CHAPTER 5

CELLULAR AUTOMATA

Cellular automata (CA) are symbolic dynamical systems with very rich and diverse behaviour. They are often used in physics and sciences as models of complex behaviour. In computer science, they are useful as models of parallel processing and give rise to several complexity classes of languages. Cellular automata have been introduced by Ulam [**152**] and von Neumann [**118**]. The latter used them as models of self-reproduction and universal computation. A well-known example of a cellular automaton is the Conway's game of Life [**60**]. Mathematical theory of cellular automata has been developed by Hedlund [**73**]. He studied them in the context of symbolic dynamics as morphisms of the full shift. Later, Wolfram [**165, 166, 167**] studied computational aspects of cellular automata. He performed extensive computer simulations and classified cellular automata into four classes according to their behaviour on finite configurations. There have been several attempts to formalize Wolfram's classification, e.g., Culik et al [**44**], Gilman [**61, 62**], Hurley [**80**], Kůrka [**88**].

Cellular automata can be characterized as homomorphism of a full shift. They have always periodic points and dense sets of preperiodic points. In cellular automata, the dichotomy between almost equicontinuous and sensitive systems is true in general and not only for transitive systems. In fact transitive cellular automata are sensitive. Equicontinuous cellular automata are rather trivial, All their points are preperiodic with the same preperiod and period. Almost equicontinuous cellular automata also act periodically at any coordinate, but the lengths of these periods might vary, so that whole state need not be periodic. Sensitive cellular automata are more diverse. They contain a special subclass of positively expansive cellular automata which are conjugate to subshifts of finite type.

Another particular property of cellular automata is surjectivity. A surjective cellular automaton has only nonwandering points and bounded number of preimages. If all points have the same number of preimages, the cellular automaton maps open sets to open sets. An intermediate class between open and surjective cellular automata are closing cellular automata. They have dense sets of periodic points.

Another classifying scheme uses the concept of attractors. If a cellular automata has two disjoint attractors, it has an infinite number of attractors and a continuum of quasi-attractors. On the other hand there exist cellular automata with a unique attractor or a sequence of decreasing attractors. We show in Section 5.8 how these classification schemes are interrelated.

Definition 5.1. — A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a **cellular automaton (CA)** if there exist integers $m \leq a$ (**memory** and **anticipation**) and a **local rule** $f : A^{a-m+1} \to A$ such that for any $x \in A^{\mathbb{Z}}$ and any $i \in \mathbb{Z}$ (Figure 1 left)

$$F(x)_i = f(x_{[i+m,i+a]}).$$

Call $r = \max\{-m, a\} \geq 0$ the **radius** of F and $d = a - m \geq 0$ its **diameter**.

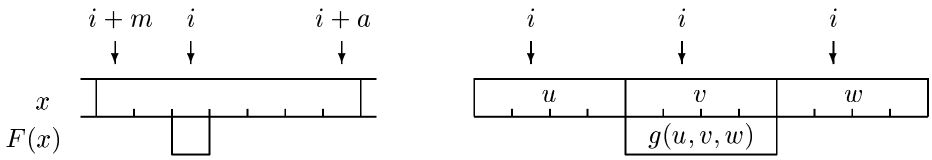


FIGURE 1. Local rule

Cellular automata can be characterized topologically as morphisms of the full shift. This is a two-sided version of Hedlund's theorem 3.56.

***Theorem 5.2* (Hedlund [73]).** — *A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton iff it is continuous and commutes with the shift, i.e., $\sigma \circ F = F \circ \sigma$.*

$$\begin{tikzcd} A^{\mathbb{Z}} \arrow[r, "F"] \arrow[d, "\sigma"'] & A^{\mathbb{Z}} \arrow[d, "\sigma"] \\ A^{\mathbb{Z}} \arrow[r, "F"] & A^{\mathbb{Z}} \end{tikzcd}$$

Proof. — 1. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA with radius $r = \max\{-m, a\}$. Since $-r \leq m \leq a \leq r$, for any $n \geq 0$ we have

$$\begin{aligned} d(x,y) < 2^{-n-r} \quad &\Rightarrow \quad x_{[-n-r,n+r]} = y_{[-n-r,n+r]} \ \Rightarrow \ x_{[-n+m,n+a]} = y_{[-n+m,n+a]} \\ &\Rightarrow \quad F(x)_{[-n,n]} = F(y)_{[-n,n]} \ \Rightarrow \ d(F(x), F(y)) < 2^{-n} \end{aligned}$$

so F is continuous. For any $i \in \mathbb{Z}$,

$$F(\sigma(x))_i = f(\sigma(x)_{[i+m,i+a]}) = f(x_{[i+m+1,i+a+1]}) = F(x)_{i+1} = \sigma(F(x))_i,$$

so F commutes with the shift. Conversely assume that F is continuous and commutes with the shift. Since F is uniformly continuous, for $\varepsilon = 1$ there exists $r \geq 0$ such that

$$\begin{aligned} d(x,y) < 2^{-r} \quad &\Rightarrow \quad d(F(x), F(y)) < 1 \\ x_{[-r,r]} = y_{[-r,r]} \quad &\Rightarrow \quad F(x)_0 = F(y)_0. \end{aligned}$$

There exists $f : A^{2r+1} \to A$ such that for any $x \in A^{\mathbb{Z}}$, $F(x)_0 = f(x_{[-r,r]})$. Since F commutes with the shift,

$$F(x)_i = \sigma^i(F(x))_0 = F(\sigma^i(x))_0 = f(\sigma^i(x)_{[-r,r]}) = f(x_{[i-r,i+r]}).$$

Thus we have a local rule with memory $m = -r$ and anticipation $a = r$. $\square$

We can assume that the local rule acts on a symmetric neighbourhood of 0, so $F(x)_i = f(x_{[i-r,i+r]})$, and $f : A^{2r+1} \to A$. There is a trade-off between the radius and the size of the alphabet.

Proposition 5.3. — *Any CA is conjugate to a CA with radius* 1.

Proof. — Let $(A^{\mathbb{Z}}, F)$ be a CA with radius $r > 1$ and $f : A^{2r+1} \to A$ its local rule. Put $B = A^r$ and define $\varphi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ by $\varphi(x)_i = x_{[ir,ir+r)}$. Let $(B^{\mathbb{Z}}, G)$ be a CA with local rule $g : B^3 \to B$ defined by $g(u, v, w)_i = f(u_{[i,r)} v w_{[0,i]})$ (Figure 1 right). Then $\varphi : (A^{\mathbb{Z}}, F) \to (B^{\mathbb{Z}}, G)$ is a conjugacy. □

Proposition 5.4. — *Any σ-periodic point of a CA $(A^{\mathbb{Z}}, F)$ is F-eventually periodic (Definition 1.13). Hence the set of eventually periodic points is dense.*

Proof. — If $\sigma^n(x) = x$, then $\sigma^n(F(x)) = F(x)$. Since σ-periodic points with period n are in one-to-one correspondence with words of A^n, there is a finite number of them and for some preperiod $m \geq 0$ and period $p > 0$, $F^{m+p}(x) = F^m(x)$. □

Thus cellular automata always have periodic points. Cellular automata with binary alphabet **2** and radius $r = 1$ are called **elementary** (Wolfram [**167**]). Their local rules are coded by numbers between 0 and 255 by

$$f(000) + 2 \cdot f(001) + 4 \cdot f(010) + \cdots + 32 \cdot f(101) + 64 \cdot f(110) + 128 \cdot f(111)$$

Example 5.5 **(P: the product rule 128)**. — $(\mathbf{2}^{\mathbb{Z}}, P)$, where $P(x)_i = x_{i-1}x_ix_{i+1}$, has an attracting fixed point 0^∞ with dense basin, and another attractor $\omega(\mathbf{2}^{\mathbb{Z}})$. It is almost equicontinuous (Figure 2).

The radius is $r = 1$ and the local rule $f : \mathbf{2}^3 \to \mathbf{2}$ is defined by the table

000	001	010	011	100	101	110	111
0	0	0	0	0	0	0	1

FIGURE 2. P: the product rule 128

The behaviour of the product CA is shown in the **space-time diagram** in Figure 2. The n-th row represents a central part of the state at time n, i.e., the word $F^n(x)_{[-m,m]}$, where $2m + 1$ is the width displayed. Ones are represented by black squares and zeros are left empty. We see that 0's propagate both to the left and to the right. If $x_i = 0$, then $F(x)_{[i-1,i+1]} = 000$ and $F^2(x)_{[i-2,i+2]} = 00000$.

$$d(x, 0^\infty) < 2^{-n} \quad \Rightarrow \quad d(F(x), 0^\infty) < 2^{-n-1}.$$

The point 0^∞ is an attracting fixed point and its basin is $\mathcal{B}(0^\infty) = \mathbf{2}^{\mathbb{N}} \setminus \{1^\infty\}$. By Proposition 2.74, the system is almost equicontinuous. The point 1^∞ is another fixed point and it is not stable. The product CA is not surjective. Suppose that $y \in \mathbf{2}^{\mathbb{Z}}$ has a preimage $F(x) = y$ and $y_{[0,2]} = 101$. Then $x_{[-1,1]} = x_{[1,3]} = 111$ but in this case $y_1 = 1$ and this is a contradiction. Similarly, y cannot contain 1001.

$$\begin{aligned} F(\mathbf{2}^{\mathbb{Z}}) &= \{x \in \mathbf{2}^{\mathbb{Z}} : 101 \not\sqsubseteq x \ \ \& \ \ 1001 \not\sqsubseteq x\} \\ F^n(\mathbf{2}^{\mathbb{Z}}) &= \{x \in \mathbf{2}^{\mathbb{Z}} : \forall m \in [1, 2n], 10^m 1 \not\sqsubseteq x\} \\ \omega(\mathbf{2}^{\mathbb{Z}}) &= \{x \in \mathbf{2}^{\mathbb{Z}} : \forall m > 0, 10^m 1 \not\sqsubseteq x\} \end{aligned}$$

Any $F^n(\mathbf{2}^{\mathbb{Z}})$ is an SFT and $\omega(\mathbf{2}^{\mathbb{Z}})$ is a sofic subshift.

***Example 5.6* (S: the sum rule 90)**. — $(\mathbf{2}^{\mathbb{Z}}, S)$, where $S(x)_i = \mathrm{mod}_2(x_{i-1}+x_{i+1})$, is chaotic. It is conjugate to the full shift on four symbols (Figure 3).

000	001	010	011	100	101	110	111
0	1	0	1	1	0	1	0

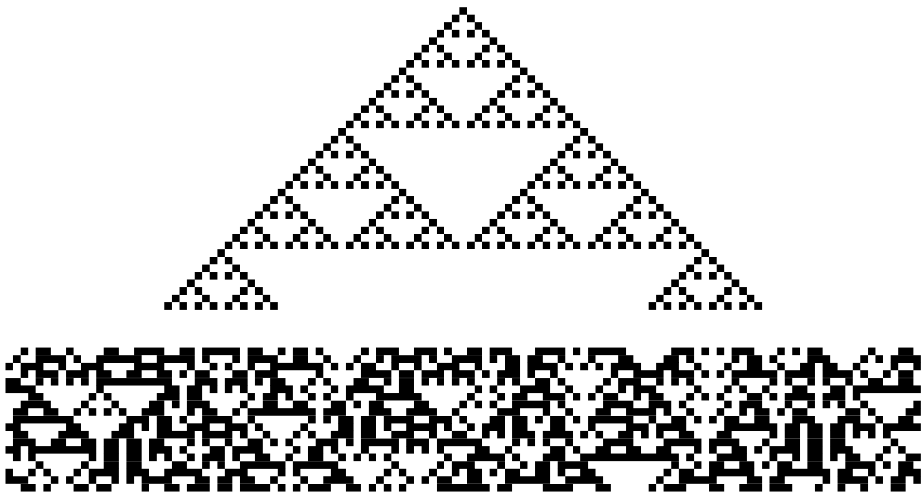


FIGURE 3. S: the sum rule 90

There is a formula for the n-th iteration. We regard $\mathbf{2}$ as a group $\mathbb{Z}(2)$.

$$\begin{aligned} S^2(x)_i &= x_{i-2} + 2x_i + x_{i+2} = x_{i-2} + x_{i+2} \\ S^n(x)_i &= \sum_{j=0}^{n} \binom{n}{j} x_{i-n+2j} \end{aligned}$$

Since $2x = 0$, we count only odd binomial coefficients. In particular, if $n = 2^m$ is a power of two, all but two binomial coefficients are even. In contrast, for $n-1 = 2^m-1$, all binomial coefficients are odd, so

$$\begin{aligned} S^{n-1}(x)_i &= x_{i-n+1} + x_{i-n+3} + \cdots + x_{i+n-3} + x_{i+n-1} \\ S^n(x)_i &= x_{i-n} + x_{i+n} \end{aligned}$$

If the initial state is $0^\infty.10^\infty$ (Figure 3 top), then $F^{2^m}(x)$ contains exactly two 1's, one at 2^{-m} and another at 2^m. If the initial state is random (Figure 3 bottom), we see no pattern in the time development.

We show that the sum CA is conjugate to the full shift on four symbols. Put $B = A^2 = \{00, 01, 10, 11\}$ and consider the clopen partition $\mathcal{V} = \{[u]_0 : u \in B\}$. Then $\mathcal{V}$ is a generating partition (Definition 2.80) and its subshift (Proposition 3.19) $\Sigma_{\mathcal{V}}(\mathbf{2}^{\mathbb{Z}}, S) = B^{\mathbb{N}}$ is the full shift. The conjugacy $\mathcal{I} : (\mathbf{2}^{\mathbb{Z}}, S) \to (B^{\mathbb{N}}, \sigma)$ constructed in Proposition 3.19) is given by $\mathcal{I}(x)_i = S^i(x)_{[0,1]}$. We show that $\mathcal{I}$ is surjective. Given a point $y \in B^{\mathbb{N}}$, let us search for a point $x \in \mathbf{2}^{\mathbb{Z}}$ such that for any $n \geq 0$, $S^n(x)_{[0,1]} = y_n$. We have

$$S^n(x)_0 + S^n(x)_2 = S^{n+1}(x)_1 \quad \Rightarrow \quad S^n(x)_2 = S^n(x)_0 + S^{n+1}(x)_1$$

0110	01	0010
111	10	110
00	10	11
1	00	1
	11	

The second column $(S^n(x)_2)_{n\geq 0}$ can be computed from the zeroth and the first columns. Similarly we may compute the third and (-1)-st columns. This shows that the map $\mathcal{I} : \mathbf{2}^{\mathbb{Z}} \to B^{\mathbb{N}}$ is bijective. Since the sum CA is conjugate to a full shift, it is chaotic and mixing. We generalize Example 5.6 to a class of permutive CA.

Definition 5.7. — Let $(A^{\mathbb{Z}}, F)$ be a CA with local rule $f : A^{d+1} \to A$.

(1) F is **left-permutive** if $\forall u \in A^d, \forall b \in A, \exists! a \in A, f(au) = b$.
(2) F is **right-permutive** if $\forall u \in A^d, \forall b \in A, \exists! a \in A, f(ua) = b$.
(3) F is **permutive** if it is either left-permutive or right-permutive.
(4) A **column subshift** of $(A^{\mathbb{Z}}, F)$ is any one-sided subshift $\Sigma_I(F) = \Sigma_{\mathcal{V}_I}(A^{\mathbb{Z}}, F)$, where $I = [a, b]$ is an interval of integers $a \leq b$ and $\mathcal{V}_I = \{[u]_a : u \in A^{b-a+1}\}$.

Here $[u]_a = \{x \in A^{\mathbb{Z}} : x_{[a,b]} = u\}$ is the cylinder of u at position a. The factor map $\mathcal{I} : (A^{\mathbb{Z}}, F) \to (\Sigma_I(F), \sigma)$ is defined by $\mathcal{I}(x)_i = F^i(x)_{[a,b]}$.

Proposition 5.8. — *If $(A^{\mathbb{Z}}, F)$ is a cellular automaton which is both left-permutive and right-permutive with memory and anticipation $m < 0 < a$, then it is conjugate to the full shift $((A^d)^{\mathbb{N}}, \sigma)$.*

Proof. — The factor map $\mathcal{I} : (A^{\mathbb{Z}}, F) \to ((A^d)^{\mathbb{N}}, \sigma)$ is bijective. ☐

$$\begin{tikzcd} A^{\mathbb{Z}} \arrow[r, "F"] \arrow[d, "\mathcal{I}"'] & A^{\mathbb{Z}} \arrow[d, "\mathcal{I}"] \\ \Sigma_I(F) \arrow[r, "\sigma"] & \Sigma_I(F) \end{tikzcd}$$

***Example 5.9* (T: the traffic rule 184)**. — $(\mathbf{2}^{\mathbb{Z}}, T)$ where

$$T(x)_i = 1 \quad \Leftrightarrow \quad x_{[i-1,i]} = 10 \;\text{ or }\; x_{[i,i+1]} = 11,$$

has a unique transitive attractor $\omega(\mathbf{2}^{\mathbb{Z}})$ (Figure 4).

000	001	010	011	100	101	110	111
0	0	0	1	1	1	0	1

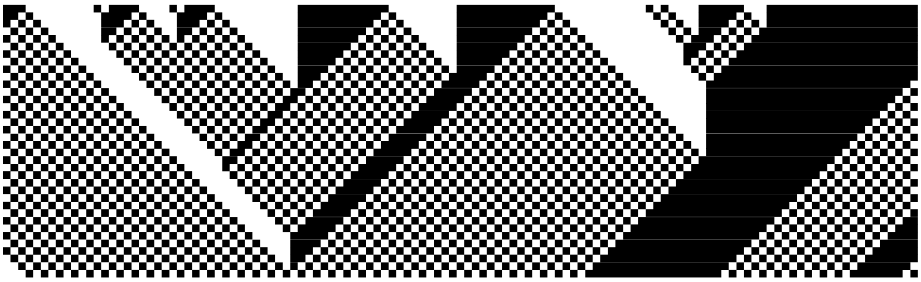


FIGURE 4. T: the traffic rule 184

This is a model of car traffic. A particle represented by 1 goes to the right if its right neighbour is empty, otherwise it rests. There are fixed points 0^∞ and 1^∞ and a periodic orbit of period two $\{(01)^\infty, (10)^\infty\}$ which represents saturated traffic. At this background, blocks of zeros (holes) travel to the right and blocks of ones (jams) travel to the left. When a hole meets a jam, they partially annihilate one another. If their lengths are not equal, the longer of them continues with length which is the difference of the original hole and jam. The omega limit set consists of configurations in which no jam is followed by a hole.

$$\omega(\mathbf{2}^{\mathbb{Z}}) = \{x \in \mathbf{2}^{\mathbb{Z}} : \forall n > 0, 1(10)^n0 \not\sqsubseteq x\}$$

This is a sofic subshift and $(\omega(\mathbf{2}^{\mathbb{Z}}), T)$ is transitive. For any $n > 0$ and any $u \in \mathcal{L}^{2n}(\omega(\mathbf{2}^{\mathbb{Z}}))$, we have $T^k([u]_0) \cap [(01)^n]_0 \neq \emptyset$ for some $k > 0$ and also $T^k([(01)^n]_0) \cap [u]_0 \neq \emptyset$. It follows that $(\mathbf{2}^{\mathbb{Z}}, T)$ is sensitive (Theorem 5.18).

