CHAPTER 4

MINIMAL SYMBOLIC SYSTEMS

There are two large classes of minimal symbolic systems. **Adding machines** are topologically characterized as minimal equicontinuous systems on symbolic spaces. The state space of an adding machine is a compact group. It can be constructed either as an infinite product of possibly different alphabets, or, more simply, as an inverse limit of a sequence of finite groups. This is a parametric construction whose parameter is a **periodic structure** - a sequence $\mathbf{p} = (p_i)_{i \geq 0}$ such that p_i divides p_{i+1}.

Minimal subshifts form another large class of symbolic systems. They are topologically characterized as positively expansive minimal systems on symbolic spaces. There is no descriptive theory which encompasses all minimal subshifts. There are, however, some distinguished classes.

A simple and effective constructions of minimal subshifts is based on **substitutions**. All substitutive subshifts are uniquely ergodic, have zero topological entropy and their characteristics of recurrence and transition are linear. An important subclass is formed by **substitutions of constant length**. In this case the maximal equicontinuous factor is an adding machine. In the general case of nonconstant substitution length, the maximal equicontinuous factor can be a rotation of the circle (Fibonacci subshift) or even the trivial system (Example 2.8).

Sturmian subshifts are extensions of irrational rotations of the circle generated by a specific interval partition. Conversely, the maximal equicontinuous factor of a Sturmian subshift is the corresponding rotation of the circle. A Sturmian subshift is constructed from the continued fraction expansion of its parameter combinatorially. Sturmian subshifts are strictly ergodic with zero topological entropy. They are **subshifts of minimal complexity** $P(n) = n + 1$ and they are **balanced**. Two words of the same length have nearly the same number of ones.

Besides Sturmian subshifts which correspond to irrational numbers there are also **skew Sturmian subshifts** which correspond to rational numbers. They are not minimal but they may be thought of as limit cases of true Sturmian subshifts. Both Sturmian and skew Sturmian subshifts can be described in a unified scheme with the use of infinite sequences of substitutions. Some Sturmian subshifts (like the Fibonacci subshift) are substitutive: those obtained from quadratic irrationals.

Toeplitz subshifts are orbit closures of **quasi-periodic sequences**. An important characteristics of a Toeplitz subshift is its periodic structure. In fact the maximal equicontinuous factor of a Toeplitz subshift is the adding machine with the same periodic structure. Moreover, Toeplitz subshifts are smallest subshift extensions of adding machines. Toeplitz subshifts are much more diversified than either substitutive or Sturmian subshifts. Their entropy may attain an arbitrary nonnegative real number and they need not be strictly ergodic. Like a Sturmian subshift, a Toeplitz subshift can be obtained from an infinite sequence of substitutions. A Toeplitz subshift is substitutive iff this sequence is periodic. For example, the Feigenbaum subshift is both substitutive and Toeplitz. There is, however, no subshift which could be both Sturmian and Toeplitz. These two classes have different types of maximal equicontinuous factors: irrational rotations of the circle for the Sturmian subshifts and adding machines for the Toeplitz subshifts.

4.1. Adding machines

By Theorem 2.42, a minimal equicontinuous system is a rotation of a compact Abelian group. If the state space is a symbolic space, the group is a group of p-adic integers.

4.1.1. Groups of p-adic numbers. — The binary adding machine of Example 1.24 can be generalized by placing in the k-th position an alphabet $A_k = \{0, 1, \dots, n_k - 1\}$ with $n_k \geq 2$. For a sequence $\mathbf{n} = (n_k)_{k\geq 0}$ of integers greater than 1 we consider the product space

$$X_{\mathbf{n}} = \prod_{k\geq 0} \{0, 1, \dots, n_k - 1\}.$$

A sequence in $x \in X_{\mathbf{n}}$ with finitely many nonzero entries represents a nonnegative integer

$$\sum_{j=0}^{\infty} n_0 \cdots n_{j-1} x_j = x_0 + n_0 x_1 + n_0 n_1 x_2 + \cdots$$

Negative integers are represented by sequences with only finitely many entries x_i different from $n_i - 1$, for example $-1 = n_0 - 1, n_1 - 1, n_2 - 1, \dots$. Sequences $x, y \in X_{\mathbf{n}}$ are added coordinatewise with the carry to the right. We denote by c_j the carry from the position j to $j + 1$. Then $c_{-1} = 0$ and for $j \geq 0$,

$$(x + y)_j = \mathrm{mod}_{n_j}(x_j + y_j + c_{j-1}), \quad c_j = \left\lfloor \frac{x_j + y_j + c_{j-1}}{n_j} \right\rfloor.$$

We have $(x+y)_j + n_j c_j = x_j + y_j + c_{j-1}$, and by induction we prove that $c_j \in \{0, 1\}$. Indeed if $0 \leq c_{j-1} \leq 1$, then

$$0 \leq x_j + y_j + c_{j-1} \leq 2(n_j - 1) + c_{j-1} \leq 2n_j - 1$$

so $0 \leq c_j \leq 1$. Since $(x + y)_j$ depends only on $x_{[0,j]}$ and $y_{[0,j]}$, the addition is continuous.

There is an alternative and simpler description of $X_{\mathbf{n}}$. Given a sequence $\mathbf{n} = (n_j)_{j\geq 0}$ with $n_j \geq 2$, define a sequence $\mathbf{p} = (p_j)_{j\geq 0}$ by

$$p_0 = n_0, \quad p_j = p_{j-1} \cdot n_j = n_0 \cdot n_1 \cdots n_j.$$

Then $\mathbf{p} = (p_j)_{j\geq 0}$ is an increasing sequence such that $p_0 > 1$, and p_j divides p_{j+1}. Conversely, any sequence with these properties yields a sequence $n_i = p_i/p_{i-1}$ of integers greater than 1 provided we set $p_{-1} = 1$. Recall that $\mathbb{Z}(p) = \{0, 1, \ldots, p-1\}$ with addition modulo p is a group. If $p|q$ (p divides q), then $\mathrm{mod}_p : \mathbb{Z}(q) \to \mathbb{Z}(p)$ is a group homomorphism.

Definition 4.1. —

(1) A periodic structure is a sequence of integers $\mathbf{p} = (p_j)_{j\geq 0}$ such that $p_0 > 1$ and p_j divides p_{j+1} (we adopt the convention that $p_{-1} = 1$).

(2) The group $\mathbb{Z}_{\mathbf{p}}$ of **p**-adic integers is the inverse limit

$$\mathbb{Z}_{\mathbf{p}} = \left\{ x \in \prod_{j=0}^{\infty} \mathbb{Z}(p_j) : \forall j, \mathrm{mod}_{p_j}(x_{j+1}) = x_j \right\}.$$

(3) The zero element of $\mathbb{Z}_{\mathbf{p}}$ is $0 = (0, 0, \ldots)$. The addition and subtraction are defined by formulas

$$(x+y)_j = \mathrm{mod}_{p_j}(x_j + y_j), \quad (-x)_j = p_j - 1 - x_j.$$

In $\mathbb{Z}_{\mathbf{p}}$, positive integers are represented by sequences which are eventually constant. Negative sequences are represented by sequences for which eventually $x_j = p_j - 1$. For $n_0 = 3$, $n_1 = 2$, $n_2 = 2$ we get following representations of integers in $X_{\mathbf{n}}$ and $\mathbb{Z}_{\mathbf{p}}$:

	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$X_{\mathbf{n}}$:	201	011	111	211	000	100	200	010	110	210	001	101
$\mathbb{Z}_{\mathbf{p}}$:	228	039	$14A$	$25B$	000	111	222	033	144	255	006	117

We show that $X_{\mathbf{n}}$ and $\mathbb{Z}_{\mathbf{p}}$ are isomorphic. Define $\varphi : X_{\mathbf{n}} \to \mathbb{Z}_{\mathbf{p}}$, $\psi : \mathbb{Z}_{\mathbf{p}} \to X_{\mathbf{n}}$ by

$$\varphi(x)_k = \sum_{j=0}^{k} p_{j-1} x_j, \quad \psi(y)_j = \left\lfloor \frac{y_j}{p_{j-1}} \right\rfloor.$$

We show first that for $x \in X_{\mathbf{n}}$, $\varphi(x)_k < p_k$. For $k = 0$ we have $\varphi(x)_0 = x_0 < n_0 = p_0$. If $\varphi(x)_{k-1} \leq p_{k-1} - 1$, then

$$\varphi(x)_k = \varphi(x)_{k-1} + p_{k-1} x_k \leq p_{k-1} - 1 + p_{k-1}(n_k - 1) = p_{k-1} n_k - 1 = p_k - 1.$$

It follows that

$$\psi(\varphi(x))_k = \left\lfloor \frac{\varphi(x)_k}{p_{k-1}} \right\rfloor = \left\lfloor \frac{\varphi(x)_{k-1}}{p_{k-1}} + x_k \right\rfloor = x_k.$$

Conversely for $y \in \mathbb{Z}_{\mathbf{p}}$,

$$\begin{aligned} y_k &= p_{k-1} \left\lfloor \frac{y_k}{p_{k-1}} \right\rfloor + \mathrm{mod}_{p_{k-1}}(y_k) = p_{k-1} \left\lfloor \frac{y_k}{p_{k-1}} \right\rfloor + y_{k-1} = \cdots \\ &= p_{k-1} \left\lfloor \frac{y_k}{p_{k-1}} \right\rfloor + p_{k-2} \left\lfloor \frac{y_{k-1}}{p_{k-2}} \right\rfloor + \cdots + p_0 \left\lfloor \frac{y_1}{p_0} \right\rfloor + y_0 = \varphi(\psi(y))_k \end{aligned}$$

Thus ψ is inverse to φ. Assume now that $x, y \in X_{\mathbf{n}}$ and $z = x + y$. Then

$$x_j + y_j + c_{j-1} = z_j + n_j c_j$$

$$\sum_{j=0}^{k} p_{j-1}(x_j + y_j) + \sum_{j=0}^{k} p_{j-1} c_{j-1} = \sum_{j=0}^{k} p_{j-1} z_j + \sum_{j=0}^{k} p_j c_j$$

$$\sum_{j=0}^{k} p_{j-1}x_j + \sum_{j=0}^{k} p_{j-1}y_j = \sum_{j=0}^{k} p_{j-1}z_j + p_k c_k$$

and $\varphi(z)_k = \mathrm{mod}_{p_k}(\varphi(x)_k + \varphi(y)_k)$.

Proposition 4.2. — *The group $\mathbb{Z}_{\mathbf{p}}$ is compact and contains a dense subgroup isomorphic to $\mathbb{Z}$.*

Proof. — By Proposition 3.3, the inverse limit construction yields a compact space. Since $(x+y)_j$ and $(-x)_j$ depend only on x_j and y_j, the addition and subtraction are continuous. We have an injective group homomorphism $\varphi_{\mathbf{p}} : \mathbb{Z} \to \mathbb{Z}_{\mathbf{p}}$ given by $\varphi_{\mathbf{p}}(x)_j = \mathrm{mod}_{p_j}(x)$ and $\varphi_{\mathbf{p}}(\mathbb{Z})$ is dense in $\mathbb{Z}_{\mathbf{p}}$. □

Recall that a character of a compact group X is a continuous homomorphism $f : X \to \mathbb{K}$ (Definition A.66). The dual group $\widehat{X}$ of X is the group of characters of X.

Proposition 4.3. — *Let $\mathbf{p} = (p_j)_{j\geq 0}$ be a periodic structure. The dual group of $\mathbb{Z}_{\mathbf{p}}$ is isomorphic to the* **group of p-adic rationals**

$$\mathbb{Q}_{\mathbf{p}} = \left\{ \frac{j}{p_k} : j \in \{0, 1, \ldots, p_k - 1\}, k > 0 \right\} \subseteq \mathbb{T}.$$

Proof. — For $j/p_k \in \mathbb{Q}_{\mathbf{p}}$ define $f_{j/p_k} : \mathbb{Z}_{\mathbf{p}} \to \mathbb{K}$ by

$$f_{j/p_k}(x) = e^{2\pi i x_k j/p_k}$$

It is easy to see that this is a character. Conversely suppose that $f : \mathbb{Z}_{\mathbf{p}} \to \mathbb{K}$ is a character. There exists k such that for any $a \in \mathbb{Z}(p_k)$, $f([a]) \subseteq B_1(1)$. Here $[a] = \{x \in \mathbb{Z}_{\mathbf{p}} : x_k = a\}$ is the cylinder of a. Denote by $\mathbf{e} = (1, 1, 1, \ldots) = \varphi_{\mathbf{p}}(1) \in \mathbb{Z}_{\mathbf{p}}$ the unit of $\mathbb{Z}_{\mathbf{p}}$. Then $[a] = [a] + p_k\mathbf{e}$, and

$$f([a]) = f([a] + p_k\mathbf{e}) = f([a]) \cdot f(\mathbf{e})^{p_k}$$

hence $f(\mathbf{e})$ is a p_k-th root of 1, $f(\mathbf{e}) = e^{2\pi i j/p_k}$. If $n \in \mathbb{Z}$ is an integer and $x = \varphi_{\mathbf{p}}(n)$ its representation in $\mathbb{Z}_{\mathbf{p}}$, then

$$f(x) = f(n\mathbf{e}) = f(\mathbf{e})^n = e^{2\pi i \mathrm{mod}_{p_k}(n) j/p_k} = e^{2\pi i x_k j/p_k}.$$

Since $\mathbb{Z}$ is dense in $\mathbb{Z}_{\mathbf{p}}$, we have $f = f_{j/p_k}$. □

By the Pontrjagin duality theorem A.69, the dual group of $\mathbb{Q}_{\mathbf{p}}$ is isomorphic to $\mathbb{Z}_{\mathbf{p}}$. If $g : \mathbb{Q}_{\mathbf{p}} \to \mathbb{K}$ is a character, then $g(1/p_k)^{p_k} = 1$, so for some $x_k < p_k$, $g(1/p_k) = e^{2\pi i x_k/p_k}$. It follows that $\mathrm{mod}_{p_k}(x_{k+1}) = x_k$, so $x \in \mathbb{Z}_{\mathbf{p}}$.

In $\mathbb{Z}_{\mathbf{p}}$ we have not only addition but also multiplication. Each $\mathbb{Z}(p_k)$ is a ring with multiplication $\mathrm{mod}_{p_k}(a \cdot b)$. Thus we have a multiplication in $\mathbb{Z}_{\mathbf{p}}$

$$(x \cdot y)_k = \mathrm{mod}_{p_k}(x_k \cdot y_k).$$

Clearly $\mathbf{e} = (1, 1, \ldots)$ is a unit for this multiplication and $(\mathbb{Z}_{\mathbf{p}}, 0, \mathbf{e}, +, \cdot)$ is a ring. We can extend this ring to a field by a method used to extend $\mathbb{Z}$ to $\mathbb{Q}$. The resulting structure is called the group of p-adic numbers or the group of **Hensel expansions**. It is no longer compact and much of analysis can be developed inside it (see Koblitz [**86**], Schikhof [**138**] or Hewitt and Ross [**76**]).

4.1.2. Adding machines. — Any periodic structure $\mathbf{p} = (p_k)_{k\geq 0}$ gives rise to an adding machine $(\mathbb{Z}_{\mathbf{p}}, R_1)$, where R_1 is the addition of $\mathbf{e} = (1, 1, \ldots)$. Since any point is quasi-periodic, adding machines are minimal and equicontinuous. On symbolic spaces, these two properties characterize adding machines completely.

Theorem 4.4. — *Let (X, F) be a minimal equicontinuous dynamical system on a totally disconnected infinite space X. Then (X, F) is conjugate to an adding machine.*

Proof. — Suppose that $x \in X$ is an isolated point. Then $\{x\}$ is a neighbourhood of x and since x is recurrent, it is periodic. Since X is minimal, it is finite and this is a contradiction. Thus X is perfect. By Theorem 2.42, we can assume that F is an isometry. For $\varepsilon > 0$ define an equivalence $\approx_\varepsilon$ on X by $x \approx_\varepsilon y$ iff

$$\exists n \geq 0, \exists x_0, x_1, \ldots, x_n, (x_0 = x \ \ \& \ \ x_n = y \ \ \& \ \ \forall i < n, d(x_i, x_{i+1}) < \varepsilon)$$

Thus $x \approx_\varepsilon y$ iff there exists an ε-chain from x to y in the dynamical system (X, Id).

The equivalence classes of $\approx_\varepsilon$ are disjoint open sets and there is only a finite number of them, so they form a clopen partition. Since X is a Cantor space, there exists ε_0 such that $\approx_{\varepsilon_0}$ has at least two equivalence classes. Let $\mathcal{V}_0 = \{V_a^0 : a \in A_0\}$ be the clopen partition of $\approx_{\varepsilon_0}$. Since (X, F) is minimal, no V_a^0 is invariant and there exists $x \in V_a^0$ such that $F(x) \in V_b^0$ with $b \neq a$. If $d(y, x) < \varepsilon_0$, then $y \in V_a$ and $d(F(y), F(x)) < \varepsilon_0$, so $F(x) \in V_b^0$. Thus $F(V_a^0) \subseteq V_b^0$. Since (X, F) is minimal, F acts cyclically on $(V_a^0)_{a\in A_0}$ (Figure 1). Thus we may assume $A_0 = \{0, 1, \ldots, p_0 - 1\}$, $p_0 \geq 2$, and

$$F(V_a^0) \subseteq V_b^0, \ \text{ where } \ b = \mathrm{mod}_{p_0}(a + 1).$$

Since X is totally disconnected, there exists $\varepsilon_1 < \varepsilon_0$ such that the number p_1 of the equivalence classes of $\approx_{\varepsilon_1}$ is strictly greater than p_0. The clopen partition $\mathcal{V}_1 = \{V_a^1 : a \in [0, p_1)\}$, which consists of the equivalence classes of $\approx_{\varepsilon_1}$, is a refinement of $\mathcal{V}_0$ and F acts cyclically on $\mathcal{V}_1$ too. Let $n_1 > 1$ be the maximal number of classes of $\approx_{\varepsilon_1}$, which contain some class V_a^0 of $\approx_{\varepsilon_0}$. Its image $F(V_a^0)$ contains also n_1 classes of $\approx_{\varepsilon_0}$ and therefore any V_a^0 contains exactly n_1 classes of $\approx_{\varepsilon_0}$. It follows that $p_1 = p_0 n_1$. We can rearrange $\mathcal{V}_1$ so that if $b = \mathrm{mod}_{p_0}(a)$, then $V_a^1 \subset V_b^0$. In this way we construct a periodic structure $p = (p_i)_{i\geq 0}$, such that the adding machine $(\mathbb{Z}_p, R_1)$ is conjugate to (X, F). □

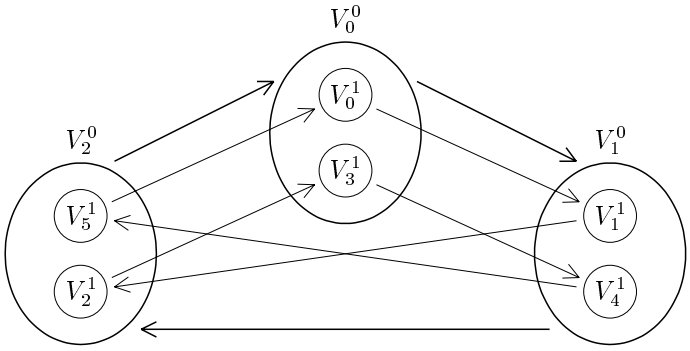

FIGURE 1. Adding machines

Two different periodic structures may yield conjugate adding machines. For example, if $\mathbf{p} = (3, 6, \ldots)$ and $\mathbf{q} = (2, 6, \ldots)$, we get trajectories

$$\begin{array}{lccccccccccccc} \mathbf{p}: & 000 & \mapsto & 111 & \mapsto & 222 & \mapsto & 033 & \mapsto & 144 & \mapsto & 255 & \mapsto & 066 & \mapsto & 117\ldots \\ \mathbf{q}: & 000 & \mapsto & 111 & \mapsto & 022 & \mapsto & 133 & \mapsto & 044 & \mapsto & 155 & \mapsto & 066 & \mapsto & 117\ldots \end{array}$$

and there is a conjugacy $\varphi : (\mathbb{Z}_{\mathbf{p}}, R_1) \to (\mathbb{Z}_{\mathbf{q}}, R_1)$ given by $\varphi(x)_0 = \mathrm{mod}_2(x_1)$, $\varphi(x)_i = x_i$ for $i \geq 1$. By Theorem 2.56, two adding machines are conjugate iff they have the same spectrum. Since the dual group of $\mathbb{Z}_{\mathbf{p}}$ is (isomorphic) to $\mathbb{Q}_{\mathbf{p}}$, the spectrum of $(\mathbb{Z}_{\mathbf{p}}, R_1)$ is

$$\mathcal{S}(\mathbb{Q}_{\mathbf{p}}) = \{e^{2\pi i j/p_k} : j \in \mathbb{Z}(p_k), k > 0\}.$$

***Proposition 4.5* (Buescu and Stewart [33]).** — *Let* $\mathbf{p}, \mathbf{q}$ *be periodic structures.*
(1) $(\mathbb{Z}_{\mathbf{p}}, R_1)$ *is a factor of* $(\mathbb{Z}_{\mathbf{q}}, R_1)$ *iff* $\mathbb{Q}_{\mathbf{q}} \subseteq \mathbb{Q}_{\mathbf{p}}$ *iff* $\forall j, \exists k, q_j | p_k$.
(2) $(\mathbb{Z}_{\mathbf{p}}, R_1)$ *and* $(\mathbb{Z}_{\mathbf{q}}, R_1)$ *are conjugate iff* $\mathbb{Q}_{\mathbf{p}} = \mathbb{Q}_{\mathbf{q}}$.

Proof. — If $\mathbb{Q}_{\mathbf{q}} \subseteq \mathbb{Q}_{\mathbf{p}}$, then for any $\frac{1}{q_j} \in \mathbb{Q}_{\mathbf{q}}$ there exists $\frac{m}{p_k} \in \mathbb{Q}_{\mathbf{p}}$ such that $\frac{1}{q_j} = \frac{m}{p_k}$, or $p_k = mq_j$. □

Define the **multiplicity** of a prime number a in a periodic structure $\mathbf{p}$ as

$$M_{\mathbf{p}}(a) = \sup\{k \geq 0 : \exists i, a^k | p_i\}$$

so $M_{\mathbf{p}}(a)$ is the highest power of a which divides some p_i. We have $0 \leq M_{\mathbf{p}}(a) \leq \infty$ and $\mathbb{Q}_{\mathbf{p}} = \mathbb{Q}_{\mathbf{q}}$ iff $M_{\mathbf{p}}(a) = M_{\mathbf{q}}(a)$ for all primes a.

We can easily construct a periodic structure all of whose multiplicities are infinite, e.g., $p_k = (k+2)!$. The corresponding adding machine is "universal" among all adding machines, i.e., any adding machine is a factor of it. More generally, universal systems occur in classes of dynamical systems defined by complexity classes of languages (see Kůrka [**90**]).

Example 4.6. — Any adding machine is a factor of the adding machine with periodic structure $p_k = (k+2)!$.

There are many equicontinuous systems on Cantor space which are not minimal, e.g., the identity. We shall see further examples in Chapter 5.

Proposition 4.7. — *Any equicontinuous system on the Cantor space has the shadowing property.*

Proof. — Let $(A^{\mathbb{N}}, F)$ be equicontinuous. By assumption, for any $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that if $x_{[0,k]} = y_{[0,k]}$, then $F^i(x)_{[0,n]} = F^i(y)_{[0,n]}$ for all $i \geq 0$. Let $(x_i \in A^{\mathbb{N}})_{i \geq 0}$ be a δ-chain. Then $F(x_i)_{[0,k]} = (x_{i+1})_{[0,k]}$, so for any $m > 0$,

$$F^m(x_0)_{[0,n]} = F^{m-1}(x_1)_{[0,n]} = \cdots = F(x_{m-1})_{[0,n]} = (x_m)_{[0,n]}.$$

Thus x_0 ε-shadows $(x_i)_{i \geq 0}$. □

4.1.3. Generalized adding machines. — Adding machines can be further generalized if we relax the condition $p_j | p_{j+1}$. This is linked to more general systems of numeration than digital expansions. In a digital expansion in base $p \geq 2$, any nonnegative integer is uniquely expressed as

$$x = x_0 + px_1 + \cdots + p^k x_k$$

with $0 \leq x_j < p$ and $x_0 + px_1 + \cdots p^k x_k < p^{k+1}$. The last property can be generalized. We say that $\mathbf{p} = (p_j)_{j \geq -1}$ is a **numeration structure**, if $p_{-1} = 1$ and $p_j > p_{j-1}$. Given a positive integer x, there exists unique k with $p_{k-1} \leq x < p_k$. Set

$$x_k = \lfloor \frac{x}{p_{k-1}} \rfloor < \frac{p_k}{p_{k-1}}$$

and continue with the number $x - p_{k-1}x_k < p_{k-1}$. We obtain a unique

$$x_{k-1} = \lfloor \frac{x - p_{k-1}x_k}{p_{k-2}} \rfloor < \frac{p_{k-1}}{p_{k-2}}$$

and $x - p_{k-1}x_k - p_{k-2}x_{k-1} < p_{k-2}$.

Proposition 4.8. — *Let* $\mathbf{p} = (p_j)_{j \geq -1}$ *be a numeration structure. For any integer* $x \geq 0$ *there exists* $k \geq 0$ *and unique integers* $x_0, \ldots, x_k$ *such that* $0 \leq x_j < p_j/p_{j-1}$, $x = p_{-1}x_0 + \cdots p_{k-1}x_k$, *and*

$$\sum_{i=0}^{j} p_{i-1}x_i < p_j, \quad 0 \leq j \leq k.$$

This condition determines symbolic space $\mathbb{Z}_{\mathbf{p}}$ which consists of all infinite integer sequences $x = (x_i)_{i \geq 0}$ which satisfy

$$0 \leq x_j < p_j/p_{j-1}, \quad \sum_{i=0}^{j} p_{i-1}x_i < p_j.$$

In some of these spaces, continuous addition can be defined (see Grabner et al. [**68**] or Barat et al. [**12**]). These generalized adding machines, however, are no longer equicontinuous. For example the Fibonacci sequence $\mathbf{p} = (1, 2, 3, 5, 8, \cdots)$ determines symbolic space $\mathbb{Z}_{\mathbf{p}}$ of those binary sequences which do not contain two consecutive ones. This is the golden mean subshift of Example 3.13. The representation of positive integers in this system is

1	2	3	4	5	6	7	8	9
1	01	001	101	0001	1001	0101	00001	10001

General minimal symbolic systems can be described by infinite graphs as in the Bratelli-Vershik models. See e.g., Giordano et al. [**63**] or Gjerde and Johansen [**64**].

4.2. Substitutive subshifts

Some minimal subshifts like Feigenbaum subshift are defined via substitutions. These are maps which preserve the algebraic structure of a **monoid**. The set A^* of finite words is a monoid with the operation of concatenation and unit element λ. For $u \in A^*$ denote by $|u|_a = \#\{i < |u| : u_i = a\}$ the number of occurrences of a letter a in a word u. Clearly $\sum_{a \in A} |u|_a = |u|$ is the length of u.

Definition 4.9. —

(1) A **monoid morphism** is a map $\vartheta : A^* \to B^*$ satisfying $\vartheta^{-1}(\lambda) = \{\lambda\}$, $\vartheta(uv) = \vartheta(u)\vartheta(v)$. Its **local rule** is its restriction $\vartheta : A \to B^+$. A monoid morphism extends to a map $\vartheta : A^{\mathbb{N}} \to B^{\mathbb{N}}$ via $\vartheta(x) = \vartheta(x_0)\vartheta(x_1)\dots$

(2) Given a monoid morphism $\vartheta : A^* \to B^*$, its **lower norm** $|\vartheta|$, **upper norm** $||\vartheta||$, and **matrix** $M(\vartheta) = (M(\vartheta)_{ab})_{a\in A, b\in B}$ are defined by

$$|\vartheta| = \min\{|\vartheta(a)| : a \in A\}, \quad ||\vartheta|| = \max\{|\vartheta(a)| : a \in A\}, \quad M(\vartheta)_{ab} = |\vartheta(a)|_b.$$

(3) A **substitution** over an alphabet A is a monoid morphism $\vartheta : A^* \to A^*$.

(4) A substitution $\vartheta : A^* \to A^*$ is **primitive**, if its matrix $M(\vartheta)$ is primitive.

(5) We say that $\vartheta : A^* \to A^*$ is a **substitution of constant length** l, if $|\vartheta| = ||\vartheta|| = l$. The local rule is then $\vartheta : A \to A^l$.

