APPENDIX A

SETS, SPACES AND NUMBERS

In this appendix we present some mathematical concepts and results which are not directly connected to dynamics but which provide the basis on which topological and symbolic dynamics is built. We start with basic theory of compact metric spaces and continuous mappings, introduce residual sets and prove the Baire category theorem. Then we introduce connected sets, connected components and totally disconnected spaces on which symbolic dynamics is built. For the spectral theory of dynamical systems we need the space of continuous complex functions, in particular the Stone-Weierstrass theorem. Some of the constructions used in topological dynamics such as the factor space require us to leave the category of compact metric spaces and work instead with the larger categories of compact Hausdorff topological spaces and uniform spaces. To be sure the spaces thus constructed are in fact metric spaces, we give some results about metrizable topological spaces. A section is devoted to the Perron-Frobenius theory of nonnegative matrices which is useful both for the entropy calculations of sofic subshifts and for the frequency calculations in substitutive subshifts. Finally we treat continued fractions and their usefulness for approximation of irrational numbers.

A.1. Sets and relations

We denote the empty set by $\emptyset$. The intersection, union and set difference of sets X, Y are $X \cap Y$, $X \cup Y$ and $X \setminus Y = \{z \in X : z \notin Y\}$ respectively. The ordered pair of x, y is (x, y) and the cartesian product of X and Y is $X \times Y = \{(x, y) : x \in X, y \in Y\}$. The disjoint union of X, Y is $X \uplus Y = (X \times \{0\}) \cup (Y \times \{1\})$. These operations are also applied to families of sets $(X_i)_{i \in I}$ indexed by possibly infinite index set I. In particular the product and disjoint union are defined by

$$x \in \prod_{i \in I} X_i \quad \Leftrightarrow \quad x = (x_i)_{i \in I}, \forall i \in I, x_i \in X_i$$

$$(x, i) \in \biguplus_{i \in I} X_i \quad \Leftrightarrow \quad i \in I, x \in X_i$$

In particular if $X_i = X$ we get the power set $X^I = \prod_{i \in I} X$.

A **relation** is a subset of a cartesian product $R \subseteq X \times Y$. The image of a set $A \subseteq X$ or a point $x \in X$ is

$$R(A) = \{y \in Y : \exists x \in A, (x, y) \in R\}, \quad R(x) = R(\{x\}) = \{y \in Y : (x, y) \in R\}.$$

The **inverse relation** of $R \subseteq X \times Y$ is

$$R^{-1} = \{(y, x) \in Y \times X : (x, y) \in R\}.$$

The **composition of relations** $R \subseteq X \times Y$ with $S \subseteq Y \times Z$ is

$$S \circ R = \{(x, z) \in X \times Z : \exists y \in Y, (x, y) \in R, (y, z) \in S\}$$

We say that $f : X \to Y$ is a map from X to Y if $f \subseteq X \times Y$ and for any $x \in X$ there exists exactly one $y \in Y$ with $(x, y) \in f$. We write $y = f(x)$ in this case. A map $f : X \to Y$ is **injective**, if $f(x_0) \neq f(x_1)$ for $x_0 \neq x_1 \in X$. It is **surjective**, if for any $y \in Y$ there exists $x \in X$ with $y = f(x)$. Finally $f : X \to Y$ is **bijective** if it is both injective and surjective.

A relation on a set X is a subset $R \subseteq X \times X$. The **diagonal of a set** X and the **diagonal of a relation** $R \subseteq X \times X$ are

$$\Delta_X = \{(x, x) : x \in X\}, \quad |R| = \{x \in X : (x, x) \in R\}.$$

Definition A.1. — Let $R \subseteq X \times X$ be a relation on X.
(1) R is reflexive if $\Delta_X \subseteq R$.
(2) R is symmetric if $R^{-1} = R$.
(3) R is antisymmetric if $R \cap R^{-1} = \Delta_X$.
(4) R is transitive if $R \circ R \subseteq R$.

A relation $R \subseteq X \times X$ is an **equivalence** if it is reflexive, symmetric and transitive. A set $A \subseteq X$ is an **equivalence class** of R if

$$R(A) = A, \quad A \times A \subseteq R.$$

Distinct equivalence classes are disjoint and for any $x \in X$ there is a unique equivalence class $\pi_R(x)$ which contains it. We denote by X/R the **factor set** of equivalence classes, so $\pi_R : X \to X/R$ is a surjective **projection** map. Conversely if $f : X \to Y$ is a surjective map, then

$$R = \{(x_0, x_1) \in X \times X : f(x_0) = f(x_1)\}$$

is an equivalence and there exists a unique bijective $h : X/R \to Y$ with $h \circ \pi_R = f$.

A relation $R \subseteq X \times X$ is an **order**, if it is antisymmetric and transitive. We say that R is a **linear order**, if $R \cup R^{-1} = X \times X$. The pair (X, R) is then called an **ordered set** or a **linearly ordered set**. We write $x \leq y$ if $(x, y) \in R$.

A linear order is called a **well order**, if any nonempty $A \subseteq X$ has a least element i.e., $\exists x \in A, \forall y \in A, x \leq y$. The Axiom of choice postulates that any set can be well ordered. An equivalent statement is Zorn's lemma.

***Theorem A.2* (Zorn's lemma)**. — *Let $(X, \leq)$ be an ordered set and assume that any nonempty subset $A \subseteq X$ which is linearly ordered has a lower bound, i.e.,*

$$\exists x \in X, \forall y \in A, x \leq y.$$

Then for any $x \in X$ there exists a minimal element $y \leq x$, i.e., $\forall z \in X, z \not< y$.

A.2. Metric spaces

Definition A.3. — A **metric space** is a pair (X, d), where X is a nonempty set, and $d : X \times X \to [0, +\infty)$ is a function called the **metric**, or the **distance** which is positive for distinct points, symmetric and satisfies the triangle inequality:
(1) $d(x, y) = 0 \Leftrightarrow x = y$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) + d(y, z) \geq d(x, z)$.

On any set X there exists a **discrete metric** defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

The set of real numbers $\mathbb{R}$ is a metric space with the metric $d(x, y) = |x - y|$. The n-dimensional Euclidean space $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ is equipped with the metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

The circle $(\mathbb{T}, d)$ is a metric space $\mathbb{T} = [0, 1)$ with the metric

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

For a point $x \in X$ of a metric space (X, d), and $r > 0$, we denote by

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

the **open ball** with center x and radius r. We say that a set $U \subseteq X$ is a **neighbourhood** of a point $x \in X$, or that x is an **inner point** of U, if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. The set of inner points of U is its **interior** U°. The dual concept is the **closure** $\overline{U}$ of a set. The interior and closure of a set $Y = (0, 1) \cup (1, 2] \cup \{3\}$ (in the space $\mathbb{R}$) are depicted in Figure 1.

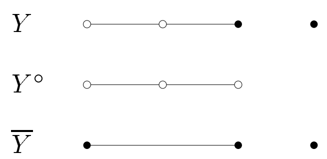

FIGURE 1. Topological operations

Definition A.4. — Let X be a metric space and $Y \subseteq X$.
(1) $Y^\circ = \{x \in X : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq Y\}$ is the **interior** of Y.
(2) $\overline{Y} = \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap Y \neq \emptyset\}$ is the **closure** of Y.
(3) Y is **open**, if $Y^\circ = Y$.
(4) Y is **closed**, if $\overline{Y} = Y$.
(5) A point $x \in X$ is **isolated**, if $\{x\}$ is an open set.

Proposition A.5. — *Let X be a metric space and $Y, Z \subseteq X$ subsets of X.*
(1) For any $x \in X$, $\varepsilon > 0$, $B_\varepsilon(x)$ is an open set.
(2) $Y^\circ \subseteq Y \subseteq \overline{Y}$.
(3) $\overline{X \setminus Y} = X \setminus Y^\circ$.

(4) Y is open iff $X \setminus Y$ is closed.
(5) If $Y \subseteq Z$ then $Y^\circ \subseteq Z^\circ$ and $\overline{Y} \subseteq \overline{Z}$.

Proof. —
(1) If $y \in B_\varepsilon(x)$ and $d(y,x) = \delta < \varepsilon$, then $B_{\varepsilon-\delta}(y) \subseteq B_\varepsilon(x)$.
(2) If $y \in Y^\circ$ then for some $\varepsilon > 0$, $y \in B_\varepsilon(y) \subseteq Y$, so $Y^\circ \subseteq Y$. If $y \in Y$ then for any $\varepsilon > 0$, $y \in B_\varepsilon(y) \cap Y$, so $Y \subseteq \overline{Y}$.
(3) $x \in \overline{X \setminus Y}$ iff for any $\varepsilon > 0$, $B_\varepsilon(x) \not\subseteq Y$ iff $x \notin Y^\circ$.
(4) follows from (3) and (5) is clear. □

Proposition A.6. — *Let X be a metric space.*
(1) $\emptyset$ and X are open sets.
(2) The intersection of two open sets is open.
(3) The union of any system of open sets is open.
(4) $\emptyset$ and X are closed sets.
(5) The union of two closed set is closed.
(6) The intersection of any system of closed sets is closed.

Proof. —
(1) is clear.
(2) If $Y_1, Y_2 \subseteq X$ are open and $x \in Y_1 \cap Y_2$ then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\varepsilon_i}(x) \subseteq Y_i$. For $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ we have $B_\varepsilon(x) \subseteq Y_1 \cap Y_2$, so $Y_1 \cap Y_2$ is open.
(3) If $x \in U = \bigcup_{i \in I} U_i$, where U_i are open sets, then for some $j \in I$, $x \in U_j$ and there exists $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq U_j \subseteq U$, so U is open.
(4),(5),(6) follow from (1),(2),(3) by duality. □

Definition A.7. — Let X be a metric space. A sequence of points $(x_n \in X)_{n \geq 0}$ **converges** to a point $x \in X$, or x is the **limit** of the sequence $(x_n)_{n \geq 0}$, if

$$\forall \varepsilon > 0, \exists n \geq 0, \forall m \geq n, d(x_m, x) < \varepsilon.$$

We write $\lim\limits_{n \to \infty} x_n = x$, or $x_n \to x$ as $n \to \infty$. We say that $(x_n)_{n \geq 0}$ is a **convergent** sequence, if it converges to some point $x \in X$.

The **topology** of a metric space is its family of open sets

$$\mathcal{T} = \{U \subseteq X : U \text{ is open}\}.$$

The properties which can be defined in terms of open sets are called **topological**. These include the operations of interior, closure and limit:

$$\begin{aligned} x \in Y^\circ \quad & \text{iff} \quad \exists U \in \mathcal{T}, x \in U \subseteq Y \\ x \in \overline{Y} \quad & \text{iff} \quad \forall U \in \mathcal{T}, (x \in U \Rightarrow U \cap Y \neq \emptyset) \\ \lim_{n \to \infty} x_n = x \quad & \text{iff} \quad \forall U \in \mathcal{T}, (x \in U \Rightarrow \exists n \geq 0, \forall m \geq n, x_m \in U) \end{aligned}$$

A topology can be also given by the convergence structure. If we know which sequences are convergent and if we know their limits, we know also which sets are open or closed:

Proposition A.8. — *A set Y of a metric space X is closed iff the limit of any convergent sequence of points of Y belongs to Y, i.e., if*

$$\forall y \in Y^{\mathbb{N}}, \forall x \in X, \left(\lim_{n\to\infty} y_n = x \;\Rightarrow\; x \in Y\right)$$

Proof. — Let Y be closed, $y_n \in Y$ and $\lim_{n\to\infty} y_n = x$. Assume that $x \notin Y$. Since $X \setminus Y$ is open, there exists n such that for all $m \geq n$, $y_m \in X \setminus Y$. This is a contradiction. Conversely assume that Y is not closed and let $x \in \overline{Y} \setminus Y$. For any $n > 0$, $B_{1/n}(x) \cap Y \neq \emptyset$, so there exists $y_n \in Y$ such that $d(y_n, x) < \frac{1}{n}$. Thus $\lim_{n\to\infty} y_n = x \notin Y$. □

Definition A.9. —

(1) A metric space (Y, d_Y) is a **subspace** of a metric space (X, d_X) if $Y \subseteq X$ and d_Y coincides with d_X on Y.

(2) If $(X_1, d_1), \ldots, (X_n, d_n)$ are metric spaces, their **product** is the metric space (X, d) where $X = X_1 \times \cdots \times X_n$, and for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$,

$$d(x, y) = \max\{d_i(x_i, y_i) : 1 \leq i \leq n\}$$

Any subset $Y \subseteq X$ of a metric space X gives rise to a subspace of X. The Euclidean metric on the space $\mathbb{R}^n$ does not coincide with the product metric (The balls in the product metric are n-dimensional cubes in $\mathbb{R}^n$). However, these two metrics are **equivalent**, i.e., they give rise to the same topology. The subspace and product can be topologically characterized.

Proposition A.10. —

(1) If $Y \subseteq X$ is a subspace of X, then a set $U \subseteq Y$ is open in Y iff there exists a set $V \subseteq X$ open in X such that $U = Y \cap V$. Thus $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\}$.

(2) If $X = X_1 \times X_2$ is a product space, then

$$\mathcal{T}_X = \{U \subseteq X : \forall x \in U, \exists U_1 \in \mathcal{T}_{X_1}, \exists U_2 \in \mathcal{T}_{X_2}, x \in U_1 \times U_2 \subseteq U\}.$$

Proof. —

(1) The ball of $y \in Y$ in Y is just the intersection of its ball in X with Y, $B^Y_\varepsilon(y) = B^X_\varepsilon(y) \cap Y$. If $U \in \mathcal{T}_X$, then clearly $U \cap Y \in \mathcal{T}_Y$. Conversely let $V \in \mathcal{T}_Y$ and set

$$U = \bigcup \{B^X_\varepsilon(y) : y \in B^Y_\varepsilon(y) \subseteq V\}.$$

Then $V = U \cap Y$.

(2) For $x = (x_1, x_2) \in X$ we have $B^X_\varepsilon(x) = B^{X_1}_\varepsilon(x_1) \times B^{X_2}_\varepsilon(x_2)$. If $x \in U \in \mathcal{T}_X$ then $x \in U_1 \times U_2 \subseteq U$, where $U_i = B^{X_i}_\varepsilon(x_i)$. Conversely, if $x \in U_1 \times U_2 \subseteq U$, then there exists $\varepsilon > 0$ such that $x_i \in B^{X_i}_\varepsilon(x_i) \subseteq U_i$. □

Definition A.11. — Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ a map.

(1) f is continuous at a point $x \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x' \in X, (d_X(x, x') < \delta \;\Rightarrow\; d_Y(f(x), f(x')) < \varepsilon).$$

(2) f is **continuous**, if it is continuous at every point $x \in X$.

(3) f is **uniformly continuous**, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, x' \in X, (d_X(x, x') < \delta \;\Rightarrow\; d_Y(f(x), f(x')) < \varepsilon).$$

(4) f is a **homeomorphism**, if it is bijective and both f and f^{-1} are continuous.
(5) X and Y are **homeomorphic**, if there exists a homeomorphism between them.
(6) f is an **isometry**, if $d_Y(f(x), f(x')) = d_X(x, x')$.

For example the map $f(x) = x^2$ on $\mathbb{R}$ is continuous but not uniformly continuous. Any open interval is homeomorphic to $\mathbb{R}$. For example $\tan x$ is a homeomorphism between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$. Continuity is a topological concept. It can be defined in terms of open sets.

Proposition A.12. — *Let $f : X \to Y$ be a map between metric spaces, and $x \in X$. The following conditions are equivalent.*
(1) f is continuous at x.
(2) For any neighbourhood V of $f(x)$, $f^{-1}(V)$ is a neighbourhood of x.
(3) For any sequence $(x_n \in X)_{n\geq 0}$, $\lim\limits_{n\to\infty} x_n = x$ implies $\lim\limits_{n\to\infty} f(x_n) = f(x)$.

Proof. —
(1) ⇒ (2): Let V be a neighbourhood of $f(x)$ so there exists $\varepsilon > 0$ with $B_\varepsilon(f(x)) \subseteq V$. Since f is continuous at x, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$, so $B_\delta(x) \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is a neighbourhood of x.
(2) ⇒ (3): Let $\lim_{n\to\infty} x_n = x$ and let V be a neighbourhood of $f(x)$. Since $f^{-1}(V)$ is a neighbourhood of x, for all n large enough, $x_n \in f^{-1}(V)$, so $f(x_n) \in V$, and $\lim_{n\to\infty} f(x_n) = f(x)$.
(3) ⇒ (1): Suppose that f is not continuous at x. There exists $\varepsilon > 0$ such that for any $\delta = \frac{1}{n}$ there exists x_n with $d(x_n, x) < \frac{1}{n}$ and $d(f(x_n), x) \geq \varepsilon$. Then $\lim_{n\to\infty} x_n = x$ but $(f(x_n))_{n\geq 1}$ does not converge to $f(x)$. □

Proposition A.13. — *Let $f : X \to Y$ be a map between metric spaces. The following conditions are equivalent.*
(1) f is continuous.
(2) For any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
(3) For any closed set $V \subseteq Y$, $f^{-1}(V)$ is closed in X.
(4) For any set $V \subseteq X$, $f(\overline{V}) \subseteq \overline{f(V)}$.

Proof. —
(1) ⇒ (2): Let $V \subseteq Y$ be open and $x \in f^{-1}(V)$. Since V is a neighbourhood of $f(x)$, $f^{-1}(V)$ is a neighbourhood of x by Proposition A.12, so $f^{-1}(V)$ is open.
(2) ⇒ (3): If $V \subseteq Y$ is closed, $Y \setminus V$ is open in Y and $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is open in X, so $f^{-1}(V) = X \setminus f^{-1}(\overline{Y \setminus V})$ is closed.
(3) ⇒ (4): By assumption, $f^{-1}(\overline{f(V)})$ is closed since $\overline{f(V)}$ is closed. It follows that

$$V \subseteq f^{-1}(f(V)) \subseteq f^{-1}(\overline{f(V)}) \ \Rightarrow\ \overline{V} \subseteq f^{-1}(\overline{f(V)}) \ \Rightarrow\ f(\overline{V}) \subseteq \overline{f(V)}$$

(4) ⇒ (1): Suppose that V is a neighbourhood of $f(x)$, so $f(x) \in V^\circ$, but $f^{-1}(V)$ is not a neighborhood of x. Then $x \notin f^{-1}(V)^\circ$, so $x \in \overline{X \setminus f^{-1}(V)}$ and

$$f(x) \in f(\overline{X \setminus f^{-1}(V)}) \subseteq \overline{f(X \setminus f^{-1}(V))} \subseteq \overline{Y \setminus V} = Y \setminus V^\circ$$

and this is a contradiction. □

A.3. Compact spaces

A **subsequence** of a sequence $(x_n)_{n\geq 0}$ is any sequence $(x_{n_i})_{i\geq 0}$, where $(n_i)_{i\geq 0}$ is an increasing sequence of indices.

Definition A.14. —
(1) A metric space X is **compact**, if every sequence of its points has a convergent subsequence.
(2) A metric space is **bounded**, if there exists $x \in X$ and $r > 0$ such that $X \subseteq B_r(x)$.
(3) A subset Y of a metric space X is compact (bounded), if it is compact (bounded) as a subspace.

The real line $\mathbb{R}$ is neither bounded nor compact. The sequence $x_n = n$ does not have any convergent subsequence. The open interval $(0, 1)$ is bounded but not compact. The sequence $x_n = \frac{1}{n}$ does not have any convergent subsequence in $(0, 1)$. Any closed bounded interval $[a, b] \subseteq \mathbb{R}$ is compact. This is a consequence of the Bolzano-Weierstrass theorem.

Theorem A.15 **(Bolzano-Weierstrass theorem).** — *Any bounded sequence of real numbers has a convergent subsequence.*

Proof. — Let $(x_n)_{n\geq 0}$ be a sequence of real numbers such that for all n, $a \leq x_n \leq b$. Construct sequences $(a_i)_{i\geq 0}$, $(b_i)_{i\geq 0}$ as follows. Put $a_0 = a$, $b_0 = b$. If a_i, b_i have been already constructed, set $c_i = (a_i + b_i)/2$, and

$$\begin{aligned} a_{i+1} &= a_i, \quad b_{i+1} = c_i \quad \text{if } \{n \in \mathbb{N} : x_n \in [a_i, c_i]\} \text{ is infinite,} \\ a_{i+1} &= c_i, \quad b_{i+1} = b_i \quad \text{otherwise.} \end{aligned}$$

Then for any i, the interval $[a_i, b_i]$ contains an infinite number of x_n. Since $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ are monotone and since $b_i - a_i \leq (b-a)2^{-i}$, these sequences have common limit $\lim_{i\to\infty} a_i = x = \lim_{i\to\infty} b_i$. Set $n_0 = 0$. For any k there exists $n_k > n_{k-1}$ such that $x_{n_k} \in [a_k, b_k]$. It follows $\lim_{k\to\infty} x_{n_k} = x$. □

Proposition A.16. —
(1) A compact subset of a metric space is closed and bounded.
(2) A closed subset of a compact metric space is compact.
(3) The product of compact metric spaces is compact.
(4) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof. —
(1) Let $Y \subseteq X$ be compact and let $(x_n \in Y)_{n\geq 0}$ be a sequence which converges to some $x \in X$. Since Y is compact, $(x_n)_{n\geq 0}$ has a convergent subsequence, $\lim_{k\to\infty} x_{n_k} = y \in Y$. But then $y = x$ so Y is closed. Suppose that Y is not bounded. Pick an arbitrary $x_0 \in X$. For $n > 0$, $X \not\subseteq B_n(x_0)$, so there exists x_n with $d(x_0, x_n) > n$. If $(x_{n_i})_{i\geq 0}$ were a convergent subsequence of $(x_n)_{n\geq 0}$, and $x_{n_i} \to x$ as $i \to \infty$, then for any $n \geq 0$, $d(x, x_0) > n$. This is a contradiction.