Example 5.10 **(M: the majority rule 232).** — $(\mathbf{2}^{\mathbb{Z}}, M)$, where

$$M(x)_i = \left\lfloor \frac{x_{i-1} + x_i + x_{i+1}}{2} \right\rfloor,$$

is almost equicontinuous and has an infinite number of attractors.

000	001	010	011	100	101	110	111
0	0	0	1	0	1	1	1

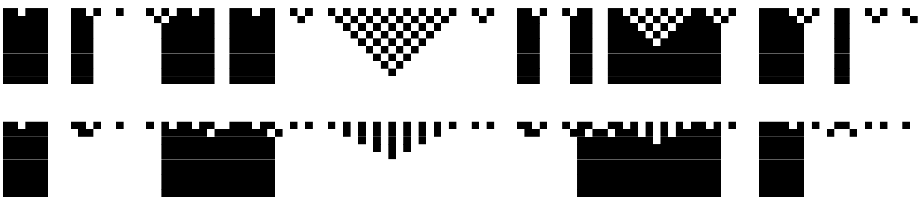


FIGURE 5. M: the majority rule 232

This is a model of opinion changes (Figure 5 top). An individual keeps his opinion (0 or 1) provided he can share it with at least one of his neighbours. We see that the cellular space is successively homogenized so that no isolated 0 or 1 remain. Both fixed points 0^∞ and 1^∞ are stable but not attracting. If $x_{[0,1]} = 00$, then $F(x)_{[0,1]} = 00$, no matter what value is at x_{-1} or x_2. Thus the cylinder set $[00]_0$ is invariant. Similarly $[11]_0$ is invariant. More generally, let

$$E = \{u \in \mathbf{2}^* : |u| \geq 2, u_0 = u_1, u_{|u|-2} = u_{|u|-1}, 010 \not\sqsubseteq u, 101 \not\sqsubseteq u\}$$

Then for any $u \in E$ and for any $i \in \mathbb{Z}$, $[u]_i$ is a clopen invariant set, so it is inward and $\omega([u]_i)$ is an attractor. These attractors are not subshifts. There are two subshift attractors, $\omega(\mathbf{2}^{\mathbb{Z}})$ and

$$\begin{aligned} Y &= \omega([00]_0 \cup [11]_0) = \{x \in \mathbf{2}^{\mathbb{Z}} : 010 \not\sqsubseteq x, 101 \not\sqsubseteq x\}, \\ \mathfrak{B}(Y) &= \mathbf{2}^{\mathbb{Z}} \setminus \{(01)^\infty, (10)^\infty\} \end{aligned}$$

Any point $x \in Y$ is a stable fixed point which is not attracting, so any point of $\mathfrak{B}(Y)$ is equicontinuous and $(\mathbf{2}^{\mathbb{Z}}, M)$ is almost equicontinuous.

The majority rule can be generalized to larger radii. In Figure 5 bottom, there is a majority CA $(\mathbf{2}^{\mathbb{Z}}, F)$ with radius 2:

$$F(x)_i = \left\lfloor \frac{x_{i-2} + x_{i-1} + x_i + x_{i+1} + x_{i+2}}{3} \right\rfloor$$

5.1. Equicontinuity

In Chapter 2 we saw that any transitive dynamical system is either sensitive or almost equicontinuous (Theorem 2.31). This dichotomy holds for all cellular automata, not only for transitive ones. In fact, transitive CA are sensitive (Corollary 5.19). Almost equicontinuous CA are characterized by the presence of **blocking words** like 00 or 11 in the majority CA of Example 5.10. All points which contain an infinite number of occurrences of blocking words in both their negative and positive parts are equicontinuous. The set of such points is residual. If a CA is equicontinuous, any sufficiently long word is blocking.

Definition 5.11. —

(1) Let $s > 0$. A word $u \in A^+$ with $|u| \geq s$ is s-**blocking** for a CA $(A^{\mathbb{Z}}, F)$, if there exists an offset $p \in [0, |u| - s]$ such that (Figure 6)

$$\forall x, y \in [u]_0, \forall n \geq 0, F^n(x)_{[p,p+s)} = F^n(y)_{[p,p+s)}.$$

(2) For $u \in A^+$, $k \geq 0$, set

$$\begin{aligned} \mathfrak{T}(u,k) &= \{x \in A^{\mathbb{Z}} : \exists i \leq -k, x_{[i,i+|u|)} = u \;\;\&\;\; \exists i \geq k, x_{[i,i+|u|)} = u\}, \\ \mathfrak{T}(u) &= \bigcap_{k>0} \mathfrak{T}(u,k), \quad \mathfrak{T} = \bigcap_{u \in A^+} \mathfrak{T}(u). \end{aligned}$$

In the product CA of Example 5.5, 0 is a 1-blocking word (with offset 0), since $x_0 = 0$ implies $F^n(x)_0 = 0$ for all $n \geq 0$. In the majority CA of Example 5.10, 00 and 11 are 2-blocking words (with offset 0). Since $[00]_0$ is invariant, $x_{[0,1]} = 00$ implies $F^n(x)_{[0,1]} = 00$ for all $n \geq 0$.

Any set $\mathfrak{T}(u,k)$ is open and dense so both $\mathfrak{T}(u)$ and $\mathfrak{T}$ are residual sets. Points of $\mathfrak{T}$ are called **bitransitive**. This is a slightly stronger condition than transitivity

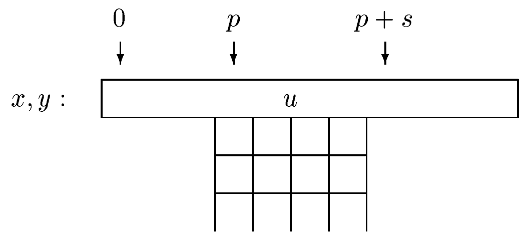


FIGURE 6. A blocking word

(Definition 2.5). A point $x \in A^{\mathbb{Z}}$ is bitransitive iff both its positive and negative orbits are dense.

$$x \in \mathfrak{T} \quad \Leftrightarrow \quad \overline{\{\sigma^i(x) : i \geq 0\}} = \overline{\{\sigma^i(x) : i \leq 0\}} = A^{\mathbb{Z}}.$$

Proposition 5.12. — *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius $r > 0$. The following conditions are equivalent.*
(1) $(A^{\mathbb{Z}}, F)$ is not sensitive.
(2) $(A^{\mathbb{Z}}, F)$ has an r-blocking word.
(3) $(A^{\mathbb{Z}}, F)$ is almost equicontinuous.

Proof. —
(1) $\Rightarrow$ (2): Suppose that F is not sensitive and let m be an integer with $2m+1 \geq r$. For $\varepsilon = 2^{-m}$ there exists $x \in A^{\mathbb{Z}}$ and $\delta > 0$, such that for all $y \in A^{\mathbb{Z}}$,

$$d(x, y) < \delta \quad \Rightarrow \quad \forall k \geq 0, d(F^k(x), F^k(y)) < \varepsilon.$$

Let $p > 0$ be such that $2^{-m-p} < \delta$ and set $u = x_{[-m-p, m+p]} \in A^{2m+2p+1}$. Then

$$\begin{aligned} y, z \in [u]_{-m-p} \quad &\Rightarrow \quad \forall k \geq 0, F^k(y)_{[-m,m]} = F^k(z)_{[-m,m]} \\ &\Rightarrow \quad \forall k \geq 0, F^k(y)_{[-m,-m+r)} = F^k(z)_{[-m,-m+r)} \\ y, z \in [u]_0 \quad &\Rightarrow \quad \forall k \geq 0, F^k(y)_{[p,p+r)} = F^k(z)_{[p,p+r)} \end{aligned}$$

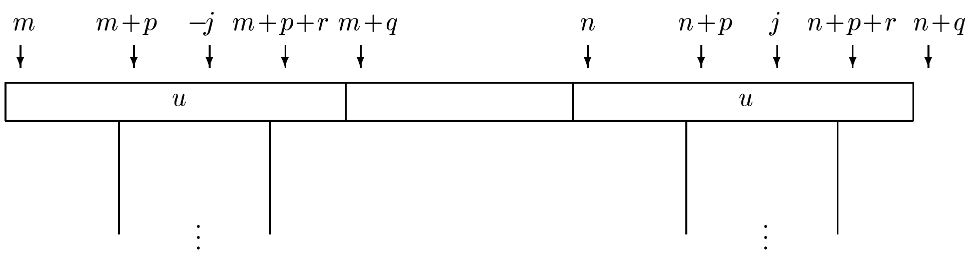


FIGURE 7. An equicontinuous point

(2) $\Rightarrow$ (3): Let $u \in A^q$ be an r-blocking word with offset $p \leq q - r$. We show that any point of $\mathfrak{T}(u)$ is equicontinuous. Let $x \in \mathfrak{T}(u)$ and $\varepsilon = 2^{-j}$. There exist $m \leq -p - j$ and $n \geq j - p - r$, such that $x_{[m,m+q)} = u = x_{[n,n+q)}$ (Figure 7). Set $\delta = \min\{2^m, 2^{-n-q}\}$ and suppose that $d(y, x) < \delta$. Then $x_{[m,n+q)} = y_{[m,n+q)}$,

so $F^k(x)_{[m+p,m+p+r)} = F^k(y)_{[m+p,m+p+r)}$ and $F^k(x)_{[n+p,n+p+r)} = F^k(y)_{[n+p,n+p+r)}$ for any $k \geq 0$. For $m+p+r < i < n+p$ we get

$$F(x)_i = f(x_{[i-r,i+r]}) = f(y_{[i-r,i+r]}) = F(y)_i$$

and by induction $F^k(x)_{[m+p,n+p+r)} = F^k(y)_{[m+p,n+p+r)}$ for any $k \geq 0$. Since $m+p \leq -j$ and $j \leq n+p+r$, we get $d(F^k(x), F^k(y)) < 2^{-j}$. $3 \Rightarrow 1$ is clear. □

In the product CA of Example 5.5, 0^∞ is an equicontinuous point but 1^∞ is not. In the majority CA of Example 5.10, both 0^∞ and 1^∞ are equicontinuous but $(01)^\infty$ is not. Both product and majority CA are almost equicontinuous but not equicontinuous. The sum CA of Example 5.6 is sensitive as it is conjugate to a full shift. The traffic CA of Example 5.9 is sensitive too.

5.1.1. Equicontinuous CA. — We have not yet encountered equicontinuous CA. There are two trivial examples.

***Example 5.13* (I: the identity rule 204).** — $(\mathbf{2}^{\mathbb{Z}}, I)$, where $I(x) = x$, is an equicontinuous CA.

***Example 5.14* (O: the zero rule 0).** — $(\mathbf{2}^{\mathbb{Z}}, O)$, where $O(x) = 0^\infty$, is an equicontinuous CA.

Proposition 5.15. — *Any CA with radius zero is equicontinuous.*

Proof. — If $r = 0$, then the local rule is a map $f : A \to A$ and $F(x)_i = f(x_i)$. Thus

$$x_{[-n,n]} = y_{[-n,n]} \quad \Rightarrow \quad F(x)_{[-n,n]} = F(y)_{[-n,n]}$$
$$d(x,y) < 2^{-n} \quad \Rightarrow \quad d(F(x), F(y)) < 2^{-n}$$

□

***Proposition 5.16* (Kůrka [88]).** — *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius $r > 0$. The following conditions are equivalent.*

(1) $(A^{\mathbb{Z}}, F)$ is equicontinuous.

(2) There exists $k > 0$ such that any $u \in A^{2k+1}$ is r-blocking.

(3) There exists a preperiod $m \geq 0$ and a period $p > 0$, such that $F^{m+p} = F^m$.

Proof. —

(1) $\Rightarrow$ (2): For $\varepsilon = 2^{-r}$ there exists $\delta = 2^{-k}$ such that for all $x, y \in A^{\mathbb{Z}}$, if $x_{[-k,k]} = y_{[-k,k]}$, then $F^n(x)_{[-r,r]} = F^n(y)_{[-r,r]}$ for all $n > 0$. Thus any $u \in A^{2k+1}$ is r-blocking.

(2) $\Rightarrow$ (3): Let $u \in A^{2k+1}$ and let $x = u^\infty \in [u]_{-k}$ be the σ-periodic point with period $2k+1$. Then $(F^n(x)_0)_{n\geq 0}$ is an eventually periodic sequence with some preperiod $m_u \geq 0$ and period $p_u > 0$. For any $y \in [u]_{-k}$, $(F^n(y)_0)_{n \geq 0}$ has preperiod m_u and period p_u. Set

$$m = \max\{m_u : u \in A^{2k+1}\}, \quad p = \mathrm{lcm}\{p_u : u \in A^{2k+1}\}$$

Here lcm is the least common multiple. For any $x \in A^{\mathbb{Z}}$, $F^m(x)_0 = F^{m+p}(x)_0$, so

$$F^m(x)_i = F^m(\sigma^i(x))_0 = F^{m+p}(\sigma^i(x))_0 = F^{m+p}(x)_i$$

and therefore $F^{m+p}(x) = F^m(x)$.

(3) $\Rightarrow$ (1): Since $F, F^2, \ldots, F^{m+p-1}$ are uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in A^{\mathbb{Z}}$,

$$d(x,y) < \delta \quad \Rightarrow \quad \forall i < m+p, d(F^i(x), F^i(y)) < \varepsilon.$$

Since any F^n is equal to some F^i with $i < m+p$, $(A^{\mathbb{Z}}, F)$ is equicontinuous. $\square$

Example 5.17 **(E: An equicontinuous rule 108)**. — $(\mathbf{2}^{\mathbb{Z}}, E)$, where

$$E(x)_i = \mathrm{mod}_2(x_i + x_{i-1}x_{i+1})$$

000	001	010	011	100	101	110	111
0	0	1	1	0	1	1	0

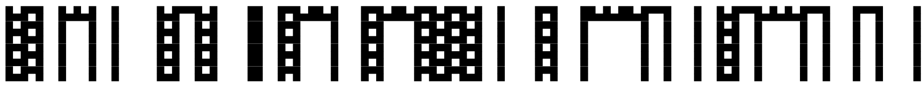


FIGURE 8. E: an equicontinuous rule 108

We show first that 01110 is a 3-blocking word with offset 1. Considering all possibilities for the left and right extensions of 01110, we get

0011100	0011101	1011100	1011101
0010100	0010111	1110100	1110111
0011100	001110	011100	01110

Thus $E^2([01110]_0) \subseteq [01110]_0$, so $E^{2n}([01110]_0) \subseteq [111]_1$ and $E^{2n+1}([01110]_0) \subseteq [101]_1$. Words in $B = \{00, 1111, 01110, 0110110, 010110, 011010, 10101\}$ are 2-blocking.

00	1111	01110	0110110	010110	011010	10101
00	00	101	11111	1111	1111	11111
00	00	01110	000	00	00	000

Any word v of length at least 7 contains at least one occurrence of some word from B, so it is also 2-blocking. It follows that the CA is equicontinuous. The preperiod and period are $m = p = 2$.

Proposition 5.18. — *Let $(A^{\mathbb{Z}}, F)$ be a CA and $\Sigma \subseteq A^{\mathbb{Z}}$ an F-invariant subshift. If (Σ, F) is transitive then it is either sensitive or consists of a single periodic orbit.*

Proof. — Suppose that (Σ, F) is transitive and not sensitive. By Theorem 2.31, it is almost equicontinuous and by Theorem 2.33, it is uniformly rigid. Thus for $\varepsilon = 1$ there exists n such that for all $x \in \Sigma$, $d(F^n(x), x) < 1$, i.e., $F^n(x)_0 = x_0$. For $y = \sigma^i(x)$ we get

$$F^n(x)_i = \sigma^i(F^n(x))_0 = F^n(\sigma^i(x))_0 = \sigma^i(x)_0 = x_i.$$

Thus F^n is identity and since (Σ, F) is transitive, it consists of a single periodic orbit. $\square$

Corollary 5.19 **(Codenotti and Margara [36])**. — *Any transitive CA is sensitive.*

Example 5.20. — $(\mathbf{2}^{\mathbb{Z}} \times \mathbf{2}^{\mathbb{Z}}, \mathrm{Id} \times \sigma)$ (i.e., $F(x,y)_i = (x_i, y_{i+1})$) is a surjective and sensitive CA which is not transitive.

5.2. Surjectivity

Let $(A^{\mathbb{Z}}, F)$ be a CA with diameter $d \geq 0$ and a local rule $f : A^{d+1} \to A$. We extend the local rule to a function $f : A^* \to A^*$ by

$$f(u)_i = \begin{cases} f(u_{[i,i+d]}) & \text{if } i < |u| - d \\ \lambda & \text{if } i \geq |u| - d \end{cases}$$

Thus $|f(u)| = \max\{0, |u| - d\}$. For example, in the sum CA of Example 5.6, the successive images of 01100011 are

$$01100011 \mapsto 111011 \mapsto 0101 \mapsto 00 \mapsto \lambda \mapsto \lambda$$

Observe that for any $n \geq 0$,

$$\sum_{u \in A^n} \#f^{-1}(u) = (\#A)^{n+d}.$$

The mean number of preimages of a word of length n is $(\#A)^d$. We show that a CA is surjective iff any nonempty word has exactly $(\#A)^d$ preimages.

