TOPOLOGICAL DYNAMICS OF CELLULAR AUTOMATA

PETR KŮRKA *

Abstract. This is an overview of some classical and recent results in topological dynamics of cellular automata on the space of twosided symbolic sequences. The concepts studied include surjectivity, transitivity, equicontinuity, closingness, openness, expansivity, attractors and the shadowing property.

Key words. equicontinuity, sensitivity, transitity, expansivity

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1. Introduction. Cellular automata (CA) are dynamical systems with very rich and diversified behaviour. They are often used in physics and sciences as models of complicated behaviour. In computer science they serve as models of parallel processing and yield several complexity classes of languages.

Cellular automata have been introduced in the fifties by Ulam [28] and von Neumann [26]. The latter used them as models of self-reproduction and universal computation. A well-known example of a CA with surprising complexity and universal computation is the Conway's Game of Life [14]. Deep mathematical theory of CA has been developed in Hedlund's pioneering paper [19]. Hedlund studied CA in the context of symbolic dynamics as homomorphisms of the shift dynamical system. Later, Wolfram [30, 31] studied computational aspects of CA. He performed extensive computer simulations and classified CA informally into four classes according to their behaviour on finite configurations. There have been several attempts to formalize Wolfram's classification, using computational, measure-theoretic and topological properties of CA, see e.g., Culik et al. [12], Gilman [16], Hurley [20] and Kůrka [21]. Simultaneously many results have been obtained interrelating these properties. While some classes of CA have been completely understood and some interesting examples have been thoroughly elucidated, many problems remain still open.

CA have been considered on many symbolic spaces: space of finite or periodic sequences, space of onesided or twosided infinite sequences, onesided or twosided mixing subshifts of finite type or configurations in n-dimensional grids. While these classes have some properties in common, there are also important differences. We concentrate here on one particular, if most familiar case, the space of twosided infinite symbolic sequences. With the product topology, this space is compact and metrizable and CA are continuous, so the concepts of topological dynamics apply. In the present study we disregard ergodic and computational aspects of CA

*Faculty of Mathematics and Physics, Charles University in Prague, Malostranské náměstí 25, CZ-118 00 Praha 1, CZECHIA

and concentrate on topological dynamics. We present both classical and recent results and sometimes complete the classification with new theorems. The concept studied include surjectivity, transitivity, equicontinuity, closingness, openness, expansivity, attractors and the shadowing property.

2. Topological dynamics. A **dynamical system** (DS) is a couple (X, F) where X is a compact metric space and $F : X \to X$ is a continuous map.

The n-th **iteration** of F is denoted by F^n, so $F^0 = \mathrm{Id}$ is the identity map and $F^{n+1} = F \circ F^n$.

A set $Y \subseteq X$ is **invariant**, if $F(Y) \subseteq Y$ and **strongly invariant** if $F(Y) = Y$. If Y is closed and invariant, (Y, F) is a DS which is called a **subsystem** of (X, F). The **orbit** of a point $x \in X$ is $\ \ (x) = \{F^n(x) :\ n > 0\}$. For every $x \in X$, the closure $\overline{\ \ (x)}$ of the orbit of x is a subsystem.

A **homomorphism** $\varphi : (X, F) \to (Y, G)$ is a continuous map $\varphi : X \to Y$ such that $\varphi \circ F = G \circ \varphi$.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "G"] & Y \end{tikzcd}$$

A **conjugacy** is a bijective homomorphism. The systems (X, F) and (Y, G) are **conjugated** if there exists a conjugacy between them. If φ is injective, $(\varphi(X), G)$ is a subsystem of (Y, G) as $\varphi(X) \subseteq Y$ is a closed invariant set. If φ is surjective, (Y, G) is a **factor** of (X, F).

A point $x \in X$ is **periodic** with period $n > 0$ if $F^n(x) = x$. If $F(x) = x$, x is a **fixed** point. A point $x \in X$ is **eventually periodic**, if $F^m(x)$ is periodic for some $m \geq 0$. If $m > 0$, x is **preperiodic** and m is its preperiod. A point is eventually periodic iff its orbit $\ \ (x)$ is finite.

A DS is **open**, if $F(U)$ is open for any open $U \subseteq X$.

A point $x \in X$ is a **transitive point**, if it has dense orbit, i.e., if $\overline{\ \ (x)} = X$. A DS is **transitive**, if it has a transitive point. In this case the set of transitive points is **residual**, i.e., it contains a countable intersection of dense open sets. Equivalently, a DS is transitive, if for any nonempty open sets $U, V \subseteq X$ there exists $n > 0$ with $F^n(U) \cap V \neq \emptyset$.

A DS is **mixing**, if for every nonempty open sets $U, V \subseteq X$, $F^n(U) \cap V \neq \emptyset$ for all sufficiently large n. Every mixing DS is transitive.

A point $x \in X$ is an **equicontinuous point**, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall n \geq 0, d(F^n(y), F^n(x)) < \varepsilon$$

Here d is the metric and $B_\delta(x) = \{y \in X : d(y,x) < \delta\}$ is the ball with center x and radius δ.

A DS is **equicontinuous**, if every its point is equicontinuous. In this case it is uniformly equicontinuous, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X, (d(x,y) < \delta \Rightarrow \forall n \geq 0, d(F^n(x), F^n(y)) < \varepsilon)$$

A DS is **almost equicontinuous**, if the set of its equicontinuous points is residual.

A DS is **sensitive**, if

$$\exists \varepsilon > 0, \forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, d(F^n(y), F^n(x)) \geq \varepsilon$$

Clearly, a sensitive system has no equicontinuous points. The converse is not true in general but holds in transitive systems. Every transitive system is either sensitive or almost equicontinuous (Akin et al. [2]). Every transitive almost equicontinuous system is **uniformly rigid** (Glasner and Weiss [18]). This means

$$\forall \varepsilon > 0, \exists n > 0, \forall x \in X, d(F^n(x), x) < \varepsilon$$

We shall see that the dichotomy between sensitive and almost equicontinuous systems holds also for all CA. Moreover, transitive CA are sensitive.