We always assume that a monoid morphism $\vartheta : A^* \to B^*$ is **nonerasing**, i.e., $\vartheta(a) \neq \lambda$ for $a \in A$. It follows that $|\vartheta(u)| \geq |u|$ for any $u \in A^*$. Moreover, for any $x \in A^{\mathbb{N}}$, $\vartheta(x) = \lim_{n\to\infty} \vartheta(x_{[0,n)})$ in the space $\overline{B^*}$. A substitution over A can be iterated and we are interested in its fixed points or periodic points in $A^{\mathbb{N}}$.

Example 4.10 **(Feigenbaum substitution)**. —

$$\vartheta = \begin{cases} 0 \mapsto 11 \\ 1 \mapsto 10 \end{cases} \qquad M(\vartheta) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad M(\vartheta)^2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

This is a primitive substitutions. It has a fixed point whose prefixes are

$$1 \mapsto 10 \mapsto 1011 \mapsto 1011\ 1010 \mapsto 1011\ 1010\ 1011\ 1011\dots$$

Example 4.11 **(Fibonacci substitution)**. —

$$\nu_1 = \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 10 \end{cases} \qquad M(\vartheta) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad M(\vartheta)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

This is a primitive substitutions with a fixed point

$$1 \mapsto 10 \mapsto 101 \mapsto 10110 \mapsto 10110\ 101 \mapsto 10110\ 101\ 10110 \mapsto \dots$$

Example 4.12 **(Morse substitution [112])**. —

$$\vartheta = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases} \qquad M(\vartheta) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This is a primitive substitution with two fixed points

$$\begin{array}{ccccccccccc} 0 & \mapsto & 01 & \mapsto & 0110 & \mapsto & 0110\ 1001 & \mapsto & 0110\ 1001\ 1001\ 011 & \cdots \\ 1 & \mapsto & 10 & \mapsto & 1001 & \mapsto & 1001\ 0110 & \mapsto & 1001\ 0110\ 0110\ 1001 & \dots \end{array}$$

Example 4.13 **(Cantor substitution)**. —

$$\vartheta = \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 111 \end{cases}$$

This substitution is not primitive. It has two fixed points 1^∞ and

$$0 \mapsto 010 \mapsto 010111010 \mapsto 010111010\ 111111111\ 010111010 \mapsto \dots$$

Proposition 4.14. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution and λ the spectral radius of its matrix (Theorem A.72). Then $M(\vartheta^k) = M(\vartheta)^k$ and there exist $0 < \alpha_0 < \alpha_1$, such that for any $k > 0$,*

$$\alpha_0 \lambda^k \leq |\vartheta^k| \leq ||\vartheta^k|| \leq \alpha_1 \lambda^k.$$

Proof. — By induction on k we get

$$\begin{aligned} M(\vartheta^{k+1})_{ab} &= |\vartheta(\vartheta^k(a))|_b = \sum_{c \in A} |\vartheta^k(a)|_c \cdot |\vartheta(c)|_b = \sum_{c \in A} M(\vartheta)^k_{ac} \cdot M(\vartheta)_{cb} \\ &= (M(\vartheta)^{k+1})_{ab}. \end{aligned}$$

Since $|\vartheta^k(a)| = \sum_{b \in A} M(\vartheta)^k_{ab}$, we get $\alpha_0 \lambda^k \leq |\vartheta^k| \leq ||\vartheta^k|| \leq \alpha_1 \lambda^k$ by the Perron-Frobenius theorem A.72. □

Recall that the language $\mathcal{L}(x)$ of $x \in A^{\mathbb{N}}$ is the set of finite subwords of x.

Proposition 4.15. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution.*

(1) $(A^{\mathbb{N}}, \vartheta)$ is equicontinuous and has a periodic point.

(2) Any two ϑ-periodic points $x, y \in A^{\mathbb{N}}$ have the same languages, $\mathcal{L}(x) = \mathcal{L}(y)$.

(3) If $x \in A^{\mathbb{N}}$ is ϑ-periodic, then there exists $r > 0$, such that for any $n > 0$ and $m \geq rn$, any $v \in \mathcal{L}^m(x)$ contains as subwords all words $u \in \mathcal{L}^n(x)$.

(4) If $x \in A^{\mathbb{N}}$ is ϑ-periodic, it is σ-almost periodic.

Proof. —

(1) If $x_{[0,n)} = y_{[0,n)}$, then $\vartheta(x)_{[0,n)} = \vartheta(y)_{[0,n)}$, so $d(\vartheta(x), \vartheta(y)) \leq d(x, y)$ and ϑ is equicontinuous. Define a map $f : A \to A$ by $f(a) = \vartheta(a)_0$. Then (A, f) is a dynamical system on a finite set, so it has a periodic point $a \in A$ with $f^p(a) = a$, i.e., $\vartheta^p(a)_0 = a$, and

$$a \sqsubseteq_p \vartheta^p(a) \sqsubseteq_p \vartheta^{2p}(a) \sqsubseteq_p \vartheta^{3p}(a) \dots$$

where $u \sqsubseteq_p v$ means that u is a prefix of v. Since ϑ is primitive, $|\vartheta^p(a)| > 1$ and $|\vartheta^{pi}(a)| > i$. The unique $x \in A^{\mathbb{N}}$ with prefixes $\vartheta^{ip}(a)$ satisfies $\vartheta^p(x) = x$. In the space $\overline{A^*}$ we have $x = \lim_{i \to \infty} \vartheta^{ip}(a)$.

(2) Let $x, y \in A^{\mathbb{N}}$ be ϑ-periodic and let u be a finite subword of x. There exists $k > 0$ such that $u \sqsubseteq \vartheta^k(x_0)$. Since ϑ is primitive, x_0 occurs in some $\vartheta^j(y_0)$ and $u \sqsubseteq \vartheta^k(x_0) \sqsubseteq \vartheta^{k+j}(y_0) \sqsubseteq y$. Thus $\mathcal{L}(x) \subseteq \mathcal{L}(y)$ and similarly, $\mathcal{L}(y) \subseteq \mathcal{L}(x)$.

(3) There exist integers q, s, r such that

$$\forall ab \in \mathcal{L}^2(x), ab \sqsubseteq \vartheta^q(x_0), \quad \forall c \in A, x_0 \sqsubseteq \vartheta^s(c), \quad r = 2\lambda^{q+s+1}\alpha_1/\alpha_0,$$

where λ is the spectral radius of $M(\vartheta)$ and α_i are the constants from Proposition 4.14. Given $n > 0$, there exists a unique k with $|\vartheta^{k-1}| < n \leq |\vartheta^k|$. Set $m_0 = 2||\vartheta^{k+q+s}||$. We show that any $v \in \mathcal{L}^{m_0}(x)$ contains as subword any $u \in \mathcal{L}^n(x)$. Given u and v, there exist $ab \in \mathcal{L}^2(x)$ and $c \in A$ with

$$u \sqsubseteq \vartheta^k(ab) \sqsubseteq \vartheta^{k+q}(x_0), \quad \vartheta^{k+q+s}(c) \sqsubseteq v.$$

Since $x_0 \sqsubseteq \vartheta^s(c)$, we get $u \sqsubseteq v$. If $m \geq rn$, then

$$m \geq rn > r\alpha_0\lambda^{k-1} = 2\alpha_1\lambda^{k+q+s} \geq 2||\vartheta^{k+q+s}|| = m_0,$$

so any word of length m contains as subwords all words of length n.

(4) If $x \in [u]$, and $m \geq r|u|$, then for any i, $u \sqsubseteq x_{[i,i+m)}$, so x is σ-almost periodic. □

Observe that the assertion (1) holds also for nonprimitive substitutions. Any substitution ϑ has a ϑ-periodic point. However, its periodic point need not be σ-almost periodic. This situation arises in the Cantor substitution in Example 4.13.

Definition 4.16. — If $\vartheta : A^* \to A^*$ is a primitive substitution, denote by $\Sigma_\vartheta \subseteq A^{\mathbb{N}}$ the subshift which has the same language as any ϑ-periodic point.

$$\Sigma_\vartheta = \overline{\mathcal{O}(x)} \Leftrightarrow \exists p > 0, \vartheta^p(x) = x \Leftrightarrow \mathcal{L}(\Sigma_\vartheta) = \mathcal{L}(x)$$

A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is **substitutive**, if it is the orbit closure of a ϑ-periodic point of some substitution.

If $\vartheta^p(x) = x$, then x is a fixed point of the substitution ϑ^p. Thus we have

Corollary 4.17. —

(1) Any substitutive subshift is the orbit closure of a fixed point of some substitution.

(2) Any substitutive subshift is minimal and its running time is sublinear, i.e., there exists r such that $T(n) \leq rn$.

(3) Any substitutive subshift has zero topological entropy.

(4) If ϑ is a primitive substitution, then $\vartheta(\Sigma_\vartheta) \subseteq \Sigma_\vartheta$.

Proof. —

(1),(2) follow from Proposition 4.15.

(3) Since $P(n) \leq T(n)$, the topological entropy is zero.

(4) If $x \in A^{\mathbb{N}}$ is a ϑ-periodic point, then $\mathcal{L}(\vartheta(x)) = \mathcal{L}(x)$ by Proposition 4.15. If $y \in \Sigma_\vartheta$, then $\mathcal{L}(y) \subseteq \mathcal{L}(x)$, so $\mathcal{L}(\vartheta(y)) \subseteq \mathcal{L}(\vartheta(x))$, and $\vartheta(y) \in \Sigma_\vartheta$. □

Proposition 4.18. — *If $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift and $\vartheta : A^* \to B^*$ is a monoid morphism, then*

$$\vartheta[\Sigma] = \{\sigma^k \vartheta(x) : x \in \Sigma, k < ||\vartheta||\} \subseteq B^{\mathbb{N}}$$

is a subshift. If Σ is minimal then so is $\vartheta[\Sigma]$.

Proof. — Since $\vartheta[\Sigma] = \vartheta(\Sigma) \cup \sigma\vartheta(\Sigma) \cup \cdots \cup \sigma^{||\vartheta||-1}\vartheta(\Sigma)$ is a finite union of closed sets, it is closed. Clearly it is σ-invariant, so $\vartheta[\Sigma]$ is a subshift. If $\vartheta[\Sigma]$ were not minimal and $\Theta \subset \vartheta[\Sigma]$, then $\vartheta^{-1}(\Theta) \subset \Sigma$ would be a proper subshift of Σ. □

If $\vartheta : A^* \to B^*$ is a monoid morphism we write

$$\vartheta[A^{\mathbb{N}}] = \{\vartheta(a_1), \ldots, \vartheta(a_n)\}^{\mathbb{N}}.$$

This is a sofic subshift. Its labelled graph has a specified vertex 0 and for any $a \in A$ a path from 0 to 0 with label $\vartheta(a)$. For example the golden mean subshift can be written as $\{0, 01\}^{\mathbb{N}}$ and the even subshift is $\{0, 11\}^{\mathbb{N}}$. If $\vartheta : A^* \to A^*$ is a primitive substitution, then $\vartheta[\Sigma_\vartheta] = \Sigma_\vartheta$ since both these subshifts are minimal and both contain a ϑ-periodic point. It follows that

$$\Sigma_\vartheta \subseteq \bigcap_{i>0} \vartheta^i[A^{\mathbb{N}}].$$

For many substitutions, these two subshifts are equal, but in general they may differ.

Example 4.19. —

$$\vartheta = \begin{cases} 0 \mapsto 220 \mapsto 021\,021\,220 \\ 1 \mapsto 122 \mapsto 122\,021\,021 \\ 2 \mapsto 021 \mapsto 220\,021\,122 \end{cases}$$

No fixed point contains word 01, so $01 \notin \mathcal{L}(\Sigma_\vartheta)$. However, all iterates of 01 do contain 01, so $01 \in \mathcal{L}(\bigcap_{i>0} \vartheta^i[A^{\mathbb{N}}])$.

4.2.1. Elementary substitutions. — A ϑ-fixed point of a primitive substitution $\vartheta : A^* \to A^*$ may be also σ-periodic. In this case Σ_ϑ is a finite subshift consisting of a single σ-periodic orbit.

Example 4.20. —

$$\vartheta = \begin{cases} 0 \mapsto 01 \mapsto 0101 \\ 1 \mapsto 01 \mapsto 0101 \end{cases}$$

This is a primitive substitution with unique fixed point $(01)^\infty$ and finite subshift $\Sigma_\vartheta = \{(01)^\infty, (10)^\infty\}$.

Example 4.21. —

$$\vartheta = \begin{cases} 0 \mapsto 01 \mapsto 012 \\ 1 \mapsto 2 \mapsto 012 \\ 2 \mapsto 012 \mapsto 012012 \end{cases}$$

This is a primitive substitution with unique fixed point $(012)^\infty$.

The question whether a ϑ-fixed point of an elementary substitution is σ-periodic or not, can be decided algorithmically. The key concept is **elementary substitution** (Rozenberg and Salomaa [**136**]). We now also consider substitutions which are not primitive.

Definition 4.22. — A substitution $\vartheta : A^* \to A^*$ is **elementary**, if it cannot be written as a composition $\vartheta = \eta \circ \zeta$ where $\zeta : A^* \to B^*$, $\eta : B^* \to A^*$ are monoid morphisms and $\#B < \#A$.

If ϑ is not elementary and $\vartheta = \eta \circ \zeta$ is its decomposition, then any word $\vartheta(a) = \eta(\zeta(a)_0) \dots \eta(\zeta(a)_{n-1})$ is a concatenation of some words from $\eta(B)$. The substitution from Example 4.20 is not elementary since it can be decomposed using alphabet $B = \{a\}$ as

$$\begin{array}{ccccc} 0 & \overset{\zeta}{\mapsto} & a & \overset{\eta}{\mapsto} & 01 \\ 1 & \overset{\zeta}{\mapsto} & a & \overset{\eta}{\mapsto} & 01 \end{array}$$

Proposition 4.23. — *The local rule $\vartheta : A \to A^+$ of an elementary substitution is injective.*

Proof. — If $\vartheta : A \to A^+$ is not injective and $\vartheta(a) = \vartheta(b)$ for some $a \neq b$, set $B = A \setminus \{a\}$, define $\eta(c) = \vartheta(c)$ for $c \in B$, and

$$\zeta(c) = \begin{cases} c \text{ for } c \neq a \\ b \text{ for } c = a \end{cases}$$

Then $\eta(\zeta(a)) = \eta(b) = \vartheta(b) = \vartheta(a)$. For $c \neq a$ we have $\eta(\zeta(c)) = \eta(c) = \vartheta(c)$, so $\vartheta = \eta \circ \zeta$. □

$$\begin{array}{ccccc} a & \overset{\zeta}{\mapsto} & b & \overset{\eta}{\mapsto} & \vartheta(b) \\ b & \overset{\zeta}{\mapsto} & b & \overset{\eta}{\mapsto} & \vartheta(b) \\ c & \overset{\zeta}{\mapsto} & c & \overset{\eta}{\mapsto} & \vartheta(c) \end{array}$$

The local rule of Example 4.21 is injective but the substitution is not elementary since $012 = \vartheta(3) = \vartheta(0)\vartheta(2)$. It can be therefore decomposed using alphabet $B = \{a, b\}$ as

$$\begin{array}{ccccc} 0 & \overset{\zeta}{\mapsto} & a & \overset{\eta}{\mapsto} & 01 \\ 1 & \overset{\zeta}{\mapsto} & b & \overset{\eta}{\mapsto} & 2 \\ 2 & \overset{\zeta}{\mapsto} & ab & \overset{\eta}{\mapsto} & 012 \end{array}$$

Proposition 4.24. — *If $\vartheta : A^* \to A^*$ is an elementary substitution, then $\vartheta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is injective.*

Proof. — Define the length of a substitution as $\ell(\vartheta) = \sum_{a \in A} |\vartheta(a)|$. Assume that the proposition does not hold and, for a given alphabet A, let $\vartheta : A^* \to A^*$ be an elementary substitution of minimal length for which ϑ is not injective. If $\ell(\vartheta) = \#A$, then the local rule $\vartheta : A \to A$ is a permutation, so $\vartheta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is also a permutation. Thus $\ell(\vartheta) > \#A$. There exist $u, v \in A^{\mathbb{N}}$ such that $\vartheta(u) = \vartheta(v)$ and $a = u_0 \neq v_0 = b$. Since $\vartheta : A \to A^+$ is one-to-one, $\vartheta(a) \neq \vartheta(b)$ and one of these words is a prefix of the other. Suppose that $\vartheta(b)$ is a prefix of $\vartheta(a)$, (Figure 2 top). Define substitutions $\zeta, \eta : A^* \to A^*$ by

$$\begin{array}{rclcrcl} \zeta(c) & = & c, & & \eta(c) & = & \vartheta(c) \quad \text{for} \quad c \neq a \\ \zeta(a) & = & ba, & & \vartheta(b)\eta(a) & = & \vartheta(a) \end{array}$$

For $c \neq a$ we have $\eta \circ \zeta(c) = \eta(c) = \vartheta(c)$ and

$$\eta \circ \zeta(a) = \eta(ba) = \eta(b)\eta(a) = \vartheta(b)\eta(a) = \vartheta(a).$$

Thus $\vartheta = \eta \circ \zeta$, so η is elementary (If it were not, its decomposition would give rise to a decomposition of ϑ). Since $\ell(\eta) < \ell(\vartheta)$, the statement holds for η, so $\eta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is injective. Since $\zeta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is injective, $\vartheta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is also injective. This is a contradiction. □

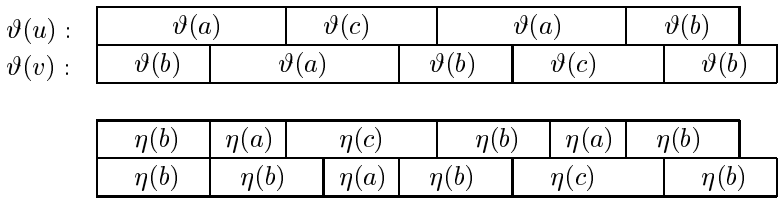

FIGURE 2. Elementary substitution

***Proposition 4.25* (Pansiot [120]).** — *Let $\vartheta : A^* \to A^*$ be an elementary substitution and $x \in A^{\mathbb{N}}$ a ϑ-fixed point, which is σ-periodic. Then $\forall a \in A, \exists! b \in A, ab \in \mathcal{L}(x)$.*

Proof. — We show more generally, that for any $n > 0$ and $u \in \mathcal{L}^n(x)$ there exists a unique $b \in A$ with $ub \in \mathcal{L}^{n+1}(x)$. Let $n > 0$ and suppose that $u \in \mathcal{L}^n(x)$ is a word with two different prolongations. Thus there exist $a, b \in A$ with $a \neq b$ and $ua, ub \in \mathcal{L}(x)$. Since any word of length n has a prolongation and u has two prolongations, $P(n+1) > P(n)$, where $P(n) = \#\mathcal{L}^n(x)$ is the complexity function. Let $i, j \in \mathbb{N}$ be such that $x_{[i,i+n]} = ua$, $x_{[j,j+n]} = ub$ and set $y = x_{[i,\infty)}$, $z = x_{[j,\infty)}$. Then $y \neq z$, so $\vartheta(y) \neq \vartheta(z)$. Let $\vartheta(u)v$ be the longest common prefix of $\vartheta(y)$ and $\vartheta(z)$, so there exist $c \neq d$ such that $\vartheta(u)vc, \vartheta(u)vd \in \mathcal{L}(x)$. If $m = |\vartheta(u)|v$, then $m > n$ and $P(m+1) > P(m)$. Repeating this argument we see that for any k, $\vartheta^k(u)$ is a prefix of some word which has two different prolongations in $\mathcal{L}(x)$, so $\lim_{n\to\infty} P(n) = \infty$. However, if $\sigma^p(x) = x$, then $P(n) \leq p$ for any n and this is a contradiction. □

Proposition 4.25 yields an upper bound for the σ-period of a ϑ-fixed point of a substitution. If ϑ is elementary and if its ϑ-fixed point is σ-periodic, then its period is at most $\#A$: Any letter in x has a unique follower. For a general (primitive) substitution we get an upper bound by a reduction to an elementary substitution.

Proposition 4.26. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution and $x \in A^{\mathbb{N}}$ a ϑ-fixed point which is σ-periodic. Then its σ-period is at most $||\vartheta||^{\#A-1}$.*

Proof. — Set $A_0 = A$, $\vartheta_0 = \vartheta$ and assume that $\zeta_0 : A_0^* \to A_1^*$, $\eta_0 : A_1^* \to A_0^*$ are monoid morphisms, $|A_1| < |A_0|$, and $\vartheta_0 = \eta_0 \circ \zeta_0$, so that $\eta_0 : A_0^* \to A_0^*$ is not elementary. If $x \in A_0^{\mathbb{N}}$ is a ϑ_0-fixed point, then $y = \zeta_0(x) \in A_1^{\mathbb{N}}$ is a fixed point of $\vartheta_1 = \zeta_0 \circ \eta_0 : A_1^* \to A_1^*$ since (Figure 3)

$$\vartheta_1(y) = \zeta_0 \circ \eta_0 \circ \zeta_0(x) = \zeta_0 \circ \vartheta_0(x) = \zeta_0(x) = y.$$

Moreover, $\eta_0(y) = \eta_0 \circ \zeta_0(x) = \vartheta_0(x) = x$, so x is σ-periodic iff y is σ-periodic, and the period of x is at most the period of y multiplied by $||\eta_0|| \leq ||\vartheta_0||$.
If $\vartheta = \vartheta_0$ is not elementary, there exist alphabets $A = A_0, A_1, \dots A_k$ with $\#A_i > \#A_{i+1}$. Further, for $0 \leq i < k$ there exist monoid morphisms $\zeta_i : A_i^* \to A_{i+1}^*$, $\eta_i : A_{i+1}^* \to A_i^*$, such that $\zeta_i \circ \eta_i = \vartheta_{i+1} = \eta_{i+1}\zeta_{i+1}$ for $i < k$, and the last substitution $\vartheta_k = \zeta_{k-1} \circ \eta_{k-1} : A_k \to A_k$ is elementary. (The last A_k may be one-element set.) Since the period of the ϑ_k-fixed point of the last substitution is at most $\#A_k$, the period of the fixed point of ϑ is at most $\#A_k \cdot ||\eta_{k-1}|| \cdots ||\eta_0|| \leq (\#A - k) \cdot ||\vartheta||^k$. Since $||\vartheta|| \geq 2$,

$$\#A \leq (\#A - 1) \cdot ||\vartheta|| \leq (\#A - 2) \cdot ||\vartheta||^2 \leq \cdots \leq 2 \cdot ||\vartheta||^{\#A-2} \leq ||\vartheta||^{\#A-1},$$

and the period of x is at most $||\vartheta||^{\#A-1}$. □

For the Feigenbaum substitution, $||\vartheta||^{\#A-1} = 2$. Since the ϑ-fixed point is not periodic with period 1 or 2, it is aperiodic. The ϑ-fixed point $(01)^\infty$ of Example 4.20 is periodic with period $2 = ||\vartheta||^{\#A-1}$. The fixed point $(012)^\infty$ of Example 4.21 is periodic with period $3 < ||\vartheta||^{\#A-1} = 3^2$.

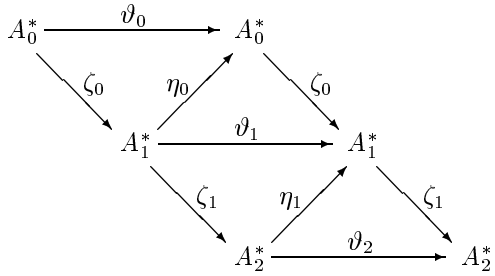

FIGURE 3. Elementary decomposition

4.2.2. Aperiodic words. — To construct an algorithm deciding whether a ϑ-fixed point is σ-periodic, we need a sufficient condition for periodicity. Call a finite word $v \in A^+$ **aperiodic**, if $v \neq u^n$ for any $n > 1$ and $u \in A^+$.

Proposition 4.27 **(Mossé [114])**. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution with a ϑ-fixed point x. Suppose that there exists $p > 0$ and an aperiodic word $v \in A^+$ such that $2|v| \leq |\vartheta^p|$ and $\forall ab \in \mathcal{L}^2(x)$, $\vartheta^p(ab) \sqsubseteq v^\infty$. Then $x = v^\infty$ is σ-periodic.*

Proof. — For any $a \in A$, $|\vartheta^p(a)| \geq 2|v|$, so we can write $\vartheta^p(a) = u_a v^{n_a} w_a$, where $n_a \geq 1$, u_a is a suffix of v and w_a is a prefix of v (Figure 4 left). For $ab \in \mathcal{L}^2(\Sigma_\vartheta)$ we get

$$v^\infty \sqsupseteq \vartheta^p(ab) = u_a v^{n_a} w_a u_b v^{n_b} w_b \sqsupseteq v w_a u_b v.$$

Since v is aperiodic, either $w_a u_b = \lambda$ or $w_a u_b = v$. It follows that for any $y \in \mathcal{L}(\Sigma_\vartheta)$, $\vartheta^p(y) \sqsubseteq v^\infty$, so $x = \vartheta^p(x) = v^\infty$ is σ-periodic. □

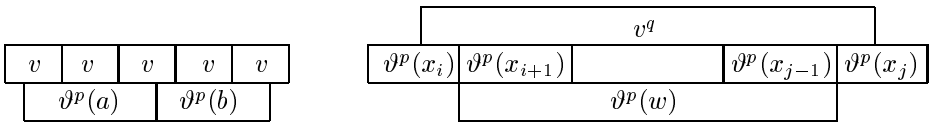

FIGURE 4. Aperiodic words

Corollary 4.28. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution, $x \in A^{\mathbb{N}}$ its ϑ-fixed point and p the first integer with $|\vartheta^p| \geq 2 \cdot ||\vartheta||^{\#A-1}$. Then x is σ-periodic iff there exists an aperiodic word $v \in A^+$ such that $|v| \leq ||\vartheta||^{\#A-1}$ and for any $ab \in \mathcal{L}^2(\Sigma_\vartheta)$, $\vartheta^p(ab) \sqsubseteq v^\infty$.*

Proof. — If $x \in A^{\mathbb{N}}$ is σ-periodic, then there exists a unique aperiodic word $v \in A^+$ with $x = v^\infty$. By Proposition 4.26, $|v| \leq ||\vartheta||^{\#A-1}$ and the condition is satisfied. Conversely, if the condition is satisfied, then $2|v| \leq |\vartheta^p|$, so x is σ-periodic by Proposition 4.27. □

Theorem 4.29 **(Mossé [114])**. — *Let ϑ be a primitive substitution with ϑ-fixed point $x \in A^{\mathbb{N}}$ which is not σ-periodic. Then there exists $m > 0$, such that for any $v \in A^+$, $v^m \notin \mathcal{L}(\Sigma_\vartheta)$.*

Proof. — Set

$$n = T(2) + 3, \quad m = \frac{\lceil 2n\alpha_1\lambda \rceil}{\alpha_0}.$$

Here $T(2)$ is the running time of words of length 2, λ is the spectral radius and α_i are numbers from Proposition 4.14. Let $v \in A^+$ and let $p > 0$ be the least integer with $2|v| \leq |\vartheta^p|$. Then $|\vartheta^{p-1}| < 2|v|$, where we set $|\vartheta^0| = 1$ for $p = 1$. Assume that there exists $q > 0$ with $v^q \in \mathcal{L}(x)$ and $|v^q| \geq n||\vartheta^p||$ (Figure 4 right). There exists $i < j$ such that $v^q \sqsubseteq \vartheta^p(x_{[i,j]})$, so

$$n \cdot ||\vartheta^p|| \leq |v^q| \leq (j - i + 1) \cdot ||\vartheta^p||.$$

Thus $T(2) + 1 = n - 2 \leq j - i - 1$ and for $w = x_{[i+1,j-1]}$ we have $|w| = T(2) + 1$ and $\vartheta^p(w) \sqsubseteq v^q$. By Proposition 3.28, v^q contains as subwords all words $\vartheta^p(ab)$ where $ab \in \mathcal{L}^2(x)$. By Proposition 4.27, x is σ-periodic and this is a contradiction. Thus if $v^q \in Ll(x)$, then $q|v| = |v^q| < n||\vartheta^p||$ and therefore

$$q < \frac{n||\vartheta^p||}{|v|} \leq \frac{n\alpha_1\lambda|\vartheta^{p-1}|}{\alpha_0|v|} \leq \frac{2n\alpha_1\lambda}{\alpha_0} \leq m,$$

so $v^m \notin \mathcal{L}(x)$. □

It can be shown that for the Morse sequence of Example 4.12, $m = 3$, so no third power of an nonempty word appears. We show that the return time of a substitutive subshift is linear with the arbitrarily large linearity constant.

