \in \Sigma : x_0 \in C\}$ is clopen and invariant, so it is inward. Since $\sigma(Y) = Y$, $Y = \omega(Y)$ is an attractor and clearly $Y = C^{\mathbb{N}} \cap \Sigma$.

If $Y \subseteq \Sigma$ is an attractor, then $Y \cap |\mathcal{N}|$ is a chain terminal set by Theorem 2.68. Thus there exist path components $B_1, \ldots, B_m$ such that $Y \cap |\mathcal{N}| = (B_1^{\mathbb{N}} \cup \cdots \cup B_m^{\mathbb{N}}) \cap \Sigma$ and clearly $Y = C^{\mathbb{N}} \cap \Sigma$. □

Corollary 3.43. — *If Y is an SFT and $C \subseteq \Sigma$ is a chain-terminal set, then $Y = \mathcal{C}(C)$ is an attractor.*

3.6.2. Topological entropy. — To evaluate the topological entropy of a Markov subshift we need a formula for its complexity function $P_\Sigma(n) = \#\mathcal{L}^n(\Sigma)$. Words in $\mathcal{L}^n(\Sigma)$ can be regarded as paths in its transition graph. The number of these paths can be evaluated if we encode the graph by a binary matrix and compute its powers.

Definition 3.44. — Let Σ be a Markov subshift and $A = \mathcal{L}^1(\Sigma)$. The **adjacency matrix** $M = (M_{ab})_{a,b \in A}$ of Σ is defined by

$$M_{ab} = \begin{cases} 1 & \text{if } ab \in \mathcal{L}(\Sigma) \\ 0 & \text{if } ab \notin \mathcal{L}(\Sigma) \end{cases}$$

Any row of M is a nonzero vector, i.e., $\forall a \in A, \exists b \in A, M_{ab} = 1$. Conversely, if $(M_{ab})_{a,b \in A}$ is a square binary matrix (i.e., matrix whose entries are zeros or ones), with nonzero rows, then it defines a Markov subshift

$$\Sigma_M = \{x \in A^{\mathbb{N}} : \forall n, M_{x_n, x_{n+1}} = 1\}.$$

Thus binary matrices with nonzero rows are in one-to-one correspondence with transition graphs. For $u \in A^{n+1}$ we get

$$u \in \mathcal{L}(\Sigma_M) \text{ iff } M_{u_0u_1} \cdot M_{u_1u_2} \cdots M_{u_{n-1}u_n} = 1.$$

This product is one of the terms in the n-th power M^n of M. For the complexity function $P(n) = \#\mathcal{L}^n(\Sigma)$ we get

$$\begin{aligned}
P(2) &= \sum_{a\in A}\sum_{b\in A} M_{ab} \\
P(3) &= \sum_{a\in A}\sum_{b\in A}\sum_{c\in A} M_{ab}M_{bc} = \sum_{a\in A}\sum_{c\in A}(M^2)_{ac} \\
P(n+1) &= \sum_{u_0\in A}\sum_{u_n\in A}(M^n)_{u_0,u_n}
\end{aligned}$$

Example 3.45. — The topological entropy of the golden mean subshift of Example 3.13 is $h(\Sigma,\sigma) = \ln\frac{\sqrt{5}+1}{2}$.

Proof. — For the powers of M we get by induction

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad M^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

where F_n are Fibonacci numbers (Example 1.39). We get

$$P(n+1) = F_{n+1} + 2F_n + F_{n-1} = F_{n+3} = (\alpha^{-n-3} - (-\alpha)^{n+3})/\sqrt{5},$$

where $\alpha = (\sqrt{5}-1)/2$ is the golden number. Thus $h(\Sigma,\sigma) = \lim_{n\to\infty}\frac{\ln P(n)}{n} = \ln\frac{1}{\alpha}$. □

To obtain a formula for the powers of binary matrices we use the Perron-Frobenius theory of nonnegative matrices (Section A.9). A matrix $M = (M_{ab})_{a,b\in A}$ is **nonnegative** ($M \geq 0$), if $M_{ab} \geq 0$ for any $a, b \in A$ and **positive** ($M > 0$), if $M_{ab} > 0$ for any $a, b \in A$. A matrix M is **irreducible**, if for any $a, b \in A$ there exists $n > 0$ with $(M^n)_{ab} > 0$. It is **primitive**, if there exists $n > 0$ such that M^n is positive. Proposition 3.39 can be reformulated

Corollary 3.46. — *A Markov subshift is mixing iff its matrix is primitive. A Markov subshift is transitive iff its matrix is irreducible.*

The largest positive eigenvalue of an irreducible matrix is called its **spectral radius**. If M is not irreducible, we have the **path relation** $\mathcal{P} \subseteq A \times A$ of M

$$(a,b) \in \mathcal{P} \quad \Leftrightarrow \quad \exists n > 0, (M^n)_{ab} > 0.$$

If $B \subseteq A$ is a path component, then $(M_{ij})_{i,j\in B}$ is an irreducible submatrix.

Definition 3.47. — The **spectral radius** of a nonnegative matrix $M = (M_{ab})_{a,b\in A}$ with nonzero rows is the maximum of spectral radii of its irreducible submatrices

$$\lambda(M) = \max\{\lambda((M_{ab})_{a,b\in B}) : B \subseteq A \text{ is a path component}\}.$$

Theorem 3.48. — *Let Σ be a Markov subshift with $\mathcal{L}^1(\Sigma) = A$, M its adjacency matrix, and λ the spectral radius of M. Then $h(\Sigma,\sigma) = \ln\lambda$.*

Proof. — Suppose first that Σ is transitive, so M is irreducible. We have $P(n) = \sum_{a,b\in A}(M^{n-1})_{ab}$. By Theorem A.73 there exist positive constants, $c_0 < c_1$ with $c_0\lambda^n \le P(n) \le c_1\lambda^n$, so

$$\ln\lambda = \lim_{n\to\infty}\frac{\ln(c_0\lambda^n)}{n} \le \liminf_{n\to\infty}\frac{\ln P(n)}{n} \le \limsup_{n\to\infty}\frac{\ln P(n)}{n} \le \lim_{n\to\infty}\frac{\ln(c_1\lambda^n)}{n} = \ln\lambda.$$