A DS is **positively expansive**, if

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists n \geq 0, d(F^n(x), F^n(y)) \geq \varepsilon$$

Every positively expansive DS on a perfect space (without isolated points) is sensitive.

A finite sequence $(x_i \in X)_{0 \leq i \leq n}$ is a **δ-chain** from x_0 to x_n, if

$$\forall i < n, d(F(x_i), x_{i+1}) < \delta.$$

A point x **ε-shadows** a sequence $(x_i)_{0 \leq i \leq n}$, if $d(F^i(x), x_i) < \varepsilon$ for all $0 \leq i \leq n$.

A DS has the **shadowing property** if for every $\varepsilon > 0$ there exists $\delta > 0$, such that every δ-chain is ε-shadowed by some point. It follows that every infinite δ-chain is ε-shadowed by some point.

A DS is **chain transitive** if for every $x, y \in X$ and for every $\varepsilon > 0$, there exists an ε-chain from x to y. Every transitive DS is chain transitive. Every chain transitive system with the shadowing property is transitive.

A point $x \in X$ is a **nonwandering point**, if for every open neighbourhood $U \subseteq X$ of x there exists $n > 0$ with $F^n(U) \cap U \neq \emptyset$. The set of nonwandering points is invariant and closed. A DS is **nonwandering**, if every its point is nonwandering.

The **omega limit** set of a closed invariant set $V \subseteq X$ is

$$\omega(V) = \bigcap_{n \geq 0} F^n(V)$$

A set $Y \subseteq X$ is an **attractor** iff there exists a nonempty open set V such that $F(\overline{V}) \subseteq V$ and $Y = \omega(V)$. In a totally disconnected space, attractors are omega limit sets of clopen invariant sets. The **basin** of an attractor $Y \subseteq X$ is the set

$$\mathcal{B}(Y) = \{x \in X;\ \lim_{n \to \infty} d(F^n(x), Y) = 0\}$$

There exists always the largest attractor $\omega(X)$. The number of attractors is at most countable.

A periodic point $x \in X$ is **attracting** if its orbit $\ (x)$ is an attractor. Any attracting periodic point is equicontinuous.

A set $Y \subseteq X$ is a **minimal attractor**, if it is an attractor and no proper subset of Y is an attractor. An attractor is minimal iff it is chain transitive. A DS has unique attractor iff $\omega(X)$ is chain-transitive (Akin [1]).

A **quasi-attractor** is a countable intersection of attractors, which is not an attractor.

3. Subshifts. We call **alphabet** any finite set with at least two elements. We frequently use alphabets $\mathbf{2} = \{0, 1\}$ and $\mathbf{3} = \{0, 1, 2\}$. The cardinality of a finite set A is denoted by $|A|$. A word over A is any finite sequence $u = u_0 \dots u_{n-1}$ of elements of A. The length of u is $|u| = n$. The **empty word** of length 0 is denoted by λ. The set of all words of length n is denoted by A^n, so $A^0 = \{\lambda\}$ and $A^1 = A$. The set of all nonzero words and the set of all words are

$$A^+ = \bigcup_{n > 0} A^n, \quad A^* = \bigcup_{n \geq 0} A^n.$$

A onesided infinite word over A is a map $x : \ \to A$, where $\ = \{0, 1, \dots\}$ is the set of non-negative integers. A twosided infinite word over A is a map $x : \ \to A$, where $\ = \{\dots, -1, 0, 1, \dots\}$ is the set of all integers. The position of the zeroth letter is denoted by a period, $x = \dots x_{-2}x_{-1}.x_0x_1 \dots$. If u is a finite or infinite word and $I = \langle i, j \rangle$ is an interval of integers on which u is defined, put $u_{\langle i,j \rangle} = u_i \dots u_j$. Similarly for open or half-open integer intervals $u_{(i,j)} = u_{i+1} \dots u_{j-1}$. We say that v is a **subword** of u

and write $v \sqsubseteq u$, if $v = u_I$ for some interval $I \subseteq$.

If $u \in A^n$, then $u^\infty \in A$ is the infinite repetition of u, i.e., $(u^\infty)_{kn+i} = u_i$. If $v \in A^m$, then $x = v^\infty.u^\infty \in A$ is the word defined by

$$\begin{aligned} x_{km+i} &= v_i \quad \text{for} \quad k < 0, 0 \le i < m \\ x_{kn+i} &= u_i \quad \text{for} \quad k \ge 0, 0 \le i < n \end{aligned}$$

In A and A we have metrics

$$d(x,y) = 2^{-n} \quad \text{where} \quad n = \min\{i \ge 0 : \ x_i \ne y_i\}, \quad x, y \in A$$

$$d(x,y) = 2^{-n} \quad \text{where} \quad n = \min\{i \ge 0 : \ x_i \ne y_i \ \vee \ x_{-i} \ne y_{-i}\}, \quad x, y \in A$$

Both these metrics satisfy a stronger form of the triangle inequality

$$d(x,z) \le \max\{d(x,y), d(y,z)\}$$

Both A and A are compact, perfect (do not contain isolated points) and totally disconnected, i.e., their clopen (closed and open) sets form a base of their topology. Every two compact, perfect and totally disconnected spaces are homeomorphic.

In A and A the cylinder sets of a word $u \in A^n$ are

$$[u] = \{x \in A \ : \ x_{\langle 0,n)} = u\}$$

$$[u]_k = \{x \in A \ : \ x_{\langle k,k+n)} = u\}, \ k \in$$

Cylinder sets are clopen and every clopen set is a finite union of cylinders.