Corollary 4.30. — *The return time (Definition 2.96) of a substitutive subshift is linear, i.e., there exists $C > 0$ such that $Q(n) \geq C \cdot n$.*

Proof. — Let m be such that for any $v \in A^+$, $v^m \notin \mathcal{L}(\Sigma)$. Then $Q(n) \geq n/m$. □

Proposition 4.31. — *For any $C > 0$ there exists a substitutive subshift such that $Q(n) \geq C \cdot n$ for any $n > 0$.*

Proof. — Let k, m be integers satisfying $1 < m < (k-1)!/k$, so $k \geq 5$. There exist km different permutations $(\pi_j)_{0 \leq j < km}$ of the set $\{0, 1, \ldots, k-1\}$ such that $\pi_j(0) = 0$. We set $A = \{0, 1, \ldots km - 1\}$ and define a substitution whose local rule $\vartheta : A \to A^{km}$ is given by

$$\vartheta(j)_n = \pi_j(n - kp) + kp, \quad 0 \leq p < m, \quad kp \leq n < kp + k.$$

Hence if $kp \leq n < k(p+1)$, then $kp \leq \vartheta(j)_n < k(p+1)$. Any interval $[kp, k(p+1))$ is subject to the same permutation as $[0, kp)$, so

$$\vartheta(a)_{[kp,kp+k)} = \vartheta(b)_{[kp,kp+k)} \;\Rightarrow\; a = b.$$

For example for $k = 5$, $m = 2$ we get

$$\vartheta = \begin{cases} 0 \mapsto 01234\ 56789 & 5 \mapsto 01432\ 56987 \\ 1 \mapsto 01243\ 56798 & 6 \mapsto 02134\ 57689 \\ 2 \mapsto 01324\ 56879 & 7 \mapsto 02143\ 57698 \\ 3 \mapsto 01342\ 56897 & 8 \mapsto 02314\ 57869 \\ 4 \mapsto 01423\ 56978 & 9 \mapsto 02341\ 57896 \end{cases}$$

Then ϑ has a unique fixed point x and $\Sigma_\vartheta = \overline{\mathcal{O}(x)}$ is a substitutive subshift. For any $a \in A$ we have $\tau([a]) \geq k(m-1)$. If $|u| \leq (2k+1)$, then $\tau([u])/|u| \geq \frac{k(m-1)}{2k+1}$.

If $(2k+1)\cdot(km)^j \le |u| < (2k+1)\cdot(km)^{j+1}$, then there exists $v \in \mathcal{L}^{2k-1}(\Sigma)$, such that $\vartheta^j(v) \sqsubseteq u$ (Figure 5). There exist a unique $a \in A$ and $0 \le p < m$ such that $\vartheta([a])_{[kp,kp+k)} \sqsubseteq v$. Since $\tau([a]) \ge k(m-1)$, $\tau([v]) \ge k(m-1)(km)$ and $\tau([u]) \ge k(m-1)(km)^{j+1}$, so

$$\frac{\tau([u])}{u} \ge \frac{k(m-1)}{2k+1} \ge \frac{5}{11}(m-1).$$

Thus for any n, $Q(n)/n \ge \frac{5}{11}(m-1)$. Since m may be arbitrary, the assertion follows. □

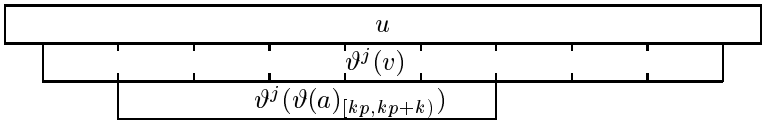

FIGURE 5. Return times

4.2.3. Frequencies. — We show that any substitutive subshift is strictly ergodic (Definition 2.98) and give some formulas for the frequencies of letters and words in its sequences. Let $\vartheta : A^* \to A^*$ be a primitive substitution and $M = M(\vartheta)$ its matrix. Then M is primitive and the Perron-Frobenius theory applies (Section A.9). Denote by $\lambda > 0$ the spectral radius of M. Let μ be the left eigenvector of M whose sum is 1 and let ν be the right eigenvector of M whose scalar product with μ is 1, so

$$\mu M = \lambda\mu, \quad M\nu = \lambda\nu, \quad \sum_{a\in A}\mu_a = 1, \quad \sum_{a\in A}\mu_a\nu_a = 1$$

By the Perron-Frobenius Theorem A.72 we have

$$\begin{aligned}
\lim_{k\to\infty}\frac{|\vartheta^k(a)|_b}{\lambda^k} &= \lim_{k\to\infty}\frac{(M^k)_{ab}}{\lambda^k} = \nu_a\mu_b \\
\lim_{k\to\infty}\frac{|\vartheta^k(a)|}{\lambda^k} &= \sum_{b\in A}\nu_a\mu_b = \nu_a \\
\lim_{k\to\infty}\frac{|\vartheta^k(a)|_b}{|\vartheta^k(a)|} &= \mu_b
\end{aligned}$$

Thus the asymptotic frequencies of letters in $\vartheta^k(a)$ are given by the left eigenvector μ of M.

Proposition 4.32. — *For any $x \in \Sigma_\vartheta$ and any $b \in A$, $\lim_{n\to\infty}\dfrac{|x_{[0,n)}|_b}{n} = \mu_b$.*

Proof. — Given $\varepsilon > 0$, there exists k such that for all $a, b \in A$,

$$|\vartheta^k(a)|\cdot(\mu_b - \varepsilon) \le |\vartheta^k(a)|_b \le |\vartheta^k(a)|\cdot(\mu_b + \varepsilon)$$

For a given n there exists $m > 0$ and $z \in \mathcal{L}^m(\Sigma_\vartheta)$, such that for $y = z_{[1,m-2]}$ we have $\vartheta^k(y) \sqsubseteq x_{[0,n)} \sqsubseteq \vartheta^k(z)$. Since $|\vartheta^k(z)| - |\vartheta^k(y)| \le 2||\vartheta^k||$ is bounded by n, both

$|\vartheta^k(z)|/n$ and $|\vartheta^k(y)|/n$ tend to 1. For all sufficiently large n we get

$$\begin{aligned} |\vartheta^k(z)|\cdot(1-\varepsilon) \;\le\; & n \;\le |\vartheta^k(y)|\cdot(1+\varepsilon) \\ n\cdot\frac{\mu_b-\varepsilon}{1+\varepsilon} \;\le\; |\vartheta^k(y)|\cdot(\mu_b-\varepsilon) \;\le\; & |x_{[0,n)}|_b \;\le |\vartheta^k(z)|\cdot(\mu_b+\varepsilon) \le n\cdot\frac{\mu_b+\varepsilon}{1-\varepsilon} \end{aligned}$$

$$\frac{\mu_b-\varepsilon}{1+\varepsilon} \;\le\; \liminf_{n\to\infty}\frac{|x_{[0,n)}|_b}{n} \;\le\; \limsup_{n\to\infty}\frac{|x_{[0,n)}|_b}{n} \;\le\; \frac{\mu_b+\varepsilon}{1-\varepsilon}$$

Since this holds for any $\varepsilon>0$, we get the result. □

For example, the Fibonacci substitution has spectral radius $\lambda=\frac{\sqrt{5}+1}{2}\doteq 1.618$ and left eigenvector $\mu=(\frac{3-\sqrt{5}}{2},\frac{\sqrt{5}-1}{2})\doteq(0.382,0.618)$.

To obtain the frequencies of words of length 2, we consider the alphabet $B=\mathcal{L}^2(\Sigma_\vartheta)$. Define a substitution $\vartheta_2: B^*\to B^*$ by

$$\vartheta_2(ab)=\vartheta(ab)_{[0,1]}\vartheta(ab)_{[1,2]}\dots\vartheta(ab)_{[|\vartheta(a)|-1,|\vartheta(a)|]}$$

so $|\vartheta_2(ab)|=|\vartheta(a)|$. Define $\varphi_2:\Sigma_\vartheta\to B^{\mathbb{N}}$ by $\varphi_2(u)_i=u_{[i,i+1]}$. Then for $u\in\Sigma_\vartheta$ we get $\vartheta_2(\varphi(u))=\vartheta_2(u_{[0,1]}u_{[1,2]}\dots)=\vartheta(u)_{[0,1]}\vartheta(u)_{[1,2]}\dots=\varphi_2(\vartheta(u))$, so $\varphi_2:(\Sigma_\vartheta,\vartheta)\to(B^{\mathbb{N}},\vartheta_2)$ is a conjugacy.

$$\begin{array}{ccc} \Sigma_\vartheta & \xrightarrow{\vartheta} & \Sigma_\vartheta \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ B^{\mathbb{N}} & \xrightarrow{\vartheta_2} & B^{\mathbb{N}} \end{array}$$

If $x\in\Sigma_\vartheta$ is a ϑ-fixed point, then $\varphi_2(x)\in B^{\mathbb{N}}$ is a ϑ_2-fixed point, so $\varphi_2(\Sigma_\vartheta)=\Sigma_{\vartheta_2}$. For the Fibonacci substitution $\vartheta(0)=1$, $\vartheta(1)=10$ we have $B=\{01,10,11\}$ and

$$\vartheta_2=\begin{cases} 01 \mapsto 11 \\ 10 \mapsto 10,01 \\ 11 \mapsto 10,01 \end{cases} \qquad M(\vartheta_2)=\begin{bmatrix} 0&0&1\\ 1&1&0\\ 1&1&0 \end{bmatrix}$$

$$\begin{array}{cccccccccccccccc} \vartheta\downarrow & 1 & 0 & 1 & 1 & 0 & 1 & \cdots & \overset{\varphi_2}{\mapsto} & 10 & 01 & 11 & 10 & 01\cdots & \downarrow\vartheta_2 \\ & 10 & 1 & 10 & 10 & 1 & 10 & \cdots & & 10,01 & 11 & 10,01 & 10,01 & 11\cdots & \end{array}$$

Proposition 4.33. — *The spectral radius of $M(\vartheta_2)$ is λ. If $(\mu'_{ab})_{ab\in B}$ is the left eigenvector of $M(\vartheta_2)$, then for any $x\in\Sigma_\vartheta$*

$$\lim_{n\to\infty}\frac{\#\{i<n: x_{[i,i+1]}=ab\}}{n}=\lim_{n\to\infty}\frac{|\varphi_2(x)_{[0,n)}|_{ab}}{n}=\mu'_{ab}.$$

Proof. — Let l be the spectral radius of $M(\vartheta_2)$ and ν' its right eigenvector. Since $|\vartheta_2^k(ab)|=|\vartheta^k(a)|$ and both limits

$$\lim_{k\to\infty}\frac{|\vartheta_2^k(ab)|}{l^k}=\nu'_{ab},\qquad \lim_{k\to\infty}\frac{|\vartheta^k(a)|}{\lambda^k}=\nu_a$$

are positive and finite, $\lambda=l$. By Proposition 4.32 the frequencies of ab in x are given by μ'_{ab}. □

Theorem 4.34 **(Michel [108])**. — *Any substitutive subshift is strictly ergodic.*

The proof is based on the construction of a substitution ϑ_p in an alphabet $\mathcal{L}^p(\Sigma_\vartheta)$ which is analogous to ϑ_2. To obtain frequencies of words of length p we have to compute the left eigenvector of $M(\vartheta_p)$. However, for $p > 2$, there is a simpler way. Let k be the least integer, such that $p \le |\vartheta^k|$. If $u \in \mathcal{L}^p(\Sigma_\vartheta)$ then there exists $ab \in B = \mathcal{L}^2(\Sigma_\vartheta)$ such that $u \sqsubseteq \vartheta^k(ab)$. We count all occurrences of u in $\vartheta^k(ab)$ which start within $\vartheta^k(a)$ and weight them by the frequencies μ'_{ab}. Set

$$m(u) = \sum_{ab \in B} \mu'_{ab} \cdot \#\{i < |\vartheta^k(a)| : \vartheta^k(ab)_{[i,i+p)} = u\}.$$

Then

$$\lim_{n\to\infty} \frac{|x_{[0,n)}|_u}{n} = m(u) / \sum_{v \in \mathcal{L}^p(\Sigma_\vartheta)} m(v).$$

For example, for the Fibonacci subshift, the left eigenvector of $M(\vartheta_2)$ is $\mu'_{00} = \sqrt{5}-2$, $\mu'_{01} = \mu'_{10} = \frac{3-\sqrt{5}}{2}$. The frequencies of words of length 3 are computed in the following table.

ab	μ'_{ab}	$\vartheta^2(ab)$	010	011	101	110	$\sum m(v)$
01	$\frac{3-\sqrt{5}}{2}$	10|101	1	0	1	0	
10	$\frac{3-\sqrt{5}}{2}$	101|10	0	1	1	1	
11	$\sqrt{5}-2$	101|101	0	1	1	1	
$m(u)$			$\frac{3-\sqrt{5}}{2}$	$\frac{\sqrt{5}-1}{2}$	1	$\frac{\sqrt{5}-1}{2}$	$\frac{3+\sqrt{5}}{2}$
$m(u)/\sum m(v)$			$\frac{7-3\sqrt{5}}{2}$	$\sqrt{5}-2$	$\frac{3-\sqrt{5}}{2}$	$\sqrt{5}-2$	

4.3. Decoding and recognizability

Let $\vartheta : A^* \to A^*$ be a primitive substitution with a fixed point $x \in \Sigma_\vartheta$. If we regard ϑ as a code, we may try to decompose a given word $u \in \mathcal{L}(\Sigma_\vartheta)$ as a concatenation of words $(\vartheta(a))_{a\in A}$. This means to search for a word v of minimal length such that $u \sqsubseteq \vartheta(v)$. Such a word v is usually not unique, but a remarkable fact is that for any primitive substitution, the central part of v is uniquely determined by u. This property is called **recognizability**. The first step in finding v is to find boundaries between successive words $\vartheta(a)$ in u. For the ϑ-fixed point $x = 1011010\ldots$ of the Fibonacci substitution $\vartheta(0) = 1$, $\vartheta(1) = 10$ we have

$$x = |10|1|10|10|1|10|1|10|10|1|10|10|1|\ldots$$

where successive $\vartheta(a)$ are separated by strokes. In this case the decoding is simple. Since the letter 0 occurs in $\vartheta(1)$ but not in $\vartheta(0)$, we put a stroke before and after any occurrence of 10. However, the first and last block of u need not be complete, so even in this case, v need not be unique. For example for $u = 101$ we have $u \sqsubseteq \vartheta(10) = 10|1$ and $u \sqsubseteq \vartheta(11) = 10|10$. For the second iteration $\vartheta^2(0) = 10$, $\vartheta^2(1) = 101$ we have

$$x = |101|10|101|101|10|101|10|101|\ldots$$

The word $u = 1101$ can be decomposed in two ways as $1|101$ and $1|10|1$ with $u \sqsubseteq \vartheta^2(11) = 10\underline{1101}$ and $u \sqsubseteq \vartheta^2(101) = 10\underline{1101}10$.

The n-th stroke in x occurs at position $f(n) = |\vartheta(x_{[0,n)})|$, so the range $f(\mathbb{N})$ of the function $f : \mathbb{N} \to \mathbb{N}$ is the set of positions of all strokes. For the Fibonacci sequence we get $f(\mathbb{N}) = \{0, 2, 3, 5, 7, 8, 10, 11, 13, 15, \ldots\}$. If a word u occurs in x at two positions $u = x_{[i,i+n)} = x_{[j,j+n)}$, we say that the intervals $[i, i+n)$ and $[j, j+n)$ are **equivalent**, if $f(\mathbb{N}) \cap [j, j+n)$ can be obtained from $f(\mathbb{N}) \cap [i, i+n)$ by a shift of $j - i$ positions, and if, at the preimages of these positions occur the same letters. Formally define $\approx$ as the smallest equivalence on the set of intervals satisfying

$$x_{[i,i+n)} = x_{[j,j+n)} \;\Rightarrow\; [f(i) + k, f(i+n) - l] \approx [f(j) + k, f(j+n) - l]$$

Here $k, l \geq 0$ and $k + l < f(n+i) - f(i) = f(n+j) - f(j)$. For the second iteration of the Fibonacci substitution we get $[2, 5] \not\approx [7, 10]$ but $[2, 5] \approx [15, 18]$

$$10\overbrace{1|10|1}^{[2,5]}0\overbrace{1|101}^{[7,10]}|10|10\overbrace{1|10|1}^{[15,18]}01|\ldots$$

***Proposition 4.35* (Mossé [114]).** — *Let $\vartheta : A \to A^+$ be a primitive substitution and $x \in A^{\mathbb{N}}$ its ϑ-fixed point which is not σ-periodic. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = |\vartheta(x_{[0,n)})|$. There exists $l > 0$ such that if $x_{[i-l,i+l]} = x_{[j-l,j+l]}$ and $i \in f(\mathbb{N})$, then $j \in f(\mathbb{N})$.*

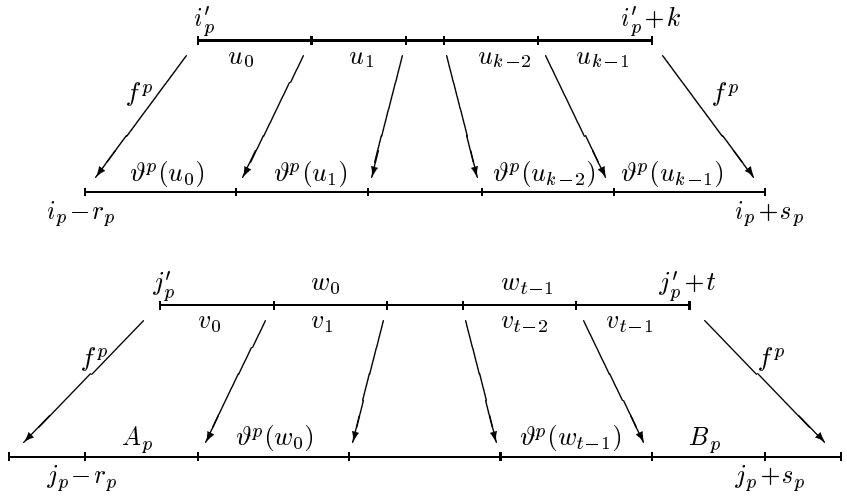

FIGURE 6. Decomposition of $\vartheta^p(x)$

Proof. — Assume the contrary and fix $k > 0$. For a given $p > 0$, let $l_p = ||\vartheta^p||k + 1$. There exist $i_p \in f(\mathbb{N})$ and $j_p \notin f(\mathbb{N})$ such that $x_{[i_p - l_p, i_p + l_p)} = x_{[j_p - l_p, j_p + l_p)}$. Let r_p, s_p be the smallest integers with

$$\#([i_p - r_p, i_p) \cap f^p(\mathbb{N})) = \lfloor \tfrac{k}{2} \rfloor, \quad \#([i_p, i_p + s_p] \cap f^p(\mathbb{N})) = k - \lfloor \tfrac{k}{2} \rfloor.$$

Then $i_p - l_p < i_p - r_p$ and $i_p + s_p \leq i_p + l_p$ so $x_{[i_p - r_p, i_p + s_p)} = x_{[j_p - r_p, j_p + s_p)}$. There exists i'_p such that $i_p - r_p = f^p(i'_p)$ and $i_p + s_p = f^p(i'_p + k)$ (Figure 6 top). For $j_p - r_p$, $j_p + s_p$ there exist j'_p, t_p such that

$$f^p(j'_p) \leq j_p - r_p < f^p(j'_p + 1), \quad f^p(j'_p + t_p - 1) < j_p + s_p \leq f^p(j'_p + t_p).$$

Since $t_p \le k||\vartheta^p||/|\vartheta^p|$ is bounded, there exists $t > 0$, $u \in A^k$, $v \in A^t$ and an infinite set $E \subseteq \mathbb{N}$ such that for any $p \in E$, $t_p = t$, $x_{[i'_p, i'_p+k)} = u$, and $x_{[j'_p, j'_p+t)} = v$ (Figure 6 bottom). Put $w = v_{[1,t-2]}$. For any $p \in E$ there exists a suffix A_p of $\vartheta^p(v_0)$ and a prefix B_p of $\vartheta^p(v_{t-1})$ such that

$$\vartheta^p(u) = x_{[i_p - r_p, i_p + s_p)} = x_{[j_p - r_p, j_p + s_p)} = A_p \vartheta^p(w) B_p.$$

For $q > p$, $q \in E$ we have

$$A_q \vartheta^q(w) B_q \quad = \quad \vartheta^{q-p}(\vartheta^p(u)) = \vartheta^{q-p}(A_p)\vartheta^q(w)\vartheta^{q-p}(B_p).$$

Assume that $A_q \neq \vartheta^{q-p}(A_p)$, so that one of them is a prefix of the other. This implies that $\vartheta^q(w) \sqsubseteq z^\infty$ for some $z \in A^*$ whose length is the difference of the lengths of A_q and $\vartheta^{q-p}(A_p)$. For sufficiently large k this is not possible by Theorem 4.29. Thus there exist $q, p \in E$ with $q > p$ for which $A_q = \vartheta^{q-p}(A_p)$ and $B_q = \vartheta^{q-p}(B_p)$ (Figure 7 bottom). Set m_q by $m_q - r_p - f^p(j'_q) = j_p - r_p - f^p(j'_p)$. Then

$$x_{[m_q - r_p, m_q + s_p)} = x_{[j_p - r_p, j_p + s_p)} = A_p \vartheta^p(w) B_p = \vartheta^p(u) = x_{[f^p(i'_q), f^p(i'_q + k))}.$$

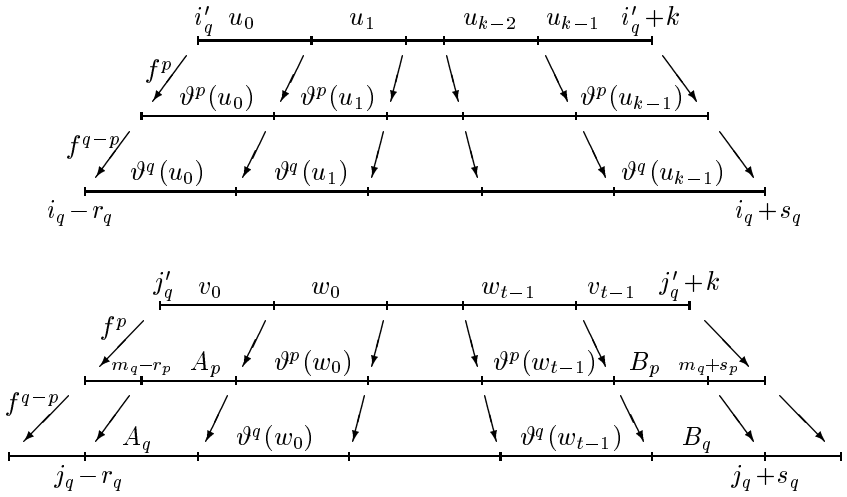

FIGURE 7. Decomposition of $\vartheta^q(x)$

Applying ϑ^{q-p}, we obtain equivalent intervals

$$\begin{aligned}[j_q - r_q, j_q + s_q) &= [f^{q-p}(m_q - r_p), f^{q-p}(m_q + s_p)) \\ &\approx [f^q(i'_q), f^q(i'_q + k)) = [i_q - r_q, i_q + s_q).\end{aligned}$$

However, by assumption $i_q \in f(\mathbb{N})$ and $j_q \notin f(\mathbb{N})$, and this is a contradiction. □

Theorem 4.36 **(Mossé [115])**. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution with a ϑ-fixed point x which is not σ-periodic and let $f : \mathbb{N} \to \mathbb{N}$ be the function from Proposition 4.35. Then ϑ is* **recognizable**, *i.e., there exists a* **context length** *$l > 0$ such that for any $u \in \mathcal{L}(\Sigma_\vartheta)$ of length at least $2l$, there exist i, j with $0 \le i \le l$, $|u| - l \le j \le |u|$ and unique $v \in \mathcal{L}(\Sigma_\vartheta)$, such that $u_{[i,j)} = \vartheta(v)$ (Figure 8 top), and whenever $x_{[m,m+|u|)} = u$, then there exist i', j' with $f(i') = m + i$, $f(j') = m + j$ and $x_{[i',j')} = v$.*

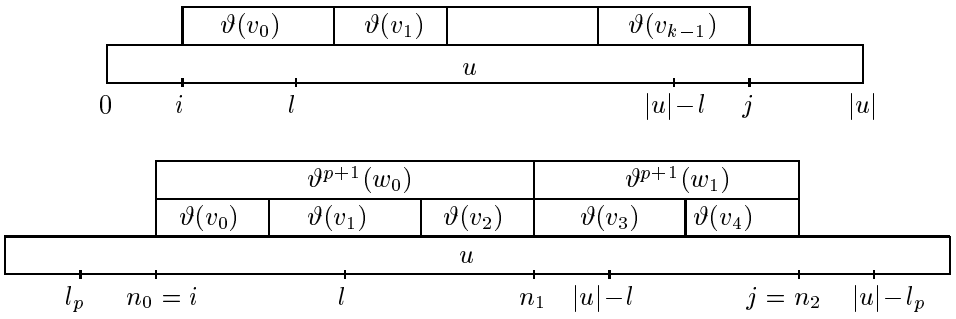

FIGURE 8. Decoding

Proof. — We define an equivalence $\sim$ on A by $a \sim b$ iff there exists n such that $\vartheta^n(a) = \vartheta^n(b)$. There exists a maximal integer p at which two letters become equivalent. Thus if $\vartheta^p(a) \neq \vartheta^p(b)$, then for all $k \geq 0$, $\vartheta^k(a) \neq \vartheta^k(b)$. By Proposition 4.35 applied to ϑ^{p+1}, there exists l_p, such that for any $q, r > l_p$, if $x_{[q-l_p,q+l_p)} = x_{[r-l_p,r+l_p)}$ and $q \in f^{p+1}(\mathbb{N})$, then $r \in f^{p+1}(\mathbb{N})$. Set $l = l_p + ||\vartheta^{p+1}||$ (Figure 8 bottom). For $u \in \mathcal{L}(\Sigma_\vartheta)$, $|u| \geq 2l$ there exists a unique k and unique numbers

$$l_p \leq n_0 < n_1 < \cdots < n_k \leq |u| - l_p,$$

such that for any q with $x_{[q,q+|u|)} = u$,

$$[q + l_p, q + |u| - l_p] \cap f^{p+1}(\mathbb{N}) = \{q + n_0, q + n_1, \ldots, q + n_k\}.$$

There exists $w \in \mathcal{L}^k(\Sigma_\vartheta)$ such that $u_{[n_t,n_{t+1})} = \vartheta^{p+1}(w_t)$ for $0 \leq t < k$. Moreover, if $w' \in \mathcal{L}^k(\Sigma_\vartheta)$ is another word with $\vartheta^{p+1}(w') = \vartheta^{p+1}(w)$, then, since p is the maximal number at which two letters become equivalent, $\vartheta^p(w) = \vartheta^p(w') = v$. Set $i = n_0$ and $j = n_k$. Then $i \leq l$, $j \geq n - l$ and v satisfies the Theorem. □

For the Fibonacci subshift, the context length is $l = 2$. If $u \in \Sigma_\vartheta$ and $|u| > 4$, then $u_{[0,2]}$ contains at least one letter 1. Theorem 4.36 yields some topological properties of $(\Sigma_\vartheta, \vartheta)$.

Corollary 4.37. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution with nonperiodic ϑ-fixed point, and l the context length given in Theorem 4.36. Then for any $x, y \in \Sigma_\vartheta$,*

$$\vartheta(x) = \vartheta(y) \quad \Rightarrow \quad x_{[l,\infty)} = y_{[l,\infty)}.$$

If for any different $a, b \in A$, $\vartheta(a)$ is not a suffix of $\vartheta(b)$, then $\vartheta : \Sigma_\vartheta \to \Sigma_\vartheta$ is injective.