(2) Let $Y \subseteq X$ be a closed subset of a compact space X and $(x_n \in Y)_{n\geq 0}$ a sequence. There exists a subsequence which converges in X, $\lim_{k\to\infty} x_{n_k} = x \in X$. Since Y is closed, $x \in Y$.

(3) Let Y and Z be compact spaces and $X = Y \times Z$. Let $(x_n = (y_n, z_n))_{n\geq 0}$ be a sequence in X. Since Y is compact, there exists a converging subsequence $y_{n_k} \to y \in Y$ as $k \to \infty$. Since Z is compact, the sequence $(z_{n_k})_{k\geq 0} \in Z$ has a converging subsequence $z_{n_{k_i}} \to z$ as $i \to \infty$. Thus $x_{n_{k_i}} \to x = (y, z)$ is a converging subsequence in X.

(4) By (1), any compact subset of $\mathbb{R}^n$ is closed and bounded. We prove the converse first for $n = 1$. Let $X \subseteq \mathbb{R}$ be closed and bounded and $x_n \in X$ for any $n \geq 0$. By Bolzano-Weierstrass theorem, $(x_n)_{n\geq 0}$ has a subsequence which converges in $\mathbb{R}$ to some $x \in \mathbb{R}$. Since X is closed, $x \in X$, so X is compact. If $n > 1$ and $X \subseteq \mathbb{R}^n$ is closed and bounded, then there exists $R > 0$ such that $X \subseteq [-R, R]^n$. Since $[-R, R]$ is compact $[-R, R]^n$ is compact by 3, and since $X \subseteq [-R, R]^n$ is closed, it is compact by (2). □

In particular, the circle $\mathbb{T}$ is compact since it is homeomorphic to a closed and bounded subspace of $\mathbb{R}^2$. The n-dimensional torus $\mathbb{T}^n$ is compact since it is a product of compact spaces. In compact metric spaces we can form even (countably) infinite products of spaces. Assume that $(X_i)_{i\geq 0}$ is a sequence of compact spaces. Since compact spaces are bounded, we can normalize the metrics so that $d_i(y, z) \leq 1$ for any $y, z \in X_i$.

Definition A.17. — Let $(X_i, d_i)_{i\geq 0}$ be a bounded sequence of metric spaces, i.e., $d_i(y, z) \leq 1$ for any $y, z \in X_i$.

(1) The product metric d on $X = \prod_{i=0}^{\infty} X_i$ is defined by

$$d(y, z) = \sum_{i=0}^{\infty} d_i(y_i, z_i) \cdot 2^{-i}.$$

(2) The sum metric on $X = \biguplus_{i=0}^{\infty} X_i$ is defined by

$$d(y, z) = \begin{cases} d_i(x, y) & \text{if } y, z \in X_i \\ 1 & \text{if } y \in X_i, z \in X_j, i \neq j \end{cases}$$

The product and sum can be topologically characterized.

Proposition A.18. — *Let $(X_i, d_i)_{i\geq 0}$ be a bounded sequence of metric spaces.*

(1) A space X is homeomorphic to $\prod_{i=0}^{\infty} X_i$ iff there exist continuous surjections $\pi_i : X \to X_i$ such that for any metric space Y and any sequence $f_i : Y \to X_i$ of continuous maps there exists a unique continuous map $f : Y \to X$ with $\pi_i \circ f = f_i$.

(2) A space X is homeomorphic to $\biguplus_{i=0}^{\infty} X_i$ iff there exist continuoous injections $\nu_i : X_i \to X$ such that for any metric space Y and any sequence $f_i : X_i \to Y$ of continuous maps there exists a unique continuous map $f : X \to Y$ with $f \circ \pi_i = f_i$.

$$\begin{tikzcd} & X \arrow[r, "\pi_i"] & X_i \\ Y \arrow[ur, "\exists! f"] \arrow[urr, "f_i"'] & & \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r, "\nu_i"] \arrow[drr, "f_i"'] & X \arrow[dr, "\exists! f"] & \\ & & Y \end{tikzcd}$$

Proof. — Set $X = \prod_{i=0}^{\infty} X_i$. The projections $\pi_i : X \to X_i$ defined by $\pi_i(x) = x_i$ are continuous as

$$d(y,z) < \delta \;\Rightarrow\; d_i(y_i, z_i) < \delta \cdot 2^i.$$

Let $f_i : Y \to X_i$ be a sequence of continuous maps defined on a metric space Y. Then there exists a unique map $f : Y \to X$ such that for any i and $y \in Y$, $f(y)_i = f_i(y)$. We show that f is continuous. Let $y \in Y$ and $\varepsilon > 0$ be given. For any i there exists δ_i such that for any $y, z \in Y$

$$d(y,z) < \delta_i \;\Rightarrow\; d_i(f_i(y), f_i(z)) < \frac{\varepsilon}{4}.$$

If $n \geq 0$ is such that $2^{-n+1} < \frac{\varepsilon}{2}$, set $\delta = \min\{\delta_0, \ldots, \delta_{n-1}\}$. Then

$$d(y,z) < \delta \;\Rightarrow\; d(f(y), f(z)) < \sum_{i=0}^{n-1} \frac{\varepsilon}{4} \cdot 2^{-i} + \sum_{i=n}^{\infty} 2^{-i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

If X' is another space with projections $\pi'_i : X' \to X_i$ which satisfy the assumptions, then there exists a unique continuous $f : X' \to X$ with $\pi_i \circ f = \pi'_i$ and a unique continuous $f' : X \to X'$ with $\pi'_i \circ f' = \pi_i$. It follows that f, f' are inverse to one another. The proof for the disjoint sum is analogous. □

Proposition A.19. —

(1) The product of a bounded sequence of compact spaces is compact.

(2) The sum of a finite number of compact spaces is compact.

Proof. —

(1) Let $(x_n)_{n\geq 0}$ be a sequence of points in $X = \prod_{i=0}^{\infty} X_i$. We denote by $(x_n)_i \in X_i$ the i-th component of the n-th point. Since X_0 is compact, there exists an increasing sequence $(n_{k,0})_{k\geq 0}$ such that $(x_{n_{k,0}})_0 \to y_0 \in X_0$ as $k \to \infty$. Since X_1 is compact, we can choose a subsequence $(n_{k,1})_{k\geq 0}$ of $(n_{k,0})_{k\geq 0}$ such that $n_{0,1} > n_{0,0}$ and $(x_{n_{k,1}})_1 \to y_1 \in X_1$ as $k \to \infty$. We continue in this way so that $(n_{k,m+1})_{k\geq 0}$ is a subsequence of $(n_{k,m})_{k\geq 0}$, $n_{0,m+1} > n_{0,m}$ and $(x_{n_{k,m}})_m \to y_m \in X_m$ as $k \to \infty$. This defines a point $y = (y_i)_{i\geq 0} \in X$. Set $p_m = n_{0,m}$. Then for any k, $(x_{p_m})_k \to y_k$ as $m \to \infty$, so $x_{p_m} \to y$ as $m \to \infty$.

(2) If $(x_i)_{i\geq 0}$ is a sequence of points in $X = X_1 \uplus \cdots \uplus X_n$, then for some $j \leq n$ the set $\{i : x_i \in X_j\}$ is infinite, so $(x_i)_{i\geq 0}$ has a convergent subsequence in X_j. □

Proposition A.20. — *Let $f : X \to Y$ be continuous map and X compact. Then*

(1) $f(X)$ is compact.

(2) If f is bijective, then f^{-1} is continuous, so f is a homeomorphism.

Proof. —

(1) Let $(y_n \in f(X))_{n\geq 0}$ be a sequence of points. There exist $x_n \in X$ such that $f(x_n) = y_n$ and $(x_n)_{n\geq 0}$ has a convergent subsequence $\lim_{k\to\infty} x_{n_k} = x$. Applying f, we get $y_{n_k} \to f(x) \in f(X)$ as $k \to \infty$.

(2) Let f be bijective and $V \subseteq X$ a closed set. Since f is continuous, $f(V)$ is compact and therefore closed. But $f(V) = (f^{-1})^{-1}(V)$ is the preimage of V by f^{-1}, so f^{-1} is continuous. □

Definition A.21. — Let X be a metric space.

(1) A set $Y \subseteq X$ is **dense**, if $\overline{Y} = X$.

(2) A set $Y \subseteq X$ is **residual**, if there exists a sequence $(U_i)_{i\geq 0}$ of open dense sets, such that $\bigcap_{i\geq 0} U_i \subseteq Y$.

(3) A family $\mathcal{B} \subseteq \mathcal{T}$ of open sets of X is called a **base**, if any open set $U \subseteq X$ is the union of some sets of $\mathcal{B}$, i.e., if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U$$

A set is dense iff it intersects any nonempty open set. Both sets of rational and irrational numbers are dense, $\overline{\mathbb{Q}} = \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$. Residual sets play in topology the role of "large" sets, similarly as sets of measure 1 do in measure theory. The set of irrational numbers is residual in $\mathbb{R}$, since $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{x\in\mathbb{Q}} \mathbb{R} \setminus \{x\}$, and any set $\mathbb{R} \setminus \{x\}$ is open and dense. We shall see that the set of rational numbers is not residual.

In $\mathbb{R}$, the set of all intervals is a base. There exists also a countable base consisting of all intervals with rational end points. In any metric space X, there is a base consisting of all balls $\{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$.

Proposition A.22. — *Any compact metric space has a countable dense set and a countable base.*

Proof. — Let X be a compact space. For a given $\varepsilon > 0$, construct a sequence $(x_n)_{n\geq 0}$ by induction. Choose arbitrary $x_0 \in X$. If $x_0, \ldots, x_{n-1}$ have been already constructed, choose x_n so that for all $i < n$, $d(x_n, x_i) \geq \varepsilon$. If this construction could be continued indefinitely, we would obtain an infinite sequence which would not have any converging subsequence. Thus, the construction stops at some n, and the set $K_\varepsilon = \{x_0, \ldots, x_{n-1}\}$ has the property that for any $y \in X$ there exists some $x \in K_\varepsilon$ for which $d(x, y) < \varepsilon$ (Figure 2 left). It follows that $K = \bigcup_{m=1}^{\infty} K_{1/m}$ is a countable dense set. To construct a countable base, set

$$\mathcal{B} = \{B_q(x) : x \in K, q > 0 \text{ rational}\}.$$

If U is an open set and $x \in U$, then there exists a rational $q \in \mathbb{Q}$ for which $B_{2q}(x) \subseteq U$ (Figure 2 right). Since K is dense, there exists $y \in K \cap B_q(x)$, and we have $x \in B_q(y) \subseteq B_{2q}(x) \subseteq U$. Since $B_q(y) \in \mathcal{B}$, the family $\mathcal{B}$ is a base. □

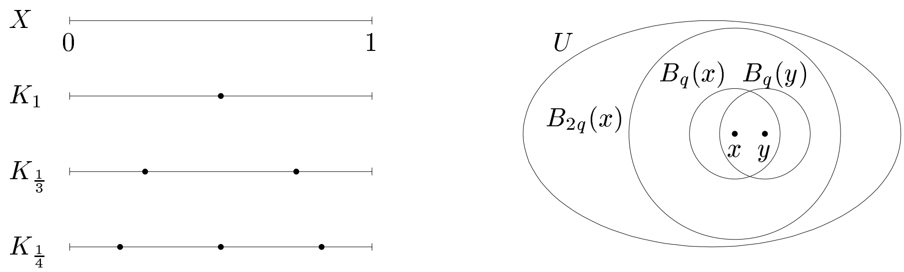

FIGURE 2. Countable base

Definition A.23. —

(1) A **cover** of a space X is a family $\mathcal{U}$ of its nonempty subsets whose union is X.

(2) A **partition** is a cover whose sets are pairwise disjoint.
(3) If all sets of $\mathcal{U}$ are open (closed), we say that $\mathcal{U}$ is an **open** (**closed**) **cover**.
(4) If $\mathcal{V} \subseteq \mathcal{U}$ are covers of X, we say that $\mathcal{V}$ is a **subcover** of $\mathcal{U}$.
(5) A cover $\mathcal{V}$ is **finer** than $\mathcal{U}$, if any set of $\mathcal{V}$ is contained in some set of $\mathcal{U}$, i.e.,

$$\mathcal{V} \succeq \mathcal{U} \quad \Leftrightarrow \quad \forall V \in \mathcal{V}, \exists U \in \mathcal{U}, V \subseteq U.$$

For any $r > 0$, $\mathcal{U}_r = \{B_r(x) : x \in X\}$ is an open cover of X, and

$$r > s \Rightarrow \mathcal{U}_s \succeq \mathcal{U}_r.$$

Proposition A.24. — *Let X be a metric space. The following conditions are equivalent.*
(1) X is compact.
(2) Any open cover of X has a finite subcover.
(3) If $(V_i)_{i\geq 0}$ are closed sets and $\emptyset \neq V_{i+1} \subseteq V_i \subseteq X$, then $V = \bigcap_i V_i$ is nonempty.

Proof. —
(1) $\Rightarrow$ (2): Let X be compact and let $\mathcal{U}$ be an open cover of X. We show first that there exists a countable subcover of $\mathcal{U}$. By Theorem A.22 there exists a countable base $\mathcal{B}$ of X. For a point $x \in X$ choose some $U \in \mathcal{U}$ such that $x \in U$, and some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then $\mathcal{V} = \{B_x : x \in X\}$ is a countable cover (some B_x repeat), which is finer than $\mathcal{U}$. For a set $V \in \mathcal{V}$ choose some $U_V \in \mathcal{U}$ such that $V \subseteq U_V$, so that $\{U_V : V \in \mathcal{V}\}$ is a countable subcover of $\mathcal{U}$. Thus any open cover has a countable subcover.

Suppose now that $(U_i)_{i\geq 0}$ is a countable cover which does not have a finite subcover. For any $n \geq 0$ there exists a point $x_n \in X \setminus (U_0 \cup \cdots \cup U_n)$. Since X is compact, $(x_n)_{n\geq 0}$ has a converging subsequence $\lim_{i\to\infty} x_{n_i} = x$. Since $(U_n)_{n\geq 0}$ is a cover, $x \in U_m$ for some m and for all sufficiently large n, $x_n \in U_m$. This is a contradiction, since $x_n \notin U_m$ if $n > m$.

(2) $\Rightarrow$ (3): Let $(V_i)_{i\geq 0}$ be closed and nonempty, $V_{i+1} \subseteq V_i$ and $V = \bigcap_{i\geq 0} V_i = \emptyset$. For $U_i = X \setminus V_i$ we have

$$\bigcup_{i\geq 0}(X \setminus V_i) = X \setminus \bigcap_{i\geq 0} V_i = X$$

so $(U_i)_{i\geq 0}$ is an a open cover and has a finite subcover. Since $U_i \subseteq U_{i+1}$, $U_n = X$ for some n and therefore $V_n = \emptyset$. This is a contradiction, so $V \neq \emptyset$.

(3) $\Rightarrow$ (1): Let $(x_n \in X)_{n\geq 0}$ be a sequence and consider sets $V_n = \{x_i : i > n\}$. Then $\overline{V_n}$ are closed and nonempty, so there exists $x \in \bigcap_n \overline{V_n}$. Since $x \in \overline{V_0}$, $B_1(x) \cap V_0 \neq \emptyset$ and there exists $n_0 > 0$ such that $x_{n_0} \in B_1(x) \cap V_0$. Suppose that we have constructed $x_{n_{k-1}}$. Since $x \in \overline{V_{n_{k-1}}}$, there exists $n_k > n_{k-1}$ such that $x_{n_k} \in B_{\frac{1}{k+1}}(x) \cap V_{n_{k-1}}$. Then $\lim_{k\to\infty} x_{n_k} = x$. $\square$

***Theorem A.25* (Baire theorem).** — *Any residual set of a compact metric space is nonempty and dense.*

Proof. — Let $(U_i \subseteq X)_{i\geq 0}$ be dense open sets and $U = \bigcap_{i\geq 0} U_i$ their intersection. Let V be a nonempty open set. We show that $U \cap V \neq \emptyset$. Set $V_0 = V$. Since U_0 is dense, $U_0 \cap V_0 \neq \emptyset$, and there exists x and $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_0 \cap V_0$.

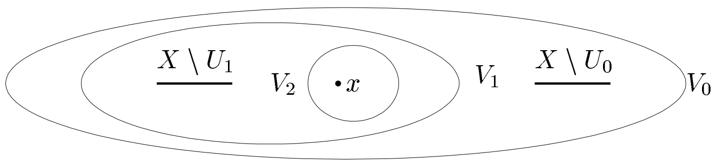

FIGURE 3. Baire theorem

Set $V_1 = B_{\frac{\varepsilon}{2}}(x)$, so $\overline{V_1} \subseteq U_0 \cap V_0$ (Figure 3). We continue by induction. If V_i is a nonempty open set, then $V_i \cap U_i \neq \emptyset$ and there exists a nonempty open set V_{i+1} such that $\overline{V_{i+1}} \subseteq U_i \cap V_i$. Thus $\overline{V_{i+1}} \subseteq \overline{V_i}$, and all V_i are nonempty. Since X is compact, there exists a point x which belongs to all the $\overline{V_i}$. We have

$$\overline{V_i} \subseteq V_{i-1} \cap U_{i-1} \subseteq V_{i-2} \cap U_{i-2} \cap U_{i-1} \subseteq \cdots \subseteq V_0 \cap U_0 \cap \cdots \cap U_{i-1}$$

Thus $x \in V_0 \cap U$ and U is dense. □

We can show now that the set $\mathbb{Q} \cap [0,1]$ of rational numbers is not residual in $[0,1]$. If it were, its intersection with the residual set $[0,1] \setminus \mathbb{Q}$ would be a residual empty set and this contradicts the Baire theorem.

Theorem A.26. — *Let X be a compact metric space and $\mathcal{U}$ its open cover. There exists $\varepsilon > 0$, such that for any $x \in X$ there exists $U \in \mathcal{U}$ such that $B_\varepsilon(x) \subseteq U$. Any $\varepsilon > 0$ with this property is called a* **Lebesgue number** *of $\mathcal{U}$.*

Proof. — If $\mathcal{U}$ contains X then arbitrary $\varepsilon > 0$ is a Lebesgue number of $\mathcal{U}$. Suppose that $\mathcal{U}$ does not contain X and define a function $f : X \to (0, \infty)$ by

$$f(x) = \sup\{\varepsilon > 0 : \exists U \in \mathcal{U} : B_\varepsilon(x) \subseteq U\}.$$

Since X is bounded, $f(x)$ is finite and positive (Figure 4). If $d(x,y) < \delta$ and $0 < \varepsilon < f(y)$, then for some $U \in \mathcal{U}$, $B_\varepsilon(x) \subseteq U$, and $B_{\varepsilon-\delta}(y) \subseteq U$, so $f(y) \geq \varepsilon - \delta$. Since this holds for any $\varepsilon < f(y)$, we get $f(y) \geq f(x) - \delta$. Interchanging x and y we get $|f(x) - f(y)| < \delta$, so f is continuous and $f(X) \subseteq \mathbb{R}$ is compact by Proposition A.20. Since $f(X)$ does not contain 0, $\varepsilon = \inf f(X) = \min f(X) > 0$ is a Lebesgue number of $\mathcal{U}$. □

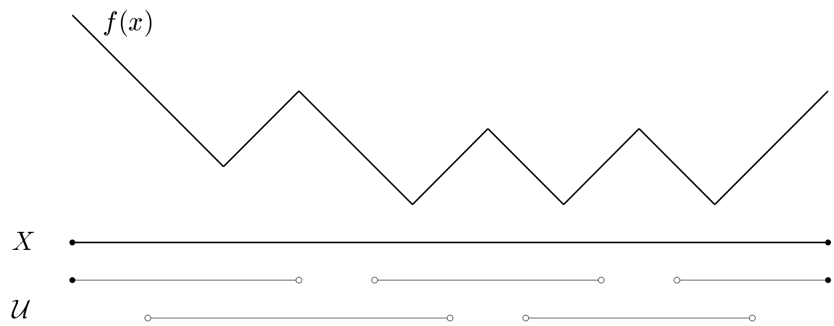

FIGURE 4. Lebesgue number of an open cover

Definition A.27. — Let X be a metric space.