The shift maps $\sigma : A \to A$ and $\sigma : A \to A$ are both defined by $\sigma(x)_i = x_{i+1}$ and they are continuous. Thus $(A \ , \sigma)$ and $(A \ , \sigma)$ are DS which are called twosided and onesided **full shifts** respectively. While any twosided full shift is bijective, onesided full shift is not, every its point has $|A|$ preimages. Full shifts are mixing and have dense sets of periodic points.

A **onesided subshift** is any subsystem of a onesided full shift, so any closed set $\Sigma \subseteq A$ with $\sigma(\Sigma) \subseteq \Sigma$. A DS on a totally disconnected space is conjugated to a onesided subshift iff it is positively expansive.

A **twosided subshift** is any closed set $\Sigma \subseteq A$ with $\sigma(\Sigma) = \Sigma$. We require strong invariance here, since the twosided subshift is bijective.

The language of a subshift Σ is the set of all subwords of points of Σ,

$$\mathcal{L}(\Sigma) = \{u \in A^* : \ \exists x \in \Sigma, u \sqsubseteq x\}$$

A subshift Σ is a **subshift of finite type** (SFT) if there exists an integer $k \geq 2$, called its order, such that for every $x \in A$, or for every $x \in A$,

$$x \in \Sigma \text{ iff } \forall n, x_{\langle n,n+k)} \in \mathcal{L}(\Sigma)$$

A subshift is a SFT iff it has the shadowing property (Walters [29]). A onesided subshift is a SFT iff it is open (Parry [27]).

A subshift Σ is **sofic**, if it is a factor of some SFT.

Topological entropy of a subshift Σ is

$$h(\Sigma) = \lim_{n\to\infty} \frac{\ln P(n)}{n}$$

where $P(n) = |\mathcal{L}(\Sigma) \cap A^n|$ is the number of its words of length n.

4. Cellular automata.

DEFINITION 4.1. *A* **cellular automaton** *(CA) is a dynamical system* $(A\ \ , F)$*, such that* $F \circ \sigma = \sigma \circ F$*.*

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\sigma"'] & A \arrow[d, "\sigma"] \\ A \arrow[r, "F"] & A \end{tikzcd}$$

THEOREM 4.1 (Hedlund-Curtis-Lyndon [19]). *A dynamical system* $(A\ \ , F)$ *is a CA iff there exist integers* $m \leq a$ *(memory and anticipation) and a* **local rule** $f : A^{a-m+1} \to A$ *such that for every* $x \in A$,

$$F(x)_i = f(x_{\langle i+m,i+a\rangle})$$

Proof. Let $(A\ \ , F)$ be a CA. For $\varepsilon = 1$ there exists $r \geq 0$ such that

$$d(x,y) < 2^{-r} \Rightarrow d(F(x), F(y)) < 1$$
$$x_{\langle -r,r\rangle} = y_{\langle -r,r\rangle} \Rightarrow F(x)_0 = F(y)_0$$

Thus there exists $f : A^{2r+1} \to A$ such that for every $x \in A$, $F(x)_0 = f(x_{\langle -r,r\rangle})$. Since F commutes with the shift,

$$F(x)_i = \sigma^i(F(x))_0 = F(\sigma^i(x))_0 = f(\sigma^i(x)_{\langle -r,r\rangle}) = f(x_{\langle i-r,i+r\rangle})$$

Thus we have a local rule with memory $m = -r$ and anticipation $a = r$. Conversely, if $F(x)_i = f(x_{\langle i+m,i+a\rangle})$ for some $m \le a$, put $r = \max\{-m, a\}$. Since $-r \le m \le a \le r$, for every $n \ge 0$,

$$\begin{aligned} d(x,y) < 2^{-n-r} &\Rightarrow x_{\langle -n-r,n+r\rangle} = y_{\langle -n-r,n+r\rangle} \\ &\Rightarrow x_{\langle -n+m,n+a\rangle} = y_{\langle -n+m,n+a\rangle} \\ &\Rightarrow F(x)_{\langle -n,n\rangle} = F(y)_{\langle -n,n\rangle} \\ &\Rightarrow d(F(x),F(y)) < 2^{-n} \end{aligned}$$

so F is continuous and clearly commutes with the shift. □

We can assume that the local rule is symmetric, so $F(x)_i = f(x_{\langle i-r,i+r\rangle})$, and $f : A^{2r+1} \to A$, where $r \ge \{-m, a\}$. Any r with this property is called a **radius** of F. Using larger alphabet, it can be easily shown that any CA is conjugated to a CA with radius 1.

A **factor subshift** of a CA is any one-sided subshift which is its factor. If $I = \langle a, b\rangle$ is an integer interval, put $B = A^{b-a+1}$ and define a map $\varphi : A \to B$ by $\varphi(x)_i = F^i(x)_{\langle a,b\rangle}$. The map φ is continuous and

$$\varphi(F(x))_i = F^{i+1}(x)_{\langle a,b\rangle} = \varphi(x)_{i+1} = \sigma(\varphi(x))_i$$

Thus $\Sigma_I(F) = \varphi(A) \subseteq B$ is a onesided subshift and $\varphi : (A, F) \to (\Sigma_I(F), \sigma)$ is a factor map

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\ \Sigma_I(F) \arrow[r, "\sigma"] & \Sigma_I(F) \end{tikzcd}$$

Let $d : A \to A$ be the homeomorphism defined by $d(x)_i = x_{-i}$. The **dual** of a CA (A, F) is $(A, \overline{F})$, where $\overline{F} = d \circ F \circ d$. Clearly $d : (A, F) \to (A, \overline{F})$ is a conjugacy.

We show some typical examples of CA.