***Theorem 5.21* (Hedlund [73])**. — *A CA $(A^{\mathbb{Z}}, F)$ with local rule $f : A^{d+1} \to A$ is surjective iff for any $u \in A^+$, $\#f^{-1}(u) = (\#A)^d$.*

Proof. — Let the condition be satisfied and $y \in A^{\mathbb{Z}}$. Let m, a be the memory and anticipation of F, so $d = a - m$. For $n \geq 0$ set

$$X_n = \{x \in A^{\mathbb{Z}} : f(x_{[-n+m, n+a]}) = y_{[-n,n]}\}.$$

By assumption, any X_n is nonempty. Moreover, the X_n are closed, and $X_{n+1} \subseteq X_n$. By compactness there exists $x \in \bigcap_{n>0} X_n$. Then $F(x) = y$, so F is surjective. Conversely, suppose that F is surjective and set

$$p = \min\{\#f^{-1}(u) : u \in A^+\}$$

If $u \in A^+$ and $y \in A^{\mathbb{Z}}$ contains u, then any preimage of y contains a preimage of u. Thus any $u \in A^+$ has at least one preimage and $p > 0$. We show

Lemma 1. — *If $\#f^{-1}(u) = p$, then for any $a \in A$, $\#f^{-1}(ua) = p$.*

Proof. — By assumption, for any $a \in A$, $\#f^{-1}(ua) \geq p$. Suppose that for some $a \in A$, $\#f^{-1}(ua) > p$. We have disjoint unions (Figure 9 left).

$$\begin{aligned} \bigcup_{b \in A} \{vb : v \in f^{-1}(u)\} &= \bigcup_{a \in A} f^{-1}(ua) \\ p \cdot \#A = \# \left(\bigcup_{b \in A} \{vb : v \in f^{-1}(u)\} \right) &= \# \left(\bigcup_{a \in A} f^{-1}(ua) \right) > p \cdot \#A \end{aligned}$$

and this is a contradiction. This proves Lemma 1. □

Lemma 2. — $p = (\#A)^d$.

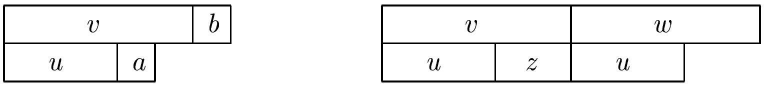


FIGURE 9. Preimages

Proof. — Let $u \in A^n$ be such that $\#f^{-1}(u) = p$. By Lemma 1, we have $\#f^{-1}(uu') = p$ for any $u' \in A^*$. We have a disjoint union (Figure 9 right)

$$\{vw : v, w \in f^{-1}(u)\} = \bigcup_{z \in A^d} f^{-1}(uzu)$$

$$p^2 = \#\left(\{vw : v, w \in f^{-1}(u)\}\right) = \#\left(\bigcup_{z \in A^d} f^{-1}(uzu)\right) = p \cdot (\#A)^d$$

so $p = (\#A)^d$. This proves Lemma 2. □

We now finish the proof of the theorem. Suppose that for some $n > 0$ and some $v \in A^n$, $\#f^{-1}(v) > (\#A)^d$. Then

$$(\#A)^{n+d} = \#\left(\bigcup_{v \in A^n} f^{-1}(v)\right) > (\#A)^n \cdot (\#A)^d$$

and this is a contradiction. Thus for any $u \in A^*$, $\#f^{-1}(u) = (\#A)^d$. □

The condition $\#f^{-1}(u) = (\#A)^d$ may be satisfied for $|u| = 1$ and fail for longer u. For example in the majority CA of Example 5.10, the condition works for $|u| = 1$. The table of the local rule contains 4 zeros and 4 ones. However, the condition is not satisfied for $|u| = 2$, since there are six preimages of 00:

$$f^{-1}(00) = \{0000, 0001, 0010, 0100, 1000, 1001\}$$

Proposition 5.22. — *Any permutive CA is surjective.*

Proof. — Let $(A^{\mathbb{Z}}, F)$ be a right-permutive CA with local rule $f : A^{d+1} \to A$. (The proof for left-permutive CA is similar.) Given $u \in A^+$ and $v \in A^d$ we construct a preimage of u with prefix v. Put $w_{[0,d)} = v$. There exists a unique w_d with $f(w_{[0,d]}) = u_0$. Similarly there exists a unique w_{d+1} with $f(w_{[1,d+1]}) = u_1$, etc. Thus the set $f^{-1}(u)$ is in one-to-one correspondence with A^d so $\#f^{-1}(u) = \#A^d$. □

Proposition 5.23. — *Any point of a surjective CA is nonwandering.*

Proof. — We use the Poincaré recurrence theorem of ergodic theory: Let $(A^{\mathbb{Z}}, F)$ be a surjective CA with local rule $f : A^{d+1} \to A$. For any clopen set U define its measure $\mu(U)$ as follows. If $u \in A^n$ and $k \in \mathbb{Z}$, then $\mu([u]_k) = (\#A)^{-n}$. If $U = V_1 \cup \cdots \cup V_m$ is a disjoint union of cylinders, set $\mu(U) = \mu(V_1) + \cdots + \mu(V_m)$. Clearly $\mu(A^{\mathbb{Z}}) = 1$. If $u \in A^n$, then $f^{-1}(u) \subseteq A^{n+d}$ and $\#f^{-1}(u) = (\#A)^d$, so

$$\mu(F^{-1}([u]_k)) = (\#A)^d \cdot (\#A)^{-n-d} = \mu([u]_k)$$

It follows that for any clopen set U, $\mu(F^{-1}(U)) = \mu(U)$. Let $x \in A^{\mathbb{Z}}$ and let U be a clopen set with $x \in U$. Suppose that all $F^{-i}(U)$ for $i \geq 0$ are pairwise disjoint. Then for any $n > 0$,

$$n \cdot \mu(U) = \mu(U \cup \cdots \cup F^{-n+1}(U)) \leq \mu(A^{\mathbb{Z}}) = 1$$

and this is a contradiction. Thus for some $i < j$, $F^{-i}(U) \cap F^{-j}(U) \neq \emptyset$ and $U \cap F^{-j+i}(U) \neq \emptyset$. □

***Theorem 5.24* (Blanchard and Tisseur [27]).** — *Any surjective almost equicontinuous CA has a dense set of periodic points.*

Proof. — Let $u \in A^m$ be arbitrary and let $v \in A^p$ be an r-blocking word with offset j, where r is the radius. Then vuv is a q-blocking word with offset j, where $q = p+m+r$ (Figure 10). Since $|\mathbf{N}| = A^{\mathbb{Z}}$, there exists $t > 0$ and $x \in [vuv]_0 \cap F^{-t}([vuv]_0)$. Set $y = (vu)^{\infty}$. Since $x, F^t(x), y \in [vuv]_0$,

$$F^t(y)_{[j,j+q)} = F^t(x)_{[j,j+q)} = x_{[j,j+q)} = y_{[j,j+q)}$$

Since y and $F^t(y)$ are σ-periodic with period $p + m \leq q$, $F^t(y) = y$ and $y \in [u]_p$ is F-periodic with period t. □

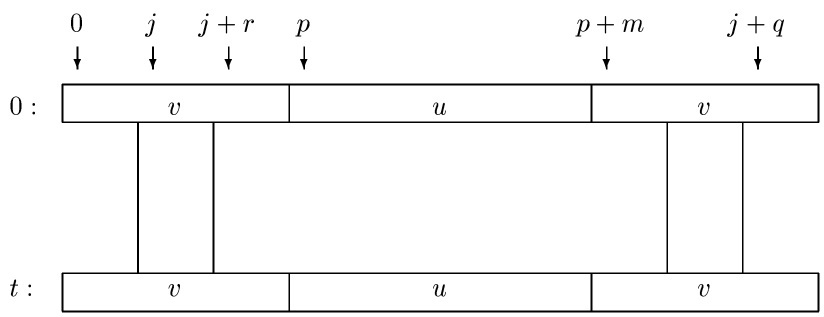


FIGURE 10. Dense periodic points

***Proposition 5.25* (Blanchard and Tisseur [27]).** — *If $(A^{\mathbb{Z}}, F)$ is an equicontinuous and surjective CA, then there exists $p > 0$ such that $F^p = \mathrm{Id}$. In particular, F is bijective.*

Proof. — By Theorem 5.16, there exist $m \geq 0$ and $p > 0$ such that $F^m = F^{p+m}$. Since any point is eventually periodic and nonwandering, it is periodic and $m = 0$. □

***Proposition 5.26*.** — *Let $(A^{\mathbb{Z}}, F)$ be a nonsurjective CA with diameter $d > 0$.*

(1) There exists a **diamond** *(Figure 11), i.e., a word $w \in A^d$ and distinct $u, v \in A^+$ with $|u| = |v|$ and $f(wuw) = f(wvw)$.*

(2) There exists a point with an uncountable number of preimages.

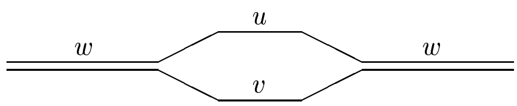
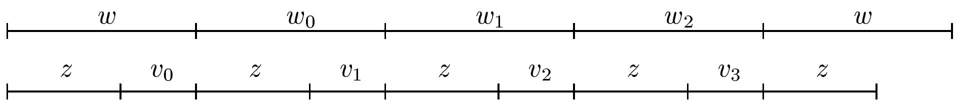


Figure 11. A diamond

Proof. —

(1) By Theorem 5.21 there exists a word $z \in A^+$ such that $\#f^{-1}(z) = k > (\#A)^d$. Choose a word $w \in f^{-1}(z)$, so $|w| = |z| + d \geq d$. For $m > 0$, consider sets

$$\begin{aligned} M_m &= \{ww_0w_1 \ldots w_{m-1}w : w_i \in f^{-1}(z)\}, \\ N_m &= \{zv_0z \ldots zv_mz : v_i \in A^d\} \end{aligned}$$



We have $f(M_m) \subseteq N_m$, $\#M_m = k^m$, $\#N_m = (\#A)^{d(m+1)}$. Since $k > (\#A)^d$, for m large enough, $k^m > (\#A)^{d(m+1)}$, so there exist distinct $u', v' \in M_n$ such that $f(u') = f(v')$. Since both u' and v' begin and end with w whose length is at least d, the statement follows.

(2) Let $w \in A^d$, $u, v \in A^n$ be such that $u \neq v$ and $f(wuw) = f(wvw)$. Set

$$\begin{aligned} M &= \{wu, wv\}^{\mathbb{Z}} \\ &= \{x \in A^{\mathbb{Z}} : \forall i \in \mathbb{Z}, x_{[i(d+n),i(d+n)+d)} = w, x_{[i(d+n)+d,(i+1)(d+n))} \in \{u, v\}\} \end{aligned}$$

Then M is uncountable and all its elements have the same image. □

Corollary 5.27. — *Any injective CA is surjective and hence bijective.*

Proposition 5.28. — *Let $(A^{\mathbb{Z}}, F)$ be a surjective CA with diameter d. Then for any $y \in A^{\mathbb{Z}}$, $\#F^{-1}(y) \leq (\#A)^d$.*

Proof. — Suppose that $x_0, x_1 \ldots x_{n-1}$ are distinct points in $F^{-1}(y)$ and $n > (\#A)^d$. There exists $p > d/2$, such that for all $i \neq j$, $(x_i)_{[-p,p]} \neq (x_j)_{[-p,p]}$, so $y_{[-p-m,p-a]}$ has at least n preimages and this is a contradiction. □

5.2.1. First image subshift. — Although any finite word has exactly $(\#A)^d$ preimages, infinite words may have strictly fewer preimages. Numbers of preimages of a CA can be determined from $F(A^{\mathbb{Z}})$. This is a continuous image of an SFT $A^{\mathbb{Z}}$, hence it is a sofic subshift. This follows from a two-sided version of Theorem 3.57.

Let $(A^{\mathbb{Z}}, F)$ be a CA with local rule $f : A^{d+1} \to A$. The labelled graph $G = (V, E, s, t, f)$ for $F(A^{\mathbb{Z}})$ has vertex set $V = A^d$, edge set $E = A^{d+1}$ and source and target functions

$$s(u) = u_{[0,d)}, \quad t(u) = u_{[1,d]}, \quad u \in A^{d+1}.$$

Thus any $u \in A^{d+1}$ gives rise to a labelled edge $u_{[0,d-1]} \xrightarrow{f(u)} u_{[1,d]}$. In Figure 12 left there is the graph of $P(\mathbf{2}^{\mathbb{Z}})$, where $(\mathbf{2}^{\mathbb{Z}}, P)$ is the product CA of Example 5.5.

Using Theorem 3.55, we can construct a right-resolving labelled graph G' for this subshift. The vertices of G' are nonempty subsets of A^d. The labelled edges are

$$M \xrightarrow{a} N \quad \Leftrightarrow \quad N = \{v_{[1,d]} : v \in A^{d+1}, v_{[0,d)} \in M, f(v) = a\}.$$

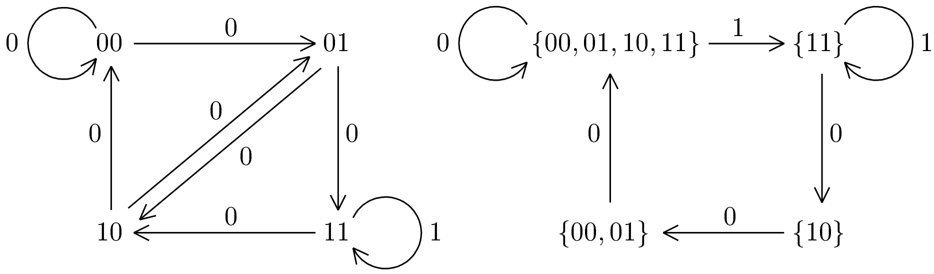


FIGURE 12. First image subshift of a CA

If there is an edge from M to N with label a, then N is uniquely determined by M and a. The vertices of G' are all subsets of A^d which are reachable from the maximal set A^d. In Figure 12 right, there is the right-resolving graph for the first image of the product CA. We see that $P(A^{\mathbb{Z}}) = \Sigma_{\{101,1001\}}$.

For the sum CA of Example 5.6, the right-resolving graph for $S(\mathbf{2}^{\mathbb{Z}})$ has unique vertex $\{00, 01, 10, 11\}$ with two edges labelled 0 and 1. Accordingly, S is surjective. Moreover, since the single vertex of the graph has four elements, any point has exactly four preimages. We consider now an example with a varying number of preimages.

Example 5.29 **(R: a right-permutive rule 106)**. — $(\mathbf{2}^{\mathbb{Z}}, R)$, where

$$R(x)_i = \mathrm{mod}_2(x_{i-1}x_i + x_{i+1})$$

is right-permutive and transitive.

000	001	010	011	100	101	110	111
0	1	0	1	0	1	1	0

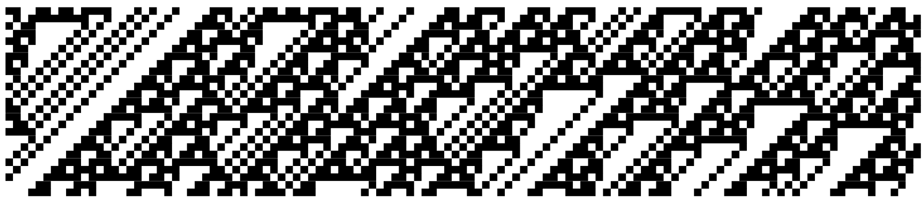


FIGURE 13. R: a right permutive rule 106

The space-time diagram of $(\mathbf{2}^{\mathbb{Z}}, R)$ is in Figure 13. Since

$$x_{i+1} = \mathrm{mod}_2(R(x)_i + x_{i-1}x_i),$$

the CA is right-permutive and by Proposition 5.22, it is surjective. We show that it is transitive. We have $R(u.10^\infty) = v.0^\infty$ for some v and $R(0^\infty.(01)^\infty) = 0^\infty.(10)^\infty$. On the other hand for any $v \in \mathbf{2}^{\mathbb{N}}$ there exists $w \in \mathbf{2}^{\mathbb{N}}$ such that $R^2((10)^\infty.10w) =$

$(10)^\infty.v$. It follows that for any $n > 0$ and any $u, v \in \mathbf{2}^n$ there exists $w \in \mathbf{2}^*$ and $k, m, p > 0$ such that

$$R^k([u0^m(10)^p w]_0) \cap [v]_0 \neq \emptyset.$$

The right-resolving graph of $R(\mathbf{2}^{\mathbb{Z}}) = \mathbf{2}^{\mathbb{Z}}$ is in Figure 14. The number of preimages of a point $y \in \mathbf{2}^{\mathbb{Z}}$ depends on the cardinality of sets through which a path with label y passes. The graph has three path components enclosed in boxes. The left path component consists of a single vertex with three elements. The unique point 1^∞ associated to this component has three preimages. The right path component consists of four vertices each with two elements. The points associated to this component have two preimages. Finally the lower path component consists of four vertices each containing a single element. The points associated to this component have unique preimages.

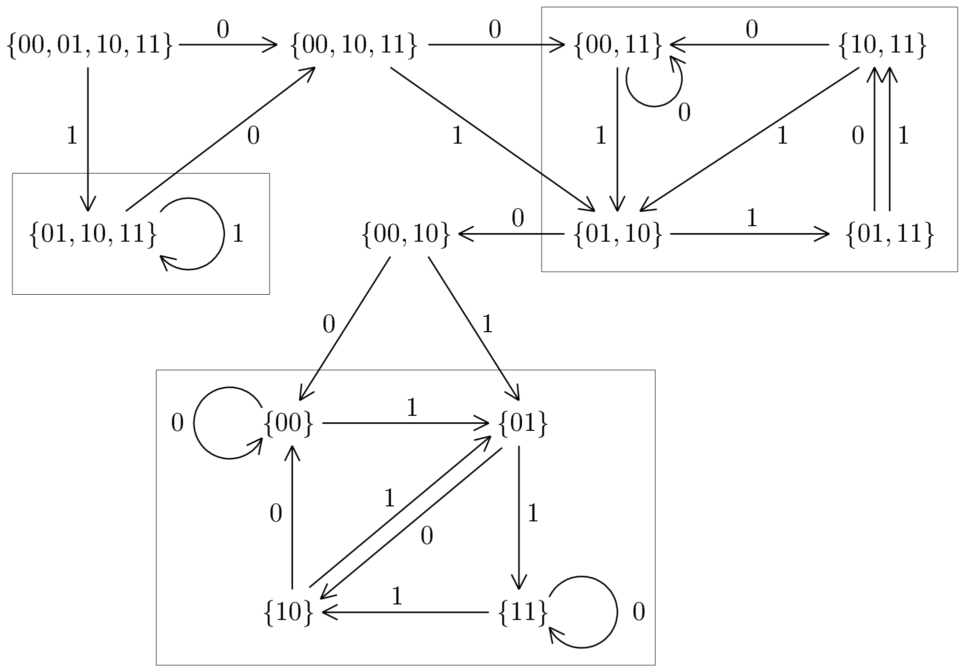


FIGURE 14. Numbers of preimages

$$\begin{aligned}
R^{-1}(1^\infty) &= \{(011)^\infty, (110)^\infty, (101)^\infty\} \\
R^{-1}(0^\infty) &= \{0^\infty, 1^\infty\} \\
R^{-1}((01)^\infty) &= \{(10)^\infty\}
\end{aligned}$$

Definition 5.30. — Let $(A^{\mathbb{Z}}, F)$ be a CA with local rule $f : A^{d+1} \to A$.