(1) The **distance of sets** $Y, Z \subseteq X$ is $d(Y,Z) = \inf\{d(y,z) : y \in Y,\ z \in Z\}$.
(2) In particular, $d(x,Y) = \inf\{d(x,y) : y \in Y\}$ for $x \in X$, $Y \subseteq X$.
(3) The **ε-neighbourhood** of $Y \subseteq X$ is $B_\varepsilon(Y) = \{x \in X : d(x,Y) < \varepsilon\}$.
(4) The **diameter** of $Y \subseteq X$ is $\operatorname{diam}(Y) = \sup\{d(x,y) : x, y \in Y\}$.

Proposition A.28. — *Let X be a compact metric space and $Y, Z \subseteq X$ nonempty.*

(1) If $\overline{Y} \cap \overline{Z} = \emptyset$ then $d(Y,Z) > 0$.
(2) If $\overline{Y} \subseteq Z^\circ$, then there exists $\varepsilon > 0$ such that $B_\varepsilon(Y) \subseteq Z$.
(3) If $V_{n+1} \subseteq V_n$ are nonempty and closed and $V = \bigcap_n V_n$, then for any $\varepsilon > 0$ there exists n with $V_n \subseteq B_\varepsilon(V)$.

Proof. —

(1) The distance $d(y,z)$ of points $y \in \overline{Y}$ and $z \in \overline{Z}$ is a continuous function defined on compact space $\overline{Y} \times \overline{Z}$, so its image is compact. Since this image does not contain 0, $\inf\{d(y,z) : (y,z) \in \overline{Y} \times \overline{Z}\}$ is positive and $d(Y,Z)$ is positive too.

(2) If $Z^\circ = X$, the assertion is trivial. If $Z^\circ \subset X$, then Y, $X \setminus Z^\circ$ are nonempty and disjoint, so their distance ε is positive and $B_\varepsilon(Y) \subseteq Z$.

(3) Suppose that there exists $\varepsilon > 0$ such that for any n there exists $x_n \in V_n$ such that $d(x_n, V) \geq \varepsilon$. By passing to a subsequence, if necessary, we can assume that there exists a limit $x_n \to x$ as $n \to \infty$ and $d(x,V) \geq \varepsilon$. For any $m \geq n$, $x_m \in V_m \subseteq V_n$, so $x \in V_n$ and therefore $x \in V$. This is a contradiction. □

Let X be a compact space and $U \subseteq X \times X$ a neighbourhood of the diagonal. This means that for any $x \in X$ there exists $\varepsilon > 0$, such that $B_\varepsilon(x) \times B_\varepsilon(x) \subseteq U$. Then there exists $\varepsilon > 0$ such that

$$\Delta_X^\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\} \subseteq U.$$

Indeed, if $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal, then $\varepsilon = d(\Delta_X, (X \times X) \setminus U)$ is positive, since both Δ_X and $(X \times X) \setminus U$ are closed in $X \times X$. This shows that uniform continuity for a map between compact spaces is a topological concept. If X, Y are compact, then $f : X \to Y$ is uniformly continuous iff for any neighbourhood $V \subseteq Y \times Y$ of the diagonal, $(f \times f)^{-1}(V)$ is a neighbourhood of the diagonal in X. We get more in next proposition.

Proposition A.29. — *If $f : X \to Y$ is a continuous map and if X is compact, then f is uniformly continuous.*

Proof. — Let $\varepsilon > 0$. For any $x \in X$ there exists $\delta_x > 0$ such that if $d(y,x) < \delta_x$, then $d(f(y), f(x)) < \frac{\varepsilon}{2}$. Let $\delta > 0$ be a Lebesgue number of the open cover

$$\{U_x = B_{\delta_x}(x) : x \in X\}.$$

If $y, z \in X$ and $d(y,z) < \delta$, then there exists $x \in X$ such that $B_\delta(y) \subseteq U_x$, so both y and z belong to U_x, and

$$d(f(y), f(z)) \leq d(f(y), f(x)) + d(f(x), f(z)) < \varepsilon.$$

□

A.4. Connected spaces

We say that a set $Y \subseteq X$ of a metric space is **clopen** if it is both closed and open. In any space X we have clopen sets $\emptyset$ and X. In spaces like $\mathbb{R}$ or $\mathbb{I}$, these are the only clopen sets. In the space $X = [0,1] \cup [2,3]$ which consists of two disjoint closed intervals, there are also clopen sets $[0,1]$ and $[2,3]$. The difference is that $\mathbb{R}$ and $\mathbb{I}$ are connected spaces whereas $X = [0,1] \cup [2,3]$ is not.

Definition A.30. — Let X be a metric space.
(1) X is **connected** iff its only clopen sets are $\emptyset$ and X.
(2) A subset $Y \subseteq X$ is connected, iff it is connected as a subspace.
(3) We say that $x, y \in X$ are **inseparable** and write $(x,y) \in \mathcal{M}$, if there is no clopen set $W \subseteq X$ such that $x \in W$ and $y \in X \setminus W$.

We show that in compact spaces the **inseparability relation** $\mathcal{M} \subseteq X \times X$ is a closed equivalence and $\mathcal{M}(x)$ is the **connected component** of $x \in X$, or the maximal connected set to which x belongs.

Proposition A.31. — *Let X be a compact metric space. Then*
(1) $\mathcal{M} \subseteq X \times X$ is a closed equivalence.
(2) For any $x \in X$, $\mathcal{M}(x)$ is closed and connected.
(3) If $Z \subseteq X$ is connected and $x \in Z$, then $Z \subseteq \mathcal{M}(x)$

Proof. —
(1) The relation $\mathcal{M}$ is clearly reflexive and symmetric. If $(x,y) \in \mathcal{M}$ and $(y,z) \in \mathcal{M}$, then for any clopen $W \subseteq X$, $x \in W$ iff $y \in W$ iff $z \in W$, so $\mathcal{M}$ is transitive. If $(x,y) \notin \mathcal{M}$, then there exist a clopen W such that

$$x \in W, \mathbf{q} y \in X \setminus W \mathbf{q}\text{and}\mathbf{q} W \times (X \setminus W) \subseteq (X \times X) \setminus \mathcal{M},$$

so $(X \times X) \setminus \mathcal{M}$ is open and $\mathcal{M}$ is closed.

(2) Since X is homeomorphic to $\{x\} \times X$ and the set $\{x\} \times \mathcal{M}(x) = \mathcal{M} \cap (\{x\} \times X)$ is closed in $\{x\} \times X$, $\mathcal{M}(x)$ is closed in X. Assume that $\mathcal{M}(x)$ is not connected. There exist disjoint closed sets Z_0, Z_1 such that $x \in Z_0$, $\mathcal{M}(x) = Z_0 \cup Z_1$. Since the distance between Z_0, Z_1 is positive, there exist disjoint open sets $U_0 \supseteq Z_0$, $U_1 \supseteq Z_1$ (Figure 5). For any $y \in X \setminus (U_0 \cup U_1)$ there exists a clopen V_y such that $x \in V_y$ and $y \in X \setminus V_y$. The open cover

$$\{X \setminus V_y : y \in X \setminus (U_0 \cup U_1)\} \cup \{U_0, U_1\}$$

has a finite subcover $\{X \setminus V_1, \ldots, X \setminus V_n, U_0, U_1\}$ and

$$\mathcal{M}(x) \subseteq V = \bigcap_{i=1}^{n} V_i \subseteq U_0 \cup U_1.$$

The set V is clopen and $\overline{U_0 \cap V} \subseteq \overline{U_0} \cap V \cap (U_0 \cup U_1) = U_0 \cap V$. Thus $U_0 \cap V$ is also clopen, so $\mathcal{M}(x) \subseteq U_0 \cap V$ and $Z_1 = \emptyset$. Thus $\mathcal{M}(x)$ is connected.

(3) Assume that $Z \subseteq X$ is connected, $x \in Z$ and $y \in Z \setminus \mathcal{M}(x)$. There exists a clopen set W such that $x \in W$ and $y \notin W$. It follows that $Z \cap W$ and $Z \setminus W$ are nonempty and clopen in Z, and this is a contradiction. $\square$

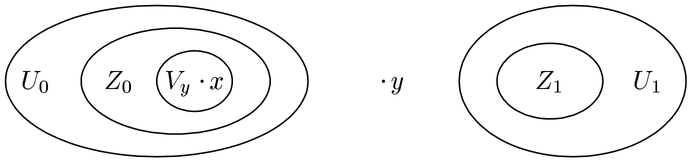

FIGURE 5. Connected components

Proposition A.32. — *Let $f : X \to Y$ be a surjective continuous map. If X is connected, then so is Y.*

Proof. — If $\emptyset \subset Z \subset f(X)$ were a clopen set, then $\emptyset \subset f^{-1}(Z) \subset X$ would be also a clopen set. □

Definition A.33. — A metric space X is **locally connected**, if any neighbourhood of any point contains a connected neighbourhood: for any $x \in U \subseteq X$ where U is open there exists a closed connected set $V \subseteq U$ such that $x \in V^\circ$.

A locally connected space need not be connected. An example is a discrete space. Conversely a connected space need not be locally connected. An example is the closure of the graph of function $y = \sin \frac{1}{x}$.

$$X = \{(x, \sin \tfrac{1}{x}) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

The line $\{0\} \times [-1, 1]$ is not separated from the graph of $\sin \frac{1}{x}$, so X is connected. On the other hand, if $\varepsilon < 1$, then $B_\varepsilon(0)$ does not contain any connected neighbourhood.

Definition A.34. — Let X be a metric space

(1) X is **totally disconnected**, if $\mathcal{M} = \Delta_X$, or if for any distinct points $x, y \in X$ there exists a clopen sets W such that $x \in W$ and $y \in X \setminus W$.
(2) X is **perfect** if it does not have isolated points. For any $x \in X$ and any $\varepsilon > 0$, the set $B_\varepsilon(x) \setminus \{x\}$ is nonempty.
(3) X is a **Cantor space**, if it is compact, perfect and totally disconnected.

By Proposition 3.6, any two Cantor spaces are homeomorphic.

A.5. Spaces of continuous functions

For a compact metric space X denote by $\mathbf{C}(X, \mathbb{R})$ the set of continuous real-valued functions $f : X \to \mathbb{R}$. By Proposition A.20, any continuous function on X is bounded. This allows us to define a metric on $\mathbf{C}(X, \mathbb{R})$ by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

It is clear that $d(f, g) = 0$ iff $f = g$ and $d(f, g) = d(g, f)$. If $f, g, h \in \mathbf{C}(X, \mathbb{R})$, then for any $x \in X$

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq d(f, g) + d(g, h)$$

so $d(f, h) \leq d(f, g) + d(g, h)$. Note that $\mathbf{C}(X, \mathbb{R})$ is not compact. The sequence of constant functions $f_n(x) = n$ does not have any convergent subsequence.

Besides being a metric space, the space $\mathbf{C}(X,\mathbb{R})$ is also an infinite-dimensional **vector space**. If $f_1,\dots,f_n \in \mathbf{C}(X,\mathbb{R})$ and $a_1,\dots,a_n \in \mathbb{R}$, then their **linear combination** $f(x) = a_1 f_1(x) + \cdots + a_n f_n(x)$ belongs to $\mathbf{C}(X,\mathbb{R})$. We say that a set $\mathcal{A} \subseteq \mathbf{C}(X,\mathbb{R})$ is **linearly independent**, if for any distinct elements $f_1,\dots,f_n \in \mathbf{C}(X,\mathbb{R})$ and any nonzero numbers $a_1,\dots,a_n \in \mathbb{R}$, the linear combination $a_1 f_1 + \cdots + a_n f_n \neq 0$ is nonzero. We say that a linearly independent set $\mathcal{A}$ is a **basis** of $\mathbf{C}(X,\mathbb{R})$, if its **linear span**

$$\mathbf{L}(\mathcal{A}) = \left\{ \sum_{i=1}^{n} a_i f_i : a_i \in \mathbb{R}, f_i \in \mathcal{A}, n > 0 \right\}$$

which consists of all linear combinations of elements of $\mathcal{A}$ is dense in $\mathbf{C}(X,\mathbb{R})$. The space $\mathbf{C}(X,\mathbb{R})$ is also an **algebra**. For any $f, g \in \mathbf{C}(X,\mathbb{R})$, their product $h(x) = f(x)g(x)$ belongs to $\mathbf{C}(X,\mathbb{R})$. This multiplication has a unit $\mathbf{e}(x) = 1$.

Definition A.35. — A subset $\mathcal{A} \subseteq \mathbf{C}(X,\mathbb{R})$ is called **subalgebra**, if

(1) $\mathbf{e} \in \mathcal{A}$,
(2) If $f \in \mathcal{A}$ and $c \in \mathbb{R}$, then $cf \in \mathcal{A}$,
(3) If $f, g \in \mathcal{A}$, then $f + g, f \cdot g \in \mathcal{A}$.

We say that $\mathcal{A}$ **separates points**, if for any distinct $x_0, x_1 \in X$ there exists $f \in \mathcal{A}$ such that $f(x_0) \neq f(x_1)$.

The classical Weierstrass theorem states that if $X = [a,b]$ is a compact real interval, then the set of polynomials is dense in $\mathbf{C}(X,\mathbb{R})$. The Stone-Weierstrass theorem is a generalization which uses the algebraic structure of $\mathbf{C}(X,\mathbb{R})$. If $\mathcal{A} \subseteq \mathbf{C}(X,\mathbb{R})$ is a subalgebra, $f \in \mathcal{A}$ and $P : \mathbb{R} \to \mathbb{R}$ a polynomial, then $P \circ f \in \mathcal{A}$. We shall prove first a special case showing that the function $f(x) = |x|$ of absolute value can be approximated by polynomials.

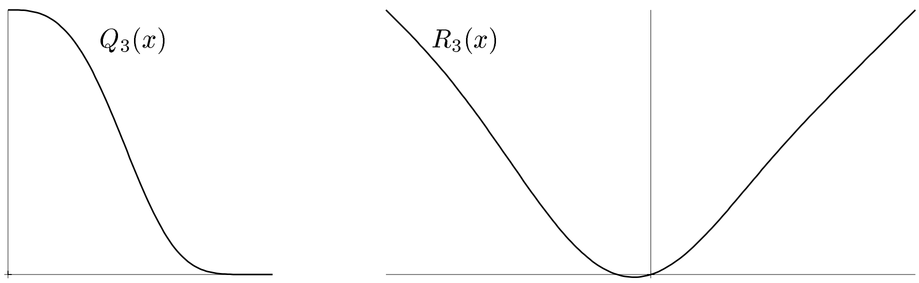

Figure 6. Approximation of the absolute value

Lemma 5. — *There exists a sequence of polynomials $(P_n(x))_{n\geq 2}$ with rational coefficients such that for $n \geq 2$*

$$-n \leq x \leq n \;\Rightarrow\; |P_n(x) - |x|\,| \leq \frac{1}{n}$$

Proof. — Set $Q_n(x) = (1-x^n)^{2^n}$, $R_n(x) = x\left(1 - 2 \cdot Q_n(\frac{x+1}{2})\right)$. Using the inequality $(1+x)^n \geq 1 + nx$ which holds for $-1 \leq x \leq 1$, we get

$$\begin{aligned}
0 \leq x \leq \tfrac{1}{2} \quad &\Rightarrow \quad 1 \geq Q_n(x) \geq 1 - (2x)^n \\
&\Rightarrow \quad -1 - 2 \cdot (2x)^n \geq 1 - 2Q_n(x) \geq -1 \\
\tfrac{1}{2} \leq x \leq 1 \quad &\Rightarrow \quad \frac{1}{Q_n(x)} = \left(1 + \frac{x^n}{1-x^n}\right)^{2^n} \geq (2x)^n \\
&\Rightarrow \quad 1 - 2 \cdot (2x)^{-n} \leq 1 - 2Q_n(x) \leq 1
\end{aligned}$$

and

$$\begin{aligned}
-1 \leq x \leq 0 \quad &\Rightarrow \quad 0 \leq \tfrac{x+1}{2} \leq \tfrac{1}{2} \quad \Rightarrow \quad -x - 2x(x+1)^n \leq R_n(x) \leq -x \\
0 \leq x \leq 1 \quad &\Rightarrow \quad \tfrac{1}{2} \leq \tfrac{x+1}{2} \leq 1 \quad \Rightarrow \quad x - 2x(x+1)^{-n} \leq R_n(x) \leq x
\end{aligned}$$

Estimating the left-hand sides of these inequalities we get $|R_n(x) - |x|\,| \leq \frac{8}{n-1}$ for $-1 \leq x \leq 1$. Then $P_n(x) = n \cdot R_{8n^2+1}(\frac{x}{n})$ satisfies the statement of the lemma. $\square$

Theorem A.36 **(Real Stone-Weierstrass theorem).** — *Let X be a compact metric space and $\mathcal{A}$ a subset of $\mathbf{C}(X,\mathbb{R})$ which satisfies*

(1) If $x_0 \neq x_1 \in X$, then there exists $f \in \mathcal{A}$ with $f(x_0) \neq f(x_1)$.

(2) $\mathbf{e} \in \mathcal{A}$.

(3) If $f \in \mathcal{A}$ and $a \in \mathbb{Q}$ then $af \in \mathcal{A}$.

(4) If $f, g \in \mathcal{A}$ then $f+g, f \cdot g \in \mathcal{A}$.

Then $\mathcal{A}$ is dense in $\mathbf{C}(X,\mathbb{R})$.

Proof. — Let $\mathcal{B} = \overline{\mathcal{A}}$ be the closure of $\mathcal{A}$. Then $\mathcal{B}$ is a closed subalgebra of $\mathbf{C}(X,\mathbb{R})$. If $f \in \mathcal{B}$ and $c \in \mathbb{R}$, then there exists a sequence $f_n \in \mathcal{A}$ such that $f_n \to f$ and a sequence $c_n \in \mathbb{Q}$ such that $c_n \to c$ and it follows that $c_n f_n \to cf$. Similarly for the other operations. We show that if $f \in \mathcal{B}$ then $|f| \in \mathcal{B}$. Let $(P_n)_{n\geq 1}$ be the polynomials from Lemma 5, which approximate the absolute value. If $f \in \mathcal{B}$, then $P_n \circ f \in \mathcal{B}$ and $\lim_{n\to\infty} P_n \circ f = |f| \in \mathcal{B}$. It follows that $\mathcal{B}$ is also closed with respect to the operations of minima and maxima. If $f, g \in \mathcal{B}$, then

$$\begin{aligned}
(f \wedge g)(x) &= \min\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|) \\
(f \vee g)(x) &= \max\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|)
\end{aligned}$$

so $f \wedge g, f \vee g \in \mathcal{B}$. Assume now that $F \in \mathbf{C}(X,\mathbb{R})$, $\varepsilon > 0$ be arbitrary. For any $x, y \in X$ there exists a function $f_{x,y} \in \mathcal{B}$ such that $f_{x,y}(x) = F(x)$, $f_{x,y}(y) = F(y)$. To see it pick a function $g \in \mathcal{B}$ with $g(x) \neq g(y)$ and set

$$f_{xy}(t) = \frac{F(x) - F(y)}{g(x) - g(y)} \cdot g(t) + \frac{F(y)g(x) - F(x)g(y)}{g(x) - g(y)}.$$

Given $y \in X$, then for each $x \in X$ there exists an open set $U_x \ni x$ such that $f_{x,y}(t) > F(t) - \varepsilon$ for $t \in U_x$. The open cover $\{U_x : x \in X\}$ has a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ and we set

$$f_y = f_{x_1,y} \vee \cdots \vee f_{x_n,y} \in \mathcal{B}$$

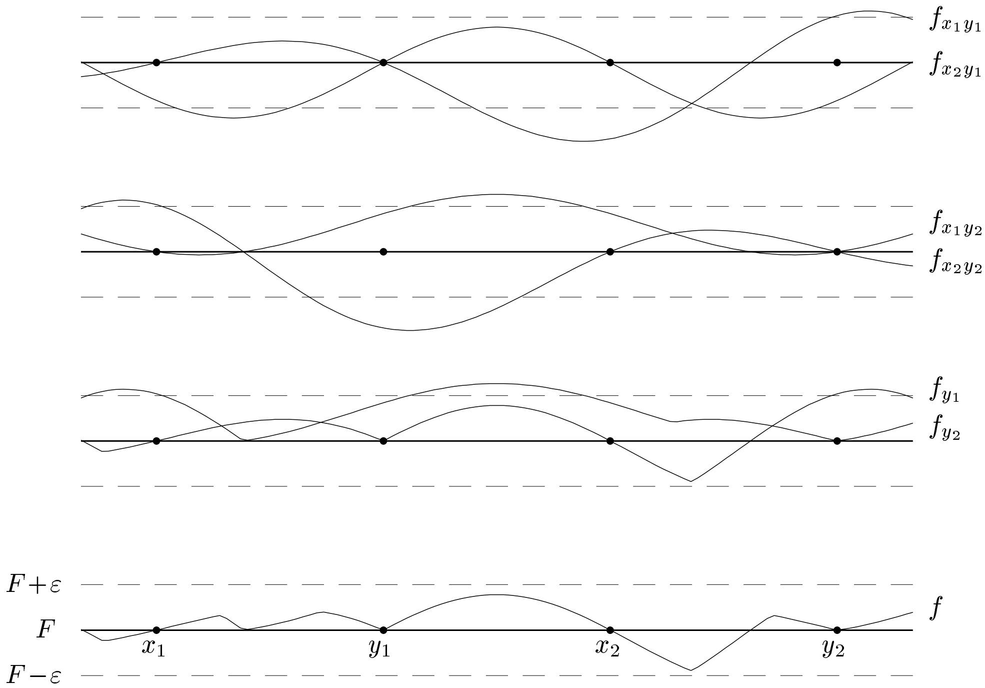

FIGURE 7. Stone-Weierstrass theorem

so $f_y(t) \geq F(t) - \varepsilon$ for all $t \in X$. Since $f_y(y) = F(y)$, there exists an open set $V_y \ni y$ such that $f_y(t) < F(t) + \varepsilon$ for $t \in V_y$. The open cover $\{V_y : y \in X\}$ has a finite subcover $\{V_{y_1}, \ldots, V_{y_m}\}$ and we set

$$f = f_{y_1} \wedge \cdots \wedge f_{y_m} \in \mathcal{B}$$

Then $f(t) < F(t) + \varepsilon$ for all $t \in X$. Since $f_{y_i}(t) > F(t) - \varepsilon$ we get also $f(t) > F(t) - \varepsilon$ and $d(F, f) < \varepsilon$ (Figure 7). $\square$

Analogously, the space $\mathbf{C}(X, \mathbb{C})$ of continuous complex-valued functions on X is a vector space over $\mathbb{C}$ and an algebra with unit element $\mathbf{e}(x) = 1$. A subspace $\mathcal{A} \subseteq \mathbf{C}(X, \mathbb{C})$ is a subalgebra, if it contains $\mathbf{e}$, and is closed with respect to the operations of multiplication and linear combinations with complex coefficients. Recall that the complex conjugate of a complex number $c = a + bi$ is $\overline{c} = a - bi$. The complex conjugate of a function $f \in \mathbf{C}(X, \mathbb{C})$ is $\overline{f}(x) = \overline{f(x)}$.