EXAMPLE 1 (P: product). $(\mathbf{2}, P)$ *where* $P(x)_i = x_{i-1}x_ix_{i+1}$ *(Figure 1).* The radius is $r = 1$ and the local rule $f : \mathbf{2}^3 \to \mathbf{2}$ is defined by the table

000	001	010	011	100	101	110	111
0	0	0	0	0	0	0	1

The behaviour of the product CA is displayed in the space time diagram in Figure 1. The n-th row represents a central part of the state at time n, i.e., the word $F^n(x)_{\langle -m,m\rangle}$, where $2m + 1$ is the width displayed. Ones

FIG. 1. *P: the product CA*

are represented by black squares and zeros are left empty. We see that 0's propagate both to the left and to the right. If $x_i = 0$, then $F(x)_{\langle i-1,i+1\rangle} = 000$ and $F^2(x)_{\langle i-2,i+2\rangle} = 00000$. This implies that 0^∞ is an attracting fixed point. If $x_0 = 0$, i.e., if $d(x, 0^\infty) < 1$, then $d(F^n(x), 0^\infty) < 2^{-n}$. The CA has another fixed point 1^∞ which is not equicontinuous. If $d(x, 1^\infty) = 2^{-n} < 1$, then $d(F(x), 1^\infty) = 2^{-n+1}$. The product CA is not surjective:

$$
\begin{aligned}
F(\mathbf{2}\quad) &= \{x \in \mathbf{2}\quad : 101 \not\sqsubseteq x \ \&\ 1001 \not\sqsubseteq x\} \\
F^n(\mathbf{2}\quad) &= \{x \in \mathbf{2}\quad : \forall m \in \langle 0, 2n\rangle, 10^m1 \not\sqsubseteq x\} \\
\omega(\mathbf{2}\quad) &= \{x \in \mathbf{2}\quad : \forall m > 0, 10^m1 \not\sqsubseteq x\}
\end{aligned}
$$

EXAMPLE 2 (M: majority). $(\mathbf{2}\quad, M)$ *where*

$$M(x)_i = \lfloor \frac{x_{i-1} + x_i + x_{i+1}}{2} \rfloor$$

000	001	010	011	100	101	110	111
0	0	0	1	0	1	1	1

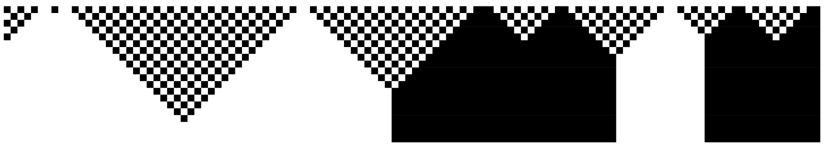


FIG. 2. *M: the majority CA*

The space time diagram of the majority CA is shown in Figure 2. We see that the cellular space is successively homogenized so that no isolated 0 or 1 remain. If $x_{\langle 0,1\rangle} = 00$, then $F(x)_{\langle 0,1\rangle} = 00$ no matter what value is in x_{-1} or x_2. Thus the cylinder set $[00]_0$, and every cylinder sets $[00]_i$ and $[11]_i$ are invariant. On the other hand, if $F(x)_{\langle -1,1\rangle} = 010$, then necessarily $x_{\langle -2,2\rangle} = 01010$. Indeed $x_{\langle -1,1\rangle}$ contains at least two ones, but $x_{\langle 0,1\rangle} = 11$ would imply $F(x)_{\langle 0,1\rangle} = 11$. By induction we get

$$
\begin{aligned}
F^n(x)_{\langle -1,1\rangle} = 010 &\Rightarrow x_{\langle -n,n\rangle} = 01\ldots10 \\
F^n(x)_{\langle -1,1\rangle} = 101 &\Rightarrow x_{\langle -n,n\rangle} = 10\ldots01
\end{aligned}
$$

The clopen set $V = [00]_0 \cup [11]_0$ is invariant and

$$x \in V \ \Rightarrow\ \forall i \in \langle -n, n+1\rangle, F^n(x)_{\langle i-1,i+1\rangle} \not\in \{010, 101\}$$

so

$$\omega(V) = \{x \in \mathbf{2} \quad : \forall i \in \quad : x_{\langle i-1,i+1\rangle} \notin \{010, 101\}\}$$

is a SFT whose every point is fixed. The set $\omega(V)$ is, however, not the largest attractor, since $\omega(A \quad)$ contains e.g., periodic points $(01)^\infty$, $(10)^\infty$. There are also many smaller attractors. If $u \in \mathcal{L}(\omega(V))$, i.e., if u does not contain neither 010 nor 101, then $[u]_i$ is an invariant set and $\omega([u]_i) = \omega(V) \cap [u]_i$ is an attractor.

EXAMPLE 3 (S: sum). ($\mathbf{2} \quad , S$), *where* $S(x)_i = x_{i-1} + x_{i+1} \bmod 2$.

000	001	010	011	100	101	110	111
0	1	0	1	1	0	1	0

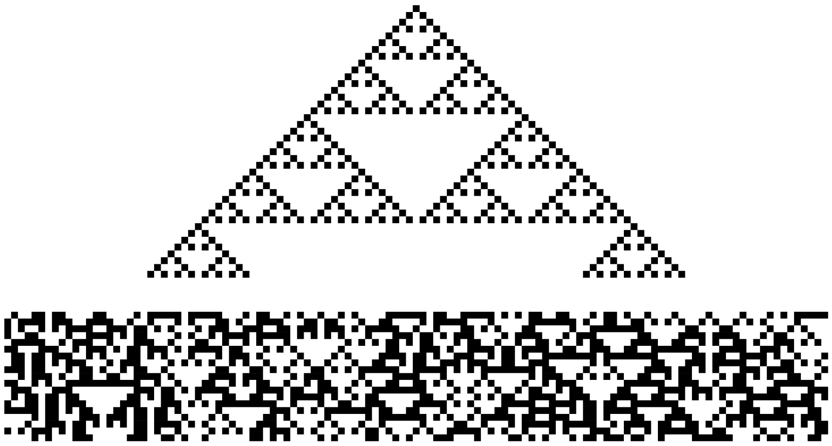


FIG. 3. *S: the sum CA*

There is a formula for the n-th iteration. We regard $\mathbf{2}$ as a group with addition modulo 2.