(1) Points $x, y \in A^{\mathbb{Z}}$ are **d-separated**, if for any $i \in \mathbb{Z}$, $x_{[i,i+d)} \neq y_{[i,i+d)}$.

(2) The **minimum preimage number** of $(A^{\mathbb{Z}}, F)$ is

$$\begin{aligned} p &= \min\{p(w) : w \in A^+\}, \text{ where} \\ p(w) &= \min\left\{\#\{u \in A^d : \exists v \in f^{-1}(w), v_{[t,t+d)} = u\} : 0 \le t \le |w|\right\} \end{aligned}$$

(3) We call any $w \in A^+$ with $p(w) = p$ a **magic word** (Figure 15 left).

0100 is a magic word of the right-permutive CA of Example 5.29. This is the label of a path in Figure 14 which leads from the initial vertex $\mathbf{2}^2 = \{00, 01, 10, 11\}$ to the path component at the bottom of the diagram which consists of one point sets. All preimages of $w = 0100$ have common suffix 00.

$$f^{-1}(0100) = \{000100, 010100, 100100, 111000\}$$

For $t = 4$, and $v \in f^{-1}(0100)$, $v_{[t,t+1]} = 00$, so $p(0100) = 1$ and $p = 1$. Another magic word is 010101, so the periodic point $(01)^\infty$ has a unique preimage.

Recall that $\mathfrak{T}(w)$ is the (residual) set of points which contain infinite number of occurrences of w in both their negative and positive parts (Definition 5.11).

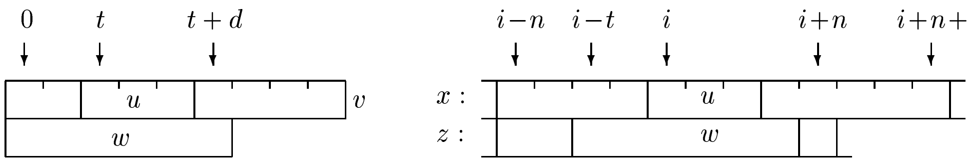


FIGURE 15. Magic word

***Proposition 5.31* (Hedlund [73], Kitchens [85]).** — *Let $(A^{\mathbb{Z}}, F)$ be a surjective CA with diameter d and minimum preimage number p.*

(1) If $w \in A^+$ is a magic word, then any $z \in \mathfrak{T}(w)$ has exactly p preimages. These preimages are pairwise d-separated.

(2) Any point $z \in A^{\mathbb{Z}}$ has at least p pairwise d-separated preimages.

Proof. —

(1) If we compose a CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with a power of σ, the numbers of preimages do not change. We can therefore assume that F is has memory 0, so $F(x)_i = f(x_{[i,i+d]})$, where $f : A^{d+1} \to A$ is the local rule. Let w be a magic word and let $t \in [0, |w|]$ be such that the set

$$B = \{u \in A^d : \exists v \in f^{-1}(w), v_{[t,t+d)} = u\}$$

has exactly p elements. Let $z \in \mathfrak{T}(w)$ and choose i such that $z_{[i-t,i-t+|w|)} = w$ (Figure 15 right). For $n > 0$ set

$$X_n = \{x \in A^{\mathbb{Z}} : f(x_{[i-n,i+n+d]}) = z_{[i-n,i+n]}\}.$$

If $i - n \le i - t$ and $i + n > i - t + |w|$ then $w \sqsubseteq z_{[i-n,i+n]}$, so

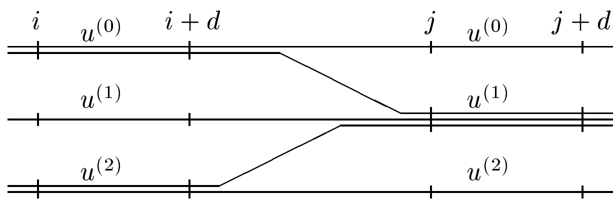
$$B_n = \{x_{[i,i+d)} : x \in X_n\} \subseteq B.$$

Since $\#B_n \ge p$, we have $B_n = B$. Thus for any $u \in B$, $X_n \cap [u]_i$ is a nonempty closed set. By compactness, these sets have nonempty intersection. We have proved

$$\forall u \in B, \forall z \in \mathfrak{T}(w), \forall i \in \mathbb{Z}, (z_{[i,i+|w|)} = w \Rightarrow \exists x \in F^{-1}(z), x_{[i,i+d]} = u).$$

Thus any $z \in \mathfrak{T}(w)$ has at least p preimages. Consider now two distinct occurrences of w in z. Let $i < j$ be such that $z_{[i-t,i-t+|w|)} = z_{[j-t,j-t+|w|)} = w$ (Figure 16). Define a binary matrix $(M_{uv})_{u,v\in B}$ by

$$M_{uv} = 1 \quad \Leftrightarrow \quad \exists x \in F^{-1}(z), x_{[i,i+d)} = u, x_{[j,j+d)} = v$$



$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

FIGURE 16. Consecutive magic words $B = \{u^{(0)}, u^{(1)}, u^{(2)}\}$

For any $u \in B$ there exists $x \in F^{-1}(z)$ with $x_{[i,i+d]} = u$ and then $v = x_{[j,j+d]} \in B$. We get

$$\forall u \in B, \exists v \in B, M_{uv} = 1, \text{ and } \forall v \in B, \exists u \in B, M_{uv} = 1.$$

Thus the sum of any row of M is at least 1 and the sum of any column of M is at least 1. Assume that for some $u \in B$, $\sum_{v\in B} M_{uv} > 1$. Then $z_{[i,j)}$ would have at least $p+1$ preimages and $(z_{[i,j)})^q$ would have at least $p+q$ preimages. If $p+q > (\#A)^d$, this is impossible, so

$$\forall x, y \in F^{-1}(z), (x_{[i,i+d]} = y_{[i,i+d]} \quad \Leftrightarrow \quad x_{[j,j+d]} = y_{[j,j+d]}).$$

Moreover, for $x, y \in F^{-1}(z)$,

$$x_{[i,i+d]} = y_{[i,i+d]} \Rightarrow x_{[i,j+d]} = y_{[i,j+d]} \quad \Leftarrow \quad x_{[j,j+d]} = y_{[j,j+d]}.$$

Otherwise, $x_{[i,j+d]}$ and $y_{[i,j+d]}$ would form a diamond. Since $z \in \mathfrak{T}(w)$, it follows that if $x_{[i,i+d]} = y_{[i,i+d]}$ then $x = y$, so z has exactly p preimages.

Assume that there exist distinct $x, y \in F^{-1}(z)$ which are not d-separated. Then there exist $k \in \mathbb{Z}$ such that $x_{[k,k+d)} = y_{[k,k+d)}$ and $i < k < j$, such that $z_{[i,i+|w|)} = z_{[j,j+|w|)} = w$. The σ-periodic point $z' = (z_{[i,j)})^\infty$ has exactly p preimages and for any $q > 0$, two of these preimages are not d-separated in the interval

$$I_q = [q(j-i) + (k-i), q(j-i) + k - i + d).$$

For some $q < q'$ the pair of points which are not d-separated in intervals I_q and $I_{q'}$ is the same, so there exists a diamond. This is a contradiction, so any point of $\mathfrak{T}(w)$ has exactly p preimages and any two of these preimages are d-separated.

(2) Now let $z \in A^{\mathbb{Z}}$ be an arbitrary point. Pick a bitransitive point $y \in \mathfrak{T}$. Then y has exactly p preimages $x_1, \ldots, x_p$ which are pairwise d-separated. There exists a sequence $(n_q)_{q>0}$ such that $\lim_{q\to\infty} d(z, \sigma^{n_q}(y)) = 0$. By passing to a subsequence if necessary we can assume that for any $1 \le i \le p$ there exists a limit $\lim_{q\to\infty} \sigma^{n_q}(x_i) = s_i$. Then $s_1, \ldots, s_p$ are pairwise d-separated and $F(s_i) = z$. $\square$

5.3. Openness

We study now cellular automata with constant number of preimages.

Proposition 5.32. — *If $(A^{\mathbb{Z}}, F)$ is a CA with diameter d which is both left-permutive and right-permutive, then any $y \in A^{\mathbb{Z}}$ has exactly $(\#A)^d$ preimages.*

Proof. — Let $m \leq a$ be the memory and anticipation respectively, and $f : A^{a-m+1} \to A$ the local map. For any $y \in A^{\mathbb{Z}}$ and $u \in A^d$ there exists a unique $x \in F^{-1}(y)$ with $x_{[0,d)} = u$. $\square$

Proposition 5.33. — *Let $(A^{\mathbb{Z}}, F)$ be a CA and $p > 0$ such that any $y \in A^{\mathbb{Z}}$ has exactly p preimages. Then there exists $n > 0$, such that any $w \in A^{2n+1}$ is magic.*

Proof. — Let $r > 0$ be the radius. For $n > 0$ set

$$Y_n = \left\{ y \in A^{\mathbb{Z}} : \#\{u \in A^{2r+1} : \exists x \in [u]_{-r}, f(x_{[-n-r,n+r]}) = y_{[-n,n]}\} > p \right\}.$$

Any Y_n is closed and $Y_{n+1} \subseteq Y_n$. If all Y_n were nonempty, their intersection would contain a point with at least $p+1$ preimages. Thus for some n, $Y_n = \emptyset$ and any $w \in A^{2n+1}$ is magic (Figure 17). $\square$

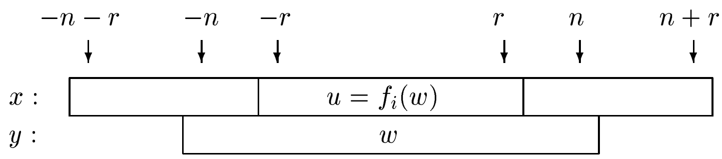


FIGURE 17. Cross sections

Recall that a dynamical system (X, F) is open, if $F(U)$ is open for any open $U \subseteq X$ (Definition 3.34). By Theorem 3.35, a one-sided subshift is open iff it is an SFT. Since the sum CA of Example 5.6 is conjugate to the full shift on four symbols, it is open. Moreover, any its point has exactly four preimages.

Definition 5.34. — A **cross section** of a CA $(A^{\mathbb{Z}}, F)$ is any continuous map $G : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with $F \circ G = \mathrm{Id}$.

If F has a cross section, it is surjective. In particular, any bijective CA has a cross section.

***Theorem 5.35* (Hedlund [73]).** — *Let $(A^{\mathbb{Z}}, F)$ be a CA. The following conditions are equivalent.*

(1) There exists $p > 0$ such that for any $x \in A^{\mathbb{Z}}$, $\#F^{-1}(x) = p$.
(2) There exists $p > 0$ and cross sections $F_1, \dots, F_p : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, such that for any $x \in A^{\mathbb{Z}}$, $F^{-1}(x) = \{F_1(x), \dots, F_p(x)\}$ and $F_i(x) \neq F_j(x)$ for $i \neq j$.
(3) $(A^{\mathbb{Z}}, F)$ is open.

Proof. —

(1) ⇒ (2): We can assume $m = -r$, $a = r$, so $d = 2r$. By Proposition 5.33 there exists n such that any $w \in A^{2n+1}$ is magic. There exist functions $f_1, \dots, f_p : A^{2n+1} \to A^{2r+1}$ such that for any $w \in A^{2n+1}$ (Figure 17),

$$\{u \in A^{2r+1} : \exists x \in [u]_{-r}, f(x_{[-n-r,n+r]}) = w\} = \{f_1(w), \dots, f_p(w)\}$$

There exist maps $F_1, \dots, F_p : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that for any $y \in A^{\mathbb{Z}}$,

$$F(F_i(y)) = y, \quad F^i(y)_{[-r,r]} = f_i(y_{[-n,n]}), \quad 1 \le i \le p$$

We show that F_i are continuous. For any $i = 1, \dots, p$, the set

$$V_i = \bigcup \{[v]_{-n-r} : v \in A^{2n+2r+1}, f_i(f(v)) = v_{[n,n+2r]}\}$$

is clopen (Figure 17). We have $x \in V_i$ iff $x = F_i(F(x))$, so $F_i^{-1} : V_i \to A^{\mathbb{Z}}$ is the restriction of F to V_i and hence continuous. By Proposition A.20, F_i is also continuous.

(2) ⇒ (3): Assume that $U \subseteq A^{\mathbb{Z}}$ is an open set such that $F(U)$ is not open, so there exists $x \in U$ such that $F(U)$ is not a neighbourhood of $F(x)$ (Figure 18 left). For some $i \le p$, $x = F_i(F(x))$. There exists a sequence $(y_n \notin F(U))_{n \ge 0}$, such that $y_n \to F(x)$ as $n \to \infty$, so $F_i(y_n) \notin U$. However, $F_i(y_n) \to F_i F(x) = x$ as $n \to \infty$ and this is a contradiction.

(3) ⇒ (1): We show first that F is surjective. Suppose that $V = F(A^{\mathbb{Z}}) \ne A^{\mathbb{Z}}$. Since F is open, V is clopen. Since $(A^{\mathbb{Z}}, \sigma)$ is transitive, there exists $n > 0$ such that $\sigma^n(V) \cap (A^{\mathbb{Z}} \setminus V) \ne \emptyset$ and this is impossible since V is σ-invariant. Thus F is surjective. Set

$$p = \min\{\#F^{-1}(x) : x \in A^{\mathbb{Z}}\}$$

and assume that for some $y \in A^{\mathbb{Z}}$, $F^{-1}(y) = \{x_1, \dots, x_m\}$, where $m > p$. Let $U_i \ni x_i$ be pairwise disjoint open sets (Figure 18 right). Then $V = F(U_1) \cap \cdots \cap F(U_m)$ is an open set containing y, and any element of V has at least m preimages. Since bitransitive points are dense, V contains a bitransitive point y' which has m preimages and this is a contradiction. □

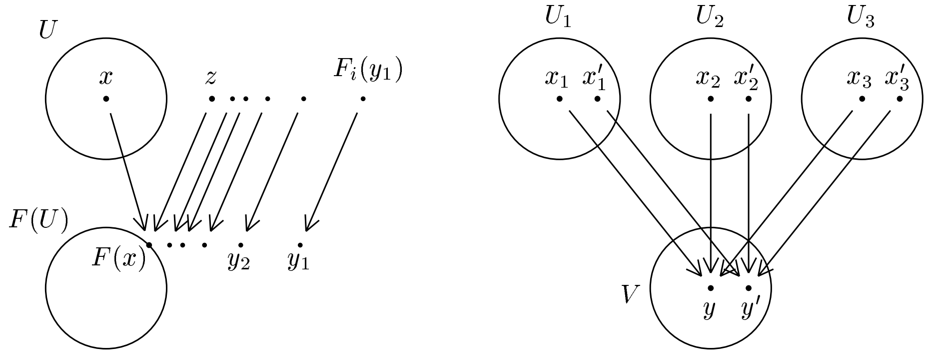


FIGURE 18. Openness

In general, the cross sections F_i are not CA and they need not commute with the shift. In the sum CA of Example 5.6, the four cross sections are given by conditions

$$F_0(x)_{[0,1]} = 00, \quad F_1(x)_{[0,1]} = 01, \quad F_2(x)_{[0,1]} = 10, \quad F_3(x)_{[0,1]} = 11.$$

Only when $p = 1$, i.e., when F is bijective, the inverse map F^{-1} is a CA. This is the case for the identity and for any almost equicontinuous open map.

Proposition 5.36. — *A CA which is open and almost equicontinuous is bijective.*

Proof. — Let r be the radius and $u \in A^m$ a $2r$-blocking word with offset j. Suppose that some $z \in [u]_0$ has two distinct preimages x, y (Figure 19). Then $v = x_{[-r,m+r)}$, $w = y_{[-r,m+r)}$ are distinct since x and y are totally $2r$-separated. Both $F([v]_{-r})$ and $F([w]_{-r})$ are contained in $[u]_0$. Moreover, v and w are $2r$-blocking words with offsets $j+r$. By Theorem 5.23, any point is nonwandering, so there exist $p, q > 0$ and $x' \in [v]_{-r} \cap F^{-p}([v]_{-r})$, $y' \in [w]_{-r} \cap F^{-q}([w]_{-r})$. It follows that

$$x_{[j,j+2r)} = F^{pq}(x')_{[j,j+2r)} = F^{pq-1}(z)_{[j,j+2r)} = F^{pq}(y')_{[j,j+2r)} = y_{[j,j+2r)},$$

so x and y are not $2r$-separated. Thus z has only one preimage, and since F is open, any point has exactly one preimage. $\square$

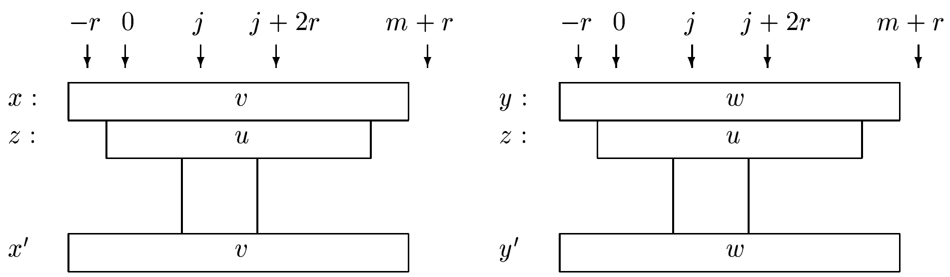


FIGURE 19. Bijective CA

Example 5.37 **(B: a bijective CA).** — $(\mathbf{5}^{\mathbb{Z}}, B)$, where $\mathbf{5} = \{000, 001, 010, 011, 100\}$, and

$$B(x,y,z)_i = (x_i, (1+x_i)y_{i+1} + x_{i-1}z_i, (1+x_i)z_{i-1} + x_{i+1}y_i)$$

is bijective and almost equicontinuous (the addition is modulo 2, Figure 20).