In the case of a general SFT we use the Bowen Theorem 2.90 which says that the entropy of Σ is concentrated on its nonwandering set. By Proposition 3.42, nonwandering set of Σ is disjoint union of subshifts $B^{\mathbb{N}}$, where $B \subseteq A$ is a path component. By Proposition 2.89, the entropy of a disjoint union of two dynamical systems is the maximum of their entropies. □

For example, the entropy of the subshift in Figure 8 is $\ln\frac{\sqrt{5}+1}{2}$, the entropy of the transitive subshift $\Sigma \cap \{3,6\}^{\mathbb{N}}$.

Proposition 3.49. — *If $\Theta \subset \Sigma$ are transitive SFT, then $h(\Theta,\sigma) < h(\Sigma,\sigma)$.*

Proof. — We can assume that both Θ and Σ are Markov subshifts and both are subsets of some $A^{\mathbb{N}}$. Let M, N be their matrices. Then $M \le N$ and there exist $a, b \in A$ with $M_{ab} = 0$ and $N_{ab} = 1$. Since both M and N are irreducible, there exists $k > 0$ such that $M^k < N^k$. If λ is the spectral radius of M and u its right eigenvector, then $\lambda^k u = M^k u < N^k u$, so the spectral radius μ of N is strictly greater than λ. □

3.7. Sofic subshifts

In computer science, the complexity of a language $L \subseteq A^*$ is thought of as the complexity of an automaton which can recognize or accept it. The lowest level in this hierarchy is represented by **regular languages** recognized by **finite automata**. Subshifts whose languages are regular are called **sofic** (Weiss [**162**]). All SFTs are sofic, but there are sofic subshifts, e.g., the even subshift, which are not SFT.

A finite automaton (over an alphabet A) is a device which successively reads letters of A and which upon reading a letter changes its inner state. The set Q of its inner states is finite. Thus a finite automaton is represented by a **transition function** $\delta : Q \times A \to Q$ or a family of functions $(\delta^a : Q \to Q)_{a\in A}$. If the automaton reads a word $u = u_0 \dots u_{n-1}$, its change of state is represented by a function $\delta^u : Q \to Q$, which is the composition $\delta^u = \delta^{u_{n-1}} \circ \cdots \circ \delta^{u_0}$, so $\delta^{uv} = \delta^v \circ \delta^u$. If we set $\delta^\lambda = \mathrm{Id}$, we see that a transition function is just a flow (Definition 1.4) with time monoid $T = A^*$ and finite state space Q.

Definition 3.50. — A **transition function** over A is a map $\delta : Q \times A \to Q$, where A is an alphabet and Q is a finite set (of states). The function δ extends to a function $\delta : Q \times A^* \to Q$ defined by

$$\delta(q,\lambda) = q, \quad \delta(q,ua) = \delta(\delta(q,u),a), \quad u \in A^*, \quad a \in A$$

Thus for any $q \in Q$ and any $u, v \in A^*$, we have $\delta(\delta(q,u),v) = \delta(q,uv)$.

A finite automaton can recognize languages. The automaton starts in a distinguished initial state $i \in Q$. If after reading a word $u \in A^*$, the automaton ends in one of the specified rejecting state, the word is rejected; otherwise it is accepted. An

accepting automaton is specified by a quadruple (Q, δ, i, R), where Q is a finite set of states, $\delta : Q \times A \to Q$ is a transition function, $i \in Q$ is an initial state, and $R \subseteq Q$ is the set of rejecting states. The language accepted by (Q, δ, i, R) is

$$\{u \in A^* : \delta(i, u) \notin R\}.$$

A language $L \subseteq A^*$ is said to be **regular**, if it is accepted by some accepting automaton.

If the language accepted by (Q, δ, i, R) is to be extendable, the automaton must satisfy certain conditions. Since $\lambda \in L$, the starting state must not be rejecting. Once the automaton enters the set R of rejecting states, it must not leave it. Any prolongation of a rejected word must be rejected. Thus we can collapse the rejecting set to a single state $R = \{r\}$. Moreover, we can assume that all nonrejecting states are reachable from the initial state, i.e.,

$$\forall q \in Q \setminus \{r\}, \exists u \in L, \delta^u(i) = q.$$

If this is not the case, we can erase all states which are not reachable without changing the language. Finally, if the accepting automaton satisfies this restriction, there is no need for the initial state, as any state from $Q \setminus R$ will do as well. We have

$$q \in Q \setminus R \ \ \& \ \ \delta^u(q) \in Q \setminus R \ \Rightarrow \ u \in L$$

Indeed since q is reachable, there exists $v \in L$ such that $\delta^v(i) = q$, so $\delta^{vu}(i) = \delta^u(q) \in Q \setminus R$, and $vu \in L$. Since L is extendable, $u \in L$. This simplifies the concept of an accepting automaton.

Definition 3.51. — An **accepting automaton** over an alphabet A is a triple (Q, δ, r), where Q is an alphabet, $r \in Q$ is a rejecting state, and $\delta : Q \times A \to Q$ is a transition function satisfying

(1) $\forall q \in Q \setminus \{r\}, \exists a \in A, \delta(q, a) \neq r.$

(2) $\forall a \in A, \delta(r, a) = r.$

The language **accepted** by (Q, δ, r) is $L = \{u \in A^* : \exists q \in Q, \delta(q, u) \neq r\}$. A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is **sofic**, if its language is accepted by some accepting automaton.

Any language accepted by an accepting automaton is extendable, so it defines some sofic subshift.

Example 3.52. — The golden mean subshift and the even subshift are sofic.