Proposition 4.38. — *Let $\vartheta : A^* \to A^*$ be a primitive substitution with a ϑ-fixed point which is not periodic. Then for any $u \in \Sigma_\vartheta$ with $|u| \geq 2l$ we have $\vartheta([u]) = [\vartheta(u)]$ in Σ_ϑ, hence $(\Sigma_\vartheta, \vartheta)$ is open.*

Proof. — Let $x \in A^{\mathbb{N}}$ be a ϑ-fixed point and let l be the context length. If $y \in \Sigma_\vartheta$, then for any n, $\vartheta(y_{[0,n)}) \in \mathcal{L}(\Sigma_\vartheta)$, so $\vartheta(y) \in \Sigma_\vartheta$ and Σ_ϑ is ϑ-invariant. Let $u \in \mathcal{L}(\Sigma_\vartheta)$ and $|u| \geq 2l$. Clearly $\vartheta([u]) \subseteq [\vartheta(u)]$. Let $y \in [\vartheta(u)]$ (Figure 9). For any n, $y_{[0,n)} \in \mathcal{L}(\Sigma_\vartheta)$, so by Theorem 4.36, there exist $i, j, k \in \mathbb{N}$ and unique $v \in \mathcal{L}^k(\Sigma_\vartheta)$ such that $y_{[i,j)} = \vartheta(v)$ and

$$0 \leq i \leq l, \quad n - l \leq j \leq n.$$

Let p be the integer such that $|\vartheta(u_{[0,p-1]})| \leq l < |\vartheta(u_{[0,p]})|$. Then $u_{[p,|u|)}$ is a prefix of v and $y \in [\vartheta(u_{[0,p)}v)]$. As $n \to \infty$, $u_{[0,p)}v$ converge to a point $z \in \Sigma_\vartheta$ such that $z \in [u]$ and $y = \vartheta(z) \in \vartheta([u])$. Thus $\vartheta([u]) = [\vartheta(u)]$ and $(\Sigma_\vartheta, \vartheta)$ is open. □

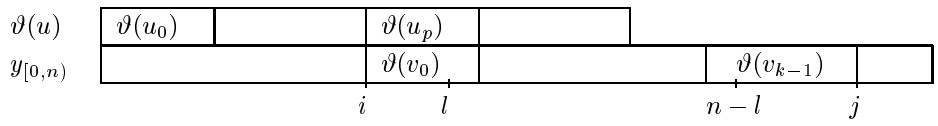

FIGURE 9. Openness

Suppose that $\vartheta : A^* \to A^*$ is a primitive substitution with a ϑ-fixed point $x \in A^{\mathbb{N}}$. Since x is σ-almost periodic, $\mathcal{L}(x)$ is a central language (Definition 3.70) and there exists a two-sided subshift $\Theta_\vartheta = \mathcal{Z}(\mathcal{L}(x)) \subseteq A^{\mathbb{Z}}$. The substitution ϑ extends to $\vartheta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ via

$$\vartheta(\ldots u_{-2}u_{-1}.u_0u_1u_2\ldots) = \ldots\vartheta(u_{-2})\vartheta(u_{-1}).\vartheta(u_0)\vartheta(u_1)\vartheta(u_2)\ldots$$

Proposition 4.39. — *Let ϑ be a primitive substitution with a ϑ-fixed point $x \in A^{\mathbb{N}}$. Then Θ_ϑ is ϑ-invariant and $\vartheta : \Theta_\vartheta \to \Theta_\vartheta$ is injective and open.*

Proof. — If $y \in \Theta_\vartheta$, then clearly $\vartheta(y) \in \Theta_\vartheta$. Let $y, z \in \Theta_\vartheta$ and $\vartheta(y) = \vartheta(z)$. For any n, $y_{[-n,n]} \in \mathcal{L}(\Theta_\vartheta) = \mathcal{L}(\Sigma_\vartheta)$ there exist $i < 0 < j$, such that

$$-|\vartheta(y_{[i,0)})| \leq -n + l < |\vartheta(y_{[i+1,0)})|, \quad |\vartheta(y_{[0j-1)})| \leq n - l < |\vartheta(y_{[0,j)})|$$

and $y_{[i,j]} = z_{[i,j]}$. As $n \to +\infty$, i tends to $-\infty$ and j tends to $+\infty$, so $y = z$. The proof of openness is similar as in Proposition 4.38.

□

4.3.1. Spectrum. — Spectra and maximal equicontinuous factors of subshifts are rather diverse. The Feigenbaum subshift has maximal equicontinuous factor $\chi : (\Sigma, \sigma) \to (\mathbf{2}^{\mathbb{N}}, R_1)$ constructed in page 37. For the kneading sequence $\mathcal{K} \in \Sigma$ we have $\chi^{-1}(\chi(\mathcal{K})) = \{\mathcal{K}\}$, so χ is a maximal equicontinuous factor by the Paul theorem 2.49. The spectrum of the Feigenbaum subshift consists therefore of numbers $e^{2\pi ij/2^k}$, where $k \geq 0$, $0 \leq j < 2^k$.

Recall that $\vartheta : A^* \to A^*$ is a **substitution of constant length** l, if $|\vartheta| = ||\vartheta|| = l$. The local rule is then $\vartheta : A \to A^l$. We show that any substitutive system of constant length has as its maximal equicontinuous factor an adding machine.

***Theorem 4.40* (Dekking[46]).** — *Let $\vartheta : A \to A^l$ be a primitive substitution with constant length l and with a ϑ-fixed point $x \in A^{\mathbb{N}}$. Define a periodic structure $\mathbf{p} = (p_j)_{j\geq 0}$ by $p_0 = d$, $p_j = dl^j$, where $d = \gcd\{a > 0 : x_a = x_0\}$. Then the maximal equicontinuous factor of Σ_ϑ is the adding machine $(\mathbb{Z}_\mathbf{p}, R_1)$*

Proof. — Let $a > 0$ be an integer with $x_a = x_0$. Assume that $f : \Sigma_\vartheta \to \mathbb{K}$ is an eigenfunction with eigenvalue $\alpha \in \mathbb{K}$. Since $x_{[al^k,(a+1)l^k)} = x_{[0,l^k)}$, and since f is continuous, we get

$$\lim_{k\to\infty} |f(x) - \alpha^{al^k} f(x)| = \lim_{k\to\infty} |f(x) - f(\sigma^{al^k}(x))| = 0,$$

and $\lim_{k\to\infty}\alpha^{al^k}=1$. If $\alpha=e^{2\pi it}$, then $\lim_{k\to\infty}\mathrm{mod}_1(tal^k)=0$ and this is possible only if $t=s/al^k$ for some integers $k,s\geq 0$. Assume now that a_0,a_1 are distinct integers with $x_{a_0}=x_{a_1}=x_0$, so $t=\frac{s_0}{a_0l^{k_0}}=\frac{s_1}{a_1l^{k_1}}$, and $s_0a_1=s_1a_0l^{k_0-k_1}$. Assume that $k_0\geq k_1$ and set $d_{01}=\gcd(a_0,a_1)$, so $a_0=d_{01}b_0$, $a_1=d_{01}b_1$, and $s_0b_1=s_1b_0l^{k_0-k_1}$. Since $\gcd(b_0,b_1)=1$, $s_0=sb_0$ for some s, $t=\frac{s_0}{a_0l^{k_0}}=\frac{s}{d_{01}l^{k_0}}$. There exist l and $a_0,\ldots,a_{l-1}$ with $d=\gcd(a_0,a_1,\ldots,a_{k-1})$ and $x_0=x_{a_0}=x_{a_1}=\cdots=x_{a_{l-1}}$. Repeating the previous argument with $a_2,\ldots,a_{l-1}$, we get that there exists s,k, such that $t=\frac{s}{dl^k}$ and $\alpha=e^{2\pi is/dl^k}$. We have proved $\mathcal{S}(\Sigma_\vartheta,\sigma)\subseteq\mathcal{S}(\mathbb{Z}_{\mathbf{p}},R_1)$.

Conversely we show that any $\alpha=e^{2\pi is/dl^k}$ is an eigenvalue of Σ_ϑ. Since x is almost periodic, there exists $p>0$ such that for any $y\in\Sigma_\vartheta$ there exists the least integer $\tau(y)<p$ with $y_{\tau(y)}=u_0$. Set $f(y)=e^{2\pi i\tau(y)/d}$. Then f is an eigenfunction with eigenvalue $e^{2\pi i/d}$. We show now that any $\alpha=e^{2\pi i/l^k}$ is an eigenvalue of Σ_ϑ. By Theorem 4.36 applied to ϑ^k, there exists $l>0$ such that for any $y\in\Sigma_\vartheta$ there exists $\tau_k(y)<l$ and $z\in\Sigma_\vartheta$, such that $y=y_{[0,\tau_k(x))}\vartheta^k(z)$. Set $f(y)=e^{2\pi i\tau_k(y)/l^k}$. Then f is an eigenfunction of Σ_ϑ with eigenvalue $e^{2\pi i/l^k}$. Thus $\mathcal{S}(\Sigma_\vartheta,\sigma)=\mathcal{S}(\mathbb{Z}_{\mathbf{p}},R_1)$, so $(\mathbb{Z}_{\mathbf{p}},R_1)$ is the maximal equicontinuous factor of $(\Sigma_\vartheta,\sigma)$. □

The integer d of Theorem 4.40 can have common divisors with the length l, Denote by $h(\vartheta)$ the **height** of ϑ defined by

$$h(\vartheta)=\max\{n\geq 1:\gcd(n,l)=1,n|d\},$$

and define a periodic structure $\mathbf{q}$ by $q_0=h(\vartheta)$, $q_j=h(\vartheta)l^j$. Then $\mathbf{p}$ and $\mathbf{q}$ are equivalent periodic structures. In the Morse subshift we have $d=h(\vartheta)=1$, so the maximal equicontinuous factor of the Morse subshift is the dyadic adding machine $(\mathbf{2}^{\mathbb{N}},R_1)$.

Example 4.41. — The substitution

$$\vartheta=\begin{cases}0&\mapsto&010\\1&\mapsto&201\\2&\mapsto&102\end{cases}$$

has height $d=h(\vartheta)=2$. The maximal equicontinuous factor is $(\mathbb{Z}_{\mathbf{p}},R_1)$, with $p_i=3\cdot 2^i$.

In substitutions of variable length there is much more diversity. We shall see that the maximal equicontinuous factor of the Fibonacci subshift is the rotation $(\mathbb{T},R_\alpha)$, where $\alpha=\frac{\sqrt{5}-1}{2}$ is the golden number, so its spectrum is $\{e^{2\pi in\alpha}:n\in\mathbb{Z}\}$. The maximal equicontinuous factor of a substitutive subshift can be even trivial.

Example 4.42 **(Queffélec [133])**. — The substitution

$$\vartheta=\begin{cases}0&\mapsto&01010\\1&\mapsto&011\end{cases}$$

has a trivial equicontinuous factor.

Proof. — Set $s_n=|\vartheta^n(0)|$, $r_n=|\vartheta^n(1)|$. Then $s_{n+1}=3s_n+2r_n$, $r_{n+1}=s_n+2r_n$,

$$s_n=\frac{4^{n+1}-1}{3},\quad r_n=\frac{2\cdot 4^n+1}{3}$$

We have a ϑ-fixed point $x = 01010011,\ldots$ and $y = \sigma^4(x) = 0011\ldots \in \Sigma_\vartheta$. It follows that $\vartheta^k(0) = \vartheta^k(y)_{[0,s_k)} = \vartheta^k(y)_{[s_k,2s_k-1)}$, so $\lim_{k\to\infty} d(\vartheta^k(y), \sigma^{s_k}(\vartheta^k(y))) = 0$. If f is an eigenfunction with eigenvalue $\alpha = e^{2\pi i t}$, then $\lim_{k\to\infty} |f(\vartheta^k(y)), \alpha^{s_k} f(\vartheta^k(y))| = 0$. and $\lim_{k\to\infty} \alpha^{s_k} = 1$. We get $\lim_{n\to\infty} \mathrm{mod}_1(\frac{4^{n+1}t}{3}) = \mathrm{mod}_1(\frac{t}{3})$ and this is possible only if $\mathrm{mod}_1(\frac{4^{n+1}t}{3}) = \mathrm{mod}_1(\frac{t}{3})$ for all sufficiently large t. It follows that $\mathrm{mod}_1(\frac{4t}{3}) = \mathrm{mod}_1(\frac{t}{3})$ and $t = 0$. Thus the only eigenvalue of Σ_ϑ is 1. □

The spectrum of general substitution subshifts has been characterized by Host [**79**] (see also Queffélec [**133**]).

4.4. Sturmian subshifts

A Sturmian subshift is an extension of an irrational rotation $(\mathbb{T}, R_\alpha)$ generated by a particular interval cover. To construct it we write α as a continued fraction (Section A.10). We consider the space $\mathcal{N}$ of unit continued fractions $a = (a_i)_{0\le i<|a|}$ where $0 < |a| \le \infty$, $a_0 = 0$ and $a_i \in \mathbb{N}^+ = \{1, 2, \ldots\}$ for $i > 0$. An irrational $\alpha \in (0, 1)$ can be expressed in a unique way by an infinite continued fraction

$$\alpha = [a] = [0, a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

The convergents of a are

$$\begin{array}{llllll} p_{-1} = 1, & p_0 = 0, & p_1 = 1, & p_2 = a_2 p_1 & \ldots & p_n = p_{n-2} + a_n p_{n-1} \\ q_{-1} = 0, & q_0 = 1, & q_1 = a_1, & q_2 = 1 + a_2 q_1 & \ldots & q_n = q_{n-2} + a_n q_{n-1} \end{array}$$

and $[0, a_1, \ldots, a_n] = p_n/q_n$. The convergents q_n are the times at which the orbit comes closest to zero. For $\eta_n = d(R_\alpha^n(0), 0)$ we have

$$\eta_n = (-1)^n(q_n\alpha - p_n), \quad \eta_{n+1} = \eta_{n-1} - a_{n+1}\eta_n < \eta_n,$$

so $\eta_{-1} = 1$, $\eta_0 = \alpha$, $\eta_1 = 1 - a_1\alpha$. By Proposition A.84, $\frac{1}{q_n+q_{n+1}} < \eta_n < \frac{1}{q_{n+1}}$ and

$$\begin{array}{rcll} q_{2n-1} < q < q_{2n+1} & \& & p > \alpha q & \Rightarrow \quad p - \alpha q > \eta_{2n}, \\ q_{2n} < q < q_{2n+2} & \& & p < \alpha q & \Rightarrow \quad \alpha q - p > \eta_{2n+1}. \end{array}$$

In particular, if $0 < q < q_n$, then $|p - \alpha q| > |p_n - \alpha q_n| = \eta_n$.

Definition 4.43. — The **Sturmian subshift** of an irrational number $\alpha \in (0, 1)$ is $\Sigma_\alpha = \Sigma_{\mathcal{V}_\alpha}(\mathbb{T}, R_\alpha)$, where

$$\mathcal{V}_\alpha = \{V_0, V_1\}, \quad V_0 = [0, 1-\alpha), \quad V_1 = [1-\alpha, 1).$$

Proposition 4.44. — *A Sturmian subshift is minimal, strictly ergodic and has zero topological entropy. Its complexity function is $P(n) = n + 1$. For any $u \in \mathcal{L}(\Sigma_\alpha)$, $x \in \Sigma_\alpha$ we have*

$$\mu([u]) = \lim_{n\to\infty} \frac{\#\{i < n : x_{[i,i+|u|)} = u\}}{n} = ||V_u||.$$

Proof. — Both of the intervals $R_\alpha^{-n}(V_0)$ and $R_\alpha^{-n}(V_1)$ have endpoints $R_\alpha^{-n}(0)$ and $R_\alpha^{-n-1}(0)$, so the cut points of $\mathcal{V}_\alpha^n$ are $\{0, R_\alpha^{-1}(0), \cdots, R_\alpha^{-n}(0)\}$, and $P(n) = |\mathcal{V}_\alpha^n| = n+1$. It follows that the topological entropy is zero. Any $V_u \in \mathcal{V}_\alpha^n$ is a semiclosed interval. Let $\varphi : (\Sigma_\alpha, \sigma) \to (\mathbb{T}, R_\alpha)$ be the factor map. If $z \in \Sigma_\alpha$ and $u \in \mathcal{L}(\Sigma_\alpha)$, then there exists $n > 0$ such that $R_\alpha^n(\varphi(z)) \in V_u$, so $\sigma^n(z) \in [u]$ and Σ_α is minimal. For $u \in \mathcal{L}(\Sigma_\alpha)$, $z \in \Sigma_\alpha$ we have $\varphi(z) \in V_u^\circ \;\Rightarrow\; z \in [u] \;\Rightarrow\; \varphi(z) \in \overline{V_u}$.

$$\begin{aligned} ||V_u|| &= \lim_{n\to\infty} \frac{\#\{i < n : R^i(\varphi(z)) \in V_u^\circ\}}{n} \le \lim_{n\to\infty} \frac{\#\{i < n : z_i \in [u]\}}{n} \\ &\le \lim_{n\to\infty} \frac{\#\{i < n : R^i(\varphi(z)) \in \overline{V_u}\}}{n} = ||V_u|| \end{aligned}$$

□

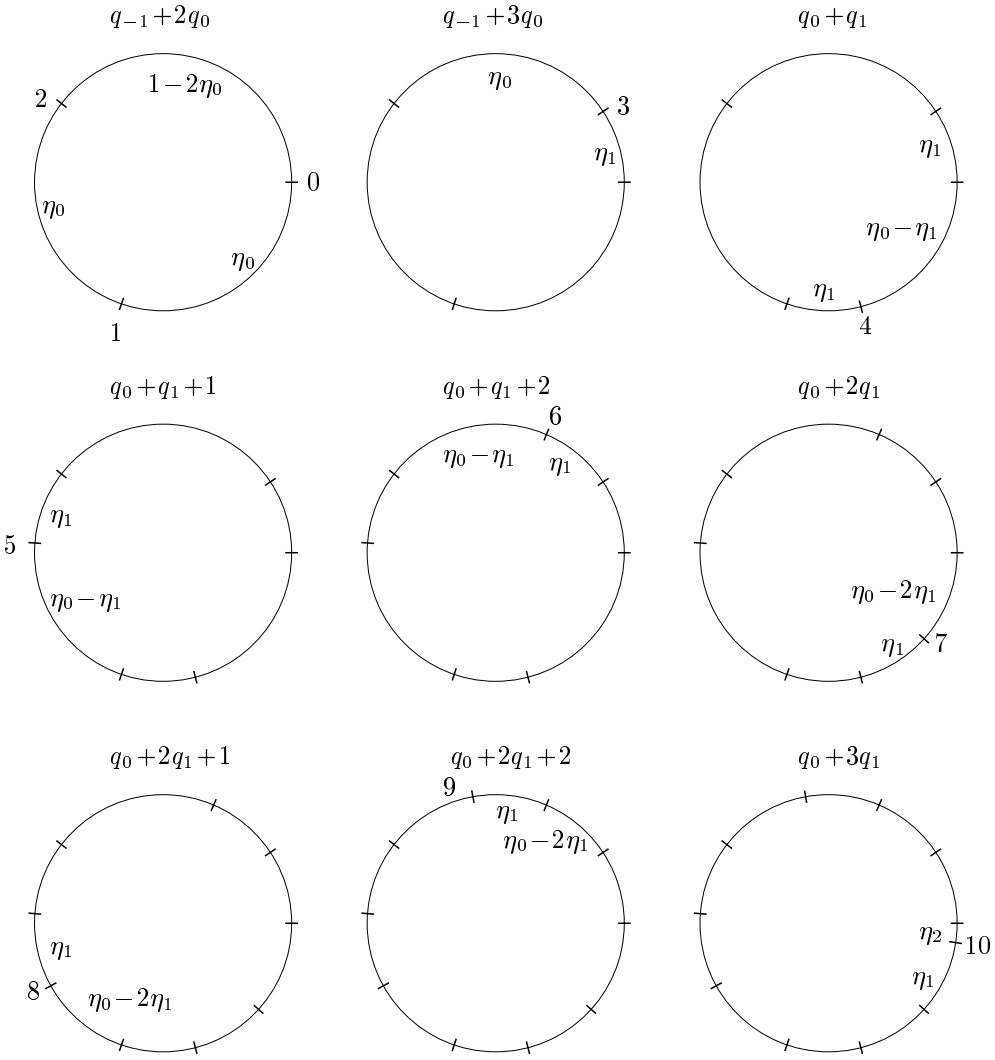

FIGURE 10. Partitions $\mathcal{V}_\alpha^n$

The powers of the partition $\mathcal{V}_\alpha$ for $\alpha = 0.85/\pi = [0, 3, 3, 3, 1, 7, \ldots]$ can be seen in Figure 10. The lengths of these intervals are η_k or $\eta_{k-1} - m\eta_k$, starting with $\eta_{-1} = 1$ and $\eta_0 = \alpha$. Each partition contains intervals of only two or three lengths.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
η_0	1	2	3	2	1										
$1-\eta_0$	1														
$1-2\eta_0$		1													
η_1			1	2	3	4	5	6	7	8	9	10	9	8	7
$\eta_0-\eta_1$				1	2	3	2	1							
$\eta_0-2\eta_1$							1	2	3	2	1				
η_2										1	2	3	4	5	6
$\eta_1-\eta_2$													1	2	3

Theorem 4.45 **(Three lengths theorem, Sós [148])**. — *Let $\alpha \in (0,1)$ be an irrational. For any $n > 0$ there exist unique integers $k \geq 0$, $1 \leq m \leq a_{k+1}$, $0 \leq r < q_k$, such that $n = q_{k-1} + mq_k + r$ and $\mathcal{V}_\alpha^n$ contains*

$$\begin{array}{lll} n_1 = n + 1 - q_k & \textit{intervals of length} & l_1 = \eta_k, \\ n_2 = r + 1 & \textit{intervals of length} & l_2 = \eta_{k-1} - m\eta_k, \\ n_3 = q_k - (r+1) & \textit{intervals of length} & l_3 = \eta_{k-1} - (m-1)\eta_k. \end{array}$$

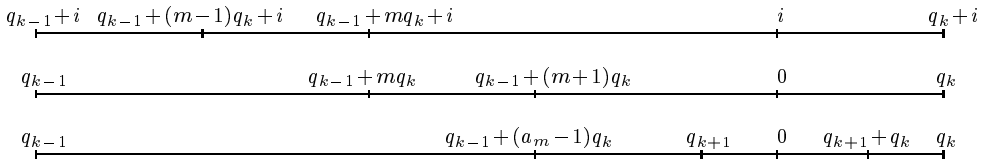

FIGURE 11. Intervals V_u

Proof. — We have $q_{-1} + q_0 = 1$. For $n > 0$ there exists a unique $k \geq 0$ with $q_{k-1} + q_k \leq n < q_k + q_{k+1}$. Since $q_{k+1} = q_{k-1} + a_k q_k$, there exists a unique $1 \leq m \leq a_{k+1}$ such that $q_{k-1} + mq_k \leq n < q_{k-1} + (m+1)q_k$. Then $r = n - q_{k-1} - mq_k < q_k$. Set $s(i) = \mathrm{mod}_1(-i\alpha)$. We show by induction that for k even, $\mathcal{V}_\alpha^n$ consists of intervals

$$\begin{array}{lll} [s(i), s(q_k + i)), & i = 0, \ldots, n - q_k, & l_1 = \eta_k \\ [s(q_{k-1} + mq_k + i), s(i)), & i = 0, \ldots, r, & l_2 = \eta_{k-1} - m\eta_k \\ [s(q_{k-1} + (m-1)q_k + i), s(i)), & i = r+1, \ldots, q_k - 1, & l_3 = \eta_{k-1} - (m-1)\eta_k \end{array}$$

For k odd we have the same intervals but but the order of the endpoints is reversed. The theorem holds for $n = 1$. We have $1 = q_{-1} + q_0$, so $k = 0$, $m = 1$, $r = 0$ and $\mathcal{V}_\alpha^1$ has intervals

$$\begin{array}{lll} [s(i + q_0), s(i)), & i = 0, & l_1 = \eta_0 = \alpha \\ [s(i), s(q_0 + i)), & i = 0, & l_2 = \eta_{-1} - \eta_0 = 1 - \alpha \end{array}$$

Assume that the theorem holds for n and let us prove it for $n + 1$. We consider three cases.

(1) If $r < q_k - 1$ then $n = q_{k-1} + mq_k + r$ and $n' = n + 1 = q_{k-1} + mq_k + r + 1$ (Figure 11 top). For $i = r + 1$ we have an interval $[s(q_{k-1} + (m-1)q_k + i), s(i)) \in \mathcal{V}_\alpha^n$ of length l_3 which in $\mathcal{V}_\alpha^{n+1}$ splits into two intervals

$$[s(q_{k-1} + (m-1)q_k + i), s(q_{k-1} + mq_k + i)), \quad [s(q_{k-1} + mq_k + i), s(i))$$

of lengths l_1 and l_2. Thus $r' = r + 1$, $n_1' = n_1 + 1$, $n_2' = n_2 + 1$, $n_3' = n_3 - 1$, while k, m, l_0, l_1, l_2 do not change.

(2) If $r = q_k - 1$, $m < a_{k+1}$, then $n = q_{k-1} + mq_k + q_k - 1$, $n' = n + 1 = q_{k-1} + (m+1)q_k$, so $n_3 = 0$ and there are only intervals of lengths l_1, l_2 (Figure 11 center). In $\mathcal{V}_\alpha^n$ there is an interval $[s(q_{k-1} + mq_k), s(0))$ of length l_2, which in $\mathcal{V}_\alpha^{n+1}$ splits into

$$[s(q_{k-1} + mq_k), s(q_{k-1} + (m+1)q_k), \quad [s(q_{k-1} + (m+1)q_k, s(0))$$

of lengths $l_1 = \eta_k$ and $l_2' = l_2 - \eta_k = \eta_k - (m+1)\eta_k$. We have therefore $l_3' = l_2$, $l_1' = l_1$, $n_1' = n_1 + 1$, $n_2' = 1 = n_3 + 1$, $n_3' = q_k - 1 = n_2 - 1$.

(3) If $r = q_k - 1$, $m = a_{k+1}$, then $n = q_{k-1} + a_{k+1}q_k + q_k - 1 = q_k + q_{k+1} - 1$, $n' = n + 1 = q_k + q_{k+1}$, so $n_1 = q_{k+1}$, $n_2 = q_k$, $n_3 = 0$, $l_1 = \eta_k$, $l_2 = \eta_{k-1} - a_{k+1}\eta_k = \eta_{k+1}$ (Figure 11 bottom). We have an interval $[s(0), s(q_k)) \in \mathcal{V}_\alpha^n$ of length l_1 which in $\mathcal{V}_\alpha^{n+1}$ splits into intervals

$$[s(0), s(q_k + q_{k+1})), \quad [s(q_k + q_{k+1}), s(q_k))$$

of lengths $l_2' = \eta_k - \eta_{k+1} = l_2$, $l_1' = \eta_{k+1}$. Then $k' = k+1$, $m' = 1$, $r' = 0$, $l_3' = \eta_k = l_1$, $n_1' = n_2 + 1$, $n_2' = 1$, $n_3' = n_1 - 1$. □

As n varies from $q_{k-1} + q_k$ to $q_k + q_{k+1}$, the lengths in $\mathcal{V}^n$ vary in the following way (recall that $q_{k+1} = q_{k-1} + a_{k+1}q_k$ and $\eta_{k+1} = \eta_{k-1} - a_{k+1}\eta_k$).

n	$q_{k-1} + q_k$	$q_{k-1} + 2q_k - 1$	$q_{k-1} + 2q_k$	$q_{k-1} + 3q_k - 1$
η_k	$q_{k-1} + 1$	$q_{k-1} + q_k$	$q_{k-1} + q_k + 1$	$q_{k-1} + 2q_k$
η_{k-1}	$q_k - 1$	0	0	0
$\eta_{k-1} - \eta_k$	1	q_k	$q_k - 1$	0
$\eta_{k-1} - 2\eta_k$	0	0	1	q_k

n	$q_{k+1} - 1$	q_{k+1}	$q_{k+1} + q_k - 2$	$q_{k+1} + q_k - 1$
η_k	$q_{k+1} - q_k$	$q_{k+1} - q_k + 1$	$q_{k+1} - 1$	q_{k+1}
$\eta_{k+1} + \eta_k$	q_k	$q_k - 1$	1	0
η_{k+1}	0	1	$q_k - 1$	q_k

We consider recurrence times in Σ_α with irrational $\alpha = [0, a_1, a_2, \ldots]$. By Proposition 3.26 for any $u \in \mathcal{L}(\Sigma_\alpha)$, $\tau([u]) = \tau(V_u)$ in $(\mathbb{T}, R_\alpha)$. The closest return times for any point $x \in \mathbb{T}$ are the convergents q_k (Proposition A.84). If $q_{k-1} < q < q_k$, then $d(R^q(0), 0) > \eta_{k-1} > \eta_k = d(R_\alpha^{q_k}(0), 0)$. If $I = [a, b) \subseteq \mathbb{T}$ is a semiclosed interval with length $||I|| = b - a$, then

$$\eta_k < ||I|| \leq \eta_{k-1} \;\Rightarrow\; \tau(I) = q_k.$$

We now calculate the recurrence characteristics of Definition 3.27.