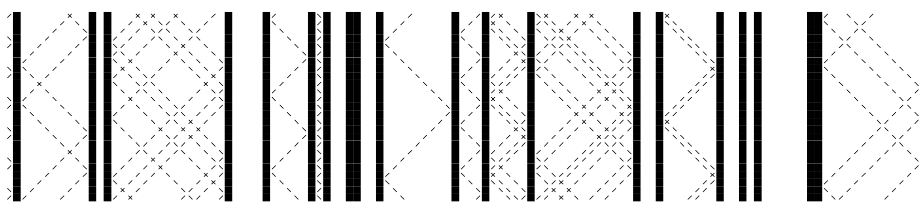


FIGURE 20. B: a bijective almost equicontinuous CA

The dynamics is conveniently described as movement of three types of particles, $1 = 001$, $2 = 010$ and $4 = 100$. Letter $0 = 000$ corresponds to empty cell and $3 = 011$ corresponds to cell occupied by both $1 = 001$ and $2 = 010$. The particle $4 = 100$ is a wall which neither moves nor changes. Particle 1 goes to the left and when it hits a wall 4, it changes to 2. Particle 2 goes to the right and when it hits a wall, it changes to 1. Clearly 4 is a 1-blocking word, so the system is almost equicontinuous. Its inverse is

$$B^{-1}(x,y,z)_i = (x_i, (1+x_i)y_{i-1} + x_{i+1}z_i, (1+x_i)z_{i+1} + x_{i-1}y_i).$$

5.4. Closingness

Closing CA are a generalization of both permutive and open CAs.

Definition 5.38. —
(1) Points $x, y \in A^{\mathbb{Z}}$ are **left-asymptotic**, if $x_{(-\infty,n)} = y_{(-\infty,n)}$ for some $n \in \mathbb{Z}$.
(2) Points $x, y \in A^{\mathbb{Z}}$ are **right-asymptotic**, if $x_{(n,\infty)} = y_{(n,\infty)}$ for some $n \in \mathbb{Z}$.
(3) A CA $(A^{\mathbb{Z}}, F)$ is **right-closing** if for any distinct left-asymptotic $x, y \in A^{\mathbb{Z}}$, $F(x) \neq F(y)$.
(4) A CA $(A^{\mathbb{Z}}, F)$ is **left-closing** if for any distinct right-asymptotic $x, y \in A^{\mathbb{Z}}$, $F(x) \neq F(y)$.
(5) A CA is **closing** if it is either left or right-closing.

The **dual** of a CA $(A^{\mathbb{Z}}, F)$ is the CA $(A^{\mathbb{Z}}, F_d)$, where $F_d = D \circ F \circ D$, and $D : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $D(x)_i = x_{-i}$. Clearly $(A^{\mathbb{Z}}, F)$ is right-permutive iff its dual $(A^{\mathbb{Z}}, F_d)$ is left-permutive (Definition 5.7), and similarly a CA is right-closing iff its dual is left-closing. If $(A^{\mathbb{Z}}, F)$ is not right-closing, then $\overline{F} = F \times F_d$ is neither right nor left-closing.

Proposition 5.39. — *Any right-permutive CA is right-closing.*

Proof. — Let $f : A^{a-m+1} \to A$ be the local rule, $x_{(-\infty,n)} = y_{(-\infty,n)}$ and $F(x) = F(y)$. Since $f(x_{[n-a+m,n]}) = F(x)_{n-a} = F(y)_{n-a} = f(y_{[n-a+m,n]})$ and f is right-permutive, $x_n = y_n$. Repeating this argument we get $x = y$. □

Example 5.40 **(L: the multiplication rule).** — $(\mathbf{4}^{\mathbb{Z}}, L)$, where $\mathbf{4} = \{0,1,2,3\}$ and

$$L(x)_i = \mathrm{mod}_4\left(2x_i + \left\lfloor \frac{x_{i+1}}{2} \right\rfloor\right)$$

is both right-closing and left-closing but neither left-permutive nor right-permutive.

00	01	02	03	10	11	12	13	20	21	22	23	30	31	32	33
0	0	1	1	2	2	3	3	0	0	1	1	2	2	3	3

000010000300001230000
000020001200003120000
000100003000012300000
000200012000031200000

FIGURE 21. L: the multiplication CA

Note that $\lfloor \frac{L(x)_i}{2} \rfloor = \mathrm{mod}_2(x_i)$ and

$$L^2(x)_i = \mathrm{mod}_4 \left(4x_i + 2 \cdot \left\lfloor \frac{x_{i+1}}{2} \right\rfloor + \mathrm{mod}_2(x_{i+1})\right) = x_{i+1}.$$

Thus $L^2(x) = \sigma(x)$, so L is a "square root" of the shift map. It expresses multiplication by two in base four. We can express the local rule equivalently as

$$L(x)_i = 2x_i + \left\lfloor \frac{x_{i+1}}{2} \right\rfloor - 4 \left\lfloor \frac{x_i}{2} \right\rfloor .$$

If $x \in \mathbf{4}^{\mathbb{Z}}$ is left-asymptotic with 0^∞, then $\varphi(x) = \sum_{i=-\infty}^{i=\infty} x_i 4^{-i}$ is finite and $\varphi(L(x)) = 2\varphi(x)$.

Suppose that x, y are distinct and left-asymptotic, i.e., $x_{(-\infty,n)} = y_{(-\infty,n)}$, $x_n \neq y_n$ and $L(x) = L(y)$. Since $x_{n-1} = y_{n-1}$ and $L(x)_{n-1} = L(y)_{n-1}$, either $\{x_n, y_n\} = \{0,1\}$ or $\{x_n, y_n\} = \{2,3\}$. In either case, $L(x)_n \neq L(y)_n$ and this is a contradiction, so $(\mathbf{4}^{\mathbb{Z}}, L)$ is right-closing. Similarly, if $x_{(n,\infty)} = y_{(n,\infty)}$, $x_n \neq y_n$ and $L(x) = L(y)$, then either $\{x_n, y_n\} = \{0,2\}$ or $\{x_n, y_n\} = \{1,3\}$. In either case, $L(x)_{n-1} \neq L(y)_{n-1}$ and this is a contradiction, so $(\mathbf{4}^{\mathbb{Z}}, L)$ is left-closing.

Proposition 5.41. — *Any closing CA is surjective.*

Proof. — Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r which is not surjective. By Proposition 5.26 there exists a diamond, i.e., words $w \in A^{2r}$ and $u \neq v \in A^+$ with $|u| = |v|$ such that $f(wuw) = f(wvw)$. Set $x = w^\infty.uw^\infty$, $y = w^\infty.vw^\infty$. Then x, y are both left and right-asymptotic and $F(x) = F(y)$, so F is neither right nor left-closing. ☐

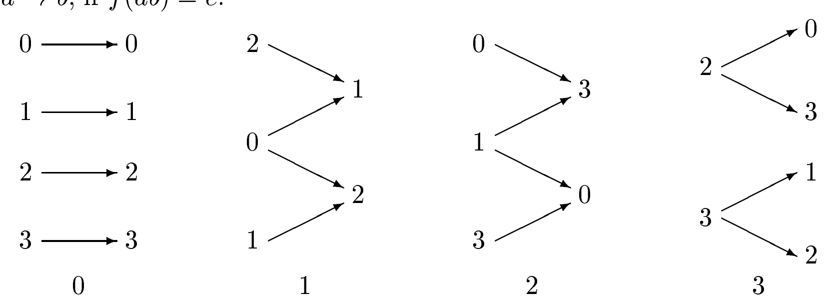
The right-permutive CA $(\mathbf{2}^{\mathbb{Z}}, R)$ of Example 5.29 is right-closing but not left-closing. The point $1^\infty.0^\infty$ has right-asymptotic preimages

$$R^{-1}(1^\infty.0^\infty) = \{(011)^\infty.0^\infty, (110)^\infty.10^\infty\}$$

***Example 5.42* (V: a surjective rule)**. — The CA $(\mathbf{4}^{\mathbb{Z}}, V)$ given by the following table is surjective but neither left nor right-closing.

00	01	02	03	10	11	12	13	20	21	22	23	30	31	32	33
0	1	1	2	2	0	1	2	3	1	0	3	2	3	3	0

We rewrite the local rule of $(\mathbf{4}^{\mathbb{Z}}, V)$, grouping together preimages of any letter. Write $a \xrightarrow{c} b$, if $f(ab) = c$.



The right-resolving graph of $V(\mathbf{4}^{\mathbb{Z}})$ is in Figure 22 top. We see that $V(\mathbf{4}^{\mathbb{Z}})$ is the full shift, so V is surjective. The point $0^\infty.1^\infty$ has left-asymptotic preimages $0^\infty.(21)^\infty$ and $0^\infty.(12)^\infty$, so V is not right-closing. This point has also right-asymptotic preimages $0^\infty.(12)^\infty$ and $2^\infty.(12)^\infty$, so V is not left-closing (Figure 22 bottom).

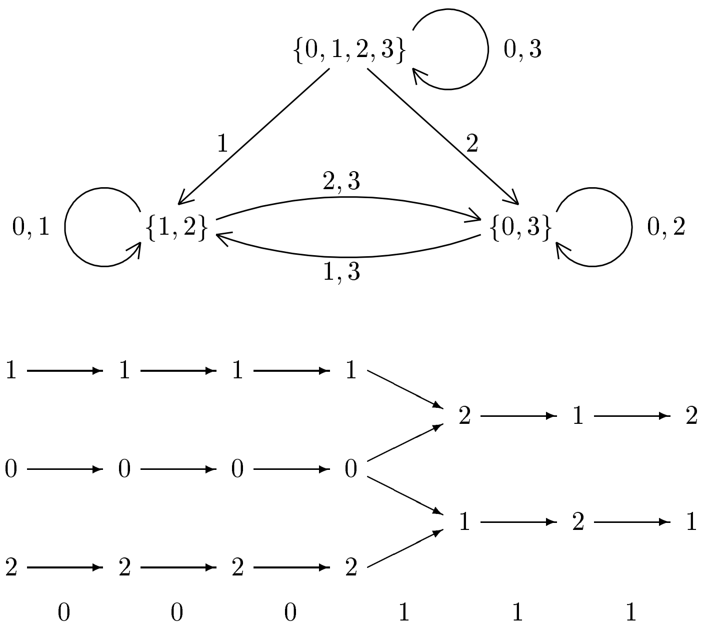


FIGURE 22. Asymptotic points

Proposition 5.43. — *A CA* $(A^{\mathbb{Z}}, F)$ *is right-closing iff there exists* $m > 0$ *such that*

$$x_{[-m,0)} = y_{[-m,0)} \ \ \& \ \ F(x)_{[-m,m]} = F(y)_{[-m,m]} \quad \Rightarrow \quad x_0 = y_0$$

Proof. — Suppose that F is right-closing (Figure 23 left). Given $m > 0$, let X_m consist of all pairs $(x, y) \in A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ such that

$$x_{[-m,0)} = y_{[-m,0)} \ \ \& \ \ F(x)_{[-m,m]} = F(y)_{[-m,m]} \quad \& \quad x_0 \neq y_0.$$

Any X_m is closed and $X_{m+1} \subseteq X_m$. If all X_m were nonempty, their intersection would contain a pair (x, y) of distinct left-asymptotic words with $F(x) = F(y)$, so F would not be right-closing. Thus for some m, $X_m = \emptyset$. Conversely, suppose that F is not right-closing, so there exist distinct left-asymptotic words x, y with $F(x) = F(y)$. Taking the shifts of x and y, if necessary, we can assume $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$. Thus for any $m > 0$, the condition is not satisfied. □

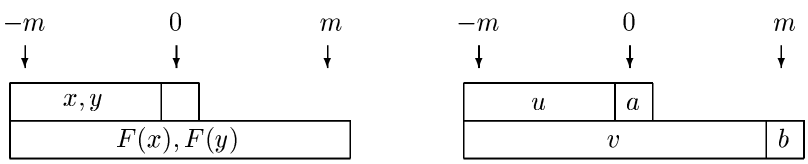


FIGURE 23. Closingness

Proposition 5.44. — *Let* $(A^{\mathbb{Z}}, F)$ *be a right-closing CA. For all sufficiently large* $m > 0$, *if* $u \in A^m$, $v \in A^{2m}$ *and* $F([u]_{-m}) \cap [v]_{-m} \neq \emptyset$, *then (Figure 23 right).*

$$\forall b \in A, \exists! a \in A, F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset.$$

Proof. — If Proposition 5.43 holds for m, it holds also for $m+1$. Thus we can assume that m is arbitrary, large enough number. Set

$$B = \{(u,v) \in A^m \times A^{2m} : F([u]_{-m}) \cap [v]_{-m} \neq \emptyset\}.$$

If $(u,v) \in B$ and $b \in A$, then there exists at most one $a \in A$ with

$$F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset.$$

Consider an oriented graph with vertices B. For any pair of words $u \in A^{m+1}$, $v \in A^{2m+1}$ such that $F([u]_{-m}) \cap [v]_{-m} \neq \emptyset$, insert an oriented edge (Figure 24 left)

$$(u_{[0,m-1]}, v_{[0,2m-1]}) \to (u_{[1,m]}, v_{[1,2m]}).$$

This graph defines an SFT $\Sigma \subseteq B^{\mathbb{Z}}$ of order two, and there is a conjugacy $\varphi : (A^{\mathbb{Z}}, \sigma) \to (\Sigma, \sigma)$ defined by $\varphi(x)_i = (x_{[i,i+m-1]}, F(x)_{[i,i+2m-1]})$. The topological entropy is $h(\Sigma,\sigma) = h(A^{\mathbb{Z}},\sigma) = \ln(\#A)$. In the graph of Σ, at most $\#A$ edges lead from any vertex $(u_{[0,m-1]}, v_{[0,2m-1]})$ (for any v_{2m} at most one). If fewer than $\#A$ edges lead from some vertex, then by Proposition 3.49, the topological entropy $h(\Sigma,\sigma)$ would be smaller than $\ln \#A$. Thus for any $(u,v) \in B$ and any $b \in A$ there exists a unique $a \in A$ such that $F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset$. □

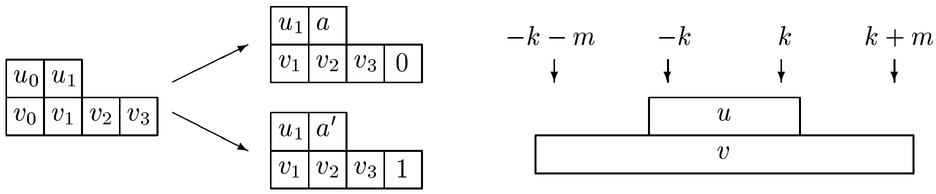


FIGURE 24. Closing and open CA

Theorem 5.45. — *A CA is open iff it is both left-closing and right-closing.*

Proof. — If $(A^{\mathbb{Z}}, F)$ is open and $z \in A^{\mathbb{Z}}$, then by Proposition 5.31 and Theorem 5.35, any two preimages of z are $2r$-separated, so they are neither left nor right-asymptotic. This means that F is both right and left-closing.

Conversely, suppose that $(A^{\mathbb{Z}}, F)$ is a CA with radius r which is both left and right-closing. Let $m > 0$ be an integer from Proposition 5.44 and $u \in A^{2k+1}$ (Figure 24 right). If $v \in A^{2k+2m+1}$ and $F([u]_{-k}) \cap [v]_{-k-m} \neq \emptyset$, then any $y \in [v]_{-k-m}$ has a preimage in $[u]_{-k}$. Thus

$$F([u]_{-k}) = \bigcup \{[v]_{-k-m} : v \in A^{2k+2m+1}, F([u]_{-k}) \cap [v]_{-k-m} \neq \emptyset\}$$

is a union of cylinders, so it is open. □

Proposition 5.46. — *If $(A^{\mathbb{Z}}, F)$ is a right-closing CA, then for all sufficiently large $p > 0$, $(A^{\mathbb{Z}}, \sigma^p \circ F)$ is a factor of a two-sided full shift.*

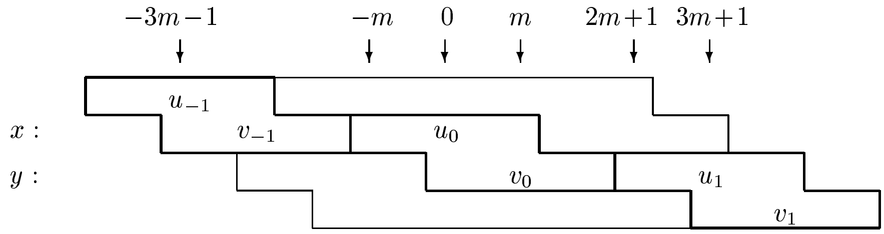


FIGURE 25. A two-sided extension of a closing CA

Proof. — Let $m > 0$ be the number from Proposition 5.44. We can assume $m \geq r$, where r is the radius. Let $p = 3m + 1$ and set

$$B = \{(u, v) \in A^{2m+1} \times A^{2m+1} : F([u]_{-m}) \cap [v]_0 \neq \emptyset\}.$$

If (u_0, v_0) and (u_1, v_1) belong to B, then there exists $y \in [u_1]_{2m+1} \cap F^{-1}([v_1]_{3m+1})$ (Figure 25). We can change y in interval $[0, 2m]$ to obtain a point $x \in [v_0 u_1]_0 \cap F^{-1}([v_1]_{3m+1})$. By Proposition 5.44, x has a preimage in $[u_0]_{-m}$, so

$$F^1([u_0]_{-m}) \cap [v_0 u_1]_0 \cap F^{-1}([v_1]_{3m+1}) \neq \emptyset.$$

If $(u, v) \in B^{\mathbb{Z}}$, then $X_1 = F^1([v_{-1} u_0]_{-3m-1}) \cap [v_0 u_1]_0 \cap F^{-1}([v_1 u_2]_{3m+1}) \neq \emptyset$, and similarly, for any $q > 0$,

$$X_q = \bigcap_{n=-q}^{q} F^{-n}([v_n u_{n+1}]_{(3m+1)n}) \neq \emptyset.$$

By compactness, $\bigcap_{q>0} X_q$ is nonempty. If $x, y \in X_q$, then

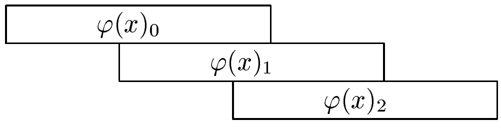
$$x_{[-q(3m+1)+mq, q(3m+1)-mq]} = y_{[-q(3m+1)+mq, q(3m+1)-mq]}.$$