***Theorem A.37* (Stone-Weierstrass theorem).** — *Let X be a compact metric space and $\mathcal{A} \subseteq \mathbf{C}(X, \mathbb{C})$ a subset which satisfies*

(1) For any distinct $x_0, x_1 \in X$ there exists $f \in \mathcal{A}$ with $f(x_0) \neq f(x_1)$.
(2) $\mathbf{e} \in \mathcal{A}$.
(3) If $f \in \mathcal{A}$ and $a, b \in \mathbb{Q}$ then $(a + bi)f \in \mathcal{A}$ and $\overline{f} \in \mathcal{A}$.
(4) If $f, g \in \mathcal{A}$ then $f + g, f \cdot g \in \mathcal{A}$.

Then $\mathcal{A}$ is dense in $\mathbf{C}(X, \mathbb{C})$.

Proof. — Let $\mathcal{A}_r$ be the set of all real-valued functions which belong to $\mathcal{A}$. Then $\mathcal{A}_r$ is a subalgebra of $\mathbf{C}(X, \mathbb{R})$. If $f = f_0 + if_1 \in \mathcal{A}$ with f_0, f_1 real, then $f_0 = (f + \overline{f})/2 \in \mathcal{A}$ and $f_1 = (f - \overline{f})/2 \in \mathcal{A}$, so they both belong to $\mathcal{A}_r$. If $f(x) \neq f(y)$ then either $f_0(x) \neq f_0(y)$ or $f_1(x) \neq f_1(y)$. Thus $\mathcal{A}_r$ separates points and is dense in $\mathbf{C}(X, \mathbb{R})$ by Theorem A.36. Since any $f \in \mathbf{C}(X, \mathbb{C})$ is a linear combination $f = f_0 + if_1$ of real-valued continuous functions, we get that $\mathcal{A}$ is dense in $\mathbf{C}(X)$. $\square$

Theorem A.38. — *If X is a compact metric space then $\mathbf{C}(X, \mathbb{C})$ has a countable dense subset.*

Proof. — Let $W \subseteq X$ be a countable dense set in X. For any pair of distinct points $y, z \in W$ there exists a continuous function $f_{y,z} \in \mathbf{C}(X, \mathbb{C})$, such that $f_{y,z}(y) \neq f_{y,z}(z)$. The family $\mathcal{A}_0$ of such functions is countable and distinguishes points. Let $\mathcal{A}$ be the smallest family containing $\mathcal{A}_0$ which satisfies the assumption of Theorem A.37. Then $\mathcal{A}$ is countable and dense in $\mathbf{C}(X, \mathbb{C})$. $\square$

A.6. Topological spaces

Topology studies more general and more abstract structures than metric spaces. Its starting point is an abstract family of subsets of the space which is required to satisfy the properties of open sets in metric spaces as shown in Proposition A.6.

Definition A.39. — A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T} \subseteq \mathcal{P}(X)$ is a family of subsets of X, which satisfies
(1) $\emptyset, X \in \mathcal{T}$.
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.
(3) If $(U_i)_{i \in I}$ is a family of sets with $U_i \in \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The family $\mathcal{T}$ is called topology and the sets of $\mathcal{T}$ are called open sets. A set $V \subseteq X$ is closed, if $X \setminus V \in \mathcal{T}$ is open. The interior and the closure of a set $Y \subseteq X$ are

$$Y^\circ = \bigcup \{U : U \subseteq Y, U \in \mathcal{T}\}, \quad \overline{Y} = \bigcap \{V : V \supseteq Y, X \setminus V \in \mathcal{T}\}.$$

A family $\mathcal{B} \subseteq \mathcal{T}$ is a base of $(X, \mathcal{T})$, if for any $U \in \mathcal{T}$ and any $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

On any set X there exists a **discrete topology** $\mathcal{P}(X)$, in which any subset of X is open and an **indiscrete topology** $\{\emptyset, X\}$, whose only open sets are $\emptyset$ and X. Topologies on a given set X can be compared by inclusion. The indiscrete topology is the smallest and the discrete topology is the largest. If $(\mathcal{T}_i)_{i \in I}$ is a family of topologies on X, then their intersection $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$ is a topology on X too.

Definition A.40. — Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces and $f : X \to Y$.
(1) f is continuous if for any $V \in \mathcal{S}$, $f^{-1}(V) \in \mathcal{T}$.
(2) f is an open map if for any $U \in \mathcal{T}$, $f(U) \in \mathcal{S}$.
(3) f is a closed map if for any closed $V \subseteq X$, $f(V)$ is a closed subset of Y.
(4) f is a homeomorphism, if it is bijective, and both f and f^{-1} are continuous.
(5) f is an embedding if it is injective and $\mathcal{T} = \{f^{-1}(V) : V \in \mathcal{S}\}$.
(6) f is a factor map, if it is surjective and $\mathcal{S} = \{V \subseteq Y : f^{-1}(V) \in \mathcal{T}\}$.

The subspace topology is the smallest topology which makes the injection map f continuous. This generalizes the concept of subspace of a metric space (Proposition A.5). If $X \subseteq Y$, then

$$\mathcal{T} = \{V \cap Y : V \in \mathcal{S}\}.$$

The factor topology is the largest topology which makes the factor map f continuous. In metric spaces the operation of factor space cannot be defined. A factor space of a metric space need not necessarily be metric. The principle used in the definition of subspace and factor space leads to a natural definition of the product topology.

Definition A.41. — Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, $X = \prod_{i\in I} X_i$ the cartesian product and $\pi_i : X \to X_i$ the projections. The product topology $\mathcal{T}$ is the smallest topology on X which makes all projections π_i continuous.

The product topology is thus well-defined. Consider the family of all topologies on X such that all projections are continuous. This family contains the discrete topology, so it is nonempty. The product topology is the intersection of all topologies in this family. The product topology $\mathcal{T}$ must contain all sets $\pi_i^{-1}(U_i)$, where $U_i \in \mathcal{T}_i$, all their finite intersections and all unions of these finite intersections. This means that the product space has a canonical base

$$\mathcal{B} = \left\{ \bigcap_{i\in K} \pi_i^{-1}(U_i) : (U_i \in \mathcal{T}_i)_{i\in K}, K \subseteq I \text{ finite} \right\}.$$

Observe that $\bigcap_{i\in K} \pi_i^{-1}(U_i) = \prod_{j\in I} V_j$, where

$$V_j = \begin{cases} U_j & \text{for } j \in K \\ X_j & \text{for } j \in I \setminus K \end{cases}$$

The following proposition is a characterization of the product space. Any space $(X, \mathcal{T})$ which satisfies it is homeomorphic to the product space. Proposition A.18 then says that the metric of the product of a bounded sequence of metric spaces yields the product topology.

Proposition A.42. — *Let $(f_i : (Y, \mathcal{S}) \to (X_i, \mathcal{T}_i))_{i\in I}$ be a family of continuous maps. Then there exists a unique continuous map $f : (Y, \mathcal{S}) \to (X, \mathcal{T})$ such that $\pi_i \circ f = f_i$ for any $i \in I$.*

Proof. — The function f is uniquely defined, since $f(y)_i = \pi_i \circ f(y) = f_i(y)$. We show that f is continuous. Let $U \subseteq X$ be open and $y \in f^{-1}(U)$. Then there exist a finite set $K \subseteq I$ and open sets $(U_i \in \mathcal{T}_i)_{i\in K}$ such that

$$f(y) \in \bigcap_{i\in K} \pi_i^{-1}(U_i) \subseteq U \;\Rightarrow\; y \in \bigcap_{i\in K} f^{-1}(\pi_i^{-1}(U_i)) \subseteq f^{-1}(U)$$

Since $f^{-1}(\pi_i^{-1}(U_i)) = f_i^{-1}(U_i)$ is open, y is an inner point of $f^{-1}(U)$. $\square$

A dual concept is the disjoint sum of spaces. The disjoint sum topology is the largest topology which makes all inclusions continuous.

Definition A.43. — Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, $X = \biguplus_{i\in I} X_i$ and $\nu_i : X_i \to X$ the inclusions. The disjoint sum topology is defined by

$$\mathcal{T} = \{U \in \mathcal{P}(X) : \forall i \in I, \nu_i^{-1}(U) \in \mathcal{T}_i\}.$$

A.6.1. Hausdorff and normal spaces. — In a metric space any singleton set $\{x\}$ is closed. This need not be so in topological spaces. In an indiscrete space the closure of any point is the whole space. Spaces in which singletons are closed are called T_1-spaces. We shall consider two stronger properties.

Definition A.44. —

(1) A topological space X is a Hausdorff space (or a T_2-space), if for any two distinct points $x_0, x_1 \in X$ there exist disjoint open sets $U_0 \ni x_0$, $U_1 \ni x_1$.

(2) A topological space X is a normal space, if its singletons are closed sets and for any two disjoint closed sets $V_0, V_1 \subseteq X$ there exist disjoint open sets $U_0 \supseteq V_0$, $U_1 \supseteq V_1$.

By passing to the complements we have an equivalent definition of normality: If U_0, U_1 are open sets such that $U_0 \cup U_1 = X$, then there exist closed sets $V_0 \subseteq U_0$ and $V_1 \subseteq U_1$ such that $V_0 \cup V_1 = X$.

Proposition A.45. —

(1) A subspace of a Hausdorff space is a Hausdorff space.

(2) The product of Hausdorff spaces is a Hausdorff space.

(3) Any metric space is normal.

Proof. —

(1) Let $f : X \to Y$ be an embedding of X into a Hausdorff space Y. If $x_0, x_1 \in X$ are distinct, then $f(x_0) \neq f(x_1)$ and there exist disjoint open $U_0 \ni f(x_0)$, $U_1 \ni f(x_1)$. Then $f^{-1}(U_0) \ni x_0$, $f^{-1}(U_1) \ni x_1$ are disjoint and open.

(2) Let $X = \prod_{i \in I} X_i$ be a product of Hausdorff spaces and $x, x' \in X$. If $x \neq x'$ then for some i, $x_i \neq x'_i$ and there exist in X_i disjoint open sets $U \ni x_i$, $U' \ni x'_i$. Then $\pi_i^{-1}(U) \ni x$, $\pi_i^{-1}(U') \ni x'$ are disjoint and open in X.

(3) If $x_0, x_1 \in X$ are two distinct points such that $d(x_0, x_1) = \varepsilon > 0$, then $x_0 \notin B_\varepsilon(x_1)$, so $X \setminus \{x_0\}$ is open and $\{x_0\}$ is closed. Let $V_0, V_0 \subseteq X$ be disjoint closed sets and define a function $f : X \to \mathbb{I}$ by

$$f(x) = \frac{d(x, V_0)}{d(x, V_0) + d(x, V_1)}$$

Then f is continuous and $U_0 = f^{-1}([0, \frac{1}{3}))$, $U_1 = f^{-1}((\frac{2}{3}, 1])$ are two disjoint open sets which contain V_0 and V_1 respectively. □

Proposition A.46 **(Urysohn lemma).** — *Let X be a normal space and $V_0, V_1 \subseteq X$ nonempty disjoint closed sets. Then there exists a continuous function $f : X \to \mathbb{I}$ such that $f(V_0) = \{0\}$, $f(V_1) = \{1\}$.*

Proof. — There exist disjoint open sets $U_i \supseteq V_i$. Set $W_0 = U_0$, $W_1 = X \setminus V_1$. Then W_0, W_1 are open sets with $\overline{W_0} \subseteq X \setminus U_1 \subseteq W_1$ (Figure 8 left). Since $\overline{W_0}$, $X \setminus W_1$ are closed and disjoint, there exist disjoint open $U'_0 \supseteq \overline{W_0}$, $U'_1 \supseteq X \setminus W_1$. Set $W_{\frac{1}{2}} = U'_0$. Then $\overline{W_{\frac{1}{2}}} \subseteq X \setminus U'_1 \subseteq W_1$. We continue with this construction using the binary dyadic numbers $i \cdot 2^{-k}$, so that for $0 \le i < j \le 2^k$ we have $\overline{W_{i2^{-k}}} \subseteq W_{j2^{-k}}$. Define a function $f : X \to \mathbb{I}$ by

$$f(x) = \begin{cases} \inf\{r > 0 : x \in W_r\} & \text{if } x \in W_1 \\ 1 & \text{if } x \in X \setminus W_1 \end{cases}$$

Since $V_0 \subseteq W_0$, we have $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$. We show that f is continuous. The inequality $f(x) < a$ holds iff there exists $r < a$ such that $x \in W_r$, so

$$f^{-1}([0,a)) = \bigcup\{W_r : r < a\}$$

is an open set. Similarly the inequality $f(x) > a$ holds iff there exists $s > a$ such that $x \notin W_s$ and this holds iff there exists $s > t > a$ such that $x \notin \overline{W_r}$, so

$$f^{-1}((a,1]) = \bigcup\{X \setminus \overline{V_r} : r > b\}$$

is open. Since the intervals $[0,a)$ and $(a,1]$ form a base in the space $\mathbb{I}$, f is continuous. □

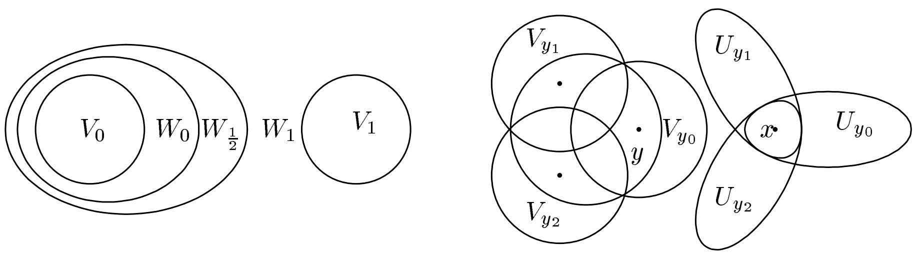

FIGURE 8. Urysohn lemma

A.6.2. Compact spaces. — The characterization of compact metric spaces given in Proposition A.24 is taken as a definition of compactness in topological spaces. Recall that a family $\mathcal{U} \subseteq \mathcal{P}(X)$ is a cover of X if its union is X and $\mathcal{U}$ is an open cover if its elements are open. We say that a family $\mathcal{F}$ has the **finite intersection property**, if any finite subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ has nonepmty intersection.

Definition A.47. — A topological space X is compact if any of its open covers has a finite subcover or, equivalently, if any its family $\mathcal{F}$ of closed sets of X with finite intersection property has a nonempty intersection.

The two definitions are equivalent. A family $\mathcal{F}$ of closed sets has nonempty intersection iff $\mathcal{U} = \{X \setminus V : V \in \mathcal{F}\}$ is not an open cover, and $\mathcal{F}$ has finite intersection property iff $\mathcal{U}$ does not have finite subcover.

Proposition A.48. —

(1) A continuous image of a compact space is compact.
(2) A closed subset of a compact space is compact.
(3) A compact subset of a Hausdorff space is closed.
(4) Any compact Hausdorff space is normal.

Proof. —

(1) Let X be compact and $f : X \to Y$ a continuous surjection. If $\mathcal{V}$ is an open cover of Y, then the open cover $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ has a finite subcover $\mathcal{U}_0$ and $\mathcal{V}_0 = \{V \in \mathcal{V} : f^{-1}(V) \in \mathcal{U}_0\}$ is a finite subcover of $\mathcal{V}$.

(2) Let X be a compact space and $Y \subseteq X$ a closed subset. Let $\mathcal{V}$ be an open cover of Y. Any $V \in \mathcal{V}$ can be written as $V = Y \cap U$, where U is open in X, so there exists a family $\mathcal{U}$ of open sets of X such that $\mathcal{V} = \{Y \cap U : U \in \mathcal{U}\}$. It follows that $\mathcal{U} \cup \{X \setminus Y\}$ is an open cover of X so it has a finite subcover $\mathcal{U}_0$ and $\{U \cap Y : U \in \mathcal{U}_0\}$ is a finite subcover of $\mathcal{V}$.

(3) Let Y be a compact subset of a Hausdorff space X. Assume that $x \in X \setminus Y$. For any $y \in Y$ there exist disjoint open sets $V_y \ni y$, $U_y \ni x$ (Figure 8 right). The family $\{V_y \cap Y : y \in Y\}$ is an open cover of Y so it has a finite subcover $\{V_y \cap Y : y \in K\}$. Then $U = \bigcap_{y \in K} U_y$ is an open set with $x \in U \subseteq X \setminus Y$, so $X \setminus Y$ is open.

(4) Let X be a compact Hausdorff space. If $Y \subseteq X$ is closed and $x \in X \setminus Y$, then by the proof of (3) there exist disjoint open sets $V = \bigcup_{y \in K} V_y \supseteq Y$, $U \ni x$. Assume now that $Y_1, Y_2 \subseteq X$ are disjoint closed sets. For any $y \in Y_2$ there exist disjoint open sets $U_y \supseteq Y_1$, $V_y \ni y$. The open cover $\{V_y \cap Y_2 : y \in Y_2\}$ has a finite subcover $\{V_y \cap Y_2 : y \in K\}$ and $U = \bigcap_{y \in K} U_y \supseteq Y_1$, $V = \bigcup_{y \in K} V_y \supseteq Y_2$ are disjoint open sets. □

The property of being a Hausdorff space requires that the topology should be sufficiently large (any discrete space is a Hausdorff space). The property of compactness on the contrary requires that the topology should be sufficiently small (any indiscrete space is compact). A topology which is both Hausdorff and compact has the right number of open sets. No smaller topology is Hausdorff and no larger topology is compact.

Proposition A.49. — *Let $f : X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y.*

(1) If f is injective, then it is an embedding.

(2) If f is surjective, then it is a factor map.

(3) If f is bijective, then it is a homeomorphism.

Proof. —

(1) If $U \subseteq X$ is closed, then it is compact, $f(U) \subseteq Y$ is compact and therefore closed. Since f is injective, $U = f^{-1}(f(U))$. Thus a set is closed in X iff it is an inverse image of a closed set in Y, so X is a subspace of Y.

(2) Let $V \subseteq Y$ and assume that $f^{-1}(V) \subseteq X$ is closed. Then it is compact, so $V = f(f^{-1}(V))$ is compact and therefore closed. Thus a set in Y is closed iff its inverse image is closed in X.