We construct automata with transition functions given by the tables

	q_0	q_1	q_2
0	q_0	q_0	q_2
1	q_1	q_2	q_2

	q_0	q_1	q_2	q_3
0	q_1	q_1	q_3	q_3
1	q_0	q_2	q_1	q_3

For the golden mean subshift, $Q = \{q_0, q_1, q_2\}$, $i = q_0$, and $r = q_2$. The automaton is in state q_1, when the last letter read is 1. For the even subshift, $Q = \{q_0, q_1, q_2, q_3\}$, $i = q_0$ and $r = q_3$. The automaton stays in the initial state q_0 till it reads 0. Then it goes to state q_1 and rests there while reading 0. On reading 1 it enters q_2. If it reads now 0, the word is rejected, otherwise the automaton returns to q_1.

We represent accepting automata with oriented graphs whose vertices are states from Q and whose edges are labelled by letters of A. There is an edge from q to q'

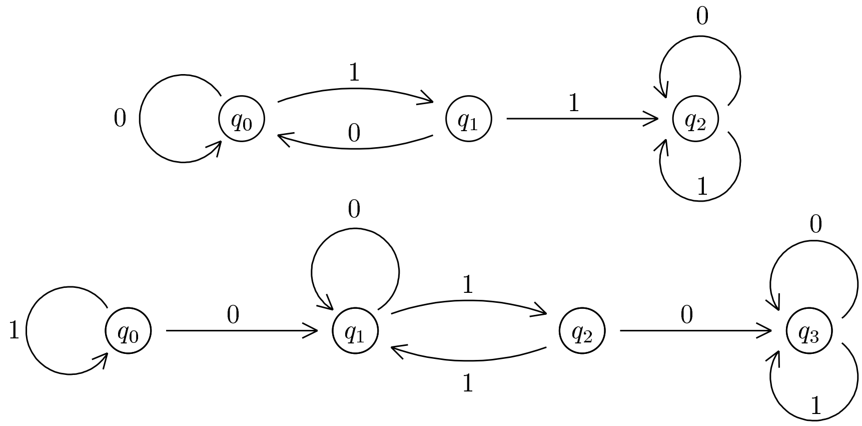


FIGURE 9. Graphs of accepting automata

labelled a, if $\delta(q,a) = q'$. Thus there are exactly $\#A$ edges which start at any vertex and these edges carry distinct labels. Figure 9 shows graphs of finite automata for the golden mean and sofic subshifts.

3.7.1. Labelled graphs. — In a graph of an accepting automaton, the rejecting state is superfluous. If we leave it out and if we leave out also all edges which lead to it, the set of accepted words is not changed. We just accept a word if it is the label of some path. This gives rise to an alternative description of sofic subshifts. A labelled graph is an oriented graph whose edges are labelled by letters of A. There may be several edges between any pair of vertices. In the graphs constructed from accepting automata, different edges going from a given vertex carry different labels. A labelled graph with this property is called **right-resolving**. However, labelled graphs which are not right-resolving also give rise to sofic subshifts.

Definition 3.53. —

(1) A **labelled graph** over an alphabet A is a quintuple $G = (V, E, s, t, l)$, where V is a finite set of vertices, E is a finite set of edges, $s, t : E \to V$ are the source and target maps such that $\forall q \in V, \exists e \in E, s(e) = q$, and $l : E \to A$ is a labeling function.
(2) A labelled graph G is **right-resolving**, if
$$\forall e_1, e_2 \in E, (s(e_1) = s(e_2) \;\&\; l(e_1) = l(e_2) \;\Rightarrow\; e_1 = e_2).$$
(3) The **edge subshift** $\Sigma_{|G|}$ of G is $\Sigma_{|G|} = \{u \in E^{\mathbb{N}} : \forall i \geq 0, t(u_i) = s(u_{i+1})\} \subseteq E^{\mathbb{N}}$.
(4) The subshift of G is $\Sigma_G = \{l(u) : u \in \Sigma_{|G|}\} = l(\Sigma_{|G|}) \subseteq A^{\mathbb{N}}$.

Clearly $\Sigma_{|G|}$ is a Markov subshift. The map $l : \overline{E^*} \to \overline{A^*}$ is the extension of $l : E \to A$ defined by $l(u)_i = l(u_i)$. The map l is continuous and $\Sigma_G = l(\Sigma_{|G|})$, so Σ_G is a subshift. Figure 10 gives some examples of labelled graphs. The first two are obtained from the graphs of accepting automata in Figure 9 by deleting some vertices. The third graph yields a skew Sturmian subshift consisting of infinite words with at most one 1. We shall treat skew Sturmian subshifts in Section 4.5.