***Proposition 4.46* (Kůrka and Kupsa [92]).** — *Let $\alpha \in (0,1)$ be an irrational number with convergents p_n, q_n. Then in the subshift Σ_α we have*

$$\begin{array}{rcccll} q_{k-1}+q_k-1 & \leq & n & < & q_k+q_{k+1}-1 & \Rightarrow \quad Q(n)=q_k, \\ q_k & \leq & n & < & q_{k+1} & \Rightarrow \quad R(n)=S(n)=q_{k+1}, \\ & & & & & \quad\ T(n)=q_k+q_{k+1}. \end{array}$$

Proof. — Denote by $s(i)=\mathrm{mod}_1(-i\alpha)=R_\alpha^{-i}(0)$. Then $d(s(q_k),0)=\eta_k$ and the cut points of $\mathcal{V}_\alpha^n$ are $\{0,s(1),\ldots,s(n)\}$.

(1) Assume that $q_{k-1}+q_k-1 \leq n \leq q_k+q_{k+1}-2$. From the proof of Theorem 4.45 we see that any interval $I \in \mathcal{V}_\alpha^n$ has length at most $||I|| \leq \eta_{k-1}$, so $\tau(I) \geq q_k$ and $Q(n) \geq q_k$. On the other hand, $\mathcal{V}_\alpha^n$ contains an interval $I=[s(q_{k-1}+mq_k), s(0))$, where $1 \leq m < a_{k+1}$. We have $||I|| = \eta_{k-1} - m\eta_k \geq \eta_{k+1}+\eta_k$ so $\tau(I)=q_k$ and $Q(n) \leq q_k$.

(2) Assume that $q_k \leq n < q_{k+1}$. Then $\mathcal{V}_\alpha^n$ contains an interval $I=[s(0),s(q_k))$ of length $||I||=\eta_k$, so $R(n) \geq \tau(I)=q_{k+1}$. On the other hand any interval $I \in \mathcal{V}_\alpha^n$ has length $||I|| \leq q_{k+1}$, so $\tau(I) \leq q_{k+1}$ and $R(n) \leq q_{k+1}$.

(3) Assume that $I=[s(i_1),s(i_2))$, $J=[s(j_1),s(j_2))$ are two distinct intervals of $\mathcal{V}^n$ where $0 \leq i_1 < j_1 \leq n$. Then $0 \leq j_1 - i_1 \leq n < q_{k+1}$, so $R_\alpha^{j_1-i_1}(J) \cap I \neq \emptyset$ and $\tau(J,I) \leq q_{k+1}$. To show $\tau(I,J) \leq q_{k+1}$, assume first that k is odd. Then $R_\alpha^{q_{k+1}-j_1+i_1}(s(i_1)) = R_\alpha^{q_{k+1}}(s(j_1)) \in J$, so $\tau(I,J) \leq q_{k+1}-j_1+i_1 \leq q_{k+1}$. If k is even, we replace the assumption $0 \leq i_1 < j_1 \leq n$ by $0 \leq i_2 < j_2 \leq n$ and apply the same argument to $s(i_2)$. Thus $S(n) \leq q_{k+1}$ and since $R(n) \leq S(n)$, we get $S(n)=R(n)=q_{k+1}$.

(4) Assume that k is even. Then $I=[s(0),s(q_k))$ is the shortest interval in $\mathcal{V}_\alpha^n$. If $\eta_{k+1} < x < s(q_k)$, then $R_\alpha^{q_{k+1}}(x) \in J$. If $0 \leq x \leq \eta_{k+1}$, then $R_\alpha^{q_{k+1}}(x) \in (1-\eta_{k+1},1)$ and $R_\alpha^{q_k+q_{k+1}}(x) \in J$, so $T(n)=q_k+q_{k+1}$. □

Observe that for $q_{k-1} \leq n < q_k$ we have $Q(T(n))=q_k$. In particular $Q(T(q_k-1))=q_k$, so the bound in Proposition 3.29 cannot be improved.

Proposition 4.47. — *Let Σ_α be a Sturmian subshift with irrational $\alpha=[0,a_1,a_2,\ldots]$. Then*

$$\begin{aligned} \liminf_{n\to\infty} \frac{Q(n)}{n} &= \liminf_{k\to\infty} \frac{q_k}{q_{k+1}+q_k+1} \leq \liminf_{k\to\infty} \frac{1}{a_k+1} \\ \limsup_{n\to\infty} \frac{Q(n)}{n} &= \limsup_{k\to\infty} \frac{q_k}{q_k+q_{k-1}+1} \geq \frac{1}{2} \\ \liminf_{k\to\infty} \frac{R(n)}{n} &= 1 \\ \limsup_{n\to\infty} \frac{R(n)}{n} &= \limsup_{k\to\infty} \frac{q_k}{q_{k-1}} \geq \limsup_{k\to\infty} a_k \\ \liminf_{n\to\infty} \frac{\ln R(n)}{n} &= \lim_{n\to\infty} \frac{\ln Q(n)}{n} = 0, \\ \limsup_{n\to\infty} \frac{\ln R(n)}{n} &= \limsup_{n\to\infty} \frac{\ln q_k}{q_{k-1}} \end{aligned}$$

The last quantity may be arbitrarily large, even infinite.

4.4.1. Kneading sequence. — We determine the structure of Σ_α from its kneading sequence $\mathcal{K}(\alpha)$, which is the itinerary of α.

$$\mathcal{K}(\alpha)_i = 0 \quad \Leftrightarrow \quad R_\alpha^i(\alpha) \in [0, 1-\alpha)$$
$$\mathcal{K}(\alpha)_i = 1 \quad \Leftrightarrow \quad R_\alpha^i(\alpha) \in [1-\alpha, 1)$$

We show how to obtain $\mathcal{K}(\alpha)$ from the continued fraction of α using a sequence of substitutions.

Proposition 4.48. — *Let $\alpha \in (0,1)$ be an irrational number. Then*

$$\alpha < \frac{1}{2} \quad \Rightarrow \quad \mathcal{K}(\alpha) = \nu_0(\mathcal{K}(\tfrac{\alpha}{1-\alpha})) \ \text{ where } \ \nu_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases}$$

$$\alpha > \frac{1}{2} \quad \Rightarrow \quad \mathcal{K}(\alpha) = \nu_1(\mathcal{K}(\tfrac{1-\alpha}{\alpha})) \ \text{ where } \ \nu_1 = \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 10 \end{cases}$$

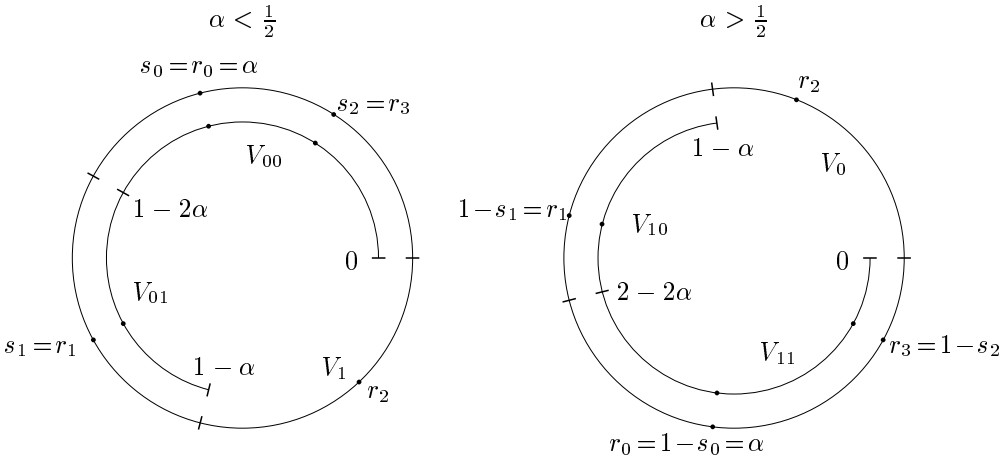

FIGURE 12. Recoding of $\mathcal{K}(\alpha)$

Proof. — Denote by $r_n = \mathrm{mod}_1((n+1)\alpha) = R_\alpha^n(\alpha)$. Assume that $\alpha < \frac{1}{2}$ and consider the integers n such that $r_n \in V_0 = [0, 1-\alpha)$ (Figure 12 left). These are defined by

$$n_0 = 0, \quad n_{i+1} = \min\{n > n_i : r_n \in V_0\}.$$

The interval V_0 is divided into $V_{00} = [0, 1-2\alpha)$ and $V_{01} = [1-2\alpha, 1-\alpha)$. Define a sequence $(x_i)_{i\geq 0}$ by $x_i = 0$ if $r_{n_i} \in V_{00}$ and $x_i = 1$ if $r_{n_i} \in V_{01}$. Since $R_\alpha(V_1) \subseteq V_0$, we have $V_1 = V_{10}$, $V_{01} = V_{010}$ and

$$\begin{array}{lllll} x_i = 0 & \Rightarrow & r_{n_i} \in V_{00} & \Rightarrow & \mathcal{K}(\alpha)_{n_i} = 0, \qquad\quad n_{i+1} = n_i + 1 \\ x_i = 1 & \Rightarrow & r_{n_i} \in V_{010} & \Rightarrow & \mathcal{K}(\alpha)_{[n_i, n_i+1]} = 01, \quad n_{i+1} = n_i + 2 \end{array}$$

so $\mathcal{K}(\alpha)_{[n_i, n_i+1]} = \nu_0(x_i)$ and therefore $\mathcal{K}(\alpha) = \nu_0(x)$. Set $s_i = r_{n_i}$. Then

$$s_i \in V_{00} \quad \Rightarrow \quad s_{i+1} = r_{n_i+1} = s_i + \alpha$$
$$s_i \in V_{01} \quad \Rightarrow \quad s_{i+1} = r_{n_i+2} = s_i + 2\alpha - 1 = s_i + \alpha - (1-\alpha)$$

Thus $s_{i+1} = \mathrm{mod}_{1-\alpha}(s_i + \alpha)$ is a rotation of α on the interval $[0, 1-\alpha)$. For $s'_i = s_i/(1-\alpha)$ and $\beta = \alpha/(1-\alpha)$ we have $s'_{i+1} = \mathrm{mod}_1(s'_i + \beta)$. Since $(1-2\alpha)/(1-\alpha) = 1-\beta$,

$$\begin{array}{rcl} s'_i \in [0, 1-\beta) & \Rightarrow & s_i \in V_{00} \quad \Rightarrow \quad x_i = 0 \\ s'_i \in [1-\beta, 1) & \Rightarrow & s_i \in V_{01} \quad \Rightarrow \quad x_i = 1. \end{array}$$

We get $\mathcal{K}(\beta) = x$ and $\mathcal{K}(\alpha) = \nu_0(\mathcal{K}(\beta))$. If $\alpha > \frac{1}{2}$, we consider integers n such that $r_n \in V_1 = [1-\alpha, 1)$ (Figure 12 right). They are defined by

$$n_0 = 0, \quad n_{i+1} = \min\{n > n_i : r_n \in V_1\}$$

The interval V_1 is divided into $V_{10} = [1-\alpha, 2-2\alpha)$ and $V_{11} = [2-2\alpha, 1)$. Since $R_\alpha(V_0) \subseteq V_1$, we have $V_{01} = V_0$ and $V_{101} = V_{10}$. Set $x_i = 0$ if $r_{n_i} \in V_{11}$ and $x_i = 1$ if $r_{n_i} \in V_{10}$. Then

$$\begin{array}{rcl} x_i = 0 & \Rightarrow & r_{n_i} \in V_{11} \quad \Rightarrow \quad \mathcal{K}(\alpha)_{n_i} = 1, \qquad\quad n_{i+1} = n_i + 1 \\ x_i = 1 & \Rightarrow & r_{n_i} \in V_{101} \quad \Rightarrow \quad \mathcal{K}(\alpha)_{[n_i, n_i+1]} = 10, \quad n_{i+1} = n_i + 2 \end{array}$$

so $\mathcal{K}(\alpha) = \nu_1(x)$. Set $s_i = 1 - r_{n_i}$. Since $1 - (2-2\alpha) = 2\alpha - 1$, we get

$$s_i \in (0, 2\alpha-1) \quad \Rightarrow \quad s_{i+1} = 1 - r_{n_{i+1}} = 1 - (r_{n_i} + \alpha - 1) = s_i + 1 - \alpha$$

$$s_i \in (2\alpha-1, \alpha) \quad \Rightarrow \quad s_{i+1} = 1 - r_{n_i+2} = 1 - (r_{n_i} + 2\alpha - 1) = s_i + 1 - \alpha - \alpha$$

Thus $s_{i+1} = \mathrm{mod}_\alpha(s_i + 1 - \alpha)$ is a rotation of $1-\alpha$ on the interval $[0, \alpha)$, so for $\beta = (1-\alpha)/\alpha$ and $s'_i = s_i/\alpha$ we have $s'_{i+1} = \mathrm{mod}_1(s'_i + \beta)$. Moreover $s_i \in (0, 2\alpha-1)$ iff $s'_i \in (0, 1-\beta)$, so $x = \mathcal{K}(\beta)$ and $\mathcal{K}(\alpha) = \nu_1(\mathcal{K}(\beta))$. □

To obtain $\mathcal{K}(\alpha)$, we use Proposition A.83. If $\alpha = [0, a_1, a_2, \ldots,]$, then

$$\begin{array}{rcl} \alpha < \frac{1}{2} & \Rightarrow & a_1 > 1, \quad \frac{\alpha}{1-\alpha} = [0, a_1 - 1, a_2, \ldots] \\ \alpha > \frac{1}{2} & \Rightarrow & a_1 = 1, \quad \frac{1-\alpha}{\alpha} = [0, a_2, a_3, \ldots] \end{array}$$

For the golden number $\alpha = \frac{\sqrt{5}-1}{2} = [0, 1, 1, 1 \ldots]$ we have $\frac{1-\alpha}{\alpha} = \alpha$, so $\mathcal{K}(\alpha) = \nu_1(\mathcal{K}(\alpha))$ is the fixed point of the Fibonacci substitution ν_1 (Example 4.11), and $\Sigma_\alpha = \Sigma_{\nu_1}$ is a substitutive system. If $\alpha = [0, 3, 1, 5, 4, \ldots]$, then

$$\mathcal{K}(\alpha) = \nu_0^2(\mathcal{K}([0, 1, 1, 5, 4, \ldots])) = \nu_0^2\nu_1^2(\mathcal{K}([0, 5, 4 \ldots])) = \cdots$$

Since $\nu_0(a)$ have common prefix 0 and $\nu_1(a)$ have common prefix 1, $\mathcal{K}(\alpha)_{[0,4)} = 0010$, the common prefix of $\nu_0^2\nu_1^2(0)$ and $\nu_0^2\nu_1^2(1)$.

A Sturmian subshift can be encoded as an infinite sequence of substitutions. Recall that for a substitution $\vartheta : \mathbf{2} \to \mathbf{2}^*$ we have (Definition 4.18)

$$\vartheta[\mathbf{2}^{\mathbb{N}}] = \{\vartheta(0), \vartheta(1)\}^{\mathbb{N}} = \{\sigma^n(\vartheta(z)) : z \in \mathbf{2}^{\mathbb{N}}, n \geq 0\}$$

We write $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ if $\eta = (\eta_i)_{i \geq 0}$ is an infinite sequence of substitutions with $\eta_i \in \{\nu_0, \nu_1\}$. For a sequence $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ we set

$$\eta[\mathbf{2}^{\mathbb{N}}] = \bigcap_{i>0} \eta_0 \cdots \eta_{i-1}[\mathbf{2}^{\mathbb{N}}].$$

Proposition 4.49. — *If $\alpha \in (0,1)$ is an irrational number, then there exists a sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$, such that for any $u \in \mathbf{2}^+$,*

$$\mathcal{K}(\alpha) = \lim_{i \to \infty} \eta_0\eta_1 \ldots \eta_{i-1}(u), \quad \Sigma_\alpha = \eta[\mathbf{2}^{\mathbb{N}}] = \bigcap_{i>0} \eta_0\eta_1 \ldots \eta_{i-1}[\mathbf{2}^{\mathbb{N}}].$$

Proof. — By Proposition 4.48, there exists a sequence of substitutions $\eta_i \in \{\nu_0, \nu_1\}$ and points $x_i \in \mathbf{2}^{\mathbb{N}}$ such that $\mathcal{K}(\alpha) = \eta_0 \cdots \eta_{i-1}(x_i)$. The sequence $(\eta_i)_{i\geq 0}$ must contain an infinite number of ν_1 (at positions $a_1 - 1$, $a_1 + a_2 - 1$, etc.) If $\eta_i = \nu_1$, then $\eta_{i-1}\eta_i$ causes a prolongation of the common prefix. Indeed we have

$$\nu_1^2 = \begin{cases} 0 \mapsto 1 \mapsto 10 \\ 1 \mapsto 10 \mapsto 101 \end{cases}, \quad \nu_0\nu_1 = \begin{cases} 0 \mapsto 1 \mapsto 01 \\ 1 \mapsto 10 \mapsto 010 \end{cases}$$

If $x, y \in \mathbf{2}^{\mathbb{N}}$ and $x_{[0,m)} = y_{[0,m)}$, then $\eta_i\nu_1(x)_{[0,m+1)} = \eta_i\nu_1(y)_{[0,m+1)}$. Thus for

$$k_n = \#\{i < n : \eta_i = \nu_1\},$$

$\eta_0 \cdots \eta_{n-1}(u)_{[0,k_n)} = \eta_0 \cdots \eta_{n-1}(v)_{[0,k_n)}$ for any two $u, v \in \mathbf{2}^+$, and it follows that $\mathcal{K}(\alpha) = \lim_{n\to\infty} \eta_0 \ldots \eta_{n-1}(u)$. Since $\mathcal{K}(\alpha) \in \Sigma_\alpha$ and Σ_α is minimal, we have

$$\Sigma_\alpha = \overline{\mathfrak{O}(\mathcal{K}(\alpha))} \subseteq \bigcap_{n>0} \eta_0 \ldots \eta_{n-1}[\mathbf{2}^{\mathbb{N}}].$$

If $x \in \bigcap_{n>0} \eta_0 \ldots \eta_{n-1}[\mathbf{2}^{\mathbb{N}}]$, then any of its prefixes is a subword of $\mathcal{K}(\alpha)$, so $x \in \Sigma_\alpha$. □

There is an alternative description of $\mathcal{K}(\alpha)$ based on a different geometric reduction. If $\alpha = [0, a_1, a_2, \ldots]$, then in time $a_1 = \lfloor \frac{1}{\alpha} \rfloor$ the system comes within $\eta_1 = 1 - a_1\alpha$ of 0. We consider the times at which the kneading sequence $\mathcal{K}(\alpha)$ visits the interval $J = (0, \alpha + \eta_1]$ and its two subintervals $J_0 = (\eta_1, \eta_1 + \alpha]$, $J_1 = (0, \eta_1]$. We use a substitution

$$\vartheta_a = \begin{cases} 0 \mapsto 0^a 1 \\ 1 \mapsto 0 \end{cases}$$

Proposition 4.50. — *Let $\alpha = [0, a_1, a_2, \ldots] \in (0, 1)$ be irrational and $\eta_1 = 1 - a_1\alpha$, Then*

$$\mathcal{K}(\alpha) = \vartheta_{a_1-1}(\mathcal{K}([0, a_2 + 1, a_3, \ldots]) = \vartheta_{a_1-1}\left(\mathcal{K}\left(\frac{\eta_1}{\eta_1 + \alpha}\right)\right).$$

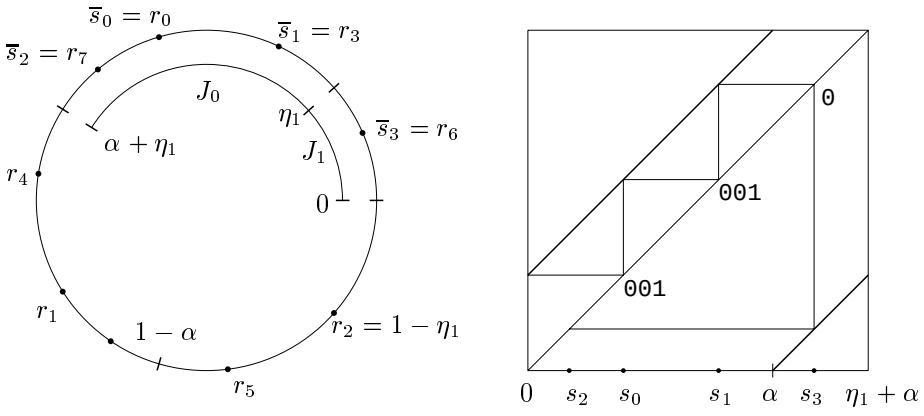

FIGURE 13. Sturmian sequence of a rotation

Proof. — Set $r_n = \mathrm{mod}_1((n+1)\alpha)$. We have $1-\alpha < a_1\alpha < 1$ and $0 < \eta_1 < \alpha$. The sequence $\mathcal{K}(\alpha)$ can be obtained from the iterates which fall into the interval $J = (0, \alpha+\eta_1]$ (Figure 13) and we decompose it into intervals $J_0 = (\eta_1, \alpha+\eta_1]$, $J_1 = (0, \eta_1]$. Define a sequence $(n_i)_{i\geq 0}$, $(x_i)_{i\geq 0}$ by

$$n_0 = 0, \quad n_{i+1} = \min\{n > n_i : r_n \in J\}, \quad x_i = \begin{cases} 0 \text{ if } r_{n_i} \in J_0 \\ 1 \text{ if } r_{n_i} \in J_1 \end{cases}$$

We show that

$$\begin{aligned} x_i = 1 \quad &\Rightarrow \quad r_{n_i} \in V_0, \quad r_{n_i+1} \in J_0 \\ x_i = 0 \quad &\Rightarrow \quad r_{n_i}, \ldots, r_{n_i+a_1-2} \in V_0, \quad r_{n_i+a_1-1} \in V_1, \quad r_{n_i+a_1} \in J \end{aligned}$$

If $x_i = 1$ then $0 < r_{n_i} \leq \eta_1 \leq 1-\alpha$, so $r_{n_i} \in V_0$. Since

$$\eta_1 < \alpha < r_{n_i} + \alpha \leq \alpha + \eta_1,$$

we get $r_{n_i+1} \in J_0$. If $x_i = 0$ then $\eta_1 < r_{n_i} \leq \eta_1 + \alpha$ and

$$0 < r_{n_i} - \eta_1 = r_{n_i} + a_1\alpha - 1 \leq \alpha,$$

so $r_{n_i+a_1} \in J$. We add $1-\alpha$ to the last inequality and get $1-\alpha < r_{n_i} + (a_1-1)\alpha \leq 1$, so $r_{n_i+a_1-1} \in V_1$. For $0 \leq j < a_1 - 1$ we have $0 < r_{n_i} + j\alpha < 1-\alpha$, so $r_{n_i+j} \in V_0$. It follows that

$$\begin{array}{lll} x_i = 0 & \Rightarrow & \mathcal{K}(\alpha)_{[n_i, n_i+a_1)} = 0^{a_1-1}1, \quad n_{i+1} = n_i + a_1 \\ x_i = 1 & \Rightarrow & \mathcal{K}(\alpha)_{n_i} = 0, \qquad\qquad\quad n_{i+1} = n_i + 1 \end{array}$$

and $\mathcal{K}(\alpha) = \vartheta_{a_1-1}(x)$. Note that this also holds for $a_1 = 1$. In this case $\alpha > \frac{1}{2}$ and $\eta_1 + \alpha = 1$. We now show that $x = \mathcal{K}(\beta)$ where $\beta = \eta_1/(\eta_1+\alpha)$. We use the transformation $x \mapsto \eta_1 + \alpha - x$ which maps intervals J_0 and J_1 to $J_0' = [0, \alpha)$ and $J_1' = [\alpha, \alpha+\eta_1)$. Set $s_i = \eta_1 + \alpha - r_{n_i}$. Then

$$\begin{aligned} s_i \in J_0' \quad &\Rightarrow \quad s_{i+1} = \eta_1 + \alpha - (r_{n_i} + a_1\alpha - 1) = s_i + \eta_1 \\ s_i \in J_1' \quad &\Rightarrow \quad s_{i+1} = \eta_1 + \alpha - (r_{n_i} + \alpha) = s_i - \alpha \end{aligned}$$

so $s_{i+1} = \mathrm{mod}_{\eta_1+\alpha}(s_i + \eta_1)$ is a rotation by η_1 on the interval $[0, \eta_1+\alpha)$. For $s_i' = s_i/(\alpha+\eta_1)$ we have $s_{i+1}' = \mathrm{mod}_1(s_i' + \beta)$, so $x = \mathcal{K}(\beta)$. We now show the action of the transformation $\alpha \mapsto \eta_1/(\eta_1+\alpha)$ on the continued fraction of α. For $\alpha_2 = [a_2, a_3, a_4, \ldots]$ we get

$$\alpha = [0, a_1, a_2, a_3, \ldots] = [0, a_1, \alpha_2] = \frac{\alpha_2}{a_1\alpha_2 + 1}, \quad \eta_1 = 1 - a_1\alpha = \frac{1}{a_1\alpha_2 + 1}$$

and

$$\frac{\eta_1}{\eta_1 + \alpha} = \frac{1}{1+\alpha_2} = \frac{1}{[a_2+1, a_3, \ldots]} = [0, a_2+1, a_3, \ldots]$$

□

We can prove Proposition 4.50 algebraically using the identity $\vartheta_a\nu_0 = (\nu_0)^a\nu_1$

$$\vartheta_a\nu_0 = \begin{cases} 0 \mapsto 0 \mapsto 0^a1 \\ 1 \mapsto 01 \mapsto 0^a10 \end{cases} \qquad (\nu_0)^a\nu_1 = \begin{cases} 0 \mapsto 1 \mapsto 0^a1 \\ 1 \mapsto 10 \mapsto 0^a10 \end{cases}$$

For $\alpha = [0, a_1, a_2, a_3, \ldots]$ we get

$$\begin{aligned}\mathcal{K}([0, a_1, a_2, \ldots]) &= (\nu_0)^{a_1-1}(\mathcal{K}([0, 1, a_2, \ldots])) = (\nu_0)^{a_1-1}\nu_1(\mathcal{K}([0, a_2, \ldots])) \\ &= \vartheta_{a_1-1}\nu_0(\mathcal{K}([0, a_2, \ldots])) = \vartheta_{a_1-1}(\mathcal{K}([0, a_2+1, \ldots])).\end{aligned}$$

Proposition 4.51. — *Let $\alpha = [0, a_1, a_2, a_3, \ldots] \in (0,1)$ be an irrational number. Define a sequence of words*

$$W_{-1} = 1, \quad W_0 = 0, \quad W_1 = 0^{a_1-1}1, \quad W_2 = (W_1)^{a_2}W_0, \ldots, W_n = (W_{n-1})^{a_n}W_{n-2}.$$

Then $|W_n| = q_n$ and $|W_n|_1 = p_n$ are the convergents of α,

$$\vartheta_{a_1-1}\vartheta_{a_2}\ldots\vartheta_{a_n} = \begin{cases} 0 \mapsto W_n \\ 1 \mapsto W_{n-1} \end{cases}$$

$$\mathcal{K}(\alpha) = \lim_{n\to\infty} W_n, \quad \Sigma_\alpha = \bigcap_{n>0} \vartheta_{a_1-1}\vartheta_{a_2}\cdots\vartheta_{a_n}[\mathbf{2}^{\mathbb{N}}] = \bigcap_{n>0}\{W_{n-1}, W_n\}^{\mathbb{N}}$$

Proof. — We have

$$\vartheta_{a_1-1} = \begin{cases} 0 \mapsto W_1 \\ 1 \mapsto W_0 \end{cases}, \quad \vartheta_{a_1-1}\vartheta_{a_2} = \begin{cases} 0 \mapsto 0^{a_2}1 \mapsto (0^{a_1-1}1)^{a_2}0 = W_2 \\ 1 \mapsto 0 \quad\quad \mapsto 0^{a_1-1}1 = W_1 \end{cases}$$

and by induction

$$\begin{aligned} W_{n+1} &= (W_n)^{a_{n+1}}W_{n-1} = (\vartheta_{a_1-1}\ldots\vartheta_{a_n}(0))^{a_{n+1}}\ \vartheta_{a_1-1}\ldots\vartheta_{a_n}(1) \\ &= \vartheta_{a_1-1}\ldots\vartheta_{a_n}(0^{a_{n+1}}1) = \vartheta_{a_1-1}\ldots\vartheta_{a_n}\vartheta_{a_{n+1}}(0) \\ W_n &= \vartheta_{a_1-1}\ldots\vartheta_{a_n}(0) = \vartheta_{a_1-1}\ldots\vartheta_{a_n}\vartheta_{a_{n+1}}(1) \end{aligned}$$

Since W_n is a prefix of W_{n+1}, we get $\mathcal{K}(\alpha) = \lim_{n\to\infty} W_n$ and $\Sigma_\alpha = \bigcap_{n>0}\{W_{n-1}, W_n\}^{\mathbb{N}}$ by Proposition 4.50. □