(3) follows from (1) and (2). □

If $f : X \to Y$ is a surjective map, then $R_f = \{(x_0, x_1) \in X \times X : f(x_0) = f(x_1)\}$ is an equivalence relation on X. Conversely, for an equivalence relation $R \subseteq X \times X$ denote by $Y = X/R$ the set of equivalence classes of R and $\pi : X \to Y$ the projection. Then $R = R_\pi$.

Proposition A.50. — *Let X, Y be compact Hausdorff spaces and $\pi : X \to Y$ a factor map. Then R_π is a closed subset of $X \times X$.*

Proof. — We show that $(X \times X) \setminus R$ is open. Let $(x_0, x_1) \in (X \times X) \setminus R$. In Y there exist disjoint open sets $V_0, V_1 \subseteq X$ such that $\pi(x_0) \in V_0$, $\pi(x_1) \in V_1$. We get

$$(x_0, x_1) \in \pi^{-1}(V_0) \times f^{-1}(V_1) \subseteq (X \times X) \setminus R,$$

so (x_0, x_1) is an inner point of $(X \times X) \setminus R$. □

Proposition A.51. — *Let X be a compact Hausdorff space, $R \subseteq X \times X$ a closed equivalence, and $\pi : X \to X/R$ the factor map. Then π is closed (Definition A.40) and X/R is a compact Hausdorff space.*

Proof. — We show first that if $W \subseteq X$ is closed, then $R(W) \subseteq X$ is closed subset of X. Let $y \in X \setminus R(W)$ (Figure 9 left). For any $x \in W$, $(x, y) \notin R$, so there exist disjoint open sets $U_x \ni x$, $V_x \ni y$ such that $(U_x \times V_x) \cap R = \emptyset$. The open cover $\{U_x : x \in W\}$ of W has a finite subcover $\{U_x : x \in K\}$. Set

$$U = \bigcup_{x \in K} U_x, \quad V = \bigcap_{x \in K} V_x$$

Then $(U \times V) \cap R = \emptyset$, $W \subseteq U$ and $y \in V \subseteq X \setminus R(W)$, so $R(W)$ is closed. Since π is a factor map, a set of X/R is closed iff its inverse image is closed in X. If $W \subseteq X$ is closed, then $R(W) = \pi^{-1}(\pi(W))$ is closed so, so $\pi(W)$ is also closed. Thus π is a closed map. If $y \in X/R$ and $x \in \pi^{-1}(y)$, then $\pi^{-1}(y) = R(x)$. Since $R(x)$ is closed, $\{y\} = \pi(R(x))$ is closed, so the singletons of X/R are closed. We show that X/R is normal using a dual formulation of this property. Let $U_0, U_1 \subseteq Y$ be open sets such that $U_0 \cup U_1 = Y$. Then $\pi^{-1}(U_0) \cup \pi^{-1}(U_1) = X$ and by normality of X, there exists closed $V_0 \subseteq \pi^{-1}(U_0)$, $V_1 \subseteq \pi^{-1}(V_1)$ such that $V_0 \cup V_1 = X$. Since π is closed, $\pi(V_0) \subseteq U_0$, $\pi(V_1) \subseteq U_1$ are closed sets such that $\pi(V_0) \cup \pi(V_1) = X/R$, so X/R is normal and therefore Hausdorff. □

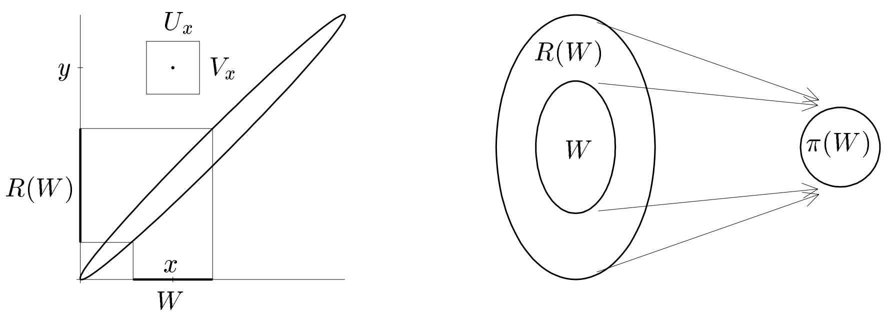

FIGURE 9. Factor space

***Theorem A.52* (Tychonoff theorem).** — *The product of compact spaces is compact.*

Proof. — Let $(X_i)_{i\in I}$ be compact spaces, $X = \prod_{i\in I} X_i$, and $\pi_i : X \to X_i$ the projections. Let $\mathcal{F}_0$ is a family of closed sets of X with the finite intersection property, i.e., every its nonempty finite subfamily has a nonempty intersection. By Zorn's lemma there exists a maximal family $\mathcal{F} \supseteq \mathcal{F}_0$ of (arbitrary) subsets of X with the finite intersection property. To prove that $\bigcap \mathcal{F}_0 \neq \emptyset$, it suffices to show that the family $\{\overline{V} : V \in \mathcal{F}\}$ has a nonempty intersection. Since $\mathcal{F}$ is maximal,

$$V_1, \dots, V_k \in \mathcal{F} \quad \Rightarrow \quad V_1 \cap \cdots \cap V_k \in \mathcal{F}$$

$$V_0 \subseteq X \;\;\&\;\; \forall V \in \mathcal{F}, V \cap V_0 \neq \emptyset \quad \Rightarrow \quad V_0 \in \mathcal{F}$$

For any $i \in I$, the family $\mathcal{F}_i = (\overline{\pi_i(V)})_{V\in\mathcal{F}}$ of closed subsets of X_i has the finite intersection property, hence it has a nonempty intersection and there exists a point

$$x_i \in \bigcap_{V\in\mathcal{F}} \overline{\pi_i(V)} \subseteq X_i$$

We show that $x = (x_i)_{i\in I}$ belongs to $\overline{V}$ for any $V \in \mathcal{F}$. Let $x \in B \in \mathcal{B}$, where B belongs to the canonical base. Then $B = \bigcap_{i\in K} \pi_i^{-1}(W_i)$, where $W_i \subseteq X_i$ are open. Since $x_i \in W_i$, $W_i \cap \pi_i(V) \neq \emptyset$ and therefore $V \cap \pi_i^{-1}(W_i) \neq \emptyset$. From the maximality of $\mathcal{F}$ we get that $\pi_i^{-1}(W_i) \in \mathcal{F}$, so $B = \bigcap_{i\in K} \pi_i^{-1}(W_i) \in \mathcal{F}$. Since $\mathcal{F}$ has the finite intersection property, we get

$$V \in \mathcal{F}, x \in B \in \mathcal{B} \;\Rightarrow\; B \cap V \neq \emptyset,$$

so $x \in \bigcap\{\overline{V} : V \in \mathcal{F}\}$. □

A.6.3. Metrizable spaces. — We say that a topological space is metrizable, if there exists a metric on the space which gives rise to the topology. For example, if (X_i, d_i) is a sequence of bounded metric spaces, then the product topological space is metrizable - by the product metric of Definition A.17. This is proved in Proposition A.18.

Proposition A.53. — *Any compact Hausdorff space with a countable base is metrizable.*

Proof. — Let $\mathcal{B}$ be a countable base of a compact Hausdorff space X. Say that a pair of sets $B_0, B_1 \in \mathcal{B}$ is compatible, if there exists a continuous function $f : X \to \mathbb{I}$ such that

$$f(B_0) \subseteq [0, \tfrac{1}{2}), \quad f(X \setminus B_1) = \{1\}$$

For any pair B_0, B_1 of compatible sets pick a function with this property. The set $\mathcal{F}$ of such functions is countable. We show that for any $x_0 \neq x_1 \in X$ there exists $f \in \mathcal{F}$ such that $f(x_0) \neq f(x_1)$. There exists $B_1 \in \mathcal{B}$ such that $x_0 \in B_1$ and $x_1 \in X \setminus B_1$. Since X is normal, there exists a function $g : X \to \mathbb{I}$ such that $g(x_0) = 0$ and $g(X \setminus B_1) = \{1\}$. There exists $B_0 \in \mathcal{B}$ such that $x_0 \in B_0 \subseteq g^{-1}([0, \frac{1}{2}))$. Thus (B_0, B_1) are compatible and there exists $f \in \mathcal{F}$ such that $f(B_0) \subseteq [0, \frac{1}{2})$, $f(X \setminus B_1) = \{1\}$, so $f(x_0) \neq f(x_1)$. Let $(f_i)_{i\geq 0}$ be a sequence of all functions of $\mathcal{F}$ and construct a continuous function $f : X \to \mathbb{I}^{\mathbb{N}}$ according to the rule $f(x)_i = f_i(x)$. Then f is injective and continuous. Since X is compact, f is an embedding by Proposition A.49. □

Theorem A.54. — *If X is a compact metric space and $R \subseteq X \times X$ a closed equivalence relation, then the factor space X/R is metrizable.*

Proof. — Denote by $Y = X/R$ the factor space and $\pi : X \to Y$ the factor map. By Proposition A.51, Y is compact and Hausdorff. We will show that Y has a countable base. Let $(B_i)_{i \geq 0}$ be a countable base of X and set

$$\mathcal{V} = \{Y \setminus \pi(X \setminus \bigcup_{i \in K} B_i) : K \text{ is a finite subset of } \mathbb{N}\}$$

Since π is closed, any $V \in \mathcal{V}$ is open and $\mathcal{V}$ is countable. We show that $\mathcal{V}$ is a base of Y. Let $W \subseteq Y$ be open and $y \in W$. Then $\pi^{-1}(y)$ is a compact subset of X, so there exists a finite set $K \subseteq \mathbb{N}$ such that $\pi^{-1}(y) \subseteq \bigcup_{i \in K} B_i \subseteq \pi^{-1}(W)$. By passing to the complements we get

$$Y \setminus W = \pi(X \setminus \pi^{-1}(W)) \subseteq \pi\left(X \setminus \bigcup_{i \in K} B_i\right) \subseteq \pi(X \setminus \pi^{-1}(y)) = Y \setminus \{y\}$$

so $y \in Y \setminus \pi(X \setminus \bigcup_{i \in K} B_i) \subseteq W$, and $\mathcal{V}$ is a base. By Proposition A.53, Y is metrizable. □

Let X be a compact metric space and $\mathfrak{M} \subseteq X \times X$ the inseparability relation of Definition A.30. By Proposition A.31, $\mathfrak{M}$ is closed, so $X/\mathfrak{M}$ is a compact metrizable space called the **space of connected components** of X.

Proposition A.55. — *If X is a compact metric space, then the space $X/\mathfrak{M}$ of connected components is totally disconnected.*

Proof. — If $y_0, y_1 \in X/\mathfrak{M}$ are distinct points, then $\pi^{-1}(x_0)$, $\pi^{-1}(x_1)$ are distinct connected components. By compactness there exists a clopen sets $W \subseteq X$ such that $\pi^{-1}(x_0) \subseteq W$ and $\pi^{-1}(x_1) \subseteq X \setminus W$. Since π is closed, $\pi(W)$ is a clopen set, $x_0 \in W$ and $x_1 \in (X/\mathfrak{M}) \setminus W$. □

A.7. Uniform spaces

Definition A.56. — A **uniform space** is a pair $(X, \mathcal{U})$, where $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ is a family of relations called uniformity with the following properties
(1) If $R \in \mathcal{U}$, $R \subseteq S \subseteq X \times X$, then $S \in \mathcal{U}$.
(2) If $R, S \in \mathcal{U}$, then $R \cap S \in \mathcal{U}$.
(3) For any $R \in \mathcal{U}$ there exists some $S \in \mathcal{U}$ with $S \circ S \subseteq R$.
(4) $\bigcap \mathcal{U} = \Delta_X = \{(x, x) : x \in X\}$.
A family $\mathcal{B} \subseteq \mathcal{U}$ is a base for $\mathcal{U}$, if for any $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$.

If X is a metric space and $\varepsilon > 0$, set

$$\Delta_X^\varepsilon = \{(x_0, x_1) \in X \times X : d(x_0, x_1) < \varepsilon\}.$$

Then

$$\mathcal{U} = \{R \in \mathcal{P}(X \times X) : \exists \varepsilon > 0, \Delta_X^\varepsilon \subseteq R\}$$

is a uniformity and $\mathcal{B} = \{\Delta_X^\varepsilon : \varepsilon > 0\}$ is its base.

Proposition A.57. — *If $(X,\mathcal{U})$ is a uniform space, then the family*

$$\mathcal{T} = \{U \subseteq X : \forall x \in U, \exists R \in \mathcal{U}, R(x) \subseteq U\}$$

is a topology on X. The interior of a set $V \subseteq X$ is $V^\circ = \{x \in X : \exists R \in \mathcal{U}, R(x) \subseteq V\}$.

Proof. — Clearly $\emptyset, X \in \mathcal{T}$. Assume that $U, V \in \mathcal{U}$ and $x \in U \cap V$. There exist $R, S \in \mathcal{U}$ such that $R(x) \subseteq U$ and $S(x) \subseteq V$, so $(R \cap S)(x) \subseteq U \cap V$. Thus $U \cap V \in \mathcal{T}$. Assume that $U = \bigcup_{i \in I} U_i$ and $U_i \in \mathcal{T}$. If $x \in U$ then there exists $i \in I$ and $R \in \mathcal{U}$ such that $R(x) \subseteq U_i \subseteq U$, so $U \in \mathcal{T}$. Given $V \subseteq X$ set

$$W = \{x \in X : \exists R \in \mathcal{U}, R(x) \subseteq V\}.$$

We show that $V^\circ = W$. If $x \in V^\circ$ then V° is open, so there is $R \in \mathcal{V}$ such that $R(x) \subseteq V^\circ \subseteq V$, and $x \in W$. Conversely if $x \in W$ and $R(x) \subseteq V$, then there exists $S \in \mathcal{U}$ such that $S \circ S \subseteq R$. If $y \in S(x)$, then $S(y) \subseteq S \circ S(x) \subseteq R(x) \subseteq V$, so $S(x) \subseteq W$ and W is open. It follows that $W = V^\circ$. □

When we speak about topological properties of a uniform space, we refer to this canonical topology. For example any uniform space is Hausdorff.

Definition A.58. — Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be uniform spaces. A map $f : X \to Y$ is uniformly continuous, if for any $S \in \mathcal{V}$ there exists $R \in \mathcal{U}$ such that $(f \times f)(R) \subseteq S$.

Any uniformly continuous map is continuous.

Definition A.59. — Let $(X_i,\mathcal{U}_i)_{i\in I}$ be a family of uniform spaces, $X = \prod_{i\in I} X_i$ and $\pi_i : X \to X_i$ the projections. The product uniformity $\mathcal{U}$ on X is the smallest uniformity which makes all π_i uniformly continuous.

If $i \in I$ and $R_i \in \mathcal{U}_i$, then $(\pi_i \times \pi_i)^{-1}(R_i) = \{(x,y) \in X \times X : (x_i, y_i) \in R_i\}$ must belong to $\mathcal{U}$. The family of all finite intersections of such sets forms the canonical base of the product uniformity

$$\mathcal{B} = \left\{ \bigcap_{i\in K} (\pi_i \times \pi_i)^{-1}(R_i) : R_i \in \mathcal{U}_i, K \subseteq I \text{ finite} \right\}.$$

Proposition A.60. — *For any family $(f_i : (Y,\mathcal{V}) \to (X_i,\mathcal{U}_i))_{i\in I}$ of uniformly continuous maps there exists a unique uniformly continuous map $f : (Y,\mathcal{V}) \to (X,\mathcal{U})$ such that $\pi_i \circ f = f_i$.*

Proof. — The uniqueness of f is clear. We show that f is uniformly continuous. For $U \in \mathcal{U}$ there exists a finite set $K \subseteq I$ and sets $U_i \in \mathcal{U}_i$, where $i \in K$, such that $\bigcap_{i\in K}(\pi_i \times \pi_i)^{-1}(U_i) \subseteq U$. Since f_i are uniformly continuous, there exists $V \in \mathcal{V}$ such that $(f_i \times f_i)(V) \subseteq U_i$. It follows $(f \times f)(V) \subseteq U$. □

Proposition A.61. — *Let X be a compact Hausdorff (topological) space. Then the family*

$$\mathcal{U} = \{R \in \mathcal{P}(X \times X) : \Delta_X \subseteq R^\circ\}$$

of neighbourhoods of diagonal is a uniformity. Moreover $\mathcal{U}$ is the the unique uniformity which gives rise to the original topology on X.

Proof. — It is clear that $\mathcal{U}$ satisfy all uniformity axioms, except possibly the third. Let $R \in \mathcal{U}$. If $(y,z) \notin R^\circ$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $|f(y) - f(z)| > 1$. Set

$$U_f = \{(u,v) \in X \times X : |f(u) - f(v)| > 1\}$$

The set of all such U_f covers $(X \times X) \setminus R^\circ$, so by compactness, there exist continuous functions $f_1, \ldots, f_n$ such that $(X \times X) \setminus R \subseteq U_{f_1} \cup \cdots \cup U_{f_n}$. Set

$$\Delta_{\varepsilon, f_1,\ldots,f_n} = \{(y,z) \in X \times X : |f_i(y) - f_i(z)| < \varepsilon \text{ for } i = 1, \ldots, n\},$$

Then $\Delta_{\varepsilon, f_1,\ldots,f_n} \in \mathcal{U}$ is an open neighbourhood of the diagonal and $\Delta_{1, f_1,\ldots,f_n} \subseteq R$. For $S = \Delta_{\frac{1}{2}, f_1,\ldots,f_n}$ we have $S \circ S \subseteq \Delta_{1, f_1,\ldots,f_n} \subseteq R$. Thus $\mathcal{U}$ is a uniformity. We show that $\mathcal{U}$ generates the topology on X. Let U be an open set and $x \in U$. For any $y \in X \setminus U$ there exists an open set $V_u \ni y$ such that $x \notin V_y$. The open cover $\{U\} \cup \{V_y : y \in X \setminus U\}$ has a finite subcover $\{U\} \cup \{V_{y_1}, \ldots, V_{y_n}\}$. Then

$$R = (U \times U) \cup (V_{y_1} \times V_{y_n}) \cup \cdots \cup (V_{y_n} \times V_{y_n}) \in \mathcal{U},$$

and $R(x) = U$, so U is $\mathcal{U}$-open. Conversely, if U is $\mathcal{U}$-open and $x \in U$, then $x \in R(x) \subseteq U$ for some $R \in \mathcal{U}$ and $x \in U^\circ$, so U is open.

Assume now that $\mathcal{V}$ is a uniformity which generates the topology of X. If $S \in \mathcal{V}$, then by Proposition A.57, $S(x)$ is a neighbourhood of x, so there exists open V_x such that $x \in V_x \subseteq S(x)$. The open cover $\{V_x : x \in X$ has a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$ and

$$(V_{x_1} \times V_{x_1}) \cup \cdots \cup (V_{x_n} \times V_{x_n})$$

is a neighbourhood of the diagonal, so $\mathcal{V} \subseteq \mathcal{U}$. To show the other inclusion assume that $S \in \mathcal{U}$. For any $x \in X$ there exists $R_x \in \mathcal{V}$ such that $R_x(x) \times R_x(x) \subseteq S$. There exists $Q_x \in \mathcal{V}$ such that $Q_x \circ Q_x \subseteq R_x$. The open cover $\{Q_x(x) : x \in X\}$ has a finite subcover $\{Q_{x_1}(x_1), \ldots, Q_{x_n}(x_n)\}$. Set $Q = \bigcap_{i=1}^n Q_{x_i} \in \mathcal{V}$. If $(y,z) \in Q$ then $y \in Q_{x_i}(x_i)$ for some $i \leq n$ and $z \in Q(y) \subseteq Q_{x_i}(y)$, so $z \in (Q_{x_i} \circ Q_{x_i})(x_i) \subseteq R_{x_i}(x_i)$. Since $y \in Q_{x_i}(x_i) \subseteq R_{x_i}(x_i)$, we get $(y,z) \in R_{x_i}(x_i) \times R_{x_i}(x_i) \subseteq S$. Thus $Q \subseteq S$ and $S \in \mathcal{V}$. □

It follows that a mapping between compact Hausdorff spaces is continuous iff it is uniformly continuous.