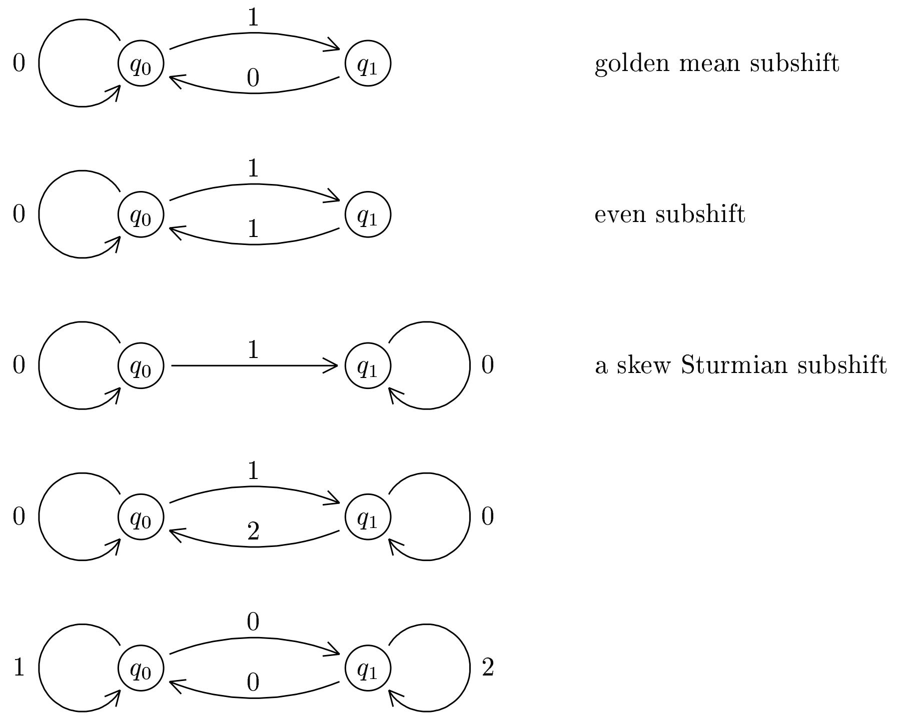


FIGURE 10. Sofic subshifts

Proposition 3.54. — *If $\Sigma \subseteq A^{\mathbb{N}}$ is a sofic subshift, then there exists a right-resolving labelled graph G such that $\Sigma = \Sigma_G$.*

Proof. — Given an accepting automaton (Q, δ, r), we construct a labelled graph $G = (V, E, s, t, l)$, where

$$V = Q \setminus \{r\}, \quad E = \{(q, a) \in V \times A : a \in A, \delta(q, a) \neq r\},$$

$s(q, a) = q$, $t(q, a) = \delta(q, a)$ and $l(q, a) = a$. □

Proposition 3.55. — *Any subshift of any labelled graph is sofic.*

Proof. — Let $G = (V, E, s, t, l)$ be a labelled graph, let Q be the set of subsets of V. Define a transition function $\delta : Q \times A \to Q$ by

$$\delta(M, a) = \{q \in Q : \exists e \in E, s(e) \in M, t(e) = q, l(e) = a\}.$$

The initial state is V and the rejecting state is $\emptyset$. We show that $(Q, \delta, V, \emptyset)$ accepts $\mathcal{L}(\Sigma_G)$. If

$$q_0 \xrightarrow{u_0} q_1 \xrightarrow{u_1} q_2 \dots q_{n-1} \xrightarrow{u_{n-1}} q_n$$

is a path in G, then $q_n \in \delta(V, u)$, so $\delta(V, u) \neq \emptyset$ and u is accepted. Conversely, if $\delta(V, u) \neq \emptyset$, then pick some $q_n \in \delta(V, u)$. There exists $q_{n-1} \in \delta(V, u_{[0,n-2]})$ such that $q_{n-1} \xrightarrow{u_{n-1}} q_n$ is a labelled edge in G. Continuing backwards, we obtain a path in G with label u. □

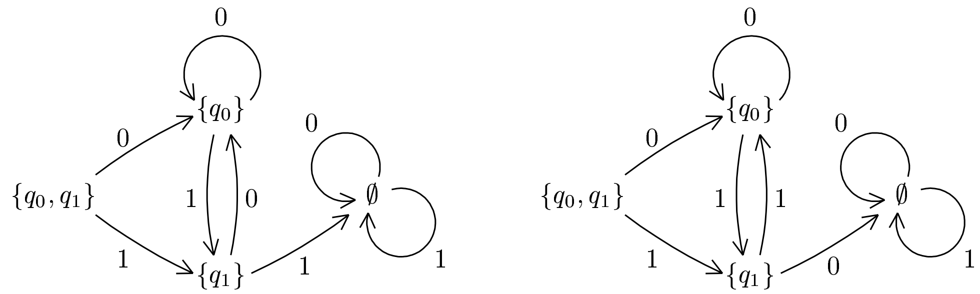


FIGURE 11. Accepting automata for labelled graphs

The accepting automata constructed from the labelled graphs of the golden mean and even subshifts from Figure 10 are shown in Figure 11.

Any sofic subshift is a factor of a subshift of finite type, since $l : (\Sigma_{|G|}, \sigma) \to (\Sigma_G, \sigma)$ is a factor map. The factor map l is obtained by extending the map $l : E \to A$ between the alphabets of these subshifts. In general, morphisms between subshifts can be defined by maps which assign letters to words. Let A, B be alphabets, $r \geq 1$ and $f : A^r \to B$ a mapping. Define a mapping $F : A^{\mathbb{N}} \to B^{\mathbb{N}}$ by

$$F(x)_n = f(x_{[n,n+r)}).$$

We shall show that $F : (A^{\mathbb{N}}, \sigma) \to (B^{\mathbb{N}}, \sigma)$ is a morphism and that any morphism between subshifts is of this form. The map f is called a **local rule** for F.

***Theorem 3.56* (Hedlund [73]).** — *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift and B an alphabet. A map $F : \Sigma \to B^{\mathbb{N}}$ is a morphism $F : (\Sigma, \sigma) \to (B^{\mathbb{N}}, \sigma)$ iff there exists $r \geq 1$ and a* **local rule** *$f : \mathcal{L}^r(\Sigma) \to B$ such that $F(x)_n = f(x_{[n,n+r)})$.*

Proof. — Let F be defined by a local rule f. Then for $x, y \in \Sigma$,

$$\begin{aligned} d(x,y) < 2^{-n-r} &\Rightarrow x_{[0,n+r]} = y_{[0,n+r]} \Rightarrow F(x)_{[0,n]} = F(y)_{[0,n]} \\ &\Rightarrow d(F(x), F(y)) < 2^{-n} \end{aligned}$$

so F is continuous. Moreover,

$$\sigma(F(x))_n = F(x)_{n+1} = f(x_{[n+1,n+r+1)}) = F(\sigma(x)_{[n,,n+r)}) = F(\sigma(x))_n,$$

so $F : (\Sigma, \sigma) \to (B^{\mathbb{N}}, \sigma)$ is a morphism. Conversely, let $F : (\Sigma, \sigma) \to (B^{\mathbb{N}}, \sigma)$ be a morphism. Since F is uniformly continuous, for $\varepsilon = 1$ there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $d(F(x), F(y)) < 1$. Let $2^{-r} < \delta$. Then

$$\begin{aligned} x_{[0,r)} = y_{[0,r)} &\Rightarrow d(x,y) \leq 2^{-r} < \delta \Rightarrow d(F(x), F(y)) < 1 \\ &\Rightarrow F(x)_0 = F(y)_0. \end{aligned}$$