For $\alpha = \sqrt{2} - 1 = [0, 2, 2, 2, \ldots] \doteq 0.414$ we have

$$W_0 = 0, \quad W_1 = 01, \quad W_2 = 01010, \quad W_3 = 010100101001, \ldots$$

For $\alpha = \sqrt{3} - 1 = [0, 1, 2, 1, 2, 1, 2, \ldots] \doteq 0.732$ we have

$$W_0 = 0, \quad W_1 = 1, \quad W_2 = 110, \quad W_3 = 1101,\ W_4 = 1101\ 1101\ 110, \ldots$$

Proposition 4.52. — *An irrational $\alpha \in (0,1)$ is quadratic iff there exist substitutions $\vartheta, \xi : \mathbf{2} \to \mathbf{2}^*$ such that $\Sigma_\alpha = \xi[\Sigma_\vartheta]$.*

Proof. — By the Lagrange theorem A.79, an irrational α is quadratic, iff its continued fraction is eventually periodic. Thus if α is quadratic, there exists an eventually periodic sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ with preperiodic part $\xi = \eta_0 \ldots \eta_{m-1}$ and periodic part $\vartheta = \eta_m \ldots \eta_{m+p-1}$ such that $\Sigma_\alpha = \eta[\mathbf{2}^{\mathbb{N}}] = \xi[\Sigma_\vartheta]$. Conversely suppose that $\Sigma_\alpha = \xi[\Sigma_\vartheta]$, where ξ, ϑ are substitutions over $\mathbf{2}$. The matrix $M(\vartheta)$ has integer entries, so its spectral radius λ is a quadratic number. By Proposition 4.32 Σ_ϑ is strictly ergodic and the frequencies are components of the normalized left eigenvector $\mu = (\mu([0]), \mu([1]))$ of λ, so they are quadratic too. The vector of frequencies of $\xi[\Sigma_\vartheta]$ is $\mu \cdot M(\xi)$, so it has quadratic entries and one of these entries is $\alpha = ||V_1||$. □

4.5. Skew Sturmian subshifts

By Proposition 4.49, for any irrational $\alpha \in (0,1)$ there exists a sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ such that $\Sigma_\alpha = \eta[\mathbf{2}^{\mathbb{N}}]$, and $\eta_i = \nu_1$ for infinitely many i. If $\eta_i = \nu_1$ only for finitely many i, we get **skew Sturmian subshifts** which may be thought of as subshifts associated to rational rotations. For example $0 = \lim_{a\to\infty}[0,a]$ gives rise to $\eta_i = \nu_0$ for all i. We have $(\nu_0)^a(0) = 0$, $(\nu_0)^a(1) = 0^a 1$, so $(\nu_0)^a[\mathbf{2}^{\mathbb{N}}] = \{\sigma^k(\nu_0)^a(u) : u \in \mathbf{2}^*, k \geq 0\}$ and this is denoted as $\{0, 0^a 1\}^{\mathbb{N}}$.

$$\Sigma_{(0)} = \bigcap_{a\geq 0} (\nu_0)^a[\mathbf{2}^{\mathbb{N}}] = \bigcap_{a\geq 0} \{0, 0^a 1\}^{\mathbb{N}} = \{0^k 10^\infty : k \geq 0\} \cup \{0^\infty\}$$

This is a sofic subshift which we have already encountered in Example 3.60 (Figure 10). We denote this subshift by 0^*10^∞ or in a shorter form by 0^*1. Similarly, $1 = [0,1] = \lim_{a\to\infty}[0,1,a]$ so

$$\Sigma_{(0,1)} = \nu_1[\Sigma_{(0)}] = \bigcap_{a>0} \{1, 1^a 0\}^{\mathbb{N}} = \{1^k 01^\infty : k \geq 0\} \cup \{1^\infty\} = 1^*0$$

While 0 and 1 have unique simple continued fraction with $a_0 = 0$, any rational number $\alpha \in (0,1)$ has two finite continued fractions in $\mathcal{N}$. For $\alpha = \frac{1}{2} = [0,2] = [0,1,1]$ we get

$$\Sigma_{(0,2)} = \nu_0[\Sigma_{(0,1)}] = \bigcap_{a>0} \{01, (01)^a 0\}^{\mathbb{N}} = (01)^*0$$

$$\Sigma_{(0,1,1)} = \nu_1[\Sigma_{(0,1)}] = \bigcap_{a>0} \{10, (10)^a 1\}^{\mathbb{N}} = (10)^*1$$

These two subshifts have the same periodic sequences $\Sigma_{(0,2)} \cap \Sigma_{(0,1,1)} = \{(01)^\infty, (10)^\infty\}$ but different preperiodic sequences. Similarly for $\alpha = \frac{1}{3}$

$$\Sigma_{(0,3)} = \nu_0[\Sigma_{(0,2)}] = (001)^*0, \quad \Sigma_{(0,2,1)} = \nu_0[\Sigma_{(0,1,1)}] = (010)^*01$$

and $\Sigma_{(0,3)} \cap \Sigma_{(0,2,1)} = \{(001)^\infty, (010)^\infty, (100)^\infty\}$. The periodic points in the intersection are itineraries of points $x \in (0,1) \setminus \{\frac{1}{3}, \frac{2}{3}\}$ with respect to the rotation $(\mathbb{T}, R_{1/3})$ and the partition $\{V_0 = [0, \frac{2}{3}), V_1 = [\frac{2}{3}, 1)\}$.

A systematic generation of all skew Sturmian subshifts from $\Sigma_{(0)}$ is shown in Figure 14. We put an arrow between subshifts

$$\eta_{n-1} \cdots \eta_0[\Sigma_{(0)}] \xrightarrow{\eta_n} \eta_n \eta_{n-1} \cdots \eta_0[\Sigma_{(0)}]$$

Because of the duplicity of continued fractions representation of rational numbers, we define Sturmian and skew Sturmian subshifts for continued fractions rather than for real numbers.

Definition 4.53. —
The Sturmian subshift of an infinite continued fraction $(0, a_1, a_2, \ldots) \in \mathcal{N}$ is

$$\Sigma_{(0,a_1,a_2,\ldots)} = \bigcap_{n>0} \vartheta_{a_1-1}\vartheta_{a_2} \cdots \vartheta_{a_n}[\mathbf{2}^{\mathbb{N}}] = \bigcap_{n>0} \{W_n, W_{n-1}\}^{\mathbb{N}}$$

The skew Sturmian subshift of a finite continued fraction $(0, a_1, \ldots, a_n) \in \mathcal{N}$ is

$$\Sigma_{(0,a_1,\ldots,a_n)} = \vartheta_{a_1-1}\vartheta_{a_2} \cdots \vartheta_{a_n}[\Sigma_{(0)}] = \bigcap_{k>0} \{(W_n)^k, W_{n-1}\}^{\mathbb{N}}$$

where W_n are words defined in Proposition 4.51.

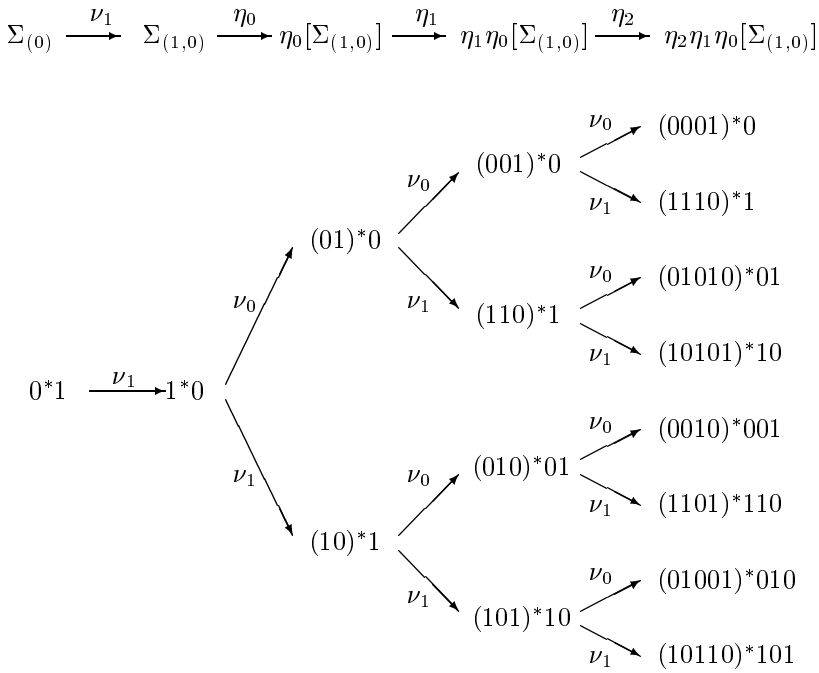

FIGURE 14. Genealogy of skew Sturmian subshifts

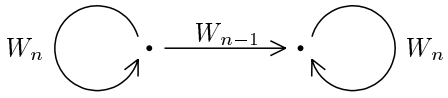

FIGURE 15. Labelled graph of a skew Sturmian subshift

Any skew Sturmian subshift $(W_n)^*W_{n-1}$ is sofic. Its labelled graph is in Figure 15. It is chain-transitive but not transitive. Since it is not minimal, neither is it strictly ergodic. However, any of its points has the same frequency of ones. For a finite continued fraction $a = (0, a_1, \ldots, a_n)$ and $x \in \Sigma_a$ we have

$$\lim_{m\to\infty} \frac{|x_{[0,m)}|_1}{m} = \frac{|W_n|_1}{|W_n|} = \frac{p_n}{q_n} = [a]$$

Now consider finite continued fractions $(0, a_1, \ldots, a_{n-1}, a_n + 1)$ and $(0, a_1, \ldots, a_n, 1)$, which correspond to the same rational number. By Proposition 4.51,

$$\begin{aligned}
\vartheta_{a_1-1}\ldots\vartheta_{a_{n-1}}\vartheta_{a_n} &= \begin{cases} 0 \mapsto W_n = (W_{n-1})^{a_n}W_{n-2} \\ 1 \mapsto W_{n-1} \end{cases} \\
\vartheta_{a_1-1}\ldots\vartheta_{a_{n-1}}\vartheta_{a_n+1} &= \begin{cases} 0 \mapsto (W_{n-1})^{a_{n+1}}W_{n-2} = W_{n-1}W_n \\ 1 \mapsto W_{n-1} \end{cases} , \\
\vartheta_{a_1-1}\ldots\vartheta_{a_{n-1}}\vartheta_{a_n}\vartheta_1 &= \begin{cases} 0 \mapsto W_nW_{n-1} \\ 1 \mapsto W_n \end{cases}
\end{aligned}$$

so

$$\begin{aligned}\Sigma_{(0,a_1,\dots,a_{n-1},a_n+1)} &= (W_{n-1}W_n)^*W_{n-1}\\ \Sigma_{(0,a_1,\dots,a_{n-1},a_n,1)} &= (W_nW_{n-1})^*W_n\end{aligned}$$

Corollary 4.54. — *A subshift $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ is Sturmian or skew Sturmian iff there exists an infinite sequence of substitutions $\eta \in \{\nu_0,\nu_1\}^{\mathbb{N}}$, such that $\Sigma = \eta[\mathbf{2}^{\mathbb{N}}]$. If $\eta,\zeta \in \{\nu_0,\nu_1\}^{\mathbb{N}}$ are different sequences of substitutions, then $\eta[\mathbf{2}^{\mathbb{N}}] \neq \zeta[\mathbf{2}^{\mathbb{N}}]$.*

Proof. — For Sturmian subshifts this follows from Proposition 4.49. For skew Sturmian subshift note that $\vartheta_a\nu_0 = \nu_0^a\vartheta_0\nu_0 = \nu_0^a\nu_1$, so

$$\begin{aligned}\vartheta_{a_1-1}\vartheta_{a_2}\cdots\vartheta_{a_{n-1}}\vartheta_{a_n}[\Sigma_{(0)}] &= \lim_{k\to\infty}\vartheta_{a_1-1}\vartheta_{a_2}\cdots\vartheta_{a_{n-1}}\vartheta_{a_n}(\nu_0)^{k+1}[\mathbf{2}^{\mathbb{N}}]\\ &= \lim_{k\to\infty}\vartheta_{a_1-1}\vartheta_{a_2}\cdots\vartheta_{a_{n-1}}(\nu_0)^{a_n}\nu_1(\nu_0)^k[\mathbf{2}^{\mathbb{N}}]\\ &= \lim_{n\to\infty}\nu_0^{a_0-1}\nu_1(\nu_0)^{a_1-1}\nu_1\cdots(\nu_0)^{a_n-1}\nu_1(\nu_0)^k[\mathbf{2}^{\mathbb{N}}]\end{aligned}$$

Assume now that $\eta,\zeta \in \{\nu_0,\nu_1\}^{\mathbb{N}}$ are different sequences of substitutions, and let i be the first integer at which they differ, say, $\eta_i = \nu_0$, $\zeta_i = \nu_1$. We have Sturmian subshifts $\Sigma_0 = \bigcap_{n>0}\eta_{i+1}\cdots\eta_{i+n}[\mathbf{2}^{\mathbb{N}}]$, $\Sigma_1 = \bigcap_{n>0}\zeta_{i+1}\cdots\zeta_{i+n}[\mathbf{2}^{\mathbb{N}}]$. Then $\eta[\mathbf{2}^{\mathbb{N}}] = \xi\nu_0[\Sigma_0]$, $\zeta[\mathbf{2}^{\mathbb{N}}] = \xi\nu_1[\Sigma_1]$ where $\xi = \eta_0\cdots\eta_{i-1}$. The language of $\nu_0[\Sigma_0]$ contains 001 but not 110, so $\eta[\mathbf{2}^{\mathbb{N}}]$ contains $\xi(001)$ but not $\xi(110)$. Similarly $\zeta[\mathbf{2}^{\mathbb{N}}]$ contains $\xi(110)$ but not $\xi(001)$, so $\eta[\mathbf{2}^{\mathbb{N}}] \neq \zeta[\mathbf{2}^{\mathbb{N}}]$. □

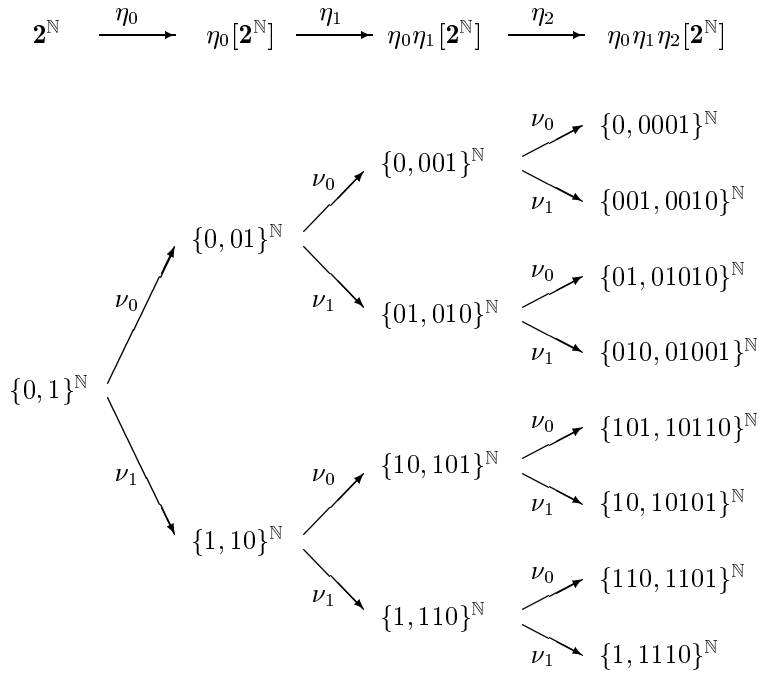

FIGURE 16. Genealogy of Sturmian subshifts

Thus Sturmian and skew Sturmian subshifts are in one-to-one correspondence with their **coding sequences** $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$. This is shown in Figure 16. In n-th column there are all subshifts $\eta[\mathbf{2}^{\mathbb{N}}]$, where $\eta \in \{\nu_0, \nu_1\}^n$. We put arrows between subshifts

$$\eta_0 \dots \eta_{n-1}[\mathbf{2}^{\mathbb{N}}] \xrightarrow{\eta_n} \eta_0 \dots \eta_{n-1}\eta_n[\mathbf{2}^{\mathbb{N}}].$$

If $\eta_0 \cdots \eta_{n-1}(0) = u$ and $\eta_0 \cdots \eta_{n-1}(1) = v$, then

$$\eta_0 \cdots \eta_{n-1}\nu_0 = \begin{cases} 0 \mapsto 0 \mapsto u \\ 1 \mapsto 01 \mapsto uv \end{cases}, \quad \eta_0 \cdots \eta_{n-1}\nu_1 = \begin{cases} 0 \mapsto 1 \mapsto v \\ 1 \mapsto 10 \mapsto vu \end{cases}$$

so we get arrows

$$\{u, v\}^{\mathbb{N}} \xrightarrow{\nu_0} \{u, uv\}^{\mathbb{N}}, \quad \{u, v\}^{\mathbb{N}} \xrightarrow{\nu_1} \{v, vu\}^{\mathbb{N}}$$

4.5.1. Balanced subshifts. —

Definition 4.55. — A subshift $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ is **balanced** if for any $k \in \mathbb{N}$ and any $u, v \in \mathcal{L}^k(\Sigma)$, $|u|_1 - |v|_1 \leq 1$. The subshift Σ is n-balanced if the condition holds for all $k \leq n$.

Finite subshifts $\{0^\infty\}$ or $\{(01)^\infty, (10)^\infty\}$ are balanced. The skew Sturmian subshifts $\Sigma_{(0)}$ or $\Sigma_{(0,1)}$ are also balanced. We shall prove that an infinite subshift is balanced iff it is either Sturmian or skew Sturmian. While this can be shown in a combinatorial manner, in case of irrational α there is a simple geometric argument.

Proposition 4.56. — *Any Sturmian subshift is balanced.*

Proof. — Let $\alpha \in (0,1)$ be irrational. We have $\beta \in V_1 = [1-\alpha, 1)$ iff $R_\alpha(\beta) = \beta + \alpha - 1$, i.e., if the line $y = \beta + \alpha x$ intersects horizontal line $y = 1$ in some $x \in (0,1)$ (Figure 17). If $u \in \mathcal{L}^n(\Sigma_\alpha)$ and $\beta \in V_u$ then $|u|_1 = \#\{i < n : R_\alpha^i(\beta) \in V_1\}$ is the number of horizontal lines, which the function $y = \beta + \alpha x$ intersects in interval $[0, n]$, so

$$R_\alpha^n(\beta) = \beta + n\alpha - |u|_1 \in (0,1).$$

If $v \in \mathcal{L}^n(\Sigma_\alpha)$ and $\gamma \in V_v$, then $R_\alpha^n(\gamma) = \gamma + n\alpha - |v|_1 \in (0,1)$, so

$$-1 < \beta - |u|_1 - \gamma + |v|_1 < 1.$$

Since $|\beta - \gamma| < 1$, we get $-1 \leq |u|_1 - |v|_1 \leq 1$. □

Proposition 4.57. — *Let $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ be a subshift and $\eta \in \{\nu_0, \nu_1\}$.*

(1) If Σ is n-balanced but not $(n+1)$-balanced, then there exists $u \in \mathcal{L}^{n-1}(\Sigma)$ such that both $0u0$ and $1u1$ belong to $\mathcal{L}^{n+1}(\Sigma)$.

(2) If Σ is n-balanced then $\eta[\Sigma]$ is $(n+1)$-balanced.

(3) If $\eta[\Sigma]$ is $2n$-balanced, then Σ is n-balanced.

Proof. —

(1) By assumption there exist $u, v \in \mathcal{L}^{n+1}(\Sigma)$ such that $|v|_1 - |u|_1 \geq 2$. Since Σ is n-balanced, $u_0 = u_n = 0$, $v_0 = v_n = 1$ and $|u_{[1,n-1]}|_1 = |v_{[1,n-1]}|_1$. Let $i \in [1, n-2]$ be the first integer for which $u_i \neq v_i$. If $u_i = 0$, $v_i = 1$, then $|v_{[0,i]}|_1 = |u_{[0,i]}|_1 + 2$. If $u_i = 1$ and $v_i = 0$ then $|u_{[i,n]}|_1 = |v_{[i,n]}|_1 + 2$. In both cases we get a contradiction so $u_{[1,n-1]} = v_{[1,n-1]}$.

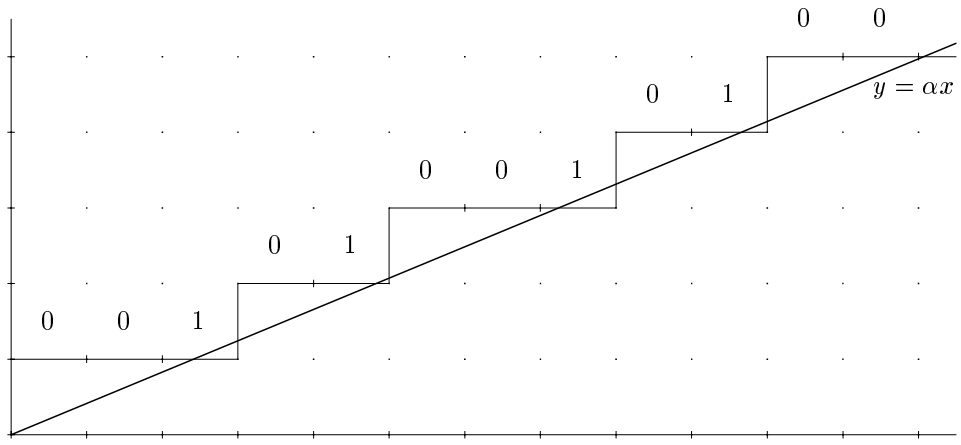

FIGURE 17. Balanced subshifts

(2) Assume that $\nu_0[\Sigma]$ is not balanced and let m be the maximal number for which it is m-balanced. There exists $u \in \mathcal{L}^{m-1}(\nu_0[\Sigma])$ such that $0u0, 1u1 \in \mathcal{L}^{m+1}(\nu_0[\Sigma])$. The last letter 1 in $1u1$ must be preceded by a 0, so there exists $v \in \mathcal{L}(\Sigma)$ such that $1u1 = 1\nu_0(v)01 \sqsubseteq \nu_0(1v1)$. Then $0u0 = 0\nu_0(v)00 \sqsubseteq \nu_0(0v01)$, so both $0v0$ and $1v1$ belong to $\mathcal{L}(\Sigma)$. Thus Σ is not $(|v|+2)$-balanced. Since $|v| \leq |\nu_0(v)| = |u| - 1$, we get $m = |u| + 1 \geq |v| + 2 \geq n + 1$.

(3) Assume that Σ is not balanced and let m be the maximal number such that it is m-balanced. Then there exists $u \in \mathcal{L}^{m-1}(\Sigma)$ such that $0u0, 1u1 \in \mathcal{L}(\Sigma)$. It follows that $0\nu_0(u)00, 01\nu_0(u)01 \in \mathcal{L}(\nu_0[\Sigma])$, so $\nu_0[\Sigma]$ is not $(|\nu_0(u)| + 3)$-balanced. Since $|\nu_0(u)| \leq 2|u|$, we get

$$2(m-1) + 3 = 2|u| + 3 \geq |\nu_0(u)| + 3 \geq 2n + 1,$$

so $m \geq n$, and Σ is n-balanced. An analogous proof works in the case $\eta = \nu_1$. □

Proposition 4.58. — *An infinite binary subshift is balanced iff it is either Sturmian or skew Sturmian.*

Proof. — If Σ is Sturmian or skew Sturmian, there exists a sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ such that for any n, $\Sigma \subseteq \Sigma_n = \eta_0\eta_1 \cdots \eta_{n-1}[\mathbf{2}^{\mathbb{N}}]$. By Proposition 4.57, Σ_n is n-balanced, so Σ is balanced. Conversely, if Σ is infinite and balanced then either $\mathcal{L}^2(\Sigma) = \{00, 01, 10\}$ or $\mathcal{L}^2(\Sigma) = \{01, 10, 11\}$. In either case there exists $\eta_0 \in \{\nu_0, \nu_1\}$ and Σ_1 such that $\Sigma = \eta_0[\Sigma_1]$. By Proposition 4.57, Σ_1 is balanced and infinite. We repeat this argument and obtain $\Sigma = \eta_0 \cdots \eta_{n-1}[\Sigma_n]$, for some subshift Σ_n and $\eta_i \in \{\nu_0, \nu_1\}$ so Σ is either Sturmian or skew Sturmian. □

4.5.2. Subshifts of minimal complexity. — By Proposition 3.23 the complexity function of an infinite subshift satisfies $P(n) \geq n+1$. We say that Σ is a **subshift of minimal complexity**, if $P(n) = n + 1$. By Proposition 4.44, any Sturmian subshift has minimal complexity.

Proposition 4.59. — *Any skew Sturmian subshift has minimal complexity* $P(n) = n + 1$.

Proof. — Let $a = (0, a_1, \ldots, a_m)$ be a finite continued fraction, so $\Sigma_a = (W_m)^* W_{m-1}$. Given $n > 0$, consider an irrational number $\beta = [0, a_1, \ldots, a_m, b_{m+1}, \ldots]$, such that $n < b_{m+1}|W_m|$. Then $\mathcal{L}^n(\Sigma_a) = \mathcal{L}^n(\Sigma_\beta)$, so $P(n) = n + 1$. □

Since $P(1) = 2$, the alphabet of a subshift of minimal complexity has necessarily two letters. There do exist subshifts of minimal complexity which are neither Sturmian nor skew Sturmian, for example

$$\Sigma = \{0^k 1^\infty : k \geq 0\} \cup \{0^\infty\}, \quad \text{or} \quad \Sigma = \{0^k (10)^\infty : k \geq 0\} \cup \{0^\infty\}$$

but these subshifts are not chain-transitive. We shall show that a chain-transitive subshift is of minimal complexity iff it is either Sturmian or skew Sturmian.

We shall investigate subshifts of minimal complexity $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ with the help of Rauzy graphs (Definition 3.61) $G_n = (V_n, E_n, s, t, l)$, where $V_n = \mathcal{L}^n(\Sigma)$, $E_n = \mathcal{L}^{n+1}(\Sigma)$, $s(u) = u_{[0,n-1]}$, $t(u) = u_{[1,n]}$, $l(u) = u_0$. By Proposition 3.62, a subshift is chain-transitive iff all its Rauzy graphs are irreducible. Define the outdegree u^+ and indegree u^- of a vertex $u \in V_n$ as

$$u^+ = \#s^{-1}(u), \quad u^- = \#t^{-1}(u)$$

Since $\#V_n = P(n)$, $\#E_n = P(n+1)$, we get

$$1 = P(n+1) - P(n) = \sum_{u \in \mathcal{L}^n(\Sigma)} (u^+ - 1) = \sum_{u \in \mathcal{L}^n(\Sigma)} (u^- - 1)$$

FIGURE 18. Partitions of Rauzy graphs

Since Σ is chain-transitive, $u^+ \geq 1$ and $u^- \geq 1$ for any vertex $u \in V_n$. Since $P(n) = n + 1$, there exists a unique **splitting vertex** $s \in V_n$ with $s^+ = 2$, and any other vertex has outdegree 1. Similarly there exists a unique **joining vertex** $j \in V_n$ with $j^- = 2$ and any other vertex has indegree 1. The joining and splitting vertices may coincide. There always exist exactly two distinct paths $U, V : s \to j$. If $s \neq j$ there exists a unique path $W : j \to s$. The labels of these paths $w = l(W)$, $u = l(U)$, $v = l(V)$ completely characterize the graph (Figure 18). We have $\Sigma_{G_n} = \{wu, wv\}^{\mathbb{N}}$ if $s \neq j$ and $\Sigma_{G_n} = \{u, v\}^{\mathbb{N}}$ if $s = j$. We say that a partition of G_n is

$$\begin{aligned} (u, v) \quad &\text{if} \quad s = j, \quad u, v : s \to j, \quad u \neq v \\ (w, u, v) \quad &\text{if} \quad s \neq j, \quad u, v : s \to j, \quad u \neq v, \quad w : j \to s \end{aligned}$$

In Figure 19 we illustrate Rauzy graphs and their partitions for the Fibonacci subshift. If $j \neq s$ in G_n, then G_{n+1} is uniquely determined. The suffix of w is prefixed to both u and v. If $j = s$, there are is a choice for G_{n+1}.