Proposition A.62. — *Let $(X_i)_{i \in I}$ be compact uniform spaces. Then the product uniformity on $X = \prod_{i \in I} X_i$ gives rise to the product topology.*

Proof. — Let $\mathcal{T}$ be the product topology and $\mathcal{T}_\mathcal{U}$ the topology of the product uniformity. Since $\pi_i : (X, \mathcal{T}_\mathcal{U}) \to X_i$ are continuous, $\mathcal{T} \subseteq \mathcal{T}_\mathcal{U}$. On the other hand, since $f : (X, \mathcal{T}) \to X_i$ are continuous, they are uniformly continuous and the unique uniformity of $\mathcal{T}$ is larger than $\mathcal{U}$, and $\mathcal{T}_\mathcal{U} \subseteq \mathcal{T}$. □

A.8. Compact groups

Definition A.63. — An **Abelian group** is a triple $(X, 0, +)$, where X is a set, $0 \in X$ is a zero element, and $+ : X \times X \to X$ is an addition operation satisfying

(1) $(x + y) + z = x + (y + z)$,

(2) $x + y = y + x$,
(3) $x + 0 = x$,
(4) For any $x \in X$ there exists a unique $y = -x$ with $x + y = 0$.

A **group homomorphism** $f : (X, 0, +) \to (Y, 0, +)$ is a map $f : X \to Y$ which preserves the group structure, i.e., $f(0) = 0$, $f(x_0 + x_1) = f(x_0) + f(x_1)$, and therefore also $f(-x) = -f(x)$. If f is bijective, we say that X and Y are **isomorphic** and write $X \approx Y$. If f is injective, we say that X is a **subgroup** of Y. For $x \in X$ and positive integer n we set

$$nx = \overbrace{x + \cdots + x}^{n}, \quad (-n)x = -(nx)$$

If X, Y are groups, their product $X \times Y$ is a group with zero element $(0, 0)$ and operations defined by

$$(x_0, y_0) + (x_1, y_1) = (x_0 + y_0, x_1, y_1), \quad -(x, y) = (-x, -y)$$

Definition A.64. —
(1) An (Abelian) **metric group** is a structure $(X, d, 0, +)$, where (X, d) is a metric space, $(X, 0, +)$ is an Abelian group and the operations $+ : X \times X \to X$ and $- : X \to X$ are continuous.
(2) A **compact group** is a metric group which is compact as a metric space.
(3) A compact group is **monothetic** if there exists $x \in X$ such that $\{nx : n \in \mathbb{Z}\}$ is dense in X.
(4) A **discrete group** is a metric group with discrete metric.

The real line $\mathbb{R}$ is a metric group which is not compact. The circle $\mathbb{T}$ is a compact group. The unit circle in the complex plane is

$$\mathbb{K} = \{z \in \mathbb{C} : |z| = 1\}.$$

With complex multiplication, $(\mathbb{K}, \cdot, 1)$ is a compact group isomorphic to $(\mathbb{T}, +, 0)$ via $\varphi(x) = e^{2\pi i x}$. The group of integers $\mathbb{Z}$ is a discrete group which is not compact. Finite groups $\mathbb{Z}(n) = \{0, 1, \ldots, n-1\}$ with addition modulo n are both discrete and compact. Groups of p-adic integers (Definition 4.1) are compact.

Proposition A.65. — *If $(X, d, 0, +)$ is a compact group, then there exists an equivalent metric ϱ on X such that for any $x, y, z \in X$, $\varrho(x + z, y + z) = \varrho(x, y)$.*

Proof. — Since the addition $+ : X \times X \to X$ is continuous, it is uniformly continuous and for any $\varepsilon > 0$ there exists δ such that for all $x, y, z \in X$,

$$d(x, y) < \delta \;\Rightarrow\; d(x + z, y + z) < \varepsilon$$

Define the metric ϱ by

$$\varrho(x, y) = \sup\{d(x + z, y + z) : z \in X\}$$

Then $d(x, y) < \delta$ implies that $\varrho(x, y) < \varepsilon$ and $d(x, y) \le \varrho(x, y)$, so ϱ is equivalent to d and $\varrho(x + z, y + z) = \varrho(x, y)$. □

Definition A.66. — Let $(X, 0, +)$ be a metric Abelian group.
(1) A **character** of X is a continuous group homomorphism $\chi : (X, 0, +) \to (\mathbb{K}, 1, \cdot)$.

(2) The unit character $\mathbf{e}$ is defined by $\mathbf{e}(x) = 1$
(3) The product of characters $\chi, \psi : X \to \mathbb{K}$ is $(\chi \cdot \psi)(x) = \chi(x) \cdot \psi(x)$.
(4) The **dual group** $(\widehat{X}, \cdot, \mathbf{e})$ of X is the group of characters of X.

For any $n \in \mathbb{Z}$, $f_n(z) = z^n$ is a character of $\mathbb{K}$. Conversely if $\chi : \mathbb{K} \to \mathbb{K}$ is a character, then for a positive integer m and $a = e^{2\pi i/m}$ we get

$$\chi(a)^m = \chi(a^m) = \chi(1) = 1$$

and this is possible only if $\chi(a) = e^{2\pi i n/m} = a^n$ for some $n \in \mathbb{Z}$. It follows that this n must be equal for all $z \in \mathbb{K}$. Thus we get $\widehat{\mathbb{K}} = \{z^n : n \in \mathbb{Z}\}$ and this is isomorphic to $(\mathbb{Z}, 0, +)$. Conversely we regard $(\mathbb{Z}, 0, +)$ as a discrete group. A character $\chi : \mathbb{Z} \to \mathbb{K}$ is determined by the element $\chi(1) \in \mathbb{K}$ since $\chi(n) = \chi(1)^n$. Thus the dual group of $\mathbb{Z}$ is $\widehat{\mathbb{Z}} \approx \mathbb{K}$.

Proposition A.67. — *If X is a compact metric group, then the dual group $\widehat{X}$ is countable.*

Proof. — The dual group $\widehat{X}$ is a subset of the space $\mathbf{C}(X, \mathbb{C})$ of continuous complex-valued functions on X. Assume that $\chi \in \widehat{X}$ is a character $\chi \neq \mathbf{e}$. Pick a point $x \in X$ such that $\chi(x) \neq 1$. There exists $n \in \mathbb{Z}$ such that $\chi(nx) = \chi(x)^n$ has negative real part, so $d(\chi, \mathbf{e}) \geq d(\chi(nx), 1) \geq d(i, 1) = \sqrt{2}$. If $\chi, \xi \in \widehat{X}$ are different characters, then

$$d(\chi, \xi) = d(\chi/\xi, \mathbf{e}) \geq \sqrt{2}$$

Since the space $\mathbf{C}(X, \mathbb{C})$ has a countable dense set (Theorem A.38), the set of characters is at most countable. □

If X is a countable group (with discrete metric), then $\widehat{X} \subseteq \mathbb{K}^X$ and $\mathbb{K}^X$ with the product metric is a compact metric space. We show that $\widehat{X}$ is then a compact metric group.

Proposition A.68. — *If X is a countable group, then $\widehat{X}$ as a subspace of $\mathbb{K}^X$ with the product metric is a compact group.*

Proof. — Since $\mathbb{K}^X$ is compact, it suffices to show that $\widehat{X}$ is a closed subset of $\mathbb{K}^X$. If $\chi \in \mathbb{K}^X \setminus \widehat{X}$, then one of the equalities $\chi(0) = 1$, $\chi(-x) = \chi(x)^{-1}$ or $\chi(x+y) = \chi(x) \cdot \chi(y)$ is violated. Then it is violated in a neighbourhood of χ, so $\mathbb{K}^X \setminus \widehat{X}$ is closed. Since $|\frac{1}{x} - \frac{1}{y}| = |x - y|$ for $x, y \in \mathbb{K}$, the map $\chi \mapsto \chi^{-1}$ is an isometry in $\widehat{X}$. Similarly we get that the addition of characters is continuous. □

Thus the dual of a compact metric group is a countable discrete group and the dual of a countable discrete group is a compact metric group. If we apply this transformation twice we get for any group X (compact or countable) a map

$$\nu_X : X \to \widehat{\widehat{X}}$$

defined by $(\nu_X(x))(\chi) = \chi(x)$. The map ν_X is a group homomorphism, and if X is compact, then ν_X is continuous. If $\chi \in \widehat{X}$ and $\pi_\chi : \widehat{\widehat{X}} \to \widehat{X}$ is the projection then $\pi_\chi \circ \nu_X = \chi$ is continuous, hence ν_X is continuous.

Theorem A.69 **(Pontrjagin theorem).** — *If X is either a compact metric group or a countable discrete group, then $\nu_X : X \to \widehat{\widehat{X}}$ is a group isomorphism.*

The proof may be found in Pontrjagin [**130**], Hofman and Morris [**78**] or Hewitt and Ross [**76**].

We have already seen that $\widehat{\mathbb{K}} \approx \mathbb{Z}$ and $\widehat{\mathbb{Z}} \approx \mathbb{K}$. The groups $\mathbb{Z}(n) = \{0, \ldots, n-1\}$ are both compact and discrete and they are self-dual, $\widehat{\mathbb{Z}(n)} \approx \mathbb{Z}(n)$. If $\chi : \mathbb{Z}(n) \to \mathbb{K}$ is a character, then $\chi(1)^n = 1$, so $\widehat{\mathbb{Z}(n)} = \{\chi_0, \ldots, \chi_{n-1}\}$, where $\chi_k(x) = \exp(2\pi i k/n)$.

A consequence of the Pontrjagin duality theorem is that the dual group of a compact group separates points (Definition A.35).

Proposition A.70. — *Let X be a compact metric group, $\widehat{X} \subseteq$ $\mathbf{C}(X, \mathbb{C})$ its dual group and $\mathbf{L}(\widehat{X})$ the linear span of characters. Then $\widehat{X}$ separates points and $\mathbf{L}(\widehat{X})$ is dense in $\mathbf{C}(X, \mathbb{C})$.*

Proof. — If $x \neq y$, then $\nu_X(x) \neq \nu_X(y)$, so there exists a character $\chi \in \widehat{X}$ such that

$$\chi(x) = (\nu_X(x))(\chi) \neq (\nu_X(y))(\chi) = \chi(y)$$

Since $\mathbf{L}(\widehat{X})$ separates points and is closed with respect to the operations of product and complex conjugation, $\mathbf{L}(\widehat{X})$ is dense in $\mathbf{C}(X, \mathbb{C})$ by the Stone-Weierstrass theorem A.37. ☐

A.9. Perron-Frobenius theory

A nonnegative square $(n \times n)$-matrix can be interpreted as a flow in a graph with n vertices. The t-th power of such a matrix then represents the flow in time t. Perron-Frobenius theory estimates this flow using the largest positive eigenvalue of the matrix. We say that a matrix $M = (M_{ab})_{a,b \in A}$ is **nonnegative** ($M \geq 0$), if $M_{ab} \geq 0$ for all $a, b \in A$ and **positive** ($M > 0$), if $M_{ab} > 0$ for all $a, b \in A$. Similarly we speak about nonnegative and positive vectors $u = (u_a)_{a \in A}$.

Definition A.71. — Let $(M_{ab})_{a,b \in A}$ be a nonnegative square matrix.
(1) M is **irreducible**, if for any $a, b \in A$ there exists $n > 0$ with $(M^n)_{ab} > 0$.
(2) M is **primitive**, if there exists $n > 0$ such that M^n is positive.
(3) The **path relation** $\mathcal{P} \subseteq A \times A$ of M is defined by

$$(a, b) \in \mathcal{P} \quad \Leftrightarrow \quad \exists n > 0, (M^n)_{ab} > 0$$

A complex number λ is an eigenvalue of $M = (M_{ab})_{a,b \in A}$, if there exists a nonzero complex (right) eigenvector $u = (u_a)_{a \in A}$ such that $Mu = \lambda u$ i.e., $\sum_{b \in A} M_{ab} u_b = \lambda u_a$ for any $a \in A$. If λ is an eigenvalue of M then $(M - \lambda I)u = 0$, where I is the identity matrix, so λ is a solution of algebraic equation $\det(M - \lambda I) = 0$ of degree $\#A$. There are at most $\#A$ eigenvalues of M. Denote by $\mathbb{R}^A$ the set of vectors indexed by A. For a vector $x \in \mathbb{R}^A$ let $|x| \in \mathbb{R}^A$ be the vector of its absolute values, $|x|_a = |x_a|$ and $||x|| = \sqrt{\sum_{a \in A} x_a^2}$ its norm.

Theorem A.72 **(Perron-Frobenius theorem).** — *Let M be a primitive matrix. There exists $\lambda = \lambda(M) > 0$ called the* **spectral radius** *of M, and a positive vector u such that*

(1) $Mu = \lambda u$ (λ is an eigenvalue of M and u is its right eigenvector).
(2) Any right eigenvector of λ is a multiple of u.
(3) If μ is an eigenvalue of M and $\mu \neq \lambda$ then $|\mu| < \lambda$.
(4) If v is the left eigenvector of λ with $\sum_{a \in A} u_a v_a = 1$, then

$$\lim_{n \to \infty} \frac{(M^n)_{ab}}{\lambda^n} = u_a v_b.$$

Proof. —

(1) Set $\lambda = \sup\{a > 0 : \exists x \in \mathbb{R}^A, x \geq 0, \ ax \leq Mx\}$. For $x = (1, \ldots, 1)$ we have $Mx > 0$ so $\lambda > 0$. There exists a sequence $(a_n)_{n \geq 0}$ converging to λ and nonnegative vectors $x^{(n)}$ for which $a_n x^{(n)} \leq M x^{(n)}$ and $||x^{(n)}|| = 1$. In the compact space $\{x \in \mathbb{R}^A : x \geq 0, ||x|| = 1\}$ there exists a convergent subsequence $x^{(n_i)} \to u$ as $i \to \infty$ and $\lambda u \leq Mu$. Suppose that $v = Mu - \lambda u \geq 0$ is nonzero. There exists $p > 0$ such that $M^p > 0$ and

$$M(M^p u) - \lambda(M^p u) = M^p v > 0,$$

so there exists $\lambda' > \lambda$ for which $\lambda'(M^p u) \leq M(M^p u)$ and this is a contradiction. Thus $\lambda u = Mu$ and u is an eigenvector. Since $\lambda^p u = M^p u > 0$, u is positive.

(2) Suppose that $Mv = \lambda v$ and v is independent of u. We can assume that v is real. There exists a linear combination $w = cu + dv$ which is nonnegative but not positive. Then $\lambda^p w = M^p w > 0$ so $w > 0$ and this is a contradiction.

(3) Let μ be an eigenvalue of M and v its eigenvector so $Mv = \mu v$. Both μ and v may be complex. We get

$$|\mu| \cdot |v_a| = \left| \sum_{b \in A} M_{ab} v_b \right| \leq \sum_{b \in A} M_{ab} |v_b|.$$

By the definition of λ, $|\mu| \leq \lambda$. Assume that $|\mu| = \lambda$. We get $\lambda |v| \leq M|v|$. As in the proof of (1) it follows that $\lambda |v| = M|v|$, so $|v|$ is an eigenvector of λ and therefore a multiple of u. Thus

$$\sum_{b \in A} (M^p)_{ab} |v_b| = \lambda^p |v_a| = \left| \sum_{b \in A} (M^p)_{ab} v_b \right|.$$

It can be proved by induction on the size of A, that this is possible only if v can be obtained from $|v|$ by multiplication by a complex unit, $v_a = |v_a| \cdot e^{i\alpha}$, so v is a multiple of u, $\lambda u = Mu = \mu u$ and $\mu = \lambda$.

(4) For a nonnegative vector x set

$$r_0(x) = \min\left\{ \frac{x_a}{u_a} : a \in A \right\}, \quad r_1(x) = \max\left\{ \frac{x_a}{u_a} : a \in A \right\}.$$

For any $a \in A$ we get

$$\frac{(Mx)_a}{\lambda u_a} = \sum_{b \in A} \frac{M_{ab} u_b}{\lambda u_a} \cdot \frac{x_b}{u_b} \geq r_0(x) \cdot \sum_{b \in A} \frac{M_{ab} u_b}{\lambda u_a} = r_0(x).$$

Similarly for $r_1(x)$, so

$$r_0(x) \leq r_0\left(\frac{Mx}{\lambda}\right) \leq r_1\left(\frac{Mx}{\lambda}\right) \leq r_1(x).$$

There exist limits of monotone sequences

$$s_0(x) = \lim_{n\to\infty} r_0\left(\frac{M^n x}{\lambda^n}\right), \quad s_1(x) = \lim_{n\to\infty} r_1\left(\frac{M^n x}{\lambda^n}\right),$$

and $s_0(x) \le s_1(x)$. We show $s_0(x) = s_1(x)$. For some p with $M^p > 0$ set

$$m = \min\left\{\frac{(M^p)_{ab}u_b}{u_a} : a, b \in A\right\} > 0$$

Let x be a nonnegative vector and $c \in A$ such that $\frac{x_c}{u_c} = r_1(x)$. Then

$$\begin{aligned}\frac{(M^p x)_a}{\lambda^p u_a} &= \sum_{b\neq c} \frac{(M^p)_{ab}u_b}{\lambda^p u_a}\cdot\frac{x_b}{u_b} + \frac{(M^p)_{ac}u_c}{\lambda^p u_a}\cdot\frac{x_c}{u_c} \\ &\ge \sum_{b\neq c} \frac{(M^p)_{ab}u_b}{\lambda^p u_a}\cdot r_0(x) + \frac{(M^p)_{ab}u_c}{\lambda^p u_a}\cdot(r_0(x) + r_1(x) - r_0(x)) \\ &\ge r_0(x) + m(r_1(x) - r_0(x))\end{aligned}$$

By a similar argument for the lower bound we obtain

$$\begin{aligned} r_0\left(\frac{M^p x}{\lambda^p}\right) &\ge (1-m)r_0(x) + mr_1(x) \\ r_1\left(\frac{M^p x}{\lambda^p}\right) &\le (1-m)r_1(x) + mr_0(x) \\ r_1\left(\frac{M^p x}{\lambda^p}\right) - r_0\left(\frac{M^p x}{\lambda^p}\right) &\le (1-2m)(r_1(x) - r_0(x))\end{aligned}$$

Thus there exists a limit

$$\lim_{n\to\infty}\frac{(M^n x)_a}{\lambda^n u_a} = s(x), \text{ or } \lim_{n\to\infty}\frac{M^n x}{\lambda^n} = s(x)u.$$

Since the transposed matrix M^T is primitive too, it has the same spectral radius λ, as M and a positive right eigenvector v, which is a left eigenvector of M. Thus $vM = \lambda M$ and we assume that $v \cdot u = 1$. For $a \in A$, let e_a be the unit vector such that $(e_a)_a = 1$, $(e_a)_b = 0$ for $b \neq a$. For $a, b \in A$ we get

$$\begin{aligned}\lim_{n\to\infty}\frac{(M^n)_{ab}}{\lambda^n} &= \lim_{n\to\infty}\frac{(M^n e_b)_a}{\lambda^n} = s(e_b)u_a \\ \lim_{n\to\infty}\frac{(M^n)_{ab}}{\lambda^n} &= \lim_{n\to\infty}\frac{(e_a M^n)_b}{\lambda^n} = s'(e_a)v_b\end{aligned}$$

Thus there exists $l > 0$ such that for any $a, b \in A$, $\frac{s'(e_a)}{u_a} = \frac{s(e_b)}{v_b} = l$. Since the trace $\mathrm{tr}(M) = \sum_{a\in A} M_{aa}$ of any matrix is the sum of its eigenvalues, we get

$$1 = \lim_{n\to\infty}\sum_{a\in A}\frac{(M^n)_{aa}}{\lambda^n} = \sum_{a\in A} s(e_a)u_a = l\sum_{a\in A} v_a u_a.$$

Thus $l = 1$ and the result follows. □

We have a weaker version of Perron-Frobenius theorem for irreducible matrices.

Theorem A.73. — *Let M be an irreducible matrix. There exists $\lambda = \lambda(M) > 0$ called the* **spectral radius** *of M, and a positive vector u such that*
(1) $Mu = \lambda u$ (λ is an eigenvalue of M and u is its right eigenvector).

(2) Any right eigenvector of λ is a multiple of u.
(3) If μ is an eigenvalue of M then $|\mu| \leq \lambda$.
(4) There exist numbers $0 < c_0 \leq 1 \leq c_1$ such that for any $n > 0$ and $a \in A$,

$$c_0\lambda^n \leq \sum_{b\in A}(M^n)_{ab} \leq c_1\lambda^n.$$

Proof. —

(1) For any $\varepsilon > 0$, $M+\varepsilon I$ is a primitive matrix. If $(M^n)_{ab} > 0$, then $(M+\varepsilon I)^{n+m}_{ab} > 0$ for any $m \geq 0$ since $M + \varepsilon I$ has positive diagonal. We have

$$Mu = \lambda u \text{ iff } (M + \varepsilon I)u = (\lambda + \varepsilon)u.$$

For $w = (1, \dots, 1)$ we have $(M + \varepsilon I)w > \varepsilon w$, so the spectral radius λ_ε of $M + \varepsilon I$ is greater than ε. Moreover, $\lambda = \lambda_\varepsilon - \varepsilon$ is a positive eigenvalue of M and u is its positive right eigenvector.