Thus $F(x)_0$ depends only on the first r letters of x, and there exists a local rule $f : A^r \to B$ such that $f(x_{[0,r)}) = F(x)_0$. Since F is a morphism, we get

$$F(x)_n = \sigma^n(F(x))_0 = F(\sigma^n(x))_0 = f(\sigma^n(x)_{[0,r)}) = f(x_{[n,n+r)}).$$

$\square$

***Theorem 3.57* (Weiss [162]).** — *A subshift is sofic iff it is a factor of an SFT.*

Proof. — If Σ is sofic, then $\Sigma = \Sigma_G$ for some labelled graph and G and $l : (\Sigma_{|G|}, \sigma) \to (\Sigma_G, \sigma)$ is a factor map with SFT $\Sigma_{|G|}$. Conversely, let $F : (\Sigma, \sigma) \to (\Theta, \sigma)$ be a factor map, $\Sigma \subseteq A^{\mathbb{N}}$ an SFT and $\Theta \subseteq B^{\mathbb{N}}$. We can assume that Σ is a Markov subshift. By Hedlund's theorem 3.56, there exists a local rule $f : \mathcal{L}^{r+1}(\Sigma) \to B$ such that $F(x)_i = f(x_{[i,i+r]})$. We can assume $r \geq 1$. Define a labelled graph $G = (V, E, s, t, f)$, where $V = \mathcal{L}^r(\Sigma)$, $E = \mathcal{L}^{r+1}(\Sigma)$,

$$\begin{aligned} s(u_0 u_1, \dots u_{r-2} u_r) &= u_0 \dots u_{r-1} \\ t(u_0 u_1, \dots u_{r-2} u_r) &= u_1 \dots u_r \end{aligned}$$

Then $\Sigma_G = \Sigma$. $\square$

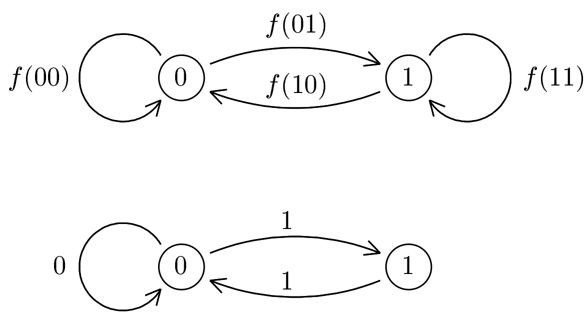


FIGURE 12. Factors of SFT

If $r = 1$ and $\Sigma = \mathbf{2}^{\mathbb{N}}$, we get a graph in Figure 12 top. Using Figure 12 bottom we obtain the even subshift as a factor of the golden mean subshift via $F : \Sigma_{\{11\}} \to \mathbf{2}^{\mathbb{N}}$ given by

$$F(x)_i = \begin{cases} 0 & \text{if } x_i = x_{i+1} = 0, \\ 1 & \text{if } x_i \neq x_{i+1}. \end{cases}$$

In particular Theorem 3.57 shows that any SFT is a sofic subshift. The accepting automata for SFT $\Sigma_{\{00,111\}}$ and $\Sigma_{\{000,111\}}$ of Figure 7 are shown in Figure 13.

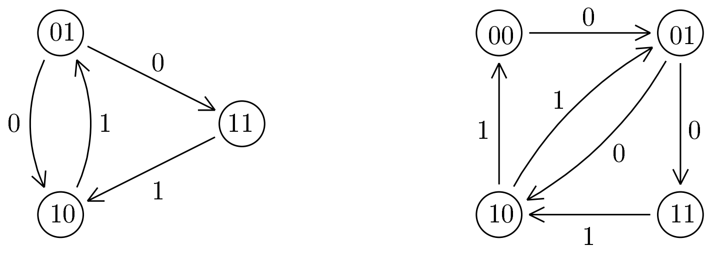


FIGURE 13. The graphs for $\Sigma_{\{00,111\}}$ and $\Sigma_{\{000,111\}}$

3.7.2. Irreducible graphs. — The theory of irreducibility developed for the transition graphs of SFT applies also to labelled graphs.

Definition 3.58. — The **path relation** $\mathcal{P} \subseteq V \times V$ of a labelled graph $G = (V, E, s, t, l)$ is defined by

$$(p, q) \in \mathcal{P} \quad \Leftrightarrow \quad \exists n \geq 1, \exists u \in \mathcal{L}^n(\Sigma_{|G|}), s(u_0) = p, t(u_{n-1}) = q.$$

The graph $G = (V, E, s, t, l)$ is **irreducible** if $\mathcal{P} = V \times V$.

Note that $\mathcal{P}$ does not depend on the labeling function l. The relation $\mathcal{P} \cap \mathcal{P}^{-1}$ is an equivalence on $|\mathcal{P}|$ and its equivalence classes are called path components. The subshift $\Sigma_{|G|} \subseteq E^{\mathbb{N}}$ is a Markov subshift and its topological properties depend on the structure of G in the same way as the properties of SFT depend on its transition graph. In particular, $\Sigma_{|G|}$ is transitive iff G is irreducible, and in this case it has a dense set of periodic points, so it is chaotic (Proposition 3.39). If $B \subseteq V$, denote by

$$\Sigma_{|G|}|B = \{x \in \Sigma_{|G|} : \forall i \geq 0, s(x_i) \in B, t(x_i) \in B\}$$

the restriction of $\Sigma_{|G|}$ to B. If $\Sigma_{|G|}|B$ is nonempty, then it is a subshift. It is easy to see that Proposition 3.42 holds for $\Sigma_{|G|}$, with $\Sigma \cap B^{\mathbb{N}}$ replaced by $\Sigma_{|G|}|B$.