Thus there exists a unique $\varphi(u, v) \in \bigcap_{q>0} X_q$ and $\varphi : (B^{\mathbb{Z}}, \sigma) \to (A^{\mathbb{Z}}, \sigma^p \circ F)$ is a factor map. □

***Theorem 5.47* (Boyle and Kitchens [31]).** — *Any closing CA $(A^{\mathbb{Z}}, F)$ has a dense set of points which are both F-periodic and σ-periodic.*

Proof. — We can assume that F is right-closing. By Proposition 5.46, $(A^{\mathbb{Z}}, \sigma^p \circ F)$ is a factor of a two-sided full shift, so it has a dense set of periodic points. We can assume that $p > m$, where m is the integer from Proposition 5.44. We show that if $x \in A^{\mathbb{Z}}$ is a $(\sigma^p \circ F)$-periodic point, then it is both σ-periodic and F-periodic. Set $B = A^{2p+1}$ and construct a map $\varphi : A^{\mathbb{Z}} \to B^{\mathbb{N}}$ given by

$$\varphi(x)_i = (\sigma^p \circ F)^i(x)_{[-p,p]} = F^i(x)_{[pi-p, pi+p]}$$



If $x \in A^{\mathbb{Z}}$ is $(\sigma^p \circ F)$-periodic, i.e., if $(\sigma^p \circ F)^q(x) = x$, then $\varphi(x)$ is σ-periodic since

$$\varphi(x)_{i+q} = (\sigma^p \circ F)^{i+q}(x)_{[-p,p]} = (\sigma^p \circ F)^i(x)_{[-p,p]} = \varphi(x)_i.$$

Because F is right-closing and $p > m$, $\varphi(\sigma(x))$ is uniquely determined by $\varphi(x)$. In fact, $\varphi(\sigma(x))_i$ depends only on $\varphi(x)_i$ and $\varphi(x)_{i+1}$. It follows that $\varphi(\sigma(x))$ is σ-periodic and also has period q. In $B^{\mathbb{N}}$, there are at most $(\#B)^q$ σ-periodic points with period q, so there exist $0 \le k < k+s < (\#B)^q$ such that $\varphi(\sigma^{k+s}(x)) = \varphi(\sigma^k(x))$ and for all $l \ge k$, $\varphi(\sigma^{l+s}(x)) = \varphi(\sigma^l(x))$. Applying this argument to $\sigma^{-j}(x)$, where $j = (\#B)^q$, we get $\varphi(\sigma^{l+s}(x)) = \varphi(\sigma^l(x))$ for all $l \ge 0$. In particular, $\varphi(\sigma^s(x)) = \varphi(x)$, so $\sigma^s(x) = x$ and

$$x = (\sigma^p \circ F)^{qs}(x) = F^{qs} \circ \sigma^{pqs}(x) = F^{qs}(x).$$

Thus x is both σ-periodic and F-periodic. $\square$

5.5. Expansivity

If a CA has negative memory and positive anticipation ($m < 0 < a$) and if it is both left and right-permutive, then it is conjugate to a full shift (Proposition 5.8) and therefore it is positively expansive. There exist positively expansive CA which are neither left nor right-permutive. An example is $(\mathbf{4}^{\mathbb{Z}} \times \mathbf{4}^{\mathbb{Z}}, \overline{L})$, where L is the multiplication CA from Example 5.40, $\overline{L} = L \times L_d$ and L_d is the dual of L.

By Theorem 3.21, any positively expansive CA is conjugate to a subshift. This subshift is identified in the following proposition.

Proposition 5.48. — *Any positively expansive CA $(A^{\mathbb{Z}}, F)$ with radius r is conjugate to $(\Sigma_{[-r,r]}(F), \sigma)$.*

Proof. — Let $\delta > 0$ be the expansivity constant. Suppose that there exist $x \ne y$ such that for all $n \ge 0$, $F^n(x)_{[-r,r]} = F^n(y)_{[-r,r]}$. Let $x_k \ne y_k$, so $|k| > r$. Assume that $k > 0$ and define $z \in A^{\mathbb{Z}}$ by $z_{(-\infty,0]} = x_{(-\infty,0]}$, $z_{[0,\infty)} = y_{[0,\infty)}$. Then $x \ne z$ and for all $n \ge 0$, $F^n(x)_{(-\infty,r]} = F^n(z)_{(-\infty,r]}$. Pick k with $2^{-(r+k)} < \delta$. Then

$$F^n(\sigma^{-k}(x))_{(-\infty,r+k]} = F^n(\sigma^{-k}(z))_{(-\infty,r+k]},$$

so $d(F^n(\sigma^{-k}(x)), F^n(\sigma^{-k}(y)) < \delta$ and this is a contradiction. Thus $\varphi : (A^{\mathbb{Z}}, F) \to (\Sigma_{[-r,r]}(F), \sigma)$ is injective, hence it is a conjugacy. $\square$

Theorem 5.49 **(Kůrka [88], Nasu[117])**. — *Any positively expansive CA is open and conjugate to a one-sided SFT.*

Proof. — We show that if $(A^{\mathbb{Z}}, F)$ is positively expansive, then it is both left and right-closing. Let $\delta = 2^{-m}$ be the constant of expansivity, so

$$x \ne y \quad \Rightarrow \quad \exists n \ge 0, F^n(x)_{[-m,m]} \ne F^n(y)_{[-m,m]}$$

Suppose that x, y are distinct left-asymptotic points with $F(x) = F(y)$. By taking a shift of x and y if necessary, we can assume $x_{(-\infty,m]} = y_{[-\infty,m]}$, so for any $n \ge 0$, $d(F^n(x), F^n(y)) < \delta$ and this is a contradiction. Thus $(A^{\mathbb{Z}}, F)$ is right-closing and similarly we show that it is left-closing. By Theorem 5.45, $(A^{\mathbb{Z}}, F)$ is open, so $(\Sigma_{[-r,r]}(F), \sigma)$ is open too. By Theorem 3.35, $\Sigma_{[-r,r]}(F)$ is an SFT. $\square$

Proposition 5.50 **(Blanchard and Maass [25])**. — *A positively expansive CA is mixing.*

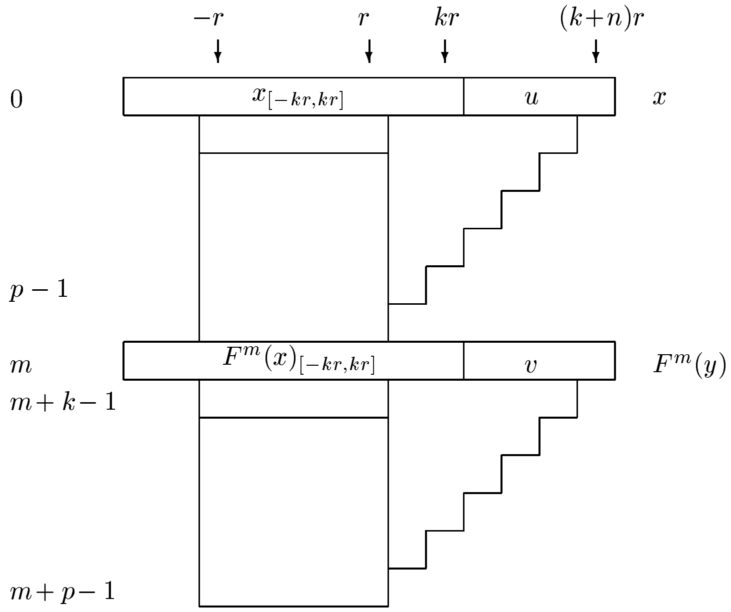


FIGURE 26. Positively expansive CA

Proof. — Let $(A^{\mathbb{Z}}, F)$ be a positively expansive CA with radius r, set $B = A^{2r+1}$ and let k be the order of the SFT $\Sigma_{[-r,r]}(F)$, so $w \in \Sigma_{[-r,r]}(F)$ iff for all $n \geq 0$, $w_{[n,n+k)} \in \mathcal{L}(\Sigma_{[-r,r]}(F))$. Let $u, v \in A^{nr}$. Since the conjugacy $\varphi^{-1} : \Sigma_{[-r,r]}(F) \to A^{\mathbb{Z}}$ is continuous, there exists $p > 0$ such that for any x, y,

$$\forall i < p, F^i(x)_{[-r,r]} = F^i(y)_{[-r,r]} \quad \Rightarrow \quad x_{[-(k+n)r,(k+n)r]} = y_{[-(k+n)r,(k+n)r]}$$

We will show that for all $m \geq p$, $F^m([u]_0) \cap [v]_0 \neq \emptyset$. Choose any $x \in A^{\mathbb{Z}}$ such that $x_{[kr+1,(k+n)r]} = u$ (Figure 26). Since F^m is surjective, there exists $y \in A^{\mathbb{Z}}$ such that

$$F^m(y)_{[-kr,kr]} = F^m(x)_{[-kr,kr]}, \quad F^m(y)_{[kr+1,(k+n)r]} = v$$

It follows that for $m \leq i < m+k$, $F^i(x)_{[-r,r]} = F^i(y)_{[-r,r]}$. Construct $w \in B^{m+p}$ by

$$w_i = \begin{cases} F^i(x)_{[-r,r]} & \text{if } 0 \leq i \leq m+k-1 \\ F^i(y)_{[-r,r]} & \text{if } m \leq i \leq m+p-1 \end{cases}$$

Since both $w_{[0,m+k)}$ and $w_{[m,m+p)}$ belong to $\mathcal{L}(\Sigma_{[-r,r]}(F))$, w belongs to $\mathcal{L}(\Sigma_{[-r,r]}(F))$ too and there exists $z \in A^{\mathbb{Z}}$ such that for $i < m+p$, $F^i(z)_{[-r,r]} = w_i$. We have

$$\begin{aligned} z_{[-(k+n)r,(k+n)r]} &= x_{[-(k+n)r,(k+n)r]} \\ F^m(z)_{[-(k+n)r,(k+n)r]} &= F^m(y)_{[-(k+n)r,(k+n)r]} \end{aligned}$$

Thus $z_{[kr+1,(k+n)r]} = u$ and $F^m(z)_{[kr+1,(k+n)r]} = v$, so $\sigma^{kr+1}(z) \in [u]_0 \cap F^{-m}([v]_0)$. □

5.6. Attractors

In totally disconnected spaces, any clopen invariant set is inward, and any attractor is omega limit of a clopen invariant set (Proposition 2.66). In the study of the attractors of CA we use the fact that $(A^{\mathbb{Z}}, \sigma)$ is mixing (Definition 2.37), i.e., for any nonempty open sets $U, V \subseteq A^{\mathbb{Z}}$ there exists $n \geq 0$, such that for all $m \geq n$, $\sigma^m(U) \cap U \neq \emptyset$.

***Proposition 5.51* (Hurley [80]).** — *Let $(A^{\mathbb{Z}}, F)$ be a CA and $x \in A^{\mathbb{Z}}$ an attracting periodic point. Then $\sigma(x) = x$ and $F(x) = x$.*

Proof. — Let p be the period of x, so $F^p(x) = x$. By assumption, there exists a clopen set U with $\mathcal{O}(x) = \omega(U)$. There exists a nonempty open set $V \subseteq U$ such that if $y \in V$, then $\lim_{n\to\infty} F^{np}(y) = x$. Since $(A^{\mathbb{Z}}, \sigma)$ is mixing, for k large enough, both $\sigma^k(V) \cap V$ and $\sigma^{k+1}(V) \cap V$ are nonempty. For $y \in \sigma^k(V) \cap V$ and $z \in \sigma^{k+1}(V) \cap V$ we get

$$\sigma^k(x) = \lim_{n\to\infty} F^{np}(y) = x = \lim_{n\to\infty} F^{np}(z) = \sigma^{k+1}(x)$$

Thus $\sigma(x) = x$, so $x = a^\infty$ for some $a \in A$. Then $F(x) = b^\infty$ for some $b \in A$ and $a^\infty = \lim_{n\to\infty} F^{np}(a^\infty.b^\infty) = b^\infty$. Thus $a = b$ and therefore $p = 1$. □

In the product CA of Example 5.5, 0^∞ is an attracting fixed point. There is also another larger attractor $\omega(\mathbf{2}^{\mathbb{Z}})$.

***Theorem 5.52* (Hurley [80]).** — *If a CA has two disjoint attractors, then any its attractor contains two disjoint attractors and a continuum of quasi-attractors.*

Proof. — Let Y_1 and Y_2 be two disjoint attractors and U_1, U_2 clopen invariant sets such that $\omega(U_1) = Y_1$, $\omega(U_2) = Y_2$. If $U_1 \cap U_2$ were nonempty, then $\omega(U_1 \cap U_2)$ would be an attractor contained in both Y_1 and Y_2. Thus $U_1 \cap U_2 = \emptyset$. Let Y be an attractor and U a clopen invariant set with $\omega(U) = Y$. Since $(A^{\mathbb{Z}}, \sigma)$ is mixing, there exists $n > 0$ such that $V_1 = \sigma^n(U_1) \cap U \neq \emptyset$ and $V_2 = \sigma^n(U_2) \cap U \neq \emptyset$ are clopen invariant sets. It follows that $\omega(V_1)$ and $\omega(V_2)$ are disjoint attractors contained in Y. □

In the majority CA of Example 5.10, $[00]_0$ and $[11]_0$ are disjoint invariant sets, so their omega-limits are disjoint attractors. While $\omega(\mathbf{2}^{\mathbb{Z}})$ and $Y = \omega([00]_0 \cup [11]_0)$ are subshifts, other attractors are not.

***Theorem 5.53* (Hurley [80]).** — *If a CA has a minimal attractor, then it is a subshift, it is contained in any other attractor and its basin of attraction is a dense open set.*

Proof. — Let Y be a minimal attractor and let Z be another attractor. If $Y \cap Z$ were empty, then Y would contain two disjoint attractors. Since Y is a minimal attractor, $Y \subseteq Z$. Let V be a clopen invariant set with $\omega(V) = Y$. Then $\sigma^{-1}(V)$ is a clopen invariant set, so $Y \subseteq \omega(\sigma^{-1}(V)) = \sigma^{-1}(Y)$. Thus $\sigma(Y) \subseteq Y$ and similarly, $\sigma^{-1}(Y) \subseteq Y$, so Y is a subshift. The basin $\mathcal{B}(Y)$ is an open σ-invariant set. Since $(A^{\mathbb{Z}}, \sigma)$ is transitive, there exists, for any nonempty open $U \subseteq A^{\mathbb{Z}}$, $n > 0$ such that $\emptyset \neq \sigma^n(\mathcal{B}(Y)) \cap U = \mathcal{B}(Y) \cap U$ and $\mathcal{B}(Y)$ is dense. □

Of course an attracting fixed point is an example of a minimal attractor. There exists also a minimal attractor which is not a fixed point.

***Example 5.54* (A minimal attractor).** — $(\mathbf{2}^{\mathbb{Z}} \times \mathbf{2}^{\mathbb{Z}}, P \times \sigma)$, where P is the product rule of Example 5.5, has a minimal attractor $\{0^\infty\} \times \mathbf{2}^{\mathbb{Z}}$.

Corollary 5.55. — *For any CA exactly one of the following statements holds.*
(1) There exist two disjoint attractors and a continuum of quasi-attractors.
(2) There exists a unique quasi-attractor. It is a subshift and it is contained in any attractor.
(3) There exists a unique minimal attractor contained in any other attractor.

Proof. — If there do not exist disjoint attractors, then any two attractors are comparable by inclusion. By Proposition 2.66, the number of attractors is at most countable, so the intersection of all attractors is either a quasi-attractor or an attractor. In either case, it is σ-invariant. □

Example 5.56 **(Q : a quasi-attractor, Hurley [80]).** — $(\mathbf{2}^{\mathbb{Z}}, Q)$ where

$$Q(x)_i = x_i x_{i+1}$$

has a quasi-attractor $\{0^\infty\}$ (Figure 27).

FIGURE 27. Q: A quasi-attractor

The fixed point 0^∞ is stable but not attracting. As in Example 5.5, we have

$$Y = \omega(\mathbf{2}^{\mathbb{Z}}) = \{x \in \mathbf{2}^{\mathbb{Z}} : \forall k > 0, 10^k 1 \not\sqsubseteq x\}.$$

For any $m \in \mathbb{Z}$, $[0]_m$ is a clopen invariant set and

$$Y_m = \omega([0]_m) = Y \cap \{x \in \mathbf{2}^{\mathbb{Z}} : \forall i \le m, x_i = 0\}$$

is an attractor. This is a countable set. For example, for $m \ge 0$,

$$Y_m = \{0^\infty.0^n 1^\infty : n \ge m\} \cup \{0^\infty.0^\infty\}$$

We have $Y_{m+1} \subset Y_m$ and $\bigcap_{m \ge 0} Y_m = \{0^\infty\}$ is the unique minimal quasi-attractor.

By Theorem 2.69, a dynamical system (X, F) has a unique attractor iff $(\omega(X), F)$ is chain transitive.

Proposition 5.57. — *An equicontinuous CA has either two disjoint attractors or a unique attractor which is an attracting fixed point.*

Proof. — Suppose that $(A^{\mathbb{Z}}, F)$ is an equicontinuous CA which has not disjoint attractors. By Proposition 5.16 there exists a preperiod $m \ge 0$ and a period $p > 0$, such that $F^{m+p} = F^m$. Assume that there exist two distinct attractors $Z \subset Y \subseteq A^{\mathbb{Z}}$. By Theorem 2.68 there exists a nonwandering point $x \in Y \setminus Z$. Since it is eventually periodic, it must be periodic and $F^p(x) = x$. There exists $\varepsilon > 0$ such that $B_\varepsilon(F^i(x)) \cap Z = \emptyset$ for all $i < p$. Since x is equicontinuous, there exists $\xi > 0$ such that if $d(z, x) < \xi$, then $d(F^n(z), F^n(x)) < \varepsilon$ for all n. By Proposition 4.7, $(A^{\mathbb{Z}}, F)$ has the shadowing property, so there exists $\delta > 0$ such that any δ-chain which starts in x, is ξ-shadowed by some point, and it remains ε-close to the orbit of x. Let $C_\delta(x)$

be the set of points which can be reached by δ-chains from x, and $V = \overline{C_\delta(x)}$ be its closure. By Proposition 2.67, V is an inward set and $\omega(V)$ is an attractor disjoint from Z. This is a contradiction, so $(A^{\mathbb{Z}}, F)$ has a unique attractor $Y = \omega(A^{\mathbb{Z}})$. Since Y is a minimal attractor, (Y, F) is chain transitive. Since it has the shadowing property, it is transitive. By Proposition 5.18, Y consists of a single periodic orbit, which must be a fixed point by Proposition 5.51. □

The zero CA of Example 5.14 is equicontinuous and has a unique attractor. The identity CA of Example 5.13 is equicontinuous and has disjoint attractors. There is also an example which is not surjective.