(2) If v is a right eigenvector of λ for M then it is also an eigenvector of $\lambda + \varepsilon$ for $M + \varepsilon I$, so it is a multiple of u.

(3) If μ is an eigenvalue of M, then $\mu+\varepsilon$ is an eigenvalue of $M+\varepsilon I$, so $|\mu+\varepsilon| < \lambda+\varepsilon$. Since this holds for any $\varepsilon > 0$, $|\mu| \leq \lambda$.

(4) Set

$$d_0 = \min\{u_a : a \in A\}, \quad d_1 = \max\{u_a : a \in A\}.$$

Then for any $a \in A$,

$$\frac{d_0}{d_1}\lambda^n \leq \frac{\lambda^n u_a}{d_1} = \sum_{b\in A} \frac{(M^n)_{ab}u_b}{d_1} \leq \sum_{b\in A}(M^n)_{ab} \leq \sum_{b\in A} \frac{(M^n)_{ab}u_b}{d_0} = \frac{\lambda^n u_a}{d_0} \leq \frac{d_1}{d_0}\lambda^n.$$

□

Definition A.74. — The **spectral radius** of a nonnegative matrix $M = (M_{ab})_{a,b\in A}$ with positive rows is the maximum of spectral radii of its irreducible components

$$\lambda(M) = \max\{\lambda((M_{ab})_{a,b\in B}) : B \subseteq A \text{ is a path component}\}.$$

A.10. Continued fractions

Continued fractions give rise to an interesting symbolic representation of real numbers. An **infinite continued fraction** is an expression

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0, a_1, a_2, a_3, \ldots]$$

We first define **finite continued fractions** by

$$[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \frac{1}{[a_1, a_2]} = [a_0, [a_1, a_2]],$$

$$[a_0, a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_n]} = [a_0, [a_1, \ldots, a_n]]$$

For example, $[0] = [-1, 1] = 0$, $[1] = [0, 1] = 1$, $[0, 2] = [0, 1, 1] = \frac{1}{2}$. A finite continued fraction can be written as a rational function of its variables $a_0, a_1, \ldots$ as

$[a_0, a_1, \ldots, a_n] = p_n(a_0, \ldots, a_n)/q_n(a_0, \ldots, a_n)$. We assume in general that a_i are real numbers.

Definition A.75. — Let $a_0, a_1, a_2, \ldots$ be real numbers. Their **convergents** $p_n(a_0, \ldots, a_n)$, $q_n(a_0, \ldots, a_n)$ are defined by recurrences

$$\begin{array}{lllll} p_{-1} = 1, & p_0 = a_0, & p_1 = 1 + a_1 a_0, & \ldots & p_n = p_{n-2} + a_n p_{n-1} \\ q_{-1} = 0, & q_0 = 1, & q_1 = a_1, & \ldots & q_n = q_{n-2} + a_n q_{n-1} \end{array}$$

Proposition A.76. — *Let $a_1, \ldots, a_n$ be positive real numbers, a_0 arbitrary and p_i, q_i their convergents. Then for any $n \geq 0$,*
(1) $[a_0, a_1, \ldots, a_n] = p_n/q_n$,
(2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$,
(3) $p_{n+1} q_{n-1} - p_{n-1} q_{n+1} = (-1)^{n+1} a_{n+1}$.

Proof. —
(1) For $n = 0, 1$ we have $[a_0] = a_0 = p_0/q_0$, $[a_0, a_1] = a_0 + 1/a_1 = p_1/q_1$. For $n \geq 2$, $[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_{n-2}, a'_{n-1}]$ where $a'_{n-1} = [a_{n-1}, a_{n-2}] = a_{n-1} + \frac{1}{a_n}$. By the induction hypothesis $[a_0, a_1, \ldots, a_{n-2}, a'_{n-1}] = \frac{p'_{n-1}}{q'_{n-1}}$, where

$$\begin{aligned} p'_{n-1} &= p_{n-3} + a'_{n-1} p_{n-2} = p_{n-3} + a_{n-1} p_{n-2} + \frac{p_{n-2}}{a_n} = p_{n-1} + \frac{p_{n-2}}{a_n} \\ q'_{n-1} &= q_{n-3} + a'_{n-1} q_{n-2} = q_{n-3} + a_{n-1} q_{n-2} + \frac{q_{n-2}}{a_n} = q_{n-1} + \frac{q_{n-2}}{a_n} \end{aligned}$$

so

$$[a_0, \ldots, a_n] = \frac{p'_{n-1}}{q'_{n-1}} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}$$

(2),(3): For $n = 0$ we have $p_0 q_{-1} - p_{-1} q_0 = -1$, $p_1 q_{-1} - p_{-1} q_1 = -a_1$. For $n > 1$

$$\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (p_{n-2} + a_n p_{n-1}) q_{n-1} - p_{n-1}(q_{n-2} + a_n q_{n-1}) = \\ &\quad -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = -(-1)^n = (-1)^{n+1} \\ p_{n+1} q_{n-1} - p_{n-1} q_{n+1} &= (p_{n-1} + a_{n+1} p_n) q_{n-1} - p_{n-1}(q_{n-1} + a_n q_n) = \\ &\quad a_{n+1}(p_n q_{n-1} - p_{n-1} q_n) = (-1)^{n+1} a_{n+1} \end{aligned}$$

□

In an infinite continued fraction $[1, 1, 1, \ldots]$ both $p_n = F_{n+2}$ and $q_n = F_{n+1}$ are Fibonacci numbers $F_n = (\alpha^{-n} - (-\alpha)^n)/\sqrt{5}$, where $\alpha = \frac{\sqrt{5}-1}{2}$ (Example 1.39). Thus the limit $\lim_{n\to\infty} p_n/q_n$ exists and is $\frac{\sqrt{5}+1}{2} \approx 1.618$. We show that this limit exists for any continued fraction whose every entry except the zeroth is a positive integer. Denote by $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ the set of positive integers.

Proposition A.77. — *Let $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}^+$ for $i > 0$. Then*
(1) p_n and q_n are relatively prime for $n \geq 0$.
(2) $q_n \geq F_{n+1}$.
(3) There exists a limit $\alpha = [a_0, a_1, a_2, \ldots] = \lim\limits_{n\to\infty} [a_0, a_1, a_2, \ldots, a_n]$.

Proof. —
(1) Since $|p_{n+1}q_n - p_n q_{n+1}| = 1$, p_n and q_n do not have common divisor.
(2) We have $q_{-1} = F_0$, $q_0 = F_1$, and $q_n \geq q_{n-2} + q_{n-1} \geq F_{n-1} + F_n = F_{n+1}$.
(3) We have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_n q_{n+1}}, \quad \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{(-1)^{n+1}a_{n+1}}{q_{n+1}q_{n-1}}\right| \leq \frac{1}{q_n q_{n-1}},$$

so

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Since $q_n \to \infty$ as $n \to \infty$, p_n/q_n is a converging sequence. □

Proposition A.78. — *Any irrational number α can be expressed in a unique way as $\alpha = [a_0, a_1, a_2, \ldots]$, where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}^+$ for $i > 0$. Moreover,*

$$\eta_n = (-1)^n (q_n \alpha - p_n)$$

satisfies

$$\frac{1}{q_n + q_{n+1}} < \eta_n < \frac{1}{q_{n+1}}, \quad \eta_{n+1} = \eta_{n-1} - a_{n+1}\eta_n, \quad \eta_{n+1} < \eta_n.$$

Proof. — Denote by $\lfloor a \rfloor$ the integer part of a, so $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$. Suppose that α is irrational and set $\alpha_0 = \alpha$, $a_0 = \lfloor \alpha_0 \rfloor$, $\alpha_1 = \frac{1}{\alpha_0 - a_0} > 1$, so $\alpha_0 = [a_0, \alpha_1]$. We continue by induction;

$$a_n = \lfloor \alpha_n \rfloor, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n} > 1, \text{ so } \alpha_n = a_n + \frac{1}{\alpha_{n+1}} = [a_n, \alpha_{n+1}]$$

and

$$\alpha_0 = [a_0, \alpha_1] = [a_0, [a_1, \alpha_2]] = [a_0, a_1, \alpha_2] = \cdots = [a_0, a_1, \ldots, a_n, \alpha_{n+1}].$$

If α is irrational this procedure never ends and for $i > 1$ we get $\alpha_i > 1$ and $a_i > 0$. We show $\alpha = [a_0, a_1, \ldots]$.

$$\begin{aligned}
\alpha &= [a_0, a_1, \ldots, a_n, \alpha_{n+1}] = \frac{p_{n-1} + \alpha_{n+1}p_n}{q_{n-1} + \alpha_{n+1}q_n} \\
\eta_n &= (-1)^n \cdot \frac{q_n(p_{n-1} + \alpha_{n+1}p_n) - p_n(q_{n-1} + \alpha_{n+1}q_n)}{q_{n-1} + \alpha_{n+1}q_n} \\
&= (-1)^n \cdot \frac{q_n p_{n-1} - p_n q_{n-1}}{q_{n-1} + \alpha_{n+1}q_n} = \frac{1}{q_{n-1} + \alpha_{n+1}q_n} \leq \frac{1}{q_{n+1}} \\
\left|\alpha - \frac{p_n}{q_n}\right| &= \frac{\eta_n}{q_n} \leq \frac{1}{q_n q_{n+1}}
\end{aligned}$$

Thus $\alpha = [a_0, a_1, \ldots] = \lim_{n\to\infty}[a_0, \ldots, a_n]$. We have already shown that $\eta_n < \frac{1}{q_{n+1}}$. On the other hand

$$\eta_n = \frac{1}{q_{n-1} + \alpha_{n+1}q_n} \geq \frac{1}{q_{n-1} + (a_{n+1}+1)q_n} = \frac{1}{q_{n+1} + q_n} \geq \frac{1}{q_{n+2}} > \eta_{n+1}.$$

The recursive formulas for p_n and q_n yield $\eta_n = \eta_{n-2} - a_n\eta_{n-1}$.

We show that the a_i are unique. Assume that $a_0 \in \mathbb{Z}$, $a_1, a_2, \ldots$ are positive integers and

$$\alpha = [a_0, a_1, \ldots] = a_0 + \frac{1}{[a_1, a_2, \ldots]}$$

Since $[a_1, a_2, \ldots] > 1$, $a_0 < \alpha < \alpha + 1$, so $a_0 = \lfloor \alpha \rfloor$ is unique. It follows that $\alpha_1 = 1/(\alpha - a_1) = [a_1, a_2, \ldots]$ is unique and by induction all a_i are unique. □

We have already seen that $\frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$. For $\alpha = \sqrt{2} + 1$ we have $a_0 = 2$, $\alpha_1 = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$, so $a_1 = 2$ and the process repeats, so $\sqrt{2} + 1 = [2, 2, 2, \ldots]$. More generally, we have

***Theorem A.79* (Lagrange theorem).** — *An irrational number α has an eventually periodic continued fraction iff it is a solution of a quadratic equations with integer coefficients.*

Proof. — Assume that $\alpha = [a_0, a_1, \ldots]$ is eventually periodic. There exists $0 < n < m$ such that $\alpha = [a_0, \ldots, a_{n-1}, \beta] = [a_0, \ldots, a_{m-1}, \beta]$, or

$$\alpha = \frac{p_{n-2} + p_{n-1}\beta}{q_{n-2} + q_{n-1}\beta} = \frac{p_{m-2} + p_{m-1}\beta}{q_{m-2} + q_{m-1}\beta}$$

This is a quadratic equation for β. Since β is a rational function of α, α is also a quadratic number. Conversely assume that $\alpha = [a_0, a_1, \ldots]$ is an irrational which satisfies quadratic equation $a\alpha^2 + b\alpha + c = 0$. There exist α_n such that

$$\alpha = [a_0, \ldots, a_{n-1}, \alpha_n] = \frac{p_{n-2} + p_{n-1}\alpha_n}{q_{n-2} + q_{n-1}\alpha_n},$$

so α_n satisfies an equation $A_n^2\alpha_n^2 + B_n\alpha_n + C_n$, where

$$\begin{aligned}
A_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2, \\
B_n &= 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2cq_{n-1}q_{n-2}, \\
C_n &= ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2.
\end{aligned}$$

We get $C_n = A_{n-1}$, and

$$B_n^2 - A_nC_n = (b^2 - 4ac)(p_{n-1}q_{n-2} - q_{n-1}p_{n-2})^2 = b^2 - 4ac$$

so the discriminant does not depend on n. Set $\delta_n = \frac{p_n}{q_n} - \alpha$, so $p_n = \alpha q_n + \frac{\delta_n}{q_n}$ and $|\delta_n| < 1$. Then

$$\begin{aligned}
A_n &= a\left(\alpha q_{n-1} + \frac{\delta_{n-1}}{q_{n-1}}\right)^2 + b\left(\alpha q_{n-1} + \frac{\delta_{n-1}}{q_{n-1}}\right) + cq_{n-1}^2 \\
&= (a\alpha^2 + b\alpha + c)q_{n-1}^2 + 2a\alpha\delta_{n-1} + a\frac{\delta_{n-1}^2}{q_{n-1}^2} + b\delta_{n-1}
\end{aligned}$$

Since $a\alpha^2 + b\alpha + c = 0$, we get

$$|A_n| < 2|a\alpha| + |a| + |b|, \quad B_n^2 < b^2 - 4ac + 4A_nC_n.$$

Thus the three sequences A_n, B_n, C_n are bounded and for some $n < m$, $\alpha_n = \alpha_m$. □

***Proposition A.80*.** — *Any rational number α can be expressed in exactly two ways as a finite continued fraction*

$$\alpha = [a_0, a_1, \ldots, a_{n-1}, a_n, 1] = [a_0, a_1, \ldots, a_{n-1}, a_n + 1]$$

where $n \geq 0$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{N}^+$ for $0 < i \leq n$.

Proof. — Suppose that $\alpha_0 = r_0/r_1$ is rational and r_0, r_1 are relatively prime. If $r_1 = 1$ then α is an integer and can be expressed in exactly two ways $\alpha = [r_0] = [r_0 - 1, 1]$. If $r_1 > 1$ then by division algorithm there exist unique $a_0 = \lfloor \frac{r_0}{r_1} \rfloor$, $0 \le r_2 < r_1$ satisfying $r_0 = a_0 r_1 + r_2$. By induction we get unique a_i, q_i, α_i such that

$$\begin{array}{rcll} r_0 & = & a_0 r_1 + r_2 & 0 < r_2 < r_1, \quad \alpha_1 = \dfrac{1}{\alpha_0 - a_0} = \dfrac{r_1}{r_0 - a_0 r_1} = \dfrac{r_1}{r_2} \\ r_1 & = & a_1 r_2 + r_3 & 0 < r_3 < r_2, \quad \alpha_2 = \dfrac{1}{\alpha_1 - a_1} = \dfrac{r_2}{r_1 - a_1 r_2} = \dfrac{r_2}{r_3} \\ & \vdots & & \\ r_{n-1} & = & a_{n-1} r_n + r_{n+1} & 0 < r_{n+1} < r_n, \quad \alpha_n = \dfrac{r_n}{r_{n+1}} = a_n + 1 \\ r_n & = & (a_n + 1) r_{n+1} & 0 = r_{n+2} \end{array}$$

Here n is the first integer for which r_{n+1} divides r_n, so $a_n + 1 = \frac{r_n}{r_{n+1}} > 1$ and $r_{n+2} = 0$. We get

$$\alpha = [a_0, \alpha_1] = \cdots = [a_0, \ldots, a_{n-1}, a_n + 1] = [a_0, \ldots, a_{n-1}, a_n, 1].$$

If α is not integer, then a_0 is unique. By induction all the a_i except the last are unique. □

The ambiguity in the expression of rational numbers in continued fractions is reflected in their approximation by longer continued fractions. A finite continued fraction may be regarded as an infinite continued fraction with infinite entries $[a_0, \ldots, a_n] = [a_0, \ldots, a_n, \infty, \infty, \ldots]$. We regard continued fractions as (finite or infinite) symbolic sequences of integers and distinguish them from their values. A suitable metric makes this assignment continuous. We say that a continued fraction $[a_0, a_1 \ldots]$ is **simple** if $a_0 \in \mathbb{Z}$ and a_i are positive integers for $i > 0$.

Definition A.81. — The space $\mathcal{Z}$ of simple continued fractions is

$$\begin{array}{rcl} \mathcal{Z} & = & \{a = (a_i)_{0 \le i < |a|} : a_0 \in \mathbb{Z}, \forall i > 0, a_i \in \mathbb{N}^+, 0 < |a| \le \infty\} \\ \mathcal{N} & = & \{a \in \mathcal{Z} : a_0 = 0\} \end{array}$$

with metric

$$d(a, b) = |a_0 - b_0| + \sum_{n=1}^{\infty} \left| \frac{1}{a_n} - \frac{1}{b_n} \right| / (F_n)^2$$

Here F_n is the n-th Fibonacci number and we set $1/a_i = 0$ when a is a finite continued fraction of length n and $i > n$. Thus $d((a_0), (b_0)) = |a_0 - b_0|$, $d((a_0), (b_0, b_1)) = |a_0 - b_0| + 1/b_1$. We therefore distinguish simple continued fractions $a = (a_0, a_1, \ldots) \in \mathcal{Z}$ from their values $\pi(a) = [a_0, a_1, \ldots] \in \mathbb{R}$. The space $\mathcal{Z}$ is totally disconnected but not compact. The space $\mathcal{N}$ is totally disconnected and compact since it is an infinite product of compact space $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \subseteq \mathbb{I}$.

Corollary A.82. — *The value assignment $\pi : \mathcal{Z} \to \mathbb{R}$ given by $\pi(a) = [a]$ is continuous. An irrational $\alpha \in \mathbb{R}$ has exactly one preimage, a rational $\alpha \in \mathbb{R}$ has exactly two preimages.*

Proof. — Let $a, b \in \mathcal{Z}$ and $d(a,b) < \varepsilon < 1$. Set $n = \min\{i \geq 0 : a_i \neq b_i\}$. If $\varepsilon < 1$ then $n \geq 1$. Assume first that the length of both a and b is at least n, so $[a] = [a_0, \ldots, a_{n-1}, \alpha_n]$, $[b] = [a_0, \ldots, a_{n-1}, \beta_n]$. We have $a_n \leq \alpha_n < a_n + 1$, $b_n \leq \beta_n < b_n + 1$, so $|\alpha_n - \beta_n| \leq |a_n - b_n| + 1$. Using the formula from the proof of Proposition A.78, we get

$$[a]q_{n-1} - p_{n-1} = \frac{(-1)^{n-1}}{\alpha_n q_{n-1} + q_{n-2}}, \quad [b]q_{n-1} - p_{n-1} = \frac{(-1)^{n-1}}{\beta_n q_{n-1} + q_{n-2}}$$

$$\begin{aligned} |[a] - [b]| &\leq \left| \frac{\alpha_n - \beta_n}{(\alpha_n q_{n-1} + q_{n-2})(\beta_n q_{n-1} + q_{n-2})} \right| \leq \frac{|a_n - b_n| + 1}{a_n b_n q_{n-1}^2} \\ &\leq \frac{2}{q_{n-1}^2} \left| \frac{1}{a_n} - \frac{1}{b_n} \right| \leq \frac{2\varepsilon F_n^2}{q_{n-1}^2} \leq 2\varepsilon \end{aligned}$$

If the length of b is n, then $[a] = [a_0, \ldots, a_{n-1}, \alpha_n]$, $[b] = [a_0, \ldots, a_{n-1}]$, and

$$\begin{aligned} |[a] - [b]| &= \left| \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}} \right| = \frac{|p_{n-2}q_{n-1} - p_{n-1}q_{n-2}|}{(\alpha_n q_{n-1} + q_{n-2})q_{n-1}} \\ &\leq \frac{1}{a_n q_{n-1}^2} \leq \frac{\varepsilon F_n^2}{q_{n-1}^2} < \varepsilon \end{aligned}$$

This shows that the function π is continuous. □

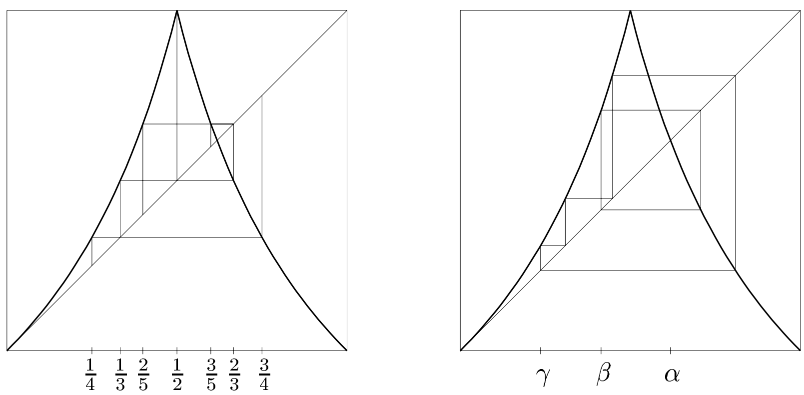

FIGURE 10. Continued fractions

On the space $\mathcal{N}$ there is a dynamical system which is analogous to the shift map on $A^{\mathbb{N}}$. Moreover its factor is an interesting dynamical system on the unit interval (Figure 10).