$$S^2(x)_i = x_{i-2} + 2x_i + x_{i+2}$$
$$S^n(x)_i = \sum_{j=0}^{n} \binom{n}{j} x_{i-n+2j}$$

Since $2x = 0$, we count only odd binomial coefficients. In particular for $n = 2^m$, all but two binomial coefficients are even, so

$$S^{2^m}(x)_i = x_{i-2^m} + x_{i+2^m}$$

If the initial state is $0^\infty.10^\infty$ (Figure 3 top), then $F^{2^m}(x)$ contains exactly two 1's, one at 2^{-m} and another at 2^m. If the initial state is random (Figure 3 bottom), we see no pattern in the time development. We show that the sum CA is conjugated to the full shift on four symbols. In fact $\Sigma_{\langle 0,1\rangle}(S) = B$ where $B = A^2 = \{00, 01, 10, 11\}$ and $\varphi : (\mathbf{2}\ , S) \to (B\ , \sigma)$ is a conjugacy. Given a point $y \in B$, let us search a point $x \in A$ such that for every $n \geq 0$, $S^n(x)_{\langle 0,1\rangle} = y_n$. We have

$$S^n(x)_0 + S^n(x)_2 = S^{n+1}(x)_1$$
$$S^n(x)_2 = S^n(x)_0 + S^{n+1}(x)_1$$

so the second column $(S^n(x)_2)_{n\geq 0}$ can be computed from the zeroth and the first columns. Similarly we compute the third and also (-1)-st columns. This shows that the map $\varphi : \mathbf{2} \to B$ is bijective. Since the sum CA is conjugated to a full shift, it is mixing, surjective, positively expansive and has a dense set of periodic points.

In general, every CA has a periodic point and the eventually periodic points are dense in it.

PROPOSITION 4.1. *Every σ-periodic point of a CA $(A\ , F)$ is F-eventually periodic. Hence the set of eventually periodic points is dense.*

Proof. If $\sigma^n(x) = x$, then $\sigma^n(F(x)) = F(x)$. Since σ-periodic points with period n are in one-to-one correspondence with words of A^n, there is a finite number of them and for some preperiod $m \geq 0$ and period $p > 0$, $F^{m+p}(x) = F^m(x)$. □

5. Equicontinuity. Almost equicontinuous systems are characterized by the presence of blocking words. In an equicontinuous system, every long enough word is blocking.

DEFINITION 5.1. *Let $s > 0$. A word $u \in A^+$ with $|u| \geq s$ is an s-blocking word for a CA $(A\ , F)$, if there exists an offset $p \in \langle 0, |u| - s\rangle$, such that*

$$\forall x, y \in [u]_0, \forall n \geq 0, F^n(x)_{\langle p,p+s)} = F^n(y)_{\langle p,p+s)}$$

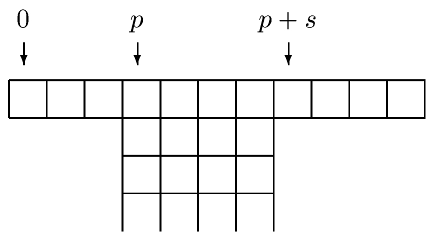


FIG. 4. *A blocking word*

In the product CA of Example 1, 0 is a 1-blocking word (with offset 0), since $x_0 = 0$ implies $F^n(x)_0 = 0$ for all $n \geq 0$. In the majority CA of

Example 2, 00 and 11 are 2-blocking words (with offset 0). Since $[00]_0$ is invariant, $x_{\langle 0,1\rangle} = 00$ implies $F^n(x)_{\langle 0,1\rangle} = 00$ for all $n \geq 0$.

THEOREM 5.1 (Kůrka [21]). *Let* $(A\ \ ,F)$ *be a CA with radius* $r > 0$. *The following conditions are equivalent.*

1. $(A\ \ ,F)$ *is not sensitive.*

2. $(A\ \ ,F)$ *has an* r*-blocking word.*

3. $(A\ \ ,F)$ *is almost equicontinuous.*

Proof. $1 \Rightarrow 2$: Suppose that F is not sensitive. Let m be an integer with $2m+1 \geq r$. For $\varepsilon = 2^{-m}$ there exists $x \in A\ \ $ and $\delta = 2^{-m-p}$, $p \geq 0$, such that for all $y \in A\ \ $,

$$d(x,y) < \delta \;\Rightarrow\; \forall k \geq 0, d(F^k(x), F^k(y)) < \varepsilon$$

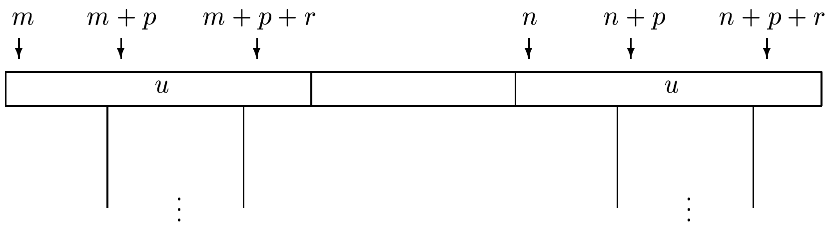
Put $u = x_{\langle -m-p, m+p\rangle} \in A^{2m+2p+1}$. Then

$$\begin{aligned} y,z \in [u]_{-m-p} &\Rightarrow \forall k, F^k(y)_{\langle -m,m\rangle} = F^k(z)_{\langle -m,m\rangle} \\ &\Rightarrow \forall k, F^k(y)_{\langle -m,-m+r)} = F^k(z)_{\langle -m,-m+r)} \\ y,z \in [u]_0 &\Rightarrow \forall k, F^k(y)_{\langle p,p+r)} = F^k(z)_{\langle p,p+r)} \end{aligned}$$

$2 \Rightarrow 3$: Let $u \in A^q$ be an r-blocking word with offset $p \leq q-r$. Let $Y_k \subseteq A\ \ $ be the set of points which contain at least k occurrences of u in both its positive and negative parts.

$$\begin{aligned} Y_k = \{y \in A\ \ :& \ |\{j \leq 0 : \ y_{\langle j,j+q)} = u\}| \geq k \ \& \\ & \ |\{j \geq 0 : \ y_{\langle j,j+q)} = u\}| \geq k\} \end{aligned}$$

Y_k is a dense open set, so $Y = \cap_{k>0} Y_k$ is residual. We show that every point of Y is equicontinuous. Let $x \in Y$ and $\varepsilon = 2^{-j}$. There exist $m \leq -j-p$ and $n \geq j-p-r$ such that $x_{\langle m,m+q)} = u = x_{\langle n,n+q)}$.