***Example 5.58* (J: disjoint attractors).** — $(\mathbf{3}^{\mathbb{Z}}, J)$ where $J(x)_i = \lfloor \frac{x_i+1}{2} \rfloor$, is equicontinuous, has disjoint attractors and is not surjective.

We have $J(\mathbf{3}^{\mathbb{Z}}) = \mathbf{2}^{\mathbb{Z}}$ and J is identity on $\mathbf{2}^{\mathbb{Z}}$, so for any $u \in \mathbf{2}^*$, $J([u])$ is an attractor.

Proposition 5.59. — *If a CA has an attracting fixed point which is a unique attractor, then it is equicontinuous.*

Proof. — Let a^∞ be the attracting fixed point, so $\omega(A^{\mathbb{Z}}) = \{a^\infty\}$. If $b \in A$, $b \neq a$, then $b \notin \mathcal{L}(\{a^\infty\})$, so for some $n > 0$, $b \notin \mathcal{L}(F^n(A^{\mathbb{Z}}))$. Taking the maximum of these n for all $b \in A \setminus \{a\}$, we get $F^n(A^{\mathbb{Z}}) = \{a^\infty\}$, so F is equicontinuous. □

Theorem 5.60. — *A surjective CA has either a unique attractor or a pair of disjoint attractors.*

Proof. — Consider the measure μ introduced in the proof of Theorem 5.23. If $u \in A^n$ and $k \in \mathbb{Z}$, then $\mu([u]_k) = (\#A)^{-n}$. If $U = V_1 \cup \cdots \cup V_m$ is a disjoint union of cylinders, put $\mu(U) = \mu(V_1) + \cdots + \mu(V_m)$. Then for any clopen set U, $\mu(F^{-1}(U)) = \mu(U)$.

Suppose that $(A^{\mathbb{Z}}, F)$ has at least two attractors. Then there exists a nonempty clopen invariant set $U \neq A^{\mathbb{Z}}$ and $U \subseteq F^{-1}F(U) \subseteq F^{-1}(U)$. The set $F^{-1}(U) \setminus U$ is clopen and if it were nonempty then it would have positive measure and this is impossible. Thus $F^{-1}(U) = U$ and $V = A^{\mathbb{Z}} \setminus U$ is a clopen invariant set too. Then $\omega(U)$ and $\omega(V)$ are disjoint attractors. □

The sum CA of Example 5.6 has a unique attractor and the identity CA has disjoint attractors. The traffic CA of Example 5.9 is an example of a nonsurjective CA with a unique transitive attractor. There exist also CA with a nontransitive unique attractor.

***Example 5.61* (U: a unique attractor, Gilman [61]).** — $(\mathbf{2}^{\mathbb{Z}}, U)$ where $U(x)_i = x_{i+1}x_{i+2}$, is sensitive and has a unique attractor which is not transitive.

FIGURE 28. U: A unique attractor

As in Example 5.5 we show

$$\omega(\mathbf{2}^{\mathbb{Z}}) = \{x \in \mathbf{2}^{\mathbb{Z}} : \forall n > 0, 10^n 1 \not\sqsubseteq x\}.$$

The subsystem $(\omega(\mathbf{2}^{\mathbb{Z}}), U)$ is not transitive. If $x \in [10]_0 \cap \omega(\mathbf{2}^{\mathbb{Z}})$, then $x_{[0,\infty)} = 10^\infty$, so for any $n > 0$, $F^n(x) \notin [11]_0$. The following ε-chains show that $\omega(\mathbf{2}^{\mathbb{Z}})$ is chain transitive

$$\begin{array}{l|l} 00000 & 11 \\ 00001 & 11 \\ 00011 & 11 \\ 00111 & 11 \\ 01111 & 11 \\ 11111 & \end{array} \qquad \begin{array}{l|l} 11111 & 0 \\ 11110 & 0 \\ 11000 & 0 \\ 00000 & 11 \\ 00001 & 10 \\ 00010 & \end{array} \qquad \begin{array}{l|l} 11111 & 0 \\ 11100 & 0 \\ 10000 & 11 \\ 00001 & 11 \\ 00011 & 10 \\ 00110 & \end{array}$$

To obtain a 2^{-n}-chain, we change the state arbitrarily outside the interval $[-n, n]$ (behind the bar). We see that any two words of length 5 can be joined by a 2^{-2}-chain. A similar argument also works for longer words.

Example 5.62 **(C: Coven CA)**. — $(\mathbf{2}^{\mathbb{Z}}, C)$ where

$$C(x)_i = \mathrm{mod}_2(x_i + x_{i+1}(x_{i+2} + 1))$$

is almost equicontinuous, has a unique attractor and does not have the shadowing property. It is left-closing but not open.

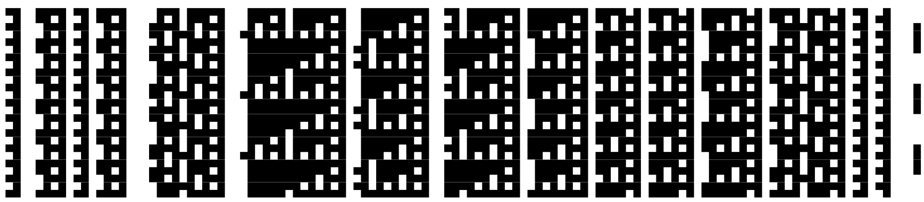


Figure 29. Coven CA

This remarkable example has been introduced in Coven and Hedlund [**40**] and studied in Blanchard and Maass [**24**]. The CA is left-closing since it is left-permutive. It is not right-closing, since it has not constant number of preimages.

$$F^{-1}(0^\infty) = \{0^\infty\}, \quad F^{-1}(1^\infty) = \{(01)^\infty, (10)^\infty\}.$$

To study its other properties, observe first that for any $a, b \in \mathbf{2}$, $f(1a1b) = 1c$ where $c = a + b + 1$ (here f is the local rule and the addition is modulo 2). Define a CA $(\mathbf{2}^{\mathbb{Z}}, G)$ by $G(x)_i = \mathrm{mod}_2(x_i + x_{i+1} + 1)$ and a map $\varphi : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ by

$$\varphi(x)_{2i} = 1, \;\; \varphi(x)_{2i+1} = x_i$$

Then $\varphi : (\mathbf{2}^{\mathbb{Z}}, G) \to (\mathbf{2}^{\mathbb{Z}}, C)$ is an injective morphism and $(\mathbf{2}^{\mathbb{Z}}, G)$ is a transitive subsystem of $(\mathbf{2}^{\mathbb{Z}}, C)$.

Lemma 3. — *For any $n \geq 0$,*

$$x_{[0,1]} = 10 \quad \Rightarrow \quad \forall i \leq n, C^i(x)_{[0,1]} \in \{10, 11\}.$$

Proof. — The statement clearly holds for $n = 0$. Assume that it holds for n and let us prove it for $n + 1$. Let $m > 0$ be the first integer for which $C^m(x)_{[0,1]} \neq 10$, so $C^m(x)_{[0,1]} = 11$ and $C^{m-1}(x)_{[0,3]} = 1010$. By the induction hypothesis

$$C^i(x)_{[2,3]} \in \{10, 11\}, \quad m - 1 \leq i \leq n$$

so $C^{n+1}(x)_{[0,1]} \in \{10, 11\}$. □

Lemma 4. — *The word* 000 *is a* 2*-blocking word with offset* 0.

Proof. — Let $x_{[0,2]} = 000$ and let $n > 0$ be the first integer such that $C^n(x)_{[0,2]} \neq 000$ so $C^n(x)_{[0,2]} = 001$ and $C^{n-1}(x)_{[0,4]} = 00010$. By Lemma 3, $C^m(x)_3 = 1$ for all $m \geq n$, so $C^k(x)_{[0,1]} = 00$ for all $k \geq 0$. □

We show now that for any $n > 0$ there exist $2^{-\lfloor \frac{n}{2} \rfloor}$-chains $1^{n+1} \to 01^n \to 1^{n+1}$. This follows from the fact that a transitive system $(\mathbf{2}^{\mathbb{Z}}, G)$ is a subsystem of $(\mathbf{2}^{\mathbb{Z}}, C)$. The chains differ slightly according to whether n is even or odd.

11111	10	01111	10	111111	0	011111	0
11110	0	01110	0	111101	11	011101	11
11010	10	01010	10	110101	0	010101	0
01111		11111		011111		111111	

Thus for any $n > 0$ and any $u \in \mathbf{2}^{2n+1}$ there exist 2^{-n}-chains $u \to 1^{2n+1} \to u$, so $(\mathbf{2}^{\mathbb{Z}}, C)$ is chain transitive. Since it is is almost equicontinuous, it is not transitive by Proposition 5.12 and Corollary 5.19. Since it is chain transitive, it does not have the shadowing property by Theorem 2.15. Its two-column factor subshift is

$$\Sigma_{[0,1]}(C) = \{10, 11\}^{\mathbb{N}} \cup \{11, 01\}^{\mathbb{N}} \cup \{01, 00\}^{\mathbb{N}}.$$

This follows from Lemma 3 and from the fact that $(\mathbf{2}^{\mathbb{Z}}, G)$ is a subsystem of $(\mathbf{2}^{\mathbb{Z}}, C)$. The factor subshift $\Sigma_{[0,1]}(C)$ is not SFT but it is sofic. Its graph is illustrated in Figure 30.

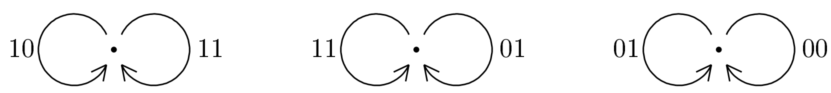


FIGURE 30. A factor subshift of the Coven CA

5.7. Factor subshifts

Definition 5.63. — A factor subshift of a CA is any one-sided subshift which is a factor of the CA in question.

In particular any column subshift (Definition 5.7) is a factor subshift. We show that any factor subshift is a factor of some column subshift.

Proposition 5.64. — *If* $\varphi : (A^{\mathbb{Z}}, F) \to (\Sigma, \sigma)$ *is a factor map where* $(A^{\mathbb{Z}}, F)$ *is a CA and* Σ *is a one-sided subshift, then there exists* $m > 0$ *such that* Σ *is a factor of* $\Sigma_{[-m,m]}(F)$.

Proof. — Since φ is uniformly continuous, there exists m and $h : A^{2m+1} \to A$ such that for any $x, y \in A^{\mathbb{Z}}$

$$x_{[-m,m]} = y_{[-m,m]} \quad \Rightarrow \quad \varphi(x)_0 = \varphi(y)_0 = h(x_{[-m,m]}).$$

Then for any $i \geq 0$ we have $\varphi(x)_i = \varphi(F^i(x))_0 = h(F^i(x)_{[-m,m]})$, so Σ is a factor of $\Sigma_{[-m,m]}(F)$. □

There are many CA, any of whose column subshifts is an SFT. This is the case for the product CA, sum CA, traffic CA, majority CA and the bijective CA of Example 5.37. Their one-column subshifts are

$$\begin{aligned}
\Sigma_0(P) &= \Sigma_{\{01\}} = \{1^n 0^\infty : n \geq 0\} \cup \{1^\infty\} \\
\Sigma_0(S) &= \mathbf{2}^{\mathbb{N}} \\
\Sigma_0(T) &= \mathbf{2}^{\mathbb{N}} \\
\Sigma_0(M) &= \Sigma_{\{001,110\}} \\
\Sigma_0(B) &= \{0,1,2,3\}^{\mathbb{N}} \cup \{4^\infty\}
\end{aligned}$$

In the Coven CA of Example 5.62 the one-column subshift $\Sigma_0(C) = \mathbf{2}^{\mathbb{N}}$ is also an SFT, but the two-culumn subshift is a sofic subshift which is not an SFT. It follows from the next theorem that any factor subshift of $(\mathbf{2}^{\mathbb{Z}}, C)$ is sofic.

***Proposition 5.65* (Blanchard and Maass [24]).** — *Let $(A^{\mathbb{Z}}, F)$ be a CA with memory $m = 0$ and local rule $f : A^{d+1} \to A$. If $\Sigma_{[0,d)}(F)$ is a sofic subshift, then any factor subshift of $(A^{\mathbb{Z}}, F)$ is sofic.*

Proof. — By Proposition 5.64, it suffices to show that for any $n \geq 0$, $\Sigma_{[0,n+d)}(F)$ is a sofic subshift. Let $G = (V, E, s, t, l)$ be a labelled graph such that $\Sigma_G = \Sigma_{[0,d)}(F)$. We construct a labelled graph G' of $\Sigma_{[0,n+d)}(F)$ as follows. Set $V' = V \cup (V \times A^{n+d})$. For any $u, v \in A^{n+d}$ such that $f(u) = v_{[0,n)}$ and for any labelled edge $q_0 \xrightarrow{v_{[n,n+d)}} q_1$ of G, we insert into G' labelled edges

$$q_0 \xrightarrow{v} (q_1, v), \quad (q_0, u) \xrightarrow{v} (q_1, v)$$

The vertices from $V \subset V'$ are initial. In G', no edges lead to them. We show that $\Sigma_{G'} = \Sigma_{[0,n+d]}(F)$. If

$$\xrightarrow{v_1} (q_1, v_1) \xrightarrow{v_2} (q_2, v_2) \xrightarrow{v_3} \cdots$$

is a path in G' whose first vertex is either q_0 or (q_0, v_0), then

$$q_0 \xrightarrow{(v_1)_{[n,n+d)}} q_1 \xrightarrow{(v_2)_{[n,n+d)}} q_2 \xrightarrow{(v_3)_{[n,n+d)}} \cdots$$

is a path in G, so there exists a point $y \in A^{\mathbb{Z}}$ such that $F^k(y)_{[n,n+d)} = (v_{k+1})_{[n,n+d)}$. Construct a point $x \in A^{\mathbb{Z}}$ such that $x_{[0,n)} = (v_1)_{[0,n)}$ and $x_{[n,\infty)} = y_{[n,\infty)}$. Then $F^k(x)_{[0,n+d)} = v_{k+1}$, so $v_1 v_2 \ldots \in \Sigma_{[0,n+d)}(F)$.
Conversely, if $v_1 v_2 \ldots \in \Sigma_{[0,n+d)}(F)$, then $(v_1)_{[n,n+d)}(v_2)_{[n,n+d)} \cdots \in \Sigma_{[n,n+d)}(F)$, so there exists a path

$$q_0 \xrightarrow{(v_1)_{[n,n+d)}} q_1 \xrightarrow{(v_2)_{[n,n+d)}} q_2 \xrightarrow{(v_3)_{[n,n+d)}} \cdots$$

in G and $q_0 \xrightarrow{v_1} (q_1, v_1) \xrightarrow{v_2} (q_2, v_2) \xrightarrow{v_3} \ldots$ is a path in G'. $\square$

Example 5.66. — The factor subshifts of the CA $(\mathbf{2}^{\mathbb{Z}}, U)$ given in the Example 5.61 are not sofic.

Proof. — If for some $x \in \mathbf{2}^{\mathbb{Z}}$, $F^n(x)_0 = 1$, then $F^{n-1}(x)_{[1,2]} = 11$ and $x_{[n,2n+1]} = 1^{n+1}$. If moreover $F^{n+1}(x)_0 = 0$, then either $x_{2n+2} = 0$ or $x_{2n+3} = 0$ and in both cases, $F^i(x)_0 = 0$ for $n < i \leq 2n+1$.

*110	*11
100	110
00	00
0	0

It follows that the factor subshift of the zeroth column is

$$\Sigma_0(U) = \{y \in \mathbf{2}^{\mathbb{N}} : \forall n \geq 0, (y_{[n,n+1]} = 10 \Rightarrow y_{[n,2n+1]} = 10^{n+1})\}$$

Assume that $G = (V, E, s, t, l)$ is a labelled graph whose subshift is $\Sigma_0(U)$, and $\#V = n$. Since $1^n 0^n 1 \in \mathcal{L}(\Sigma_0(U))$, there exists a path $u \in E^{2n+1}$ such that $l(u) = 1^n 0^n 1$. Since V has n elements, there exist $0 \leq i < j \leq n$ such that $s(u_i) = s(u_j)$. Let v be the path obtained from u by repeating $u_{[i,j)}$, so $v = u_{[0,j)} u_{[i,2n]}$. Then $l(v) = 1^{n+j-i} 0^n 1 \notin \mathcal{L}(\Sigma_0(U))$. This is a contradiction, so $\Sigma_0(U)$ is not sofic. $\square$

5.7.1. Shadowing property.

— If $(x_i)_{i \geq 0}$ is a 2^{-m}-chain in a CA $(A^{\mathbb{Z}}, F)$, then for all i, $F(x_i)_{[-m,m]} = (x_{i+1})_{[-m,m]}$, so $u_i = (x_i)_{[-m,m]}$ satisfy $F([u_i]_{-m}) \cap [u_{i+1}]_{-m} \neq \emptyset$. Conversely, if a sequence $(u_i \in A^{2m+1})_{i \geq 0}$ satisfies this property and $x_i \in [u_i]_{-m}$, then $(x_i)_{i \geq 0}$ is a 2^{-m}-chain. Some ε-chains can be seen in Example 5.61.

***Proposition 5.67* (Kůrka [88]).** — *Let $(A^{\mathbb{Z}}, F)$ be a CA. If $\Sigma_{[-n,n]}(F)$ is an SFT for any $n \geq 0$, then $(A^{\mathbb{Z}}, F)$ has the shadowing property.*

Proof. — For a given $\varepsilon = 2^{-n} \leq 2^{-r}$, let $m+1 > 1$ be the order of $\Sigma_{[-n,n]}(F)$. Set $k = n + rm$ and $\delta = 2^{-k}$. We show that any δ-chain is ε-shadowed by some point (Figure 31). Let $(x_i)_{i \geq 0}$ be a δ-chain, so $F(x_i)_{[-k,k]} = (x_{i+1})_{[-k,k]}$. We get

$$F^2(x_i)_{[-k+r,k-r]} = F(x_{i+1})_{[-k+r,k-r]} = (x_{i+2})_{[-k+r,k-r]}$$

and $\forall j \leq m, F^j(x_i)_{[-n,n]} = (x_{i+j})_{[-n,n]}$, so the sequence $((x_{i+j})_{[-n,n]})_{0 \leq j \leq m}$ of length $m+1$ belongs to $\mathcal{L}(\Sigma_{[-n,n]}(F))$. Since $\Sigma_{[-n,n]}(F)$ is an SFT of order $m+1$, the sequence $((x_i)_{[-n,n]})_{i \geq 0}$ belongs to $\mathcal{L}(\Sigma_{[-n,n]}(F))$, so there exists a point $x \in A^{\mathbb{Z}}$ such that for all $i \geq 0$, $F^i(x)_{[-n,n]} = (x_i)_{[-n,n]}$. Thus x ε-shadows the sequence $(x_i)_{i \geq 0}$. $\square$

Any positively expansive CA has the shadowing property since it is conjugate to an SFT. It is easy to see that the factor subshifts of the product CA and the majority CA are SFT, so these CA have the shadowing property. The following example shows that the converse of Proposition 5.67 is false. There exist CA with the shadowing property whose factor subshifts are not SFT.