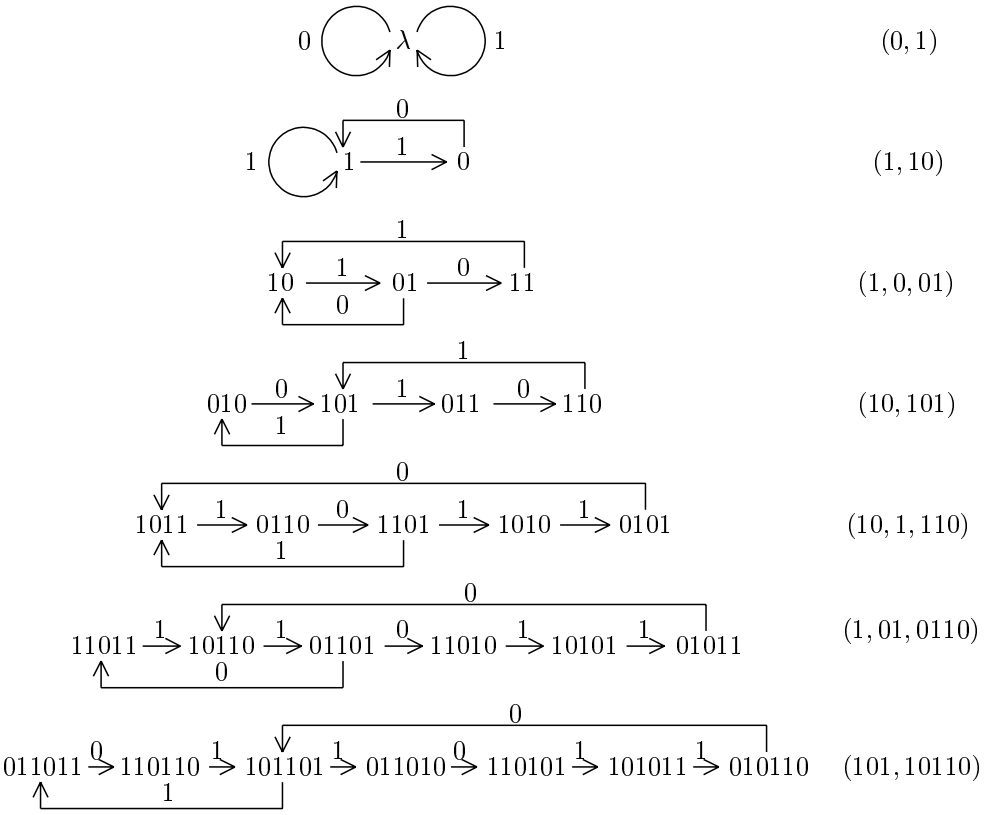

FIGURE 19. Rauzy graphs and their partitions

Proposition 4.60. — *Let G_n, G_{n+1} be the Rauzy graphs of a chain-transitive subshift of minimal complexity, $u, v, w \in \mathbf{2}^*$ and $a, b \in \mathbf{2}$.*

(1) If G_n has partition (wa, u, v), then G_{n+1} has partition (w, au, av).

(2) If G_n has partition (ua, vb) then G_{n+1} has partition (u, a, avb) or (v, b, bua).

(3) If G_n has partition (u, v), then either $G_{n+|u|}$ has partition (u, uv) or $G_{n+|v|}$ has partition (v, vu).

Proof. —

(1) We have $u_0 = v_0 = b$ and there exist $c \in \mathbf{2}$ and $z \in \mathbf{2}^*$ such that $abz \xrightarrow{a} bzc$ in G_n and bzc is the splitting vertex (Figure 20 top). Denote by w' the joining vertex of G_n. It follows that in G_{n+1}, the splitting vertex is $abzc$, the label of the path from the joining vertex w'' to $abzc$ is w and the labels of the two paths from $abzc$ to w'' are au and av.

(2) If G_n has partition (ua, vb), there exist $c, d \in \mathbf{2}$, $z \in \mathbf{2}^*$ such that czd is both joining and splitting in G_n, $(ua)_0 = (vb)_0 = c$, and $a \neq b$. (If $u = \lambda$, then $(ua)_0 = a = c$ and similarly if $v = \lambda$.) In G_n there are edges and paths

$$acz \xrightarrow{a} czd \xrightarrow{u} acz, \quad bcz \xrightarrow{b} czd \xrightarrow{v} bcz.$$

It follows that the splitting vertex of G_{n+1} is either $aczd$ or $bczd$. Likewise, the joining vertex of G_{n+1} is either $czd0$ or $czd1$, so there are four possibilities. In two of these possibilities we have both the top horizontal arrow $aczd \to czd0$ and the bottom horizontal arrow $bczd \to czd1$, and either $bczd \to czd0$ or $aczd \to czd1$. In both these cases, the graph G_{n+1} is not irreducible, so these two possibilities are excluded. The remaining two possibilities are shown in Figure 20 bottom. In the first case, the splitting and joining vertices are $s = aczd$, $j = czd0$ and the partition is (u, a, avb). In the second case we have $s = bczd$, $j = czd1$ and the partition is (v, b, bua). □

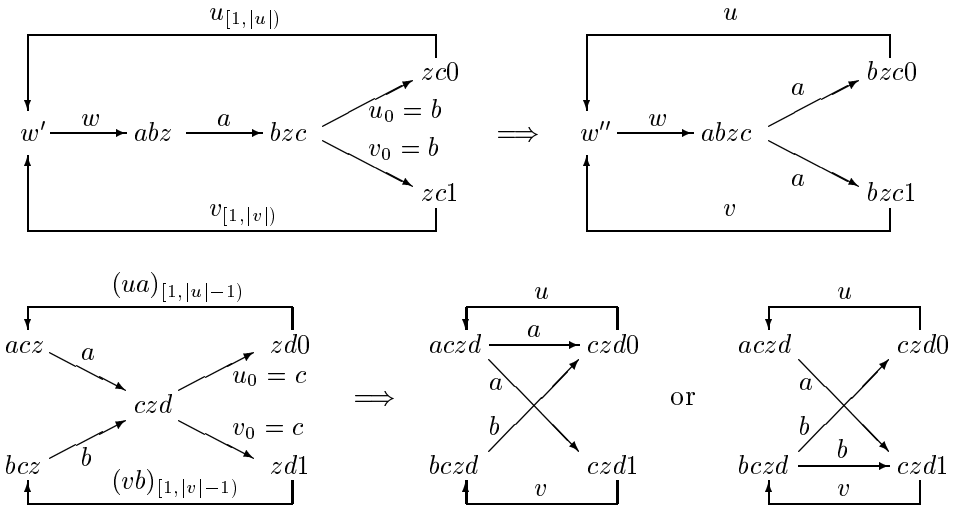

FIGURE 20. Splitting and joining vertices

If G_n has partition (w, u, v), then $\Sigma_{G_n} = \{wu, wv\}^{\mathbb{N}}$. If $|w| = m$, then G_{n+m} has partition $\{wu, wv\}$ and the same subshift $\Sigma_{G_{n+m}} = \Sigma_{G_n}$. Thus it suffices to consider only Rauzy graphs whose splitting and joining vertices coincide. The transition from a graph G_n with partition (u, v) to $G_{n+|u|}$ with partition (u, uv) or to $G_{n+|v|}$ with partition (v, vu) corresponds to a choice of one of the substitutions ν_0 or ν_1.

Proposition 4.61. — *Let $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ be a chain-transitive subshift with minimal complexity. There exists a sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ such that $\Sigma = \eta[\mathbf{2}^{\mathbb{N}}]$.*

Proof. — Since Σ is chain-transitive and $P(2) = 3$, either $\mathcal{L}^2(\Sigma) = \{00, 01, 10\}$ or $\mathcal{L}^2(\Sigma) = \{01, 10, 11\}$. In former case G_1 has partition $(0, 01)$ and $\Sigma \subseteq \nu_0[\mathbf{2}^{\mathbb{N}}]$. In latter case, G_1 has partition $(1, 10)$ and $\Sigma \subseteq \nu_1[\mathbf{2}^{\mathbb{N}}]$. If for some n, G_n has partition (u, v), then $\Sigma \subseteq \eta[\mathbf{2}^{\mathbb{N}}]$, where $\eta(0) = u$ and $\eta(1) = v$. Then we have

$$\eta\nu_0 = \begin{cases} 0 \mapsto 0 \mapsto u \\ 1 \mapsto 01 \mapsto uv \end{cases}, \quad \eta\nu_1 = \begin{cases} 0 \mapsto 1 \mapsto v \\ 1 \mapsto 10 \mapsto vu \end{cases}$$

so either $\Sigma \subseteq \eta\nu_0[\mathbf{2}^{\mathbb{N}}]$ or $\Sigma \subseteq \eta\nu_1[\mathbf{2}^{\mathbb{N}}]$ (Figure 16). Thus there exists a sequence of substitutions $\eta \in \{\nu_0, \nu_1\}^{\mathbb{N}}$ such that $\Sigma \subseteq \eta[\mathbf{2}^{\mathbb{N}}]$. By Corollary 4.54, $\eta[\mathbf{2}^{\mathbb{N}}]$ is either Sturmian or skew Sturmian. In the former case it is minimal so $\Sigma = \eta[\mathbf{2}^{\mathbb{N}}]$. In the latter case the only proper subshift of $\eta[\mathbf{2}^{\mathbb{N}}]$ is finite and its complexity is bounded by $P(n) \leq C$, so $\Sigma = \eta[\mathbf{2}^{\mathbb{N}}]$. □

Corollary 4.62. — *Let $\Sigma \subseteq \mathbf{2}^{\mathbb{N}}$ be an infinite, chain-transitive subshift. The following statements are equivalent.*
(1) Σ is Sturmian or skew Sturmian.
(2) Σ is balanced.
(3) Σ has complexity function $P(n) = n + 1$.

Proposition 4.63. — *If Σ is a Sturmian or skew Sturmian subshift then there exists a point $x \in \Sigma$ with exactly two σ-preimages. Any point $y \in \Sigma \setminus \{x\}$ has exactly one σ-preimage.*

Proof. — For any n there exists a unique word $u_n \in \mathcal{L}^n(\Sigma)$ which has indegree 2, and u_n is a prefix of u_{n+1}, so there exists a limit $x = \lim_{n\to\infty} u_n$. Thus any prefix $x_{[0,n)}$ of x has indegree 2, so $0x, 1x \in \Sigma$, since any subword of these points belongs to $\mathcal{L}(\Sigma)$. Conversely suppose that $y \in \Sigma$ has two preimages $0y, 1y \in \Sigma$. Then any prefix $y_{[0,n)}$ of y has indegree 2, so $y_{[0,n)} = x_{[0,n)}$ and $x = y$. □

4.5.3. Grand Sturmian subshift. — We say that a sequence $x \in \mathbf{2}^{\mathbb{N}}$ is Sturmian, if it belongs to some Sturmian or skew Sturmian subshift.

Definition 4.64. — The Grand Sturmian subshift $\Psi \subseteq \mathbf{2}^{\mathbb{N}}$ consists of all Sturmian sequences.

$$\Psi = \bigcup_{a\in\mathcal{N}} \Sigma_a = \bigcap_{n>0} \Psi_n, \text{ where } \Psi_n = \bigcup \{\eta[\mathbf{2}^{\mathbb{N}}] : \eta \in \{\nu_0, \nu_1\}^n\}$$

To obtain the maximal equicontinuous factor of the Grand Sturmian subshift, we first consider maximal equicontinuous factors of Sturmian and skew-Sturmian subshifts.

Proposition 4.65. — *The maximal equicontinuous factor of a Sturmian subshift Σ_α is $(\mathbb{T}, R_\alpha)$. A skew Sturmian subshift has the trivial maximal equicontinuous factor.*

Proof. — Any point of $x \in \mathbb{T}$ whose trajectory never visits 0 has a singleton preimage in Σ_α so $(\mathbb{T}, R_\alpha)$ is the maximal equicontinuous factor of (Σ_α, σ) by the Paul theorem 2.49. Consider the skew Sturmian subshift $W_n^* W_{n-1}$. It has q_n periodic points $s(i) = \sigma^i(W_n^\infty)$, $i = 0, \ldots, q_n - 1$. Any point $x \in \Sigma$ is regionally proximal to some $s(i)$, so the maximal equicontinuous factor has at most q_n points. Consider a sequence of points $y_k = \sigma^{q_n-1}(W_n^k W_{n-1} W_n^\infty)$. Then $\lim_{k\to\infty} y_k = s(q_{n-1})$ and for any k there exists m such that $\sigma^m(y_k) = \sigma^m(s(0))$. Thus $s(0) \sim s(q_{n-1})$ are regionally proximal and

$$s(q_{n-2}) \sim s(q_{n-2} + q_{n-1}) \sim \cdots \sim s(q_{n-2} + a_{n-1}q_{n-1}) = s(q_n) \sim s(0).$$

We continue by downwards induction and obtain $s(1) \sim s(0)$, so the minimal equicontinuous factor is trivial. □

Proposition 4.66. — *The Grand Sturmian subshift has zero topological entropy. Its maximal equicontinuous factor is $\delta : (\Psi, \sigma) \to (\mathbb{I}, \mathrm{Id})$, where δ assigns to a Sturmian sequence $x \in \Psi$ its frequency of ones.*

$$\delta(x) = \lim_{n\to\infty} \frac{|x_{[0,n)}|_1}{n}.$$

Proof. — For any n, Ψ_n is a sofic subshift, so $\Psi = \bigcap_{n>0} \Psi_n$ is a subshift too. Any Ψ_n is a union of sofic subshifts $\{u, v\}^{\mathbb{N}}$, where $|u| < |v|$ and $|v| > n$. The topological entropy of $\{u, v\}^{\mathbb{N}}$ is $\ln \lambda$, where λ is the unique number greater then 1 which satisfies the equation

$$\lambda^{|u|+|v|} = \lambda^{|u|} + \lambda^{|v|}.$$

If $\kappa > 1$ is the unique number which satisfies the equation $\kappa^{1+|v|} = \kappa + \kappa^{|v|}$, then a little of calculus shows that

$$\lambda \leq \kappa < 1 + \frac{1}{\sqrt{|v|} - 1} < 1 + \frac{1}{\sqrt{n} - 1},$$

and we get $h(\Psi_n) \leq \ln(1 + \frac{1}{\sqrt{n}-1})$, so $h(\Psi) = 0$.

If α is irrational, then Σ_α is strictly ergodic and for any $x \in \Sigma_\alpha$, $\delta(x) = \alpha$. If $a = (0, a_1, \ldots, a_n)$ is a finite continued fraction, and $x \in \Sigma_a = (W_n)^* W_{n-1}$, then $\delta(x) = |W_n|_1/|W_n| = p_n/q_n = [0, a_1, \ldots, a_n]$. We show that $\delta : \Psi \to [0, 1]$ is continuous. Let $x, y \in \Psi$, $d(x, y) \leq 2^{-m}$. If $km \leq n < (k+1)m$, then

$$k\left(|x_{[0,m)}|_1 - 1\right) \leq |x_{[0,n)}|_1 \leq (k+1)\left(|x_{[0,m)}|_1 + 1\right)$$

since $\overline{\mathcal{O}(x)}$ is balanced. Since $x_{[0,m)} = y_{[0,m)}$, the same inequality holds for $|y_{[0,n)}|_1$, so

$$\frac{|x_{[0,n)}|_1 - |y_{[0,n)}|_1}{n} \leq \frac{|x_{[0,m)}|_1 + 2k + 1}{km}$$

When n tends to ∞, k tends to ∞ too and the left hand side tends to $1/2m$. Thus $|\delta(x) - \delta(y)| \leq 1/2m$ and $\delta : (\Psi, \sigma) \to (\mathbb{I}, \mathrm{Id})$ is a factor map. If $\gamma : (\Psi, \sigma) \to (Y, G)$ is a maximal equicontinuous factor, then any skew Sturmian subshift factorizes to a singleton (Proposition 4.65). If α is irrational and $x, y \in \Sigma_\alpha$, then there exist convergent sequences $x_n \to x$, $y_n \to y$ such that any pair (x_n, y_n) belongs to the same skew Sturmian subshift. Since $\gamma(x_n) = \gamma(y_n)$, we get $\gamma(x) = \gamma(y)$. It follows that $\delta : (\Psi, \psi) \to (\mathbb{I}, \mathrm{Id})$ is the maximal equicontinuous factor. □

4.6. Toeplitz subshifts

Recall that a point $x \in X$ of a dynamical system (X, F) is quasi-periodic (Definition 1.25), if for any of its neighbourhoods $U \ni x$ there exists a period $p > 0$ such that $F^{np}(x) \in U$ for all $n \in \mathbb{N}$. In an adding machine, any point is quasi-periodic. A quasi-periodic nonperiodic sequence of $(A^{\mathbb{N}}, \sigma)$ is called a **Toeplitz sequence**. Some Toeplitz sequences can be constructed as fixed points of substitutions. An example is the Feigenbaum sequence $x = 1011\ 1010\ 1011\ 1011\ldots$ which is the unique fixed point of the substitution

$$\vartheta = \begin{cases} 0 \mapsto 11 \mapsto 1010 \mapsto 1011\ 1011\cdots \\ 1 \mapsto 10 \mapsto 1011 \mapsto 1011\ 1010\cdots \end{cases}$$

Since $\vartheta(0)$ and $\vartheta(1)$ have a common prefix 1, we have $x_{2n} = 1$ for any n. Similarly, since $\vartheta^2(0)$ and $\vartheta^2(1)$ have a common prefix 101, we have $x_{[4n,4n+2]} = 101$ for any n. This leads to a stepwise construction of x. In the first step we fill the even positions with 1s and leave odd positions unfilled, marked by stars. In the second step we replace every second star with 0, and leave stars at positions divisible by 4. In the third step we replace every second star with 1 etc. At step n we fill all positions

which are known in $\vartheta^n(x)$. We get sequences in the alphabet $\{0,1,*\}$ which are called **skeletons**.

$$\begin{aligned}
S_1(x) &= **********\\
S_2(x) &= 1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*\\
S_4(x) &= 101*101*101*101*101*101*101*101*101*101*101*101*101*101*\\
S_8(x) &= 1011101*1011101*1011101*1011101*1011101*1011101*1011101*\\
S_{16}(x) &= 101110101011101*101110101011101*101110101011101*
\end{aligned}$$

Definition 4.67. — Let $x \in A^{\mathbb{N}}$ and $p > 0$.

(1) The p-**periodic part** of x is

$$\mathrm{Per}_p(x) = \{k \in \mathbb{N} : \forall n \in \mathbb{N}, x_{k+pn} = x_k\}.$$

(2) The p-**skeleton** $S_p(x) \in (A \cup \{*\})^{\mathbb{N}}$ is obtained from x by replacing x_i by $*$ for any $i \notin \mathrm{Per}_p(x)$

$$S_p(x)_i = \begin{cases} x_i & \text{if } i \in \mathrm{Per}_p(x) \\ * & \text{if } i \notin \mathrm{Per}_p(x) \end{cases}$$

(3) We say that p is an **essential period** of x, if $\sigma^q(S_p(x)) \neq S_p(x)$ for $q < p$.

(4) The **period of a coordinate** $i \in \mathbb{N}$ is $r_i(x) = \min\{p : i \in \mathrm{Per}_p(x)\}$.

For the Feigenbaum sequence we have

$$\mathrm{Per}_2(x) = \{2k : k \in \mathbb{N}\}, \quad \mathrm{Per}_3(x) = \emptyset, \quad \mathrm{Per}_4(x) = \{4k + i : k \in \mathbb{N}, i = 0,1,2\}, \ldots$$

The periods of coordinates are

i	0	1	2	3	4	5	6	7
$r_i(x)$	2	4	2	8	2	4	2	16

Clearly $\sigma^p(S_p(x)) = S_p(x)$, so p is an essential period of x iff p is the least period of $S_p(x)$. For example the periodic point $x = (01220221022201330331033 3)^{\infty}$ has skeletons

$$\begin{aligned}
S_4(x) &= 0***0***0***0***0***0***0***0*** \ldots\\
S_6(x) &= *1*****1*****1*****1*****1*****1 \ldots\\
S_8(x) &= 0***0***0***0***0***0***0***0*** \ldots\\
S_{12}(x) &= 01**0**10***01**0**10***01**0*** \ldots\\
S_{24}(x) &= 012202210222013303310333012202 22 \ldots
\end{aligned}$$

Thus $4, 6, 12$ and 24 are essential periods of x while 8 is not.

Definition 4.68. —

(1) A sequence $x \in A^{\mathbb{N}}$ is a **Toeplitz sequence**, if it is quasi-periodic but not periodic.

(2) A **periodic structure** for a Toeplitz sequence $x \in A^{\mathbb{N}}$ is an increasing sequence $\mathbf{p} = (p_i)_{i \geq 0}$ of integers such that $p_0 > 1$, any p_i is an essential period of x, $p_i | p_{i+1}$ and $\forall n \geq 0, \exists i \geq 0, n \in \mathrm{Per}_{p_i}(x)$.

(3) A subshift is a **Toeplitz subshift**, if it is the orbit closure of a Toeplitz sequence.

We have already used periodic structures for the construction of adding machines (Section 4.1.2). We shall show that the adding machine with periodic structure $\mathbf{p}$

is the maximal equicontinuous factor of the Toeplitz subshift with the same periodic structure.

Proposition 4.69. — *Let $x \in A^{\mathbb{N}}$ be a Toeplitz sequence. Then*

(1) $n + p \in \mathrm{Per}_p(x) \Rightarrow n \in \mathrm{Per}_p(x)$

(2) $p|q \ \Rightarrow \ \mathrm{Per}_p(x) \subseteq \mathrm{Per}_q(x)$

(3) There exists a periodic structure $p = (p_i)_{i\geq 0}$ for x.

(4) In the space $(A \cup \{*\})^{\mathbb{N}}$, $x = \lim_{i\to\infty} S_{p_i}(x)$.

Proof. —

(1) Let $n+p \in \mathrm{Per}_p(x)$ and let q be the period of n. Since $n+qp = n+p+(q-1)p$, we get $x_n = x_{n+qp} = x_{n+p}$.

(2) If $k \in \mathrm{Per}_p(x)$, then for any j, $x_{k+jq} = x_k$, so $k \in \mathrm{Per}_q(x)$.

(3) Let p_0 be the least essential period. Since x is not periodic, $p_0 > 1$. If p_i is an essential period, set $q = \min\{r_j(x) : j \notin \mathrm{Per}_{p_i}(x)\}$ be the minimal period of a coordinate which is not p_i-periodic. Then $p_{i+1} = \mathrm{lcm}(p_i, q)$ (the least common multiple) is an essential period.

(4) For any n there exists i with $n \in \mathrm{Per}_{p_i}(x)$, so $d(x, S_{p_i}(x)) < 2^{-n}$. □

Proposition 4.69 holds trivially for periodic sequences but their periodic structure is a finite sequence. A periodic structure is not uniquely determined. For example, if the essential periods are $4, 6, 12, \ldots$, there are distinct periodic structures $\mathbf{p} = (4, 12, \ldots)$, $\mathbf{q} = (6, 12, \ldots)$, or $\mathbf{s} = (12, \ldots)$. Since quasi-periodic points are almost periodic, each Toeplitz subshift is minimal. If x is a Toeplitz sequence then any $\sigma^i(x)$ is a Toeplitz sequence. However, $\overline{\mathcal{O}(x)}$ contains also points which are not Toeplitz sequences. For example for the Feigenbaum subshift we may construct skeletons

*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*1*
*101*101*101*101*101*101*101*101*101*101*101*101*101*101*
*1011101*1011101*1011101*1011101*1011101*1011101*1011101*
*101110101011101*101110101011101*101110101011101*

The limit of these points contains the only star at position 0. If we replace this star by either 0 or 1, we get points 0 1011 1010 1011 1011 ... and 1 1011 1010 1011 1011 ... which belong to the Feigenbaum subshift. Indeed, for any n, both $\vartheta^n(1)\vartheta^n(1)$ and $\vartheta^n(0)\vartheta^n(1)$ belong to $\mathcal{L}(x)$, so both $0\vartheta^n(0)$ and $1\vartheta^n(0)$ belong to $\mathcal{L}(x)$ and both $0x$ and $1x$ belong to $\overline{\mathcal{O}(x)}$. These points are, however, not Toeplitz, since the coordinate 0 has no period.

4.6.1. Substitutions with common prefix. — We have seen that there are substitutive subshifts which are Toeplitz. This happens whenever $\vartheta : A^* \to A^*$ is a constant length substitution with **common prefix**, i.e., $||\vartheta|| = |\vartheta| = l > 1$ and $\vartheta(a)_0 = \vartheta(b)_0$ for any $a, b \in A$. In this case, all $\vartheta^2(a)$ have a common prefix of length at least $l+1$, and $\vartheta^i(a)$ have common prefix of length $l^{i-1} + l^{i-2} + \cdots + 1$. It follows that there exists a unique fixed point with $\vartheta(x) = x$ and x is a Toeplitz sequence with periodic structure $p_i = l^{i+1}$. More generally we can use a sequence of substitutions.

Consider for example substitutions

$$\xi_0 = \begin{cases} 0 \mapsto 00 \\ 1 \mapsto 01 \end{cases} \qquad \xi_1 = \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 11 \end{cases}$$

Then for any sequence of substitutions $\eta \in \{\xi_0, \xi_1\}^{\mathbb{N}}$, $\eta[\mathbf{2}^{\mathbb{N}}]$ is a Toeplitz subshift with periodic structure $p_i = 2^{i+1}$. Moreover, $\eta[\mathbf{2}^{\mathbb{N}}]$ is strictly ergodic, and for any of its points

$$\lim_{n\to\infty} \frac{|x_{[0,n)}|_1}{n} = \sum_{k=0}^{\infty} 2^{-k-1} \eta_k(0)_0$$

We can also construct Toeplitz subshifts using a sequence of alphabets and monoid morphisms $\vartheta_k : A_{k+1}^* \to A_k^*$.

Proposition 4.70. — *A sequence $x \in A^{\mathbb{N}}$ is a Toeplitz sequence with periodic structure $\mathbf{p} = (p_i)_{i\geq 0}$ iff there exists a sequence of alphabets $A = A_0, A_1, \ldots$, a sequence of monoid morphisms $\vartheta_i : A_{i+1}^* \to A_i^*$ with common prefixes and $||\vartheta_i|| = |\vartheta_i| = p_{i+1}/p_i$, such that for any sequence $a_i \in A_i$,*

$$x = \lim_{i\to\infty} \vartheta_0 \cdots \vartheta_{i-1}(a_i), \quad \overline{\mathcal{O}(x)} = \bigcap_{i>0} \eta_0 \cdots \eta_{i-1}[A_i^{\mathbb{N}}]$$

Proof. — If the condition is satisfied, then $\vartheta_0 \cdots \vartheta_{i-1}$ has a common prefix, whose length tends to ∞ with $i \to \infty$. It follows that $\lim_{n\to\infty} \vartheta_0 \cdots \vartheta_{i-1}(a_i)$ exists and does not depend on a_i. Conversely suppose that $x \in A^{\mathbb{N}}$ is a Toeplitz sequence. Let $p_0 = r_0(x)$ be the period of 0 and set

$$A_1 = \{x_{[kp_0, kp_0+k)} : k \geq 0\}.$$

Then the monoid morphism $\vartheta_0 : A_1^* \to A_0^*$, whose local rule is the identity $\vartheta_0 : A_1 \to A_0^{p_0}$, has common prefix and there exists a unique $x_1 \in A_1^{\mathbb{N}}$ such that $x = \vartheta_0(x_1)$. Since x is Toeplitz, x_1 is also Toeplitz and we apply the same construction to x_1. □

4.6.2. Clopen partitions. — Let x be a Toeplitz sequence with periodic structure $\mathbf{p}$. Given a p_i, we can assign to any $y \in \Sigma = \overline{\mathcal{O}(x)}$ its **phase** with respect to p_i which is one of the integers $0, 1, \ldots, p_i - 1$. For the Feigenbaum subshift and $p_0 = 2$, the points with phase 0 have ones at all even coordinates. The points with phase 1 have ones at all odd coordinates. Thus the phase of a point can be determined from its first zero. Similarly for higher powers of 2:

$$\begin{aligned} x_k = 0 \quad &\Rightarrow \quad \mathrm{mod}_2(k) = 1 \\ x_{[k,k+2]} = 111 \quad &\Rightarrow \quad \mathrm{mod}_4(k) = 2 \\ x_{[k,k+4]} = 01010 \quad &\Rightarrow \quad \mathrm{mod}_8(k) = 5 \end{aligned}$$

Denote by A_n^i the set of points with phase $n < p_i$ with respect to p_i. For the Feigenbaum subshift,

$$\begin{aligned} A_0^0 &= [10] \cup [111] : \quad 1*1*1*1*1*\ldots \\ A_1^0 &= [0] \cup [110] : \quad *1*1*1*1*1\ldots \end{aligned}$$

We have $\sigma(A_0^0) \subseteq A_1^0$ and $\sigma(A_1^0) \subseteq A_0^0$. The constituting cylinders are mapped according to

$$[111] \xrightarrow{\sigma} [110] \xrightarrow{\sigma} [10] \xrightarrow{\sigma} [0] \xrightarrow{\sigma} [111] \cup [10]$$

Similarly for $p_1 = 4$ we have

$$\begin{aligned} A_0^1 &= [1011] \cup [101010] : \quad 101*101*101*\dots \\ A_1^1 &= [011] \cup [01010] : \quad 01*101*101*1\dots \\ A_2^1 &= [111] \cup [101011] : \quad 1*101*101*10\dots \\ A_3^1 &= [110] \cup [01011] : \quad *101*101*101\dots \end{aligned}$$

$$101010 \to 01010 \to 101011 \to 01011 \to 1011 \to 011 \to 111 \to 110 \to 1011 \cup 101010$$

There are only cycles of length 4 and 8 in this graph and $\sigma(A_i^1) \subseteq A_{i+1}^1$, (with addition modulo 4).