Put $\delta = \min\{2^m, 2^{-n-q}\}$ and suppose that $d(y,x) < \delta$. Then $x_{\langle m,n+q\rangle} = y_{\langle m,n+q\rangle}$. For $m+p+r < i < n+p$ we get

$$F(x)_i = f(x_{\langle i-r,i+r\rangle}) = f(y_{\langle i-r,i+r\rangle}) = F(y)_i$$

and by induction $F^k(x)_{\langle m+p,n+p+r)} = F^k(y)_{\langle m+p,n+p+r)}$ for every $k \geq 0$. Since $m+p \leq -j < j \leq n+p+r$, $d(F^k(x), F^k(y)) < 2^{-j}$.

$3 \Rightarrow 1$ is clear. □

In the product CA of Example 1, 0^∞ is an equicontinuous point but 1^∞ is not. In the majority CA, both 0^∞ and 1^∞ are equicontinuous but $(01)^\infty$ is not. Both product and majority CA are almost equicontinuous but not equicontinuous. The sum CA of Example 3 is sensitive as it is conjugate to a full shift. There are two trivial examples of equicontinuous CA:

EXAMPLE 4 (I: identity). *$(A\ \ , I)$, where A is any alphabet and $I(x) = x$, is an equicontinuous CA.*

EXAMPLE 5 (O: zero). *$(A\ \ , O)$, where A is any alphabet, $a \in A$ and $O(x) = a^\infty$ for all $x \in A\ \ $, is an equicontinuous CA.*

PROPOSITION 5.1. *Every CA with radius zero is equicontinuous.*

Proof. If $r = 0$, then the local rule is a map $f : A \to A$ and $F(x)_i = f(x_i)$. Thus

$$x_{\langle -n,n\rangle} = y_{\langle -n,n\rangle} \Rightarrow F(x)_{\langle -n,n\rangle} = F(y)_{\langle -n,n\rangle}$$
$$d(x,y) < 2^{-n} \Rightarrow d(F(x),F(y)) < 2^{-n}$$

□

However, not all equicontinuous CA have radius zero.

EXAMPLE 6 (E: An equicontinuous CA). *$(\mathbf{2}\ \ , E)$, where*

$$E(x)_i = x_i + x_{i-1}x_{i+1} \bmod 2$$

000	001	010	011	100	101	110	111
0	0	1	1	0	1	1	0

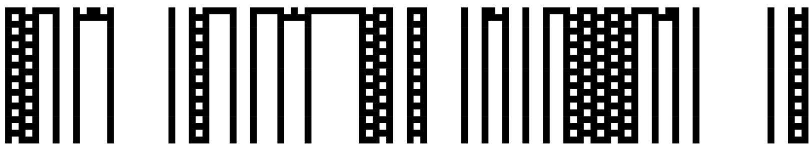


FIG. 5. *E: An equicontinuous CA*

We show first that 01110 is a 3-blocking word with offset 1. Considering all possibilities for the left and right extensions of 01110, we get

0011100	0011101	1011100	1011101
0010100	0010111	1110100	1110111
0011100	001110	011100	01110

Thus $F^2[01110] \subseteq [01110]$, so if $x \in [01110]_{-2}$, then $F^{2n}(x)_{\langle -1,1\rangle} = 111$ and $F^{2n+1}(x)_{\langle -1,1\rangle} = 101$. By inspection we verify that every word in the set $B = \{00, 1111, 01110, 0110110, 010110, 011010, 10101\}$ is 2- blocking.

00	1111	01110	0110110	010110	011010	10101
00	00	101	11111	1111	1111	11111
00	00	01110	000	00	00	0000

Every word v of length at least 10 contains at least one occurrence of some word from B, so it is also 2-blocking. It follows that the CA is equicontinuous.

THEOREM 5.2 (Kůrka [21]). *A CA $(A\;\;, F)$ is equicontinuous iff there exists a preperiod $m \geq 0$ and a period $p > 0$, such that $F^{m+p} = F^m$.*

Proof. For $\varepsilon = 1$ there exists $\delta = 2^{-k}$ such that for all $x, y \in A\;\;$, if $x_{\langle -k,k \rangle} = y_{\langle -k,k \rangle}$, then $F^n(x)_0 = F^n(y)_0$ for all $n > 0$. Let $u \in A^{2k+1}$ and let $x \in [u]_{-k}$ be the σ-periodic point with period $2k+1$. Then $(F^n(x)_0)_{n \geq 0}$ is an eventually periodic sequence with some preperiod $m_u \geq 0$ and period $p_u > 0$. For every $y \in [u]_{-k}$, $(F^n(y)_0)_{n \geq 0}$ has preperiod m_u and period p_u. Put

$$m = \max\{m_u : \; u \in A^{2k+1}\}, \;\; p = \mathrm{lcm}\{p_u : \; u \in A^{2k+1}\}$$