Proposition 3.59. — *Let $G = (V, E, s, t, l)$ be a labelled graph.*

(1) If $B \subseteq |\mathcal{P}|$ is a path component then $\Sigma_{|G|}|B$ is a chain component. Any chain component of $\Sigma_{|G|}$ is ofthis form.

(2) If $B_1, \dots, B_m$ are path components and $B = B_1 \cup \dots \cup B_m$ is a path terminal set, then $(\Sigma_{|G|}|B_1) \cup \dots \cup (\Sigma_{|G|}|B_m)$ is a chain terminal set. Any chain terminal set is of this form.

(3) If B is a chain terminal set and $C = \{c \in V : \exists b \in B, (b, c) \in \mathcal{P}\}$, then $\Sigma_{|G|}|C$ is an attractor. Any attractor is of this form.

If G is an irreducible labelled graph, then both $\Sigma_{|G|}$ and Σ_G are transitive. However, Σ_G might be transitive even if G is not irreducible. It suffices to join two disjoint copies of a graph which gives rise to a given sofic subshift. The topological properties of sofic subshifts cannot be therefore determined from the structure of their labelled graphs. We show that for sofic subshifts, the relations $\mathcal{N}$ and $\mathcal{C}$ in general do not coincide.

Example 3.60. — The skew Sturmian subshift $\Sigma_{(0)} = \{0^k 1 0^\infty : k \geq 0\} \cup \{0^\infty\}$ given in Figure 10 is chain-transitive but not transitive.

Proof. — For any $n > 0$, $\sigma^n([1]) \cap [1] = \emptyset$, so Σ is not transitive. On the other hand, for any $n > 0$,

$$0^\infty \to 0^{n-1} 1 0^\infty \to 0^{n-2} 1 0^\infty \to \dots \to 0^\infty$$

is a 2^{-n}-chain which visits all cylinders of length n, so $\Sigma_{(0)}$ is chain-transitive. □

For any subshift $\Sigma \subseteq A^{\mathbb{N}}$ we construct a sequence of labelled graphs called Rauzy graphs, which approximate Σ.

Definition 3.61. — Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift. Its n-th Rauzy graph is a labelled graph $G_n = (V_n, E_n, s, t, l)$, where $V_n = \mathcal{L}^n(\Sigma)$, $E_n = \mathcal{L}^{n+1}(\Sigma)$, $s(u) = u_{[0,n-1]}$, $t(u) = u_{[1,n]}$, $l(u) = u_0$.

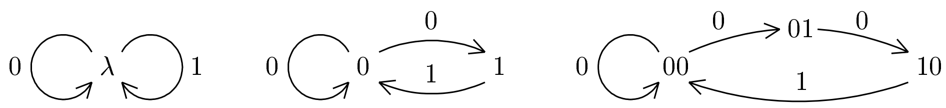


FIGURE 14. Rauzy graphs of a skew Sturmian subshift

The graph G_n gives rise to a sofic subshift Σ_{G_n} which approximates the original subshift Σ. In Figure 14 we see Rauzy graphs G_0, G_1 and G_2 for the skew Sturmian subshift $\Sigma_{(0)}$.

Proposition 3.62. — *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift and G_n its n-th Rauzy graph.*
(1) $\mathcal{L}^{n+1}(\Sigma) = \mathcal{L}^{n+1}(\Sigma_{G_n})$.
(2) Σ is chain-transitive iff all G_n are irreducible.

Proof. — The paths in G_n correspond to 2^{-n}-chains of the subshift. A sequence $(x_i \in \Sigma)_{i\geq 0}$ is a 2^{-n}-chain iff the point $y \in A^{\mathbb{N}}$ defined by $y_i = (x_i)_0$ is a label of a path in G_n, i.e., iff it belongs to the sofic subshift Σ_{G_n}. □

3.7.3. Topological entropy. — Topological entropy of a sofic subshift can be calculated from a nonnegative matrix associated to some of its right-resolving graph. The matrix gives the number of edges going from a vertex to another vertex. It does not depend on the labeling function l.

Definition 3.63. — The **adjacency matrix** $M = (M_{ab})_{a,b\in V}$ of a labelled graph $G = (V, E, s, t, l)$ is defined by

$$M_{ab} = \#\{e \in E : s(e) = a, t(e) = b\}.$$

Theorem 3.64. — *Let $G = (V, E, s, t, l)$ be a right-resolving graph and λ the spectral radius of its adjacency matrix. Then $h(\Sigma_G, \sigma) = \ln \lambda$.*

Proof. — We show first that the entropy of the edge shift is $h(\Sigma_{|G|}, \sigma) = \ln \lambda$. For any $a, b \in V$ and $n > 0$ we have

$$(M^n)_{ab} = \sum_{c_1,\dots,c_{n-1}} M_{a,c_1} \cdots M_{c_{n-1}b} = \#\{u \in \mathcal{L}^n(\Sigma_{|G|}) : s(u_0) = a,\ t(u_{n-1}) = b\}$$

For the complexity of the path subshift we get $P_{\Sigma_{|G|}}(n) = \sum_{a,b\in V}(M^n)_{ab}$. If $\Sigma_{|G|}$ is irreducible, by the Perron-Frobenius theorem A.73 we see that $h(\Sigma_{|G|}, \sigma) = \ln \lambda$. If $\Sigma_{|G|}$ is not irreducible we proceed as in Theorem 3.48 using Proposition 3.59 and Bowen's theorem 2.90.