***Example 5.68* (H: even factor subshift).** — The CA $(\mathbf{5}^{\mathbb{Z}}, H)$ given by the following table has the shadowing property but its factor subshift is not SFT.

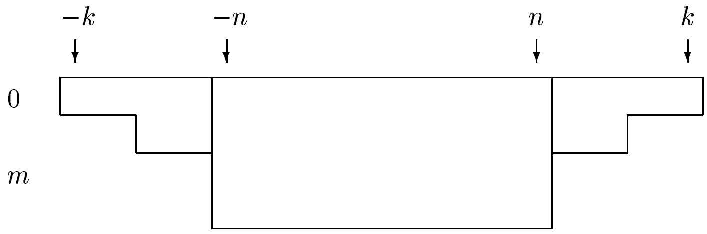


FIGURE 31. Shadowing property

02	12	13	14	2	34	3
1	1	1	0	3	4	2

The table specifies a local rule with diameter $d = 1$. At any position, the first applicable column is used, and when there is none, the letter is left unchanged. Sample trajectories are

024	0234	0224	02324
134	1344	1334	13234
144	1444	1244	12344
044	0444	1344	13444
044	0444	1444	14444
044	0444	0444	04444

We see that the one-column subshift $\Sigma_0(H) \subseteq \{0,1\}^{\mathbb{N}} \cup \{2,3,4\}^{\mathbb{N}}$ consists of two disjoint invariant subshifts.

$$\begin{aligned}\Theta_1 &= \Sigma_0(H) \cap \{2,3,4\}^{\mathbb{N}} = \Sigma_{\{22,24,33,42,43\}} \\ \Theta_0 &= \Sigma_0(H) \cap \mathbf{2}^{\mathbb{N}} = \{x \in \mathbf{2}^{\mathbb{N}} : \forall n \geq 0, 01^{2n+1}0 \not\sqsubseteq x\}\end{aligned}$$

The subshift Θ_1 is a countable SFT containing only eventually periodic points such as $(23)^n 4^\infty$ (4 is a blocking word). The subshift Θ_0 is the even subshift of Example 3.14. Thus $\Sigma_0(H)$ is a sofic subshift which is not SFT.

We show that the CA has the shadowing property. For $\varepsilon = 2^{-n}$, set $\delta = 2^{-n-1}$. Let $(x_i \in \mathbf{5}^{\mathbb{Z}})_{i \geq 0}$ be a δ-chain. If $(x_0)_{n+1} \in \{0,1\}$, then $(x_i)_n = (x_0)_n$ for all $i \geq 0$ and if $y \in [(x_0)_{[-n-1,n+1]}]$, then it ε-shadows $(x_i)_{i\geq 0}$. If $(x_0)_{n+1} \in \{2,3,4\}$, then $((x_i)_{n+1})_{i\geq 0} \in \Theta_1$ so there exists $y \in [(x_0)_{[-n-1,n+1]}$ which ε-shadows $(x_i)_{i\geq 0}$.

Proposition 5.69 **(Kůrka [88])**. — *If $(A^{\mathbb{Z}}, F)$ is a CA with the shadowing property, then any its factor subshift is sofic.*

Proof. — Let $(A^{\mathbb{Z}}, F)$ be a CA. We show that for any $n \geq 0$, $\Sigma_{[-n,n]}(F)$ is a sofic subshift. For $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that any δ-chain is ε-shadowed by some point. We can assume $k \geq n$. Set $B = A^{2k+1}$, $C = A^{2n+1}$ and define an SFT $\Sigma \subseteq B^{\mathbb{N}}$ of order 2 such that for $u, v \in B$,

$$uv \in \mathcal{L}(\Sigma) \quad \Leftrightarrow \quad F([u]_{-k}) \cap [v]_{-k} \neq \emptyset.$$

Define a map $\varphi : \Sigma \to C^{\mathbb{N}}$ by $\varphi(u)_i = (u_i)_{[k-n,k+n]}$. For $u \in \Sigma$ consider any sequence of points $(x_i \in A^{\mathbb{Z}})_{i\geq 0}$ such that $x_i \in [u_i]_{-k}$. Then $(x_i)_{i\geq 0}$ is a δ-chain so there exists a point $x \in A^{\mathbb{N}}$ which ε-shadows it. We get

$$F^i(x)_{[-n,n]} = (x_i)_{[-n,n]} = (u_i)_{[k-n,k+n]},$$

so $\varphi(u) \in \Sigma_{[-n,n]}(F)$. Thus $\varphi(\Sigma) \subseteq \Sigma_{[-n,n]}(F)$. Conversely, if $v \in \Sigma_{[-n,n]}(F)$, then there exists $x \in A^{\mathbb{Z}}$ such that $v_i = F^i(x)_{[-n,n]}$ and $v = \varphi(u)$, where $u_i = F^i(x)_{[-k,k]}$. Thus $\Sigma_{[-n,n]}(F) = \varphi(\Sigma)$ is sofic. ☐

The converse of Proposition 5.69 is false. The Coven CA of Example 5.62 does not have the shadowing property. Its subshift $\Sigma_{[0,1]}(C)$ is sofic, and by Proposition 5.65, any its factor subshift is sofic. The CA $(\mathbf{2}^{\mathbb{Z}}, U)$ of Example 5.61 does not have sofic factors, so it does not have the shadowing property.

5.7.2. Topological entropy. — The topological entropy of a CA can be calculated from its factor subshifts.

Proposition 5.70. — *If $(A^{\mathbb{Z}}, F)$ is a CA, then*

$$h(A^{\mathbb{Z}}, F) = \lim_{n\to\infty} h(\Sigma_{[-n,n]}(F), \sigma).$$

Proof. — For $n > 0$ let $\mathcal{V}_n = \{[u]_{-n} : u \in A^{2n+1}\}$. Then $|\mathcal{V}_n^k| = \#\mathcal{L}^k(\Sigma_{[-n,n]}(F))$, so $H(A^{\mathbb{Z}}, F, \mathcal{V}_n) = h(\Sigma_{[-n,n]}(F), \sigma)$ and the statement follows by Proposition 2.85. ☐

Proposition 5.71. — *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r. Then*

$$h(A^{\mathbb{Z}}, F) \leq 2 \cdot h(\Sigma_{[0,r)}(F), \sigma) \leq 2r \cdot h(\Sigma_0(F), \sigma) \leq 2r \cdot \ln \#A.$$

Proof. — Since $F^k(x)_{[-n,n]}$ depends only on $(F^j(x)_{[-n,-n+r)})_{j\leq k}$, $(F^j(x)_{(n-r,n]})_{j\leq k}$, and $x_{[-n+r,n-r]}$, we have

$$\#\mathcal{L}^k(\Sigma_{[-n,n]}(F)) \leq \#\mathcal{L}^k(\Sigma_{[-n,n+r)}(F)) \cdot \#\mathcal{L}^k(\Sigma_{(n-r,n])}(F)) \cdot (\#A)^{2(n-r)}$$

and $h(A^{\mathbb{Z}}, F) \leq 2 \cdot h(\Sigma_{[0,r)}(F), \sigma)$. Since $\#\mathcal{L}^k(\Sigma_{[0,r)}(F)) \leq (\#\mathcal{L}^k(\Sigma_0(F))^r$, we get that $h(\Sigma_{[0,r)}(F), \sigma) \leq r \cdot h(\Sigma_0(F), \sigma)$. ☐

Proposition 5.72. — *Let $(A^{\mathbb{Z}}, F)$ be a CA with nonnegative memory and anticipation, $0 \leq m \leq a$. Then*

$$h(A^{\mathbb{Z}}, F) = h(\Sigma_{[0,a)}(F), \sigma) \leq a \cdot h(\Sigma_0(F), \sigma).$$

Proof. — For any $n \geq a$, $F^k(x)_{[-n,n]}$ depends only on $x_{[-n,n]}$ and $(F^j(x)_{(n-a,n]})_{j<k}$. Thus

$$\#\mathcal{L}^k(\Sigma_{[-n,n]}(F)) \leq \#\mathcal{L}^k(\Sigma_{(n-a,n])}(F)) \cdot (\#A)^{2n+1}$$

and $h(A^{\mathbb{Z}}, F) \leq h(\Sigma_{[0,a)}(F), \sigma) \leq h(A^{\mathbb{Z}}, F)$. ☐

Proposition 5.73. — *Any equicontinuous CA has zero topological entropy*

Proof. — For any n, $\#\mathcal{L}^k(\Sigma_{[-n,n]}(F))$ is a bounded sequence by Proposition 5.16. Thus any $\Sigma_{[-n,n]}(F)$ has zero topological entropy. ☐

Example 5.74. — There exists an almost equicontinuous CA with positive topological entropy.

Proof. — The bijective CA $(A^{\mathbb{Z}}, B)$ of Example 5.37 is almost equicontinuous. Its restriction to the alphabet $\mathbf{4} = \{0,1,2,3\}$ is conjugate to $(\mathbf{2}^{\mathbb{Z}} \times \mathbf{2}^{\mathbb{Z}}, \sigma \times \sigma^{-1})$ with topological entropy $\ln 4$. □

Example 5.75. — There exists a sensitive CA with zero topological entropy.

Proof. — The CA $(\mathbf{2}^{\mathbb{Z}}, U)$ from Example 5.61 is sensitive. By Example 5.66, its one-column subshift is

$$\Sigma_0(U) = \{y \in \mathbf{2}^{\mathbb{N}} : \forall n \geq 0, (y_{[n,n+1]} = 10 \;\Rightarrow\; y_{[n,2n+1]} = 10^{n+1})\}$$

For any $m > 0$, any binary word $u \in \mathbf{2}^m$ can be encoded by a sequence $n_0, \dots, n_{k-1}$ of nonnegative integers

$$\begin{aligned} n_0 &= \min\{j \geq 0 : u_j = 1\} \\ n_{i+1} &= \min\{j > n_i : u_j \neq u_{n_i}\} \end{aligned}$$

The i-th element of the sequence is defined if its defining set is not empty. For example, for $u = 0^m$ we have $k = 0$. For $u = 1^m$ we have $k = 1$ and $n_0 = 0$. If $u \in \mathcal{L}^m(\Sigma_0(U))$, then for odd i we get $u_{[n_i-1,n_i]} = 10$, so $u_{[n_i,2n_i)} = 0^{n_i}$ and therefore $n_{i+1} \geq 2n_i$. It follows that $n_i \geq 2^{i/2}$ and $m \geq 2^{k/2}$. Since for any n_i there are at most m possibilities, the complexity function of the subshift satisfies

$$P(m) \leq 2m \cdot \log_2 m,$$

and $h(\Sigma_0(U), \sigma) = \lim_{m\to\infty} \dfrac{\ln P(m)}{m} = 0$. By Proposition 5.72, $h(\mathbf{2}^{\mathbb{Z}}, U) = 0$. □

5.8. Classification

We summarize now how dynamical and topological properties of cellular automata are interrelated. The relationships between various classes of cellular automata are conveniently depicted in Figures 32 and 33.

Corollary 5.76. —

(1) A positively expansive CA is open and transitive (Theorems 5.49, 5.50).

(2) A transitive CA is surjective and sensitive (Corollary 5.19).

(3) An open CA is closing (Theorem 5.45).

(4) A closing CA is surjective (Proposition 5.41).

(5) An open and almost equicontinuous CA is bijective (Proposition 5.36).

(6) A surjective equicontinuous CA is bijective (Proposition 5.25).

(7) A CA is either sensitive or almost equicontinuous (Proposition 5.12).

(8) A CA with an attracting fixed point is almost equicontinuous (Theorems 5.53 and 2.74).

(9) A CA whose unique attractor is an attracting fixed point is equicontinuous (Proposition 5.59).

(10) An equicontinuous CA has either two disjoint attractors or an attracting fixed point which is the unique attractor (Proposition 5.57).

(11) *A surjective CA has either a unique attractor or a pair of disjoint attractors (Theorem 5.60).*

(12) *A CA has either two disjoint attractors or a unique minimal attractor or a unique minimal quasi-attractor (Corollary 5.55)*

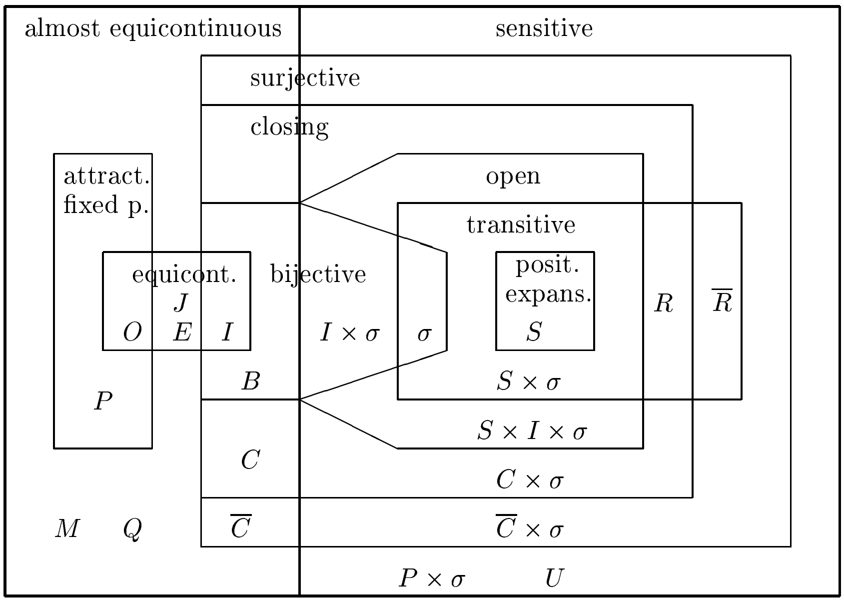


FIGURE 32. Equicontinuity classes

The nonemptiness of various CA classes is shown by the following examples:

Corollary 5.77. —

P: The product CA of Example 5.5 has an attracting fixed point, it is almost equicontinuous but not equicontinuous.

M: The majority CA of Example 5.10 is almost equicontinuous but not equicontinuous. It has disjoint attractors.

S: The sum CA of Example 5.6 is positively expansive.

I: The identity CA of Example 5.13 is equicontinuous, bijective and has disjoint attractors.

O: The zero CA of Example 5.14 is equicontinuous and has an attracting fixed point.

E: Example 5.17 is equicontinuous, not surjective and has disjoint attractors.

R: Example 5.29 is right-closing and transitive but not left-closing, so it is not open.

B: Example 5.37 is bijective and almost equicontinuous but not equicontinuous. It has disjoint attractors.

J: Example 5.58 has disjoint attractors, it is equicontinuous and not surjective.

U: Example 5.61 is sensitive and has a unique attractor. It is not surjective, so it is not chain transitive.

C: The Coven CA of Example 5.62 is closing and almost equicontinuous, has a unique attractor, it is surjective but not transitive.

$\overline{C} = C \times C_d$: *is almost equicontinuous and surjective but not closing.*
σ: *The shift CA is bijective and transitive.*
$\overline{R} = R \times R_d$: *is transitive and not closing.*
$C \times \sigma$: *is sensitive and closing but not open.*
$S \times \sigma$: *is transitive but not positively expansive.*
$I \times \sigma$: *is bijective and sensitive but not transitive.*
Q: *Example 5.56 has a quasi-attractor.*
$P \times \sigma$: *has a minimal attractor but neither an attracting fixed point nor unique attractor.*

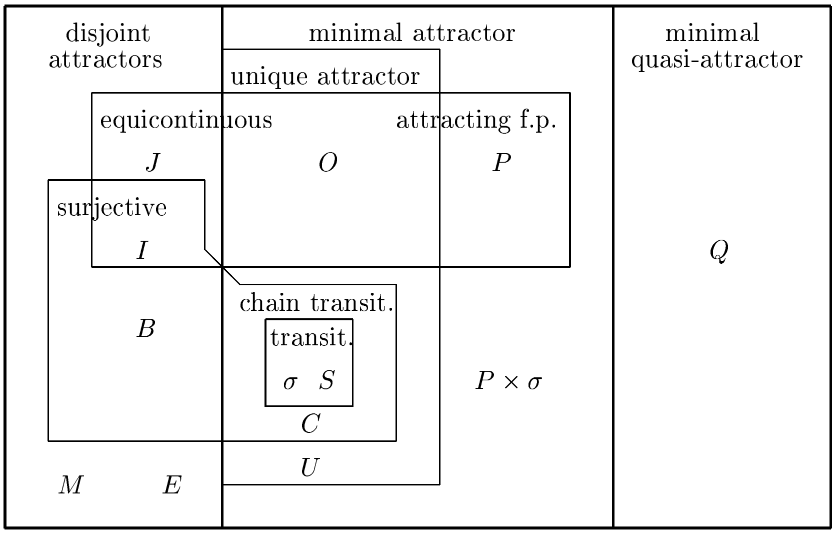


FIGURE 33. Attractor classes

Exercise 5.1. — Determine topological properties of the following CA:

1. Multiplication by two in base six: $(\mathbf{6}^{\mathbb{Z}}, F)$, where
$$F(x)_i = 2x_i + \mathrm{mod}_6\left(\left\lfloor \frac{x_{i+1}}{3} \right\rfloor\right).$$
2. Redundant coding $(\mathbf{3}^{\mathbb{Z}}, F)$ which preserves $\varphi(x) = \sum_{i=-\infty}^{i=\infty} x_i 3^{-i}$.

00	01	02	10	11	12	20	21	22
0	0	1	1	1	2	0	0	1

3. Addition of one in the redundant positional system (see page 104)

00	01	02	10	11	12	20	21	22
0	0	1	1	1	2	1	0	1