Here lcm is the least common multiple. For every $x \in A\;\;$, $F^m(x)_0 = F^{m+p}(x)_0$, so

$$F^m(x)_i = F^m(\sigma^i(x))_0 = F^{m+p}(\sigma^i(x))_0 = F^{m+p}(x)_i$$

and therefore $F^{m+p}(x) = F^m(x)$.
Conversely, suppose that there exist $m \geq 0$ and $p > 0$ such that for every x $F^{m+p}(x) = F^m(x)$. For $\varepsilon = 2^{-k}$ put $\delta = 2^{-k-r(m+p)}$. If $d(x,y) < \delta$, then

$$\begin{aligned} & d(F(x), F(y)) < 2^{-k-r(m+p-1)} < 2^{-k} \\ i \leq m+p \;\; \Rightarrow \;\; & d(F^i(x), F^i(y)) < 2^{-k-r(m+p-i)} \leq 2^{-k} \end{aligned}$$

Since both $F^i(x)_{\langle -k,k \rangle}$ and $F^i(y)_{\langle -k,k \rangle}$ are eventually periodic with preperiod m and period p, and since the first $m+p$ their elements are equal, we get $d(F^n(x), F^n(y)) < 2^{-k}$ for every $n \geq 0$. □

In Example 6, the preperiod and period are $m = p = 2$.

THEOREM 5.3. *Let $(A\;\;, F)$ be a CA and $\Sigma \subseteq A\;\;$ an F- invariant subshift. If (Σ, F) is transitive then it is either sensitive or consists of a single periodic orbit.*

Proof. Suppose that (Σ, F) is transitive and not sensitive. By a theorem of Glasner and Weiss [18], it is uniformly rigid, so for $\varepsilon = 1$ there exists n such that for all $x \in \Sigma$, $d(F^n(x), x) < 1$, i.e., $F^n(x)_0 = x_0$. For $y = \sigma^i(x)$ we get

$$F^n(x)_i = \sigma^i(F^n(x))_0 = F^n(\sigma^i(x))_0 = \sigma^i(x)_0 = x_i$$

Thus F^n is identity and since (Σ, F) is transitive, it consists of a single periodic orbit. □

COROLLARY 5.1 (Kůrka [21]). *Every transitive CA is sensitive.* The sum CA of Example 3 is transitive, and therefore sensitive.

EXAMPLE 7. $(\mathbf{2}\ \ \times \mathbf{2}\ \ , \mathrm{Id} \times \sigma)$ *(i.e., $F(x,y)_i = (x_i, y_{i+1})$) is a surjective and sensitive CA which is not transitive.*

6. Surjectivity. Let $(A\ \ , F)$ be a CA with radius $r \geq 0$ and a local rule $f : A^{2r+1} \to A$. We extend it to a function $f : A^* \to A^*$ by

$$f(u)_i = \begin{cases} f(u_{\langle i, i+2r \rangle}) & \text{if} \quad 0 \leq i < |u| - 2r \\ \lambda & \text{if} \quad i \geq |u| - 2r \end{cases}$$

Thus $|f(u)| = \max\{0, |u| - 2r\}$. For example the successive images of a word 01100011 of the sum CA of Example 3 are

$$01100011 \mapsto 111011 \mapsto 0101 \mapsto 00 \mapsto \lambda \mapsto \lambda$$

Observe that for every $n \geq 0$,

$$\sum_{u \in A^n} |f^{-1}(u)| = |A^{n+2r}| = |A|^{n+2r}$$

The mean number of preimages of a word of length n is $|A|^{2r}$. We show that a CA is surjective iff every nonempty word has exactly $|A|^{2r}$ preimages.

THEOREM 6.1 (Hedlund [19]). *Let $(A\ \ , F)$ be a CA with local rule $f : A^{2r+1} \to A$. Then F is surjective iff for every $u \in A^+$, $|f^{-1}(u)| = |A|^{2r}$.*

Proof. Let the condition be satisfied and $y \in A\ \ $. For $n \geq 0$ put

$$X_n = \{x \in A\ \ : \ f(x_{\langle -n-r, r+n \rangle}) = y_{\langle -n, n \rangle}\}$$

By the assumption every X_n is nonempty, closed and $X_{n+1} \subseteq X_n$. By compactness there exists $x \in \bigcap_{n>0} X_n$ and $F(x) = y$, so F is surjective. Conversely, suppose that F is surjective and put

$$p = \min\{|f^{-1}(u)| : \ u \in A^+\}$$

Since F is surjective, every word has at least one preimage, so $p > 0$. Indeed, if $y \in A\ \ $ contains u, then every preimage of y contains some preimage of u. We show

LEMMA 6.1. *If $|f^{-1}(u)| = p$, then for every $a \in A$, $|f^{-1}(ua)| = p$.*

Proof. By the assumption, for every $a \in A$, $f^{-1}(ua) \geq p$. Suppose that for some $a \in A$, $|f^{-1}(ua)| > p$. We have disjoint unions

$$\bigcup_{b \in A} \{vb : v \in f^{-1}(u)\} = \bigcup_{a \in A} f^{-1}(ua)$$

$$p|A| = \left| \bigcup_{b \in A} \{vb : v \in f^{-1}(u)\} \right| = \left| \bigcup_{a \in A} f^{-1}(ua) \right| > p|A|$$

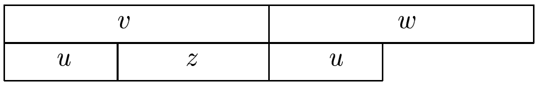
and this is a contradiction. This proves Lemma 6.1. □

LEMMA 6.2. $p = |A|^{2r}$.