We show now, that if G is right-resolving, then $h(\Sigma_G, \sigma) = h(\Sigma_{|G|}, \sigma)$. Let $k = \#V$ be the size of the set of vertices. For any $u \in \mathcal{L}(\Sigma_G)$ there are at most k words $v \in \mathcal{L}(\Sigma_{|G|})$ with $l(v) = u$. Thus

$$P_{\Sigma_{|G|}}(n) \leq P_{\Sigma_G}(n) \leq k \cdot P_{\Sigma_{|G|}}(n)$$

and $h(\Sigma_G, \sigma) = h(\Sigma_{|G|}, \sigma) = \ln \lambda$. □

In sofic subshifts, the entropy can be also calculated from the number of periodic points. For a dynamical system (X, F) and $n > 0$, set

$$\mathrm{Per}(n) = \#\{x \in X : F^n(x) = x\}.$$

Theorem 3.65. — *For any subshift* $\Sigma \subseteq A^{\mathbb{N}}$, $h(\Sigma,\sigma) \geq \limsup_{n\to\infty} \dfrac{\ln \mathrm{Per}(n)}{n}$. *For sofic subshifts the equality holds.*

Proof. —

(1) If $x,y \in \Sigma$, $x \neq y$, $F^n(x) = x$ and $F^n(y) = y$, then $x_{[0,n)} \neq y_{[0,n)}$, so the number of points $x \in \Sigma$ with $F^n(x) = x$ is at most the number of cylinders of length n, $\mathrm{Per}(n) \leq P(n)$. It follows

$$h(\Sigma,\sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{n} \geq \limsup_{n\to\infty} \frac{\ln \mathrm{Per}(n)}{n}$$

(2) Let $G = (V,E,s,t,l)$ be a labelled graph such that $\Sigma = \Sigma_G$. There exists a path component $V_0 \subseteq V$ such that $h(\Sigma,\sigma)$ is equal to the entropy of subshift $\Sigma_0 = \Sigma_G|V_0$ generated by graph $G_0 = (V_0,E_0,s,t,l)$, where $E_0 = \{e \in E : s(e),t(e) \in V_0\}$. We denote by $k = |V_0|$ the size of the set of vertices. Let $n > 0$ and $x \in \mathcal{L}^n(\Sigma_{G_0})$, so that $x = l(u)$ for some $u \in \mathcal{L}(\Sigma_{|G_0|})$. Since G_0 is irreducible, there exists a path $v \in \mathcal{L}(\Sigma_{|G_0|})$ from u_{n-1} to u_0 of length at most $k-1$. If $y = l(v)$, then $(xy)^\infty \in \Sigma_0$ is a periodic point with period at most $n+k-1$. Moreover, distinct $x,x' \in \mathcal{L}^n(\Sigma_{|G_0|})$ determine distinct xy and $x'y'$. Thus for $P(n) = \#\mathcal{L}^n(\Sigma_{G_0})$ we get

$$P(n) \leq \mathrm{Per}(n) + \mathrm{Per}(n+1) + \cdots + \mathrm{Per}(n+k-1).$$

Let $r(n) \in [n,n+k)$ be such that $\mathrm{Per}(r(n)) = \max\{\mathrm{Per}(n),\ldots,\mathrm{Per}(n+k-1)\}$, so $P(n) \leq k\cdot \mathrm{Per}(r(n))$ and $\lim_{n\to\infty} r(n)/n = 1$. Then

$$h(\Sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{r(n)} \cdot \frac{r(n)}{n} \leq \limsup_{n\to\infty} \frac{\ln k + \ln \mathrm{Per}(r(n))}{r(n)} \leq \limsup_{m\to\infty} \frac{\ln \mathrm{Per}(m)}{m}$$

□

In Theorem 4.77 we show that there exist minimal subshifts with positive topological entropy, so the inequality in Theorem 3.65 may be strict.

3.8. Two-sided subshifts

A biinfinite word in an alphabet A is a map $x : \mathbb{Z} \to A$, where $\mathbb{Z} = \{\ldots,-1,0,1,\ldots\}$ is the set of integers. We denote by x_i the letter on position i and x_I the restriction of x to an interval $I \subseteq \mathbb{Z}$. The point x is written as

$$x = \ldots x_{-2}x_{-1}.x_0x_1x_2\ldots$$

with dot immediately before the zero coordinate. For example, $x = 0^\infty.1^\infty$ is a point such that $x_i = 0$ iff $i < 0$. We consider a metric on $A^{\mathbb{Z}}$ given by

$$d(x,y) = 2^{-n} \text{ where } n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

With this metric, $A^{\mathbb{Z}}$ is a compact, totally disconnected and perfect metric space. For $u \in A^n$ and $m \in \mathbb{Z}$ denote by

$$[u]_m = \{x \in A^{\mathbb{Z}} : x_{[m,m+n)} = u\}$$

the cylinder of u at position m. The cylinders form a basis of the topology of $A^{\mathbb{Z}}$. The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by

$$\sigma_i(x) = x_i, \quad \sigma(\ldots x_{-2}x_{-1}.x_0x_1\ldots) = \ldots x_{-1}x_0.x_1x_2\ldots$$

In contrast to the one-sided case, σ is bijective. Its inverse is $\sigma^{-1}(x)_i = x_{i-1}$.

Definition 3.66. — A **two-sided subshift** is a closed subset $\Sigma \subseteq A^{\mathbb{Z}}$ such that $\sigma(\Sigma) = \Sigma$.

A closed set $\Sigma \subseteq A^{\mathbb{Z}}$ is a subshift iff $\sigma(\Sigma) \subseteq \Sigma$ and $\sigma^{-1}(\Sigma) \subseteq \Sigma$. Two-sided subshifts can be characterized topologically. If (X, F) is a bijective dynamical system, then there exist negative iterations $F^{-n} = (F^{-1})^n$.

Definition 3.67. — A bijective dynamical system (X, F) is **expansive** if

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists n \in \mathbb{Z}, d(F^n(x), F^n(y)) \geq \varepsilon.$$

Proposition 3.68. —

(1) Any expansive dynamical system is a factor of a two-sided subshift.

(2) A dynamical system (X, F) is conjugate to a two-sided subshift iff it is expansive and X is totally disconnected.