Proposition A.83. — *The map π is a factor map $\pi : (\mathcal{N}, \Phi) \to (\mathbb{I}, \varphi)$, where*

$$\Phi(0, a_1, a_2, \ldots) = \begin{cases} (0, a_1 - 1, a_2, \ldots) & \text{if } a_1 > 1 \\ (0, a_2, a_3, \ldots) & \text{if } a_1 = 1 \end{cases}, \quad \varphi(\alpha) = \begin{cases} \frac{\alpha}{1-\alpha} & \text{if } \alpha \leq \frac{1}{2} \\ \frac{1-\alpha}{\alpha} & \text{if } \alpha \geq \frac{1}{2} \end{cases}$$

Proof. — Let $a = (0, a_1, a_2, \ldots)$. Then

$$\alpha = \pi(a) = [0, a_1, \alpha_2] = \frac{1}{a_1 + 1/\alpha_2} = \frac{\alpha_2}{a_1 \alpha_2 + 1}, \quad 1 - \alpha = \frac{1 + (a_1 - 1)\alpha_2}{a_1 \alpha_2 + 1}$$

If $a_1 > 1$, then $\alpha \le \frac{1}{2}$ and

$$\varphi\pi(a) = \varphi(\alpha) = \frac{\alpha}{1-\alpha} = \frac{1}{a_1 - 1 + 1/\alpha_2} = [0, a_1 - 1, \alpha_2] = \pi\Phi(a).$$

If $a_1 = 1$, then $\alpha \ge \frac{1}{2}$ and

$$\varphi\pi(a) = \varphi(\alpha) = \frac{1-\alpha}{\alpha} = \frac{1}{\alpha_2} = [0, \alpha_2] = \pi\Phi(a)$$

so $\varphi\pi = \pi\Phi$. It is clear that both Φ and ϕ are continuous. □

Observe that $\{x \in \mathbb{I} : \exists i \ge 0 : \varphi^i(x) = 0\} = \mathbb{Q}$ is just the set of rational numbers (Figure 10 left). Other periodic or preperiodic points of $(\mathbb{I}, \varphi)$ are quadratic numbers - solutions of quadratic equations with integer coefficients. In Figure 10 right we see the fixed point $\alpha = [0, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2}$ and periodic points $\beta = [0, 2, 2, \ldots] = \sqrt{2} - 1$ and $\gamma = [0, 4, 4, \ldots] = \sqrt{5} - 2$.

Given $\alpha = [0, a_1, a_2, \ldots]$, the convergents p_n/q_n are best approximations to α. In Figure 11 we see the convergents of $\alpha = [0, 2, 2, 2, \ldots] = \sqrt{2} - 1 \doteq 0.414$. The set of points with integer coordinates is divided into two parts according to whether they lie above or below the line $y = \alpha x$

$$E_0 = \{(q, p) \in \mathbb{N}^2 : p < \alpha q\}, \quad E_1 = \{(q, p) \in \mathbb{N}^2 : p > \alpha q\}$$

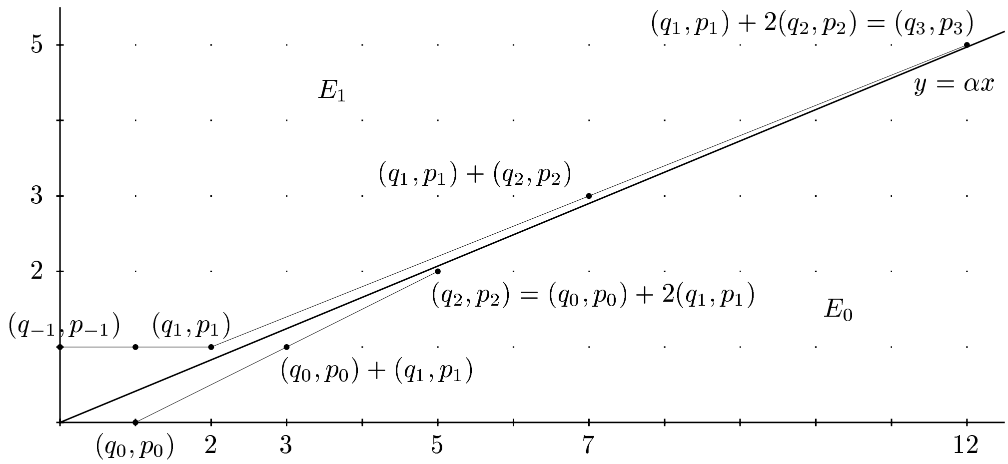

FIGURE 11. Klein Theorem

All $(q, p) \in \mathbb{N}^2$ except $(0, 0)$ belong to one of the sets E_0 or E_1. The even convergents (q_{2n}, p_{2n}) belong to E_0 and odd convergents (q_{2n-1}, p_{2n-1}) belong to E_1. The **convex hull** of E_1 (the minimal convex set which contains E_1) is bounded by the axis y and by the graph of a piecewise linear function f_1 which joins odd convergents $(q_{-1}, p_{-1}) \to (q_1, p_1) \to (q_3, p_3) \to \cdots$. Similarly the convex hull of E_0 consists of the axis x and a piecewise linear function f_0 which joins even convergents

$(q_0, p_0) \to (q_2, p_2) \to (q_4, p_4) \to \cdots$.

$$\begin{aligned} f_0(x) &= p_{2n} + \frac{p_{2n+1}}{q_{2n+1}}(x - q_{2n}), \qquad q_{2n} \le x \le q_{2n+2} \\ f_1(x) &= p_{2n-1} + \frac{p_{2n}}{q_{2n}}(x - q_{2n-1}), \quad q_{2n-1} \le x \le q_{2n+1} \end{aligned}$$

The graphs of these functions contain also some integer points such as $(q_{-1}, p_{-1}) + (q_0, p_0) = (1, 1)$ or $(q_1, p_1) + (q_2, p_2) = (7, 3)$. The differences between $f_i(x)$ and αx are bounded by $|f_i(q_{2n+i}) - \alpha q_{2n+i}| = \eta_{2n+i}$. We show that there are no integer points between these two functions except $(0, 0)$.

Proposition A.84. — *Let $\alpha = [a_0, a_1, \ldots]$ be a positive irrational number, p_i, q_i its convergents and $p, q, n \in \mathbb{N}$. Then*

$$\begin{aligned} q_{2n-1} < q < q_{2n+1} \quad \& \quad p > \alpha q \quad &\Rightarrow \quad p - \alpha q > \eta_{2n} \quad \& \quad \frac{p - p_{2n-1}}{q - q_{2n-1}} \ge \frac{p_{2n}}{q_{2n}} \\ q_{2n} < q < q_{2n+2} \quad \& \quad \alpha q > p \quad &\Rightarrow \quad \alpha q - p > \eta_{2n+1} \quad \& \quad \frac{p - p_{2n}}{q - q_{2n}} \le \frac{p_{2n+1}}{q_{2n+1}} \end{aligned}$$

and the equality holds only if $(q, p) = (q_{n-1}, p_{n-1}) + k(q_n, p_n)$, $0 < k < a_{n+1}$.

Proof. — Let $p > \alpha q$. We have $p/q > \alpha > p_{2n}/q_{2n}$, so $pq_{2n} - p_{2n}q > 0$. Since all these numbers are integers, $pq_{2n} - p_{2n}q \ge 1$. Since $p_{2n-1}q_{2n} - p_{2n}q_{2n-1} = 1$, we get

$$(p - p_{2n-1})q_{2n} - p_{2n}(q - q_{2n-1}) = pq_{2n} - p_{2n}q - 1 \ge 0.$$

Since p_{2n}, q_{2n} are relatively prime, the equality holds only if

$$q - q_{2n-1} = kq_{2n}, \quad p - p_{2n-1} = kp_{2n}$$

for some k. Since $q < q_{2n+1}$, we get $k < a_{2n+1}$. Since $pq_{2n} - p_{2n}q \ge 1$,

$$\frac{p}{q} \ge \frac{p_{2n}}{q_{2n}} + \frac{1}{q \cdot q_{2n}} > \frac{p_{2n}q_{2n+1} + 1}{q_{2n+1}q_{2n}} = \frac{p_{2n+1}q_{2n}}{q_{2n+1}q_{2n}} = \frac{p_{2n+1}}{q_{2n+1}},$$

so $pq_{2n+1} - p_{2n+1}q \ge 1 = p_{2n+1}q_{2n} - p_{2n}q_{2n+1}$ and $\frac{p+p_{2n}}{q+q_{2n}} \ge \frac{p_{2n+1}}{q_{2n+1}} > \alpha$. It follows $p - \alpha q > \alpha q_{2n} - p_{2n} = \eta_{2n}$. Similarly we get inequalities for the even convergents. $\square$

Theorem A.85 **(Klein theorem).** — *Let $\alpha > 0$ be irrational and p_n, q_n its convergents.*

(1) The extremal points of the convex hull of E_0 are (q_{2n}, p_{2n}), $n \ge 0$.

(2) The extremal points of the convex hull of E_1 are (q_{2n-1}, p_{2n-1}), $n \ge 0$.

(3) If $0 < q < q_n$ is an integer, then $|p - \alpha q| > |p_n - \alpha q_n| = \eta_n$.

Proof. — Proposition A.84 shows that if (q, p) is not a convergent, it is an extremal point of neither E_0 nor E_1. The line which joins consecutive even convergents (q_{2n}, p_{2n}) and (q_{2n+2}, p_{2n+2}) has slope $\frac{p_{n+1}}{q_{n+1}}$. Since these slopes are decreasing, (q_{2n}, p_{2n}) are extremal points. (3) follows from Proposition A.84, since $\eta_{n-1} > \eta_n$. $\square$

APPENDIX B

MAIN THEOREMS

Theorem 2.9 A dynamical system is transitive iff it has a transitive point iff its transitive set is residual.

Theorem 2.19 (Birkhoff [**18**]) A point is almost periodic iff its orbit closure is a minimal system.

Corollary 2.21 (Birkhoff [**18**]) Any dynamical system has an almost periodic point.

Theorem 2.31 (Akin, Auslander, Berg [**4**]) A transitive system is either sensitive or almost equicontinuous.

Theorem 2.33 (Glasner and Weiss [**65**], Akin et al.[**4**]) A transitive almost equicontinuous system is a uniformly rigid homeomorphism and its equicontinuity points coincide with transitive points.

Corollary 2.34 A transitive equicontinuous system is minimal. A minimal almost equicontinuous system is equicontinuous.

Theorem 2.35 (Glasner and Weiss [**65**]) A transitive system with a dense set of almost periodic points is either minimal or sensitive.

Theorem 2.42 A minimal equicontinuous system is conjugate to an isometric rotation on a compact group.

Theorem 2.44 Any dynamical system has a maximal equicontinuous factor.

Theorem 2.49 (Paul [**123**]) An equicontinuous factor of a minimal system which is an almost one-to-one extension is the maximal equicontinuous factor.

Theorem 2.55 (Halmos and von Neumann) A dynamical system is transitive with discrete spectrum iff it is conjugate to a minimal rotation on a compact group.

Theorem 2.56 Two minimal equicontinuous systems have the same spectrum iff they are conjugate. Two transitive dynamical systems have the same spectrum iff their maximal equicontinuous factors are conjugate.

Theorem 2.68 (Akin [**3**]) An attractor is chain-invariant. Two attractors are equal iff they contain the same chain-recurrent points.

Theorem 2.69 An attractor is a minimal attractor iff it is chain-transitive.

Theorem 2.76 (Buescu and Stewart [**33**]) A stable set in a locally compact dynamical system has as its maximal totally disconnected factor an adding machine.

Theorem 2.90 (Bowen [**28**]) The topological entropy of a dynamical system is concentrated on its nonwandering set.

Theorem 3.8 (Weihrauch [**161**]) Any compact metric space has a symbolic extension with the extension property.

Corollary 3.9 (Balcar and Simon [**10**]) Any dynamical system has a symbolic extension.

Theorems 3.33, 3.35 (Walters [**158**], Parry [**122**]) A subshift is an SFT iff it has the shadowing property iff it is open.

Theorem 3.48 The entropy of a subshift of finite type is the logarithm of the spectral radius of its adjacency matrix.

Theorem 3.57 (Weiss [**162**]) A subshift is sofic iff it is a factor of a subshift of finite type.

Theorem 3.64 The entropy of a sofic subshift is the logarithm of the spectral radius of the adjacency matrix of its right-resolving graph.

Theorem 4.4 A minimal equicontinuous dynamical system on a totally disconnected space is conjugate to an adding machine.

Theorem 4.34 (Michel [**108**]) A substitutive subshift is strictly ergodic.

Theorem 4.36 (Mossé [**115**]) A substitutive system is recognizable.

Theorem 4.40 (Dekking [**46**]) The maximal equicontinuous factor of a constant length substitutive subshift is an adding machine.

Theorem 4.45 (Sós [**148**]) A finite trajectory of an irrational rotation divides the circle into intervals of at most three lengths.

Theorem 4.62 An infinite chain-transitive subshift is ballanced iff it has minimal complexity iff it is either Sturmian or skew Sturmian.

Theorem 4.72 (Williams [**163**]) The maximal equicontinuous factor of a Toeplitz subshift is conjugate to an adding machine.

Theorem 4.73 (Downarowicz and Lacroix [**52**]) A minimal almost one-to-one subshift extension of an adding machine is conjugate to a Toeplitz subshift.

Theorem 4.76 (Jacobs and Keane [**82**]) A regular Toeplitz subshift is strictly ergodic and has zero topological entropy.

Theorems 4.77,4.78 (Williams [**163**]) There exist minimal dynamical systems with positive entropy which are not strictly ergodic.

Corollary 5.19 (Codenotti and Margara [**36**]) A transitive cellular automaton is sensitive.

Theorem 5.21 (Hedlund [**73**]) A cellular automaton is surjective iff any finite word has the same number of preimages.

Theorem 5.24 (Blanchard and Tisseur [**27**]) A surjective almost equicontinuous cellular automaton has a dense set of periodic points.

Theorem 5.35, 5.45 (Hedlund [**73**]) A cellular automaton is open iff it is n-to-one iff it has n crossections iff it is both left-closing and right-closing.

Theorem 5.47 (Boyle and Kitchens [**31**]) A closing cellular automaton has a dense set of periodic points.

Theorems 5.49,5.50 (Kůrka [**90**], Nasu [**117**], Blanchard and Maass [**25**]) A positively expansive cellular automaton is open and conjugate to a mixing subshift of finite type.

Theorem 5.52 (Hurley [**80**]) A cellular automaton with two disjoint attractors has an infinite number of attractors and a continuum of quasi-attractors.

Theorem 5.53 (Hurley [**80**]) A minimal attractor of a cellular automaton is a subshift and its basin of attraction is a dense open set.

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NOTATION

$\#A$	number of elements of a finite set	100
A^*	set of finite words	100
A^+	set of nonempty finite words	100
$A^{\mathbb{N}}$	symbolic power space (one-sided sequences)	100
$A^{\mathbb{Z}}$	space of two-sided sequences	132
$\mathcal{A}^{\mathbb{N}}$	space of one-sided sequences	100
$\mathfrak{B}(Y)$	basin of an attractor	77
$B_r(x)$	open ball with center x	231
$B_\delta(Y)$	δ-neighbourhood of a set	24
$\mathcal{B}$	base of a topology	238
$\mathbf{C}(X, \mathbb{C})$	space of complex continuous functions	71
$\mathbb{C}$	complex numbers	71
$\mathfrak{C}$	chain relation	49
$d(x, Y)$	distance of a point from a set	24
$d(x, y)$	distance of points	231
$\mathrm{diam}(\mathcal{U})$	diameter of a cover	86
$\mathfrak{E}$	set of equicontinuous points	61
F^n	n-th iteration of F	9
$F^{-1}(\mathcal{U})$	inverse image of a cover	87
F_d	dual of a CA	210
F_n	n-th Fibonacci number	43
G_n	n-th Rauzy graph	173
$H(X, F, \mathcal{V})$	entropy of a cover	88
$h(X, F)$	topological entropy of a system	88
$\mathbb{I}$	unit interval $[0, 1]$	14
$(\mathbb{I}, Q_r)$	quadratic system	14
$\|\|I\|\|$	length of an interval	96
$\mathbb{K}$	unit circle in the complex plane	72
$\mathbf{L}(\mathcal{A})$	linear span in a vector space	244
$\mathcal{L}(\Sigma), \mathcal{L}^n(\Sigma)$	language of a subshift	108
$\mathcal{L}(x), \mathcal{L}^n(x)$	language of a sequence	108
λ	empty word	100
$\lambda(M)$	spectral radius of a matrix	123
$M(\vartheta)$	matrix of a substitution	141

$\mathfrak{M}$	inseparability relation	242
$\mathfrak{M}(x)$	connected component of a point x	242
$M_{\mathbf{p}}(a)$	multiplicity of a periodic structure	140
$\mathrm{mod}_a(x)$	modulo function	18
$\mu(U)$	measure of a set	95
$\mathbb{N}$	nonnegative integers	21
$\mathfrak{N}$	nonwandering relation	49
$\mathcal{N}, \mathcal{Z}$	spaces of continued fractions	266
$\mathfrak{O}$	orbit relation	49
$\mathfrak{O}(x)$	orbit of a point	20
$\omega(Y)$	omega limit of a set	75
$\overline{A^*}$	set of words	100
$\overline{F} = F \times F_d$	symmetrization of a CA	210
$\overline{Y}$	closure of a set	231
$\overline{\mathfrak{O}(x)}$	orbit closure of a point	49
$P(n)$	complexity function of a subshift	111
$\mathrm{Per}(n)$	number of periodic points	131
$\mathrm{Per}_p(x)$	p-periodic part	178
Ψ	grand Sturmian subshift	176
$\mathbf{p} = (p_i)_{i \geq 0}$	periodic structure	137
$Q(\mathcal{U})$	return time	94
$\mathbb{Q}_{\mathbf{p}}$	group of p-adic rationals	138
$R(\mathcal{U})$	recurrence time	94
$\mathbb{R}$	space of real numbers	231
$\mathfrak{R}$	recurrence relation	49
$r_i(x)$	period of a coordinate i	178
$S(\mathcal{U})$	transition time	94
$\mathcal{S}(X, F)$	spectrum of (X, F)	71
$S_p(x)$	p-skeleton of a sequence	178
Σ_E	Markov subshift of a transition graph	117
Σ_F	subshift of a forbidden set of words	107
Σ_G	subshift of a labelled graph	126
$\Sigma_I(F)$	column subshift of a CA	193
Σ_M	Markov subshift of a binary matrix	122
Σ_ϑ	subshift of a substitution	144
$\Sigma_{\lvert G \rvert}$	edge subshift of a labelled graph	126
$\Sigma_{\lvert G \rvert} \lvert B$	restriction of a sofic subshift	130
$\mathscr{S}(L)$	subshift of an extendable language	108
σ	shift map	107
$T(\mathcal{U})$	running time	94
$\mathbb{T}$	circle	231
$(\mathbb{T}, D)$	angle doubling system	18
$(\mathbb{T}, R_a)$	rotation of the circle	38
$\mathfrak{T}$	set of bitransitive points	195
$\mathfrak{T}$	set of transitive points	52
$\mathbb{T}^n$	n-dimensional torus	37
$\tau(U)$	recurrence time	94
$\tau(U, V)$	transition time	94
$\vartheta[\Sigma]$	image subshift of a substitution	144

$\lvert\vartheta\rvert$	lower norm of a substitution	141
$\lvert\lvert\vartheta\rvert\rvert$	upper norm of a substitution	141
$\mathbf{2}$	binary alphabet	21
$\mathfrak{U}_X$	system of open covers of X	86
$\mathcal{U} \succeq \mathcal{V}$	finer cover	86
$\mathcal{U} \vee \mathcal{V}$	join of covers	86
$\mathcal{U}^n$	power of a cover	87
$\lvert\mathcal{U}\rvert$	size of a cover	86
$u \sqsubseteq v$	subword	100
$u \sqsubseteq_p v$	prefix	100
$u \sqsubseteq_s v$	suffix	100
$[u]_m$	cylinder of a word at position m	133
u^∞	infinite concatenation of a word	100
$u_{[i,j]}$	subword	100
$\lvert u\rvert$	length of a word	100
$\lvert u\rvert_a$	number of occurrences of a in u	141
Y°	interior of a set	231
$\mathbb{Z}$	set of integers	132
$\mathbb{Z}(n)$	group of integers modulo n	53
$\mathbb{Z}_{\mathbf{p}}$	group of p-adic integers	137
$\mathcal{Z}(L)$	subshift of a central language	133

INDEX