Proposition 4.71 **(Williams [163])**. — *Let $x \in A^{\mathbb{N}}$ be a Toeplitz sequence with periodic structure $\mathbf{p} = (p_i)_{i \geq 0}$. Set*

$$A_n^i = \overline{\{\sigma^{n+mp_i}(x) : m \in \mathbb{N}\}}, \quad i > 0, n < p_i.$$

Then for $y \in \overline{\mathfrak{O}(x)}$,
(1) $y \in A_n^i$ iff $S_{p_i}(y) = S_{p_i}(\sigma^n(x))$.
(2) $\{A_n^i : 0 \leq n < p_i\}$ is a clopen partition of $\overline{\mathfrak{O}(x)}$.
(3) $A_m^j \subseteq A_n^i$ for $j > i$ and $n = \mathrm{mod}_{p_i}(m)$.
(4) $\sigma(A_n^i) = A^i_{\mathrm{mod}_{p_i}(n+1)}$

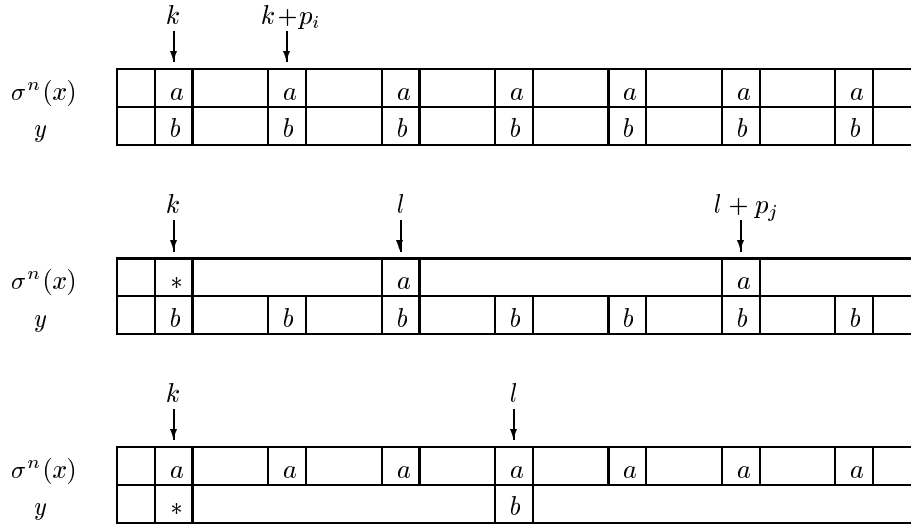

FIGURE 21. Clopen partition of a Toeplitz subshift

Proof. —

(1) Let $S_{p_i}(y) \neq S_{p_i}(\sigma^n(x))$, so there exists $k \geq 0$ with $S_{p_i}(y)_k \neq S_{p_i}(\sigma^n(x))_k$. We distinguish three cases.

(1a) $S_{p_i}(\sigma^n(x))_k = \sigma^n(x)_i = a \in A$, $S_{p_i}(y)_k = y_k = b \in A$ and $b \neq a$ (Figure 21 top). Then for any m, $\sigma^{n+mp_i}(x)_k = a$, so $d(y, \sigma^{n+mp_i}(x)) \geq 2^{-k}$ and $y \notin A_n^i$.

(1b) $S_{p_i}(\sigma^n(x))_k = *$, $S_{p_i}(y)_k = y_k = b \in A$ (Figure 21 center). There exists $l = k + m_0 p_i$ such that $\sigma^n(x)_l = a \neq b$ (otherwise $S_{p_i}(\sigma^n(x))_k$ would be equal to b).

There exists $j > i$ such that $l \in \mathrm{Per}_{p_j}(\sigma^n(x))$. For any $m \geq 0$ there exists $q < \frac{p_j}{p_i}$ such that $(m+q)p_i$ is divisible by p_j and

$$\sigma^{n+mp_i}(x)_{l+qp_i} = \sigma^n(x)_{l+(m+q)p_i} = a \neq b = y_{k+(m_0+q)p_i} = y_{l+qp_i}$$

Since $l + qp_i < l + p_j$, $d(\sigma^{n+mp_i}(x), y) \geq 2^{-l-p_j}$ and $y \notin A^i_n$.

(1c) $S_{p_i}(\sigma^n(x))_k = \sigma^n(x)_k = a \in A$, $S_{p_i}(y)_k = *$ (Figure 21 bottom). There exists $l = k + m_0 p_i$ such that $y_l = b \neq a$ (otherwise $S_{p_i}(y)$ would be equal to a). For any $m \geq 0$, $\sigma^{n+mp_i}(x)_l = \sigma^n(x)_{k+(m+m_0)p_i} = a$, so $d(\sigma^{n+mp_i}(x), y) \geq 2^{-l}$ and $y \notin A^i_n$. Thus we have proved that if $S_{p_i}(\sigma^n(x)) \neq S_{p_i}(y)$, then $y \notin A^i_n$.

Conversely suppose that $S_{p_i}(\sigma^n(x)) = S_{p_i}(y)$. Since $\overline{\mathcal{O}(x)} = \bigcup_{m<p_i} A^i_m$, there exists $m < p_i$ such that $y \in A^i_m$. By the above proof, $S_{p_i}(\sigma^m(x)) = S_{p_i}(y) = S_{p_i}(\sigma^n(x))$. Since p_i is an essential period, $n = m$ and $y \in A^i_n$.

(2) Since A^i_n are closed and disjoint, they form a clopen partition.

(3) If $y \in A^j_m$ then $S_{p_j}(y) = S_{p_j}(\sigma^m(x))$ and $S_{p_i}(y) = S_{p_i}(\sigma^m(x)) = S_{p_i}(\sigma^n(x))$, so $y \in A^i_n$. □

4.6.3. Adding machines. — The decomposition of Σ into clopen partitions $\{A^i_n : n < p_i\}$ enables us the construction of a factor map from a Toeplitz subshift to the adding machine $(\mathbb{Z}_{\mathbf{p}}, R_1)$ with periodic structure $\mathbf{p}$ (Section 4.1.2). Recall that $\varphi : (X, F) \to (Y, G)$ is an almost one-to-one extension (Definition 2.48), if it is an extension and there exists $y \in Y$ such that $\varphi^{-1}(y)$ is a singleton.

***Theorem 4.72* (Williams [163]).** — *Let Σ be a Toeplitz subshift with periodic structure $\mathbf{p}$. Then the maximal equicontinuous factor of Σ is the adding machine $(\mathbb{Z}_{\mathbf{p}}, R_1)$. The factor map $\pi : (\Sigma, \sigma) \to (\mathbb{Z}_{\mathbf{p}}, F)$ is an almost one-to-one extension and satisfies $\pi(x) = \pi(y)$ iff x, y have the same p_i-skeletons for all i. In particular, x is a Toeplitz sequence iff $\pi^{-1}(\pi(x)) = \{x\}$.*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{Z}_{\mathbf{p}} & \xrightarrow{R_1} & \mathbb{Z}_{\mathbf{p}} \end{array}$$

Proof. — Define $\pi : \Sigma \to \mathbb{Z}_{\mathbf{p}}$ by $\pi(x)_i = n$ iff $x \in A^i_n$. For a given i there exists k, such that for any $n < p_i$, A^i_n is a union of cylinders of length k. If $d(x, y) < 2^{-k}$, then x, y belong to the same A^i_n, so $d(\pi(x), \pi(y)) < 2^{-i}$. Thus π is continuous.

If $\pi(x)_i = n$, then $x \in A^i_n$, $\sigma(x) \in A^i_{\mathrm{mod}_{p_i}(n+1)}$, so $\pi\sigma(x)_i = \mathrm{mod}_{p_i}(n+1) = R_1(\pi(x))_i$, and $\pi \circ \sigma = R_1 \circ \pi$. For $x \in \mathbb{Z}_{\mathbf{p}}$, $\pi^{-1}(x) = \bigcap_{i>0} A^i_{x_i}$ is an intersection of a decreasing chain of nonempty closed sets, so it is nonempty. Moreover, $y, z \in \pi^{-1}(x)$ iff $S_{p_i}(y) = S_{p_i}(z)$ for all i and in particular, $\pi^{-1}(x) = \{y\}$ is a singleton iff y is a Toeplitz sequence. Thus $\pi : (\Sigma, \sigma) \to (X_p, R_1)$ is the maximal equicontinuous factor which is also an almost one-to-one extension. □

In the Feigenbaum subshift, we have $\pi^{-1}(0) = \{x\}$ where $x = 10111010\ldots$ is the Feigenbaum sequence and $0 = (0, 0, \ldots)$ is the zero element of X_p. For $-1 =$

$(1, 3, 7, 15, \ldots)$ and other negative integers we have

$$\pi^{-1}(-1) = \{0x, 1x\}, \quad \pi^{-1}(-2) = \{10x, 11x\}, \quad \pi^{-1}(-3) = \{010x, 011x\}$$

We now prove a converse of Theorem 4.72 which yields a topological characterization of Toeplitz subshifts.

***Theorem 4.73* (Downarowicz and Lacroix[52]).** — *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a minimal subshift such that there exists an almost one-to-one extension $\pi : (\Sigma, \sigma) \to (\mathbb{Z}_{\mathbf{p}}, R_1)$. Then Σ is a Toeplitz subshift.*

Proof. — Let $z \in \mathbb{Z}_{\mathbf{p}}$ be such that $x \in \Sigma$ is the unique point with $\pi(x) = z$. We show that x is a Toeplitz sequence. For any $k > 0$ there exists $i > 0$ such that the π-preimage of the cylinder $[z_i] = \{w \in \mathbb{Z}_{\mathbf{p}} : w_i = z_i\}$ is contained in the cylinder $[x_{[0,k)}]$, so for any $y \in \Sigma$, if $\pi(y)_i = \pi(x)_i$, then $y_{[0,k)} = x_{[0,k)}$. For any $m > 0$ we have

$$\pi\sigma^{mp_i}(x)_i = R_1^{mp_i}\pi(x)_i = \pi(x)_i,$$

so $\sigma^{mp_i}(x)_{[0,k)} = x_{[0,k)}$, and $[0, k) \subseteq \mathrm{Per}_{p_i}(x)$. Thus x is a Toeplitz sequence, and since Σ is minimal, $\Sigma = \overline{\mathcal{O}(x)}$. □

The maximal equicontinuous factor of the Morse subshift is the dyadic adding machine $(\mathbb{Z}_2, R_1)$, but the inverse images of points are not singletons. If x is a point in the Morse subshift and $y_i = 1 - x_i$ is its negation, then x and y factor to the same point.

4.6.4. Regular Toeplitz subshifts. — In the Feigenbaum subshift, the stars occur in $S_{p_i}(x)$ with low frequencies. As i grows to infinity, the frequency of stars in $S_{p_i}(x)$ tends to zero. Toeplitz subshifts with this property are called **regular**.

***Definition 4.74*.** — Let x be a Toeplitz sequence with periodic structure $\mathbf{p}$. Set

$$q_i = |S_{p_i}(x)|_* = \#\{j < p_i : j \notin \mathrm{Per}_{p_i}(x)\}.$$

We say that x is a **regular** Toeplitz sequence, if $\delta(x) = \lim_{i\to\infty} \dfrac{q_i}{p_i} = 0$.

Since $q_{i+1} < \frac{p_{i+1}}{p_i} q_i$, the sequence $(\frac{q_i}{p_i})_{i\geq 0}$ is decreasing and has a limit. For the Feigenbaum sequence, $p_i = 2^{i+1}$, $q_{i+1} = 2q_i - 1 = 1$, so it is regular.

***Example 4.75* (Nonregular Toeplitz sequence).** — Consider a Toeplitz sequence x with periodic structure $p_i = 3^i$

```
0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**
01*0**0**01*0**0**01*0**0**01*0**0**01*0**0**01*0**0**01*0**0**
0100**0**01*0**0**01*0**0**0100**0**01*0**0**01*0**0**0100**0**
01001*0**01*0**0**01*0**0**0100**0**01*0**0**01*0**0**0100**0**
```

In any step, only the first undetermined position is filled, alternatively with zeros and ones. The periods of positions are

i	0	1	2	3	4	5	6	7	8	9	10
r_i	3	3^2	3^3	3	3^4	3^5	3	3^6	3^7	3	3^2

We have $p_i = 3^{i+1}$, $q_{i+1} = 3q_i - 1$, so $\frac{q_{i+1}}{p_{i+1}} = \frac{q_i}{p_i} - \frac{1}{p_{i+1}}$ and

$$\delta(x) = 1 - \frac{1}{3} - \frac{1}{9} - \frac{1}{27} - \cdots = \frac{1}{2}.$$

Theorem 4.76 **(Jacobs and Keane [82])**. — *If x is a regular Toeplitz sequence then $\overline{\mathfrak{O}(x)}$ has zero topological entropy and it is strictly ergodic.*

Proof. — The skeleton $S_{p_i}(x) = w^{p_i}$ is a periodic sequence. The word w has length p_i and contains q_i stars. If $u \in \mathcal{L}^{p_i}(x)$ then $u = x_{[n,n+p_i)}$ for some n, and $p_i - q_i$ letters in u are determined by $\mathrm{mod}_{p_i}(n)$. It follows $P(p_i) \leq p_i \cdot (\#A)^{q_i}$ and

$$h(\overline{\mathfrak{O}(x)}, \sigma) = \lim_{i\to\infty} \frac{\ln P(p_i)}{p_i} \leq \lim_{i\to\infty} \frac{\ln p_i + q_i \ln \#A}{p_i} = 0$$

We show that Σ is strictly ergodic. Fix $u \in \mathcal{L}(x)$, $i \geq 0$ and set

$$\mu_i(u) = \frac{\#\{j < p_i : S_{p_i}(x)_{[j,j+|u|)} = u\}}{p_i}$$

Then $\mu_i(u)$ is a nondecreasing sequence and has a limit $\mu(u)$. We show that $\mu(u)$ is the frequency of occurrence of u. For a given $y \in \overline{\mathfrak{O}(x)}$ set

$$a_m = \#\{i < m : y_{[i,i+|u|)} = u\}.$$

For any $m > 0$ there exists q such that $y_{[0,m+|u|)} = x_{[q,q+m+|u|)}$. In any subinterval of $[q, q+m+|u|]$ of length p_i, there are at least $p_i\mu_i(u)$ and at most $p_i\mu_i(u) + q_i|u|$ occurrences of u. Let k satisfy $kp_i \leq m < (k+1)p_i$. Then

$$\begin{aligned} kp_i\mu_i(u) &\leq a_m \leq (k+1)(p_i\mu_i(u) + q_i|u|) \\ \frac{k}{k+1}\mu_i(u) &\leq \frac{a_m}{m} \leq \frac{k+1}{k}\left(\mu_i(u) + \frac{q_i}{p_i}|u|\right) \end{aligned}$$

Letting m and k go to infinity, we get

$$\mu_i(u) \leq \liminf_{m\to\infty} \frac{a_m}{m} \leq \limsup_{m\to\infty} \frac{a_m}{m} \leq \mu_i(u) + \frac{q_i}{p_i}|u|$$

and $\lim_{m\to\infty} \frac{a_m}{m} = \lim_{i\to\infty} \mu_i(u) = \mu(u)$. □

Theorem 4.77 **(Williams [163])**. — *For any real number $\alpha \geq 0$ there exists a Toeplitz subshift with topological entropy α.*

Proof. — We construct a family of Toeplitz subshifts depending on certain parameters. Consider an alphabet A, integers $1 \leq q_0 < p_0$, and a sequence of integers $k_n \geq 2$. Construct sequences $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ by the recursive formula

$$p_{n+1} = p_n(\#A)^{q_n} k_n, \quad q_{n+1} = q_n(\#A)^{q_n}(k_n - 1)$$

We define a sequence of words $(V(n))_{n>0}$ with $|V(n)| = p_n$, $|V(n)|_* = q_n$ and a Toeplitz sequence x with skeletons $S_{p_n}(x) = (V(n))^\infty$. $V(0)$ is a word in $(A \cup \{*\})^{p_0}$ which has q_0 stars and $V(0)_0 \in A$. Given $V(n)$, construct $W(n)$ as the concatenation of all words obtained from $V(n)$ by replacements of stars by letters of A. There are $(\#A)^{q_n}$ such words, so $|W(n)| = p_n(\#A)^{q_n}$. Finally set

$$V(n+1) = W(n)V(n)^{(\#A)^{q_n}(k_n-1)},$$

so $|V(n+1)| = k_n|W(n)| = p_{n+1}$. For example with $k_i = 2$ and binary alphabet, we get

$$p_1 = 3, \quad p_2 = 3 \cdot 2^2 \cdot 2 = 24, \quad p_3 = 24 \cdot 2^8 \cdot 2$$
$$q_1 = 2, \quad q_2 = 2 \cdot 2^2 \cdot 1 = 8, \quad q_3 = 8 \cdot 2^8$$

If we replace the stars by digits in the lexicographic order, we get

$$V(0) = 0{**}, \quad W(0) = 000001010011, \quad V(1) = 0000010100110{**}0{**}0{**}$$

The skeletons are

0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0
0000010100110**0**0**0**0000010100110**0**0**0**0000010100
0000010100110000000000000000010100110000000000010000010100

We have

$$\delta(x) = \lim_{n\to\infty} \frac{q_n}{p_n} = \frac{q_0}{p_0} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{1}{k_n}\right)$$

so the limit can be an arbitrary number $0 \leq \delta(x) < 1$. We evaluate the topological entropy. Any word obtained from $V(n)$ by replacing its stars by digits belongs to the language. There are $(\#A)^{q_n}$ such words so $P(p_n) \geq (\#A)^{q_n}$. On the other hand, if $u \in \mathcal{L}^{p_n}(\Sigma)$, then there exists a word $v \sqsubseteq V(n)V(n)$ of length p_n and u is obtained from v by replacing its stars. There are q_i stars in v and its binary digits are determined by its position within $V(n)V(n)$. This position is a number between 0 and p_n. It follows $(\#A)^{q_n} \leq P(p_n) \leq p_n(\#A)^{q_n}$, so

$$\frac{q_n \ln(\#A)}{p_n} \leq \frac{\ln P(p_n)}{p_n} \leq \frac{\ln p_n + q_n \ln(\#A)}{p_n}$$

and $h(\Sigma) = (\#A) \cdot \lim_{n\to\infty} \frac{q_n}{p_n} = \delta(x) \cdot \ln(\#A)$. This can be an arbitrary nonnegative real number. □

Theorem 4.78. — *There exists a Toeplitz subshift which is not strictly ergodic.*

Proof. — We use the construction from the preceding theorem with binary alphabet and specifically chosen parameters. Pick some real number $\frac{1}{2} < c < 1$ and p_1, q_1 with $q_1/p_1 > c$. If p_n, q_n have been constructed, choose k_n so that $\frac{q_{n+1}}{p_{n+1}} = \frac{q_n}{p_n} \cdot \frac{k_n - 1}{k_n} > c$. In the construction of $V(n)$ we oriceed differently at even and odd steps. For n odd we construct $W(n+1)$ so that in its prefix of length p_n, all stars have been replaced by zeros, so this prefix contains at least q_n zeros. For n even, all stars are replaced by ones. In our example, the third step is modified in the following way

0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0**0
0000010100110**0**0**0**0000010100110**0**0**0**0000010100
0000010100110110110110110000010100110000000000000000010100

We get $|x_{[0,p_{2n+1})}|_0 > q_{2n+1} \geq c \cdot p_{2n+1}$ and $|x_{[0,p_{2n})}|_1 > q_{2n} \geq c \cdot p_{2n}$, so

$$\liminf_{n\to\infty} \frac{|x_{[0,n)}|_0}{n} \leq 1 - c < c \leq \limsup_{n\to\infty} \frac{|x_{[0,n)}|_0}{n}$$

and the subshift is not strictly ergodic. □

Recall that lower and upper polynomial entropies are defined by

$$h_p(\Sigma) = \liminf_{n\to\infty} \frac{\ln P(n)}{\ln n}, \quad h^p(\Sigma) = \limsup_{n\to\infty} \frac{\ln P(n)}{\ln n}$$

Proposition 4.79. — *For any $1 \leq \alpha \leq \beta \leq \infty$ there exists a Toeplitz subshift with*

$$h_p(\Sigma) = \alpha, \quad h^p(\Sigma) = \beta$$

Proof. — Let $(n_k)_{k\geq 0}$ be a sequence satisfying $n_{k+1} \leq (n_k - 1)^2$ and $n_k \geq 3$. Set $A_k = \{0, 1, \ldots, n_k - 1\}$ and construct injective monoid homomorphisms $\vartheta_k : A_{k+1} \to A_k^5$ such that for any $a \in A_{k+1}$, $\vartheta_k(a) = 00b0c$ for some $b, c \in A_k \setminus \{0\}$. This is possible since $n_{k+1} \leq (n_k - 1)^2$. Set $\Sigma_k = \vartheta_0\vartheta_1 \cdots \vartheta_{k-1}[(A_k)^{\mathbb{N}}]$ and $\Sigma = \bigcap_k \Sigma_k$. Then Σ is a Toeplitz subshift with periodic structure $p_k = 5^{k+1}$. For example for $n_k = k + 3$ we have $A_0 = \{0, 1, 2\}$, $A_1 = \{0, 1, 2, 3\}$, $A_2 = \{0, 1, 2, 3, 4\}$ and we construct the substitutions as

$$\begin{array}{lllll} \vartheta_0 : & 0 \mapsto 00101 & 1 \mapsto 00102 & 2 \mapsto 00201 & 3 \mapsto 00202 \\ \vartheta_1 : & 0 \mapsto 00101 & 1 \mapsto 00102 & 2 \mapsto 00103 & 3 \mapsto 00201 \quad 4 \mapsto 00202 \end{array}$$

We obtain skeletons

$$\begin{array}{l} 00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*}00{*}0{*} \\ 001010010100{*}0{*}0010100{*}0{*}001010010100{*}0{*}0010100{*}0{*} \end{array}$$

If $u \in \mathcal{L}(\Sigma)$ and $|u| = 5^k$, then $u \sqsubseteq \vartheta_0 \ldots \vartheta_{k-1}(ab)$ for some $a, b \in A_k$ and either $a = 0$ or $b = 0$. Thus we have at most $2n_k$ possibilities for ab and 5^k possibilities for the phase. On the other hand, there are at least n_k distinct words of the form $\vartheta_0 \cdots \vartheta_{k-1}(a)$ and at least $2 \cdot 5^{k-1}$ phases for them, in which the distinguishing part $\vartheta_0 \ldots \vartheta_{k-2}(b0c)$ occurs. Thus we get

$$2 \cdot 5^{k-1} \cdot n_k \leq P(5^k) \leq 2 \cdot 5^k \cdot n_k$$

If $5^{k-1} \leq m < 5^k$, then

$$\begin{array}{rcl} 2 \cdot 5^{k-2} \cdot n_{k-1} \leq & P(m) & \leq 2 \cdot 5^k \cdot n_k \\ \dfrac{\ln 2 + (k-2)\ln 5 + \ln n_{k-1}}{k \ln 5} \leq & \dfrac{\ln P(m)}{\ln m} & \leq \dfrac{\ln 2 + k \ln 5 + \ln n_k}{(k-1)\ln 5} \end{array}$$

so

$$h_p(\Sigma) = 1 + \liminf_{k\to\infty} \frac{\ln n_k}{k \ln 5}, \quad h^p(\Sigma) = 1 + \limsup_{k\to\infty} \frac{\ln n_k}{k \ln 5}$$

If $n_{k+1} = (n_k - 1)^2$, then there exists $1 < a < 2$, such that $n_{k+1} \geq n_k^a$, so $n_k \geq (n_0)^{a^k}$ and

$$\lim_{k\to\infty} \frac{\ln n_k}{k} \geq \lim_{k\to\infty} \frac{a^k \ln n_0}{k} = \infty$$

This gives rise to a subshift such that $h_p(\Sigma) = h^p(\Sigma) = \infty$. For $1 \leq \alpha < \beta = \infty$, we can define $(n_k)_{k\geq 0}$ so that $\frac{\ln n_k}{k}$ is unbounded but its lower limit is $\alpha - 1$. Finally, since n_k can grow sufficiently fast, the sequence $\frac{\ln n_k}{k}$ can be made to oscillate between $\alpha - 1$ and $\beta - 1$. □

We show now that there can be great difference between the return time and recurrence time of Definition 3.27.

Proposition 4.80. — *There exists a Toeplitz subshift such that*

$$\liminf_{n\to\infty} \frac{Q(n)}{n} = 0, \quad \limsup_{n\to\infty} \frac{\ln R(n)}{n} = \infty$$

Proof. — We use a sequence of substitutions in ternary alphabet $A = \{0,1,2\}$. Let $(n_i)_{i\geq 0}$ be a sequence of integers with $n_i \geq 3$. Construct a sequence of substitutions $(\vartheta_i : A \to A^{n_i})_{i\geq 0}$

$$\vartheta_i = \begin{cases} 0 \mapsto 01^{n_i-1} \\ 1 \mapsto 0(12)^{n_i/2} \\ 2 \mapsto 02^{n_i-1} \end{cases}$$

The construction differs slightly for n_i even and odd. For $n_0 = 3,4,5$ we have

$$\begin{array}{lllll} \vartheta_0 : & 0 \mapsto 011 & 1 \mapsto 012 & 2 \mapsto 022 \\ \vartheta_1 : & 0 \mapsto 0111 & 1 \mapsto 0121 & 2 \mapsto 0222 \\ \vartheta_2 : & 0 \mapsto 01111 & 1 \mapsto 01212 & 2 \mapsto 02222 \end{array}$$

Any $\vartheta_k(a)$ begins with 0 and there are no other occurrences of zero. For any n, $\Sigma_n = \vartheta_0 \cdots \vartheta_{n-1}[A_n^{\mathbb{N}}]$ is a sofic subshifts and $\Sigma = \bigcap_{n\geq 0} \Sigma_n$ is a Toeplitz subshift with periodic structure $p_k = n_0 \ldots n_k$.

$$\begin{array}{l} 0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**}0{**} \\ 0110{**}0{**}0110{**}0{**}0110{**}0{**}0{**}0110{**}0{**}0{**}0110{**}0{**}0{**} \\ 0110120120120110{**}0{**}0{**}0110{**}0{**}0{**}0110{**}0{**}0{**}011012012012 \end{array}$$

For $u = \vartheta_0 \ldots \vartheta_{k-1}(1^{n_k-2})$ we have $|u| = (n_k-2)n_{k-1}\cdots n_0$. Since $\vartheta_0 \ldots \vartheta_{k-1}(1^{n_k-1})$ belongs to the language, $\tau([u],[u]) = n_{k-1}\cdots n_0$. For $v = \vartheta_0 \ldots \vartheta_{k-1}(0)$ we have $|u| = n_{k-1}\cdots n_0$ and $\tau([v],[v]) = n_k n_{k-1}\cdots n_0$. It follows that

$$\begin{aligned} Q((n_k-2)n_{k-1}\cdots n_0) &\leq n_{k-1}\cdots n_0 \\ R(n_{k-1}\cdots n_0) &\geq n_k n_{k-1}\cdots n_0 \end{aligned}$$

We can choose n_i so that $\lim_{k\to\infty} \dfrac{\ln n_k}{n_{k-1}\cdots n_0} = \infty$. Then

$$\liminf_{n\to\infty} \frac{Q(n)}{n} = 0, \quad \limsup_{n\to\infty} \frac{\ln R(n)}{n} = \infty$$

□

Exercise 4.1. —

(1) For the following substitutions evaluate their context length (Theorem 4.36) and letter frequencies.

$$\begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}, \quad \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 10 \end{cases}, \quad \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 110 \end{cases}, \quad \begin{cases} 0 \mapsto 03 \\ 1 \mapsto 0 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{cases}$$

(2) Evaluate the entropy of the subshift of Theorem 4.80.

(3) Show that the grand Sturmian subshift is not chain-transitive and has an infinite number of attractors.