Proof. — Let (X, F) be an expansive dynamical system whose constant of expansivity is $\varepsilon > 0$, and $\mathcal{V} = \{V_a : a \in A\}$ a closed cover of X with $\operatorname{diam}(X) < \varepsilon$. For $u \in A^{2n+1}$, $x \in A^{\mathbb{Z}}$, set

$$V_u = \bigcap_{i=-n}^{n} F^{-i}(V_{u_i}), \quad V_x = \bigcap_{i=-\infty}^{\infty} F^{-i}(V_{u_i}).$$

Then $\Sigma = \{x \in A^{\mathbb{Z}} : V_x \neq \emptyset\}$ is a subshift and $\varphi : (\Sigma, \sigma) \to (X, F)$ defined by $\varphi(x) \in V_x$ is a factor map. If X is totally disconnected, we choose $\mathcal{V}$ to be a clopen partition and φ is then a conjugacy. $\square$

Definition 3.69. — A nonempty language $L \subseteq A^*$ is **central** (or biextendable), if

(1) For any $u \in L$ and any $v \sqsubseteq u$, $v \in L$.

(2) For any $u \in L$ there exist $a, b \in A$ such that $aub \in L$.

Definition 3.70. —

(1) The **language of a sequence** $x \in A^{\mathbb{Z}}$ is

$$\mathcal{L}(x) = \{u \in A^* : u \sqsubseteq x\}, \quad \mathcal{L}^n(x) = \mathcal{L}(x) \cap A^n.$$

(2) The **language of a subshift** $\Sigma \subseteq A^{\mathbb{Z}}$ is

$$\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}, \quad \mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n.$$

(3) The **subshift of a central language** is

$$\mathcal{Z}(L) = \{x \in A^{\mathbb{Z}} : \forall n \geq 0, x_{[-n,n)} \in L\}.$$

Proposition 3.71. —

(1) If $x \in A^{\mathbb{N}}$ is a transitive point, then $\mathcal{L}(x)$ is a central language.

(2) If $\Sigma \subseteq A^{\mathbb{N}}$ is a transitive subshift, then $\mathcal{L}(\Sigma)$ is a central language.

(3) If $x \in A^{\mathbb{Z}}$ is a point, then $\mathcal{L}(x)$ is a central language.

(4) If $\Sigma \subseteq A^{\mathbb{Z}}$ is a subshift, then $\mathcal{L}(\Sigma)$ is a central language and $\mathcal{Z}(\mathcal{L}(\Sigma)) = \Sigma$.

(5) If $L \subseteq A^*$ is central, then $\mathcal{Z}(L) \subseteq A^{\mathbb{Z}}$ is a subshift and $\mathcal{L}(\mathcal{Z}(L)) = L$.

A subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is of **finite type** (SFT), if there is a finite set $F \subseteq A^*$ such that

$$\Sigma = \Sigma_F = \{x \in A^{\mathbb{Z}} : \forall u \sqsubseteq x, u \notin F\}.$$

Given a set F of forbidden words, its one-sided and two-sided subshifts may differ considerably. For example, the set $F = \{00, 10\}$ of forbidden blocks gives rise to a one-sided subshift $\{01^\infty, 1^\infty\}$ and a two-sided subshift $\{1^\infty\}$.

Any two-sided SFT is conjugate to an SFT of order 2. The SFT of order two is defined by an oriented graph with vertex set A or a binary matrix $M = (M_{ab})_{a,b\in A}$. The matrix has a 1 in any row and also in any column. Conversely any such matrix determines a two-sided SFT.

A subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is **sofic**, if its language is regular. Two-sided sofic subshifts are defined by labelled graphs (V, E, s, t, l) satisfying

$$\forall i \in V, \exists e \in E, s(e) = i \ \ \& \ \ \forall i \in V, \exists e \in E, t(e) = i.$$

Equivalently, the matrix of a sofic subshift has a nonzero entry in any row and in any column.

There are several generalizations of sofic subshifts. If $K \subseteq A^+$ is any language, then

$$K^* = \{u \in A^* : \exists v_1, \dots, v_n \in K, u \sqsubseteq v_1 v_2 \dots v_n\}$$

is a central language. A subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is a **coded subshift**, if $\Sigma = \mathcal{Z}(K^*)$ for some nonempty $K \subseteq A^+$ (Blanchard and Hansel [**21**]). If K is a finite set, then $\mathcal{Z}(K^*)$ is a sofic subshift. Its labelled graph has a distinguished vertex 0 and for any $u \in K$, there is a path from 0 to 0 with label u. If K is infinite, $\mathcal{Z}(K^*)$ is not sofic. However, it shares with sofic subshifts some dynamical properties.

***Proposition 3.72* (Blanchard and Hansel [21]).** — *If $K \subseteq A^+$ contains at least two elements, then $\mathcal{Z}(K^*)$ is chaotic.*

Another generalization of sofic subshifts is based on the complexity hierarchy of formal languages. A subshift Σ is a **realtime subshift**, if its language is accepted by some Turing machine in real time (Kůrka and Maass [**93**]).

Exercise 3.1. —
(1) Find symbolic extensions for $(\mathbb{T}, R_{\frac{1}{2}})$, $(\mathbb{I}, x^2)$.
(2) Determine $\Sigma_{\mathcal{V}}(\mathbb{I}, Q_r)$ for the binary cover $\mathcal{V}$ and $2 \leq 3 \leq \sqrt{6} + 1 \doteq 3.4495$.
(3) Show that any finite subshift is an SFT.
(4) Show that an infinite countable subshift cannot be transitive.
(5) Find a countable subshift which is not sofic.
(6) Show that any subshift has finite topological entropy.
(7) Evaluate topological entropy of $\Sigma_{\{00,111\}}$ and $\Sigma_{\{000,111\}}$.
(8) Evaluate topological entropy of the sofic subshifts in Figure 10.
(9) Find the recurrence and transition characteristics of the binary adding machine.