Proof. Let $u \in A^n$ be such that $|f^{-1}(u)| = p$. By Lemma 1, we have $|f^{-1}(uw)| = p$ for every $w \in A^*$. We have a disjoint union

$$\{vw:\ v, w \in f^{-1}(u)\} = \bigcup_{z \in A^{2r}} f^{-1}(uzu)$$

$$p^2 = \left|\{vw:\ v, w \in f^{-1}(u)\}\right| = \left|\bigcup_{z \in A^{2r}} f^{-1}(uzu)\right| = p|A|^{2r}$$



so $p = |A|^{2r}$. This proves Lemma 6.2. □

We finish now the proof of the theorem. Suppose that for some $n > 0$ and some $u \in A^n$, $|f^{-1}(u)| > |A|^{2r}$. Then

$$|A|^{n+2r} = \left|\bigcup_{v \in A^n} f^{-1}(v)\right| > |A|^n \cdot |A|^{2r}$$

and this is a contradiction. Thus for every $u \in A^*$, $|f^{-1}(u)| = |A|^{2r}$. □

The condition $|f^{-1}(u)| = |A|^{2r}$ may be satisfied for $|u| = 1$ and fail for longer u. For example in the majority CA of Example 2, the condition works for $|u| = 1$ but does not work for $|u| = 2$, since there are six preimages of 00:

$$f^{-1}(00) = \{0000, 0001, 0010, 0100, 1000, 1001\}$$

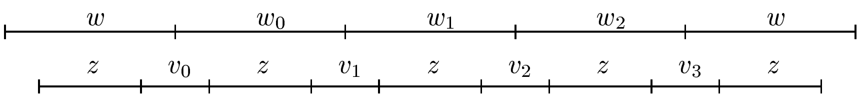
THEOREM 6.2 (Hedlund [19]). *Let $(A\quad, F)$ be a CA with radius $r > 0$ which is not surjective. Then*

1. There exists $w \in A^{2r}$ and distinct $u, v \in A^+$ with $|u| = |v|$ and $f(wuw) = f(wvw)$.

2. There exists a point $x \in A$ with an infinite (continuum) number of preimages.

Proof. 1. By Theorem 6.1 there exists a word $z \in A^+$ such that $|f^{-1}(z)| = k > |A|^{2r}$. Choose a word $w \in f^{-1}(z)$, so $|w| = |z| + 2r \geq 2r$. For $m > 0$ consider sets

$$M_m = \{ww_0w_1 \dots w_{m-1}w:\ w_i \in f^{-1}(z)\},$$
$$N_m = \{zv_0z \dots zv_mz:\ v_i \in A^{2r}\}$$



We have $f(M_m) \subseteq N_m$, $|M_m| = k^m$, $|N_m| = |A|^{2r(m+1)}$. Since $k > |A|^{2r}$, for m large enough, $k^m > |A|^{2r(m+1)}$, so there exist distinct $u', v' \in M_n$ with $f(u') = f(v')$. Since both u' and v' begin and end with w of length at least $2r$, the statement follows.
2. Let $w \in A^{2r}$, $u, v \in A^n$ be such that $u \neq v$ and $f(wuw) = f(wvw)$. Put $M = \{wu, wv\}$, i.e., $x \in M$ iff for all $i \in$, $x_{\langle i(2r+n), i(2r+n)+2r)} = w$ and $x_{\langle i(2r+n)+2r, (i+1)(2r+n))} \in \{u, v\}$. Then M is uncountable and all its elements have the same image. □

A couple of words wuw, wvw with $f(wuw) = f(wvw)$ is called a diamond.

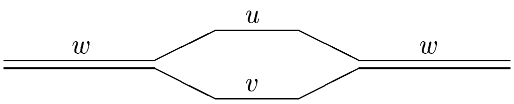


FIG. 6. *A diamond*

COROLLARY 6.1. *Every injective CA is surjective and hence bijective.*

THEOREM 6.3. *Every surjective CA is nonwandering.*

Proof. We use the Poincaré recurrence theorem: Let $(A\ , F)$ be a CA with local rule $f : A^{2r+1} \to A$. For every clopen set U define its measure $\mu(U)$ as follows. If $u \in A^n$ and $k \in$, then $\mu([u]_k) = |A|^{-n}$. If $U = V_1 \cup \cdots \cup V_m$ is a disjoint union of cylinders, put $\mu(U) = \mu(V_1) + \cdots + \mu(V_m)$. Clearly $\mu(A\ ) = 1$. If $u \in A^n$, then $f^{-1}(u) \subseteq A^{n+2r}$ and $|f^{-1}(u)| = |A|^{2r}$, so

$$\mu(F^{-1}([u]_k)) = |A|^{2r} \cdot |A|^{-n-2r} = \mu([u]_k)$$

so for every clopen set U, $\mu(F^{-1}(U)) = \mu(U)$.
Let U be a clopen set and suppose that all $F^{-i}(U)$ are pairwise disjoint. Then for every $n > 0$,

$$n \cdot \mu(U) = \mu(U \cup \cdots \cup F^{-n+1}(U)) \leq \mu(A\ ) = 1$$

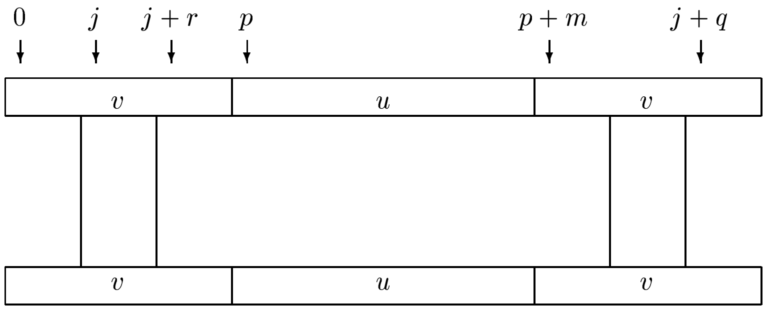
and this is a contradiction. Thus for some $i < j$, $F^{-i}(U) \cap F^{-j}(U) \neq \emptyset$ and $U \cap F^{-j+i}(U) \neq \emptyset$. □

THEOREM 6.4 (Blanchard and Tisseur [6]). *Any surjective almost equicontinuous CA has a dense set of periodic points.*

Proof. Let $u \in A^m$ be arbitrary and let $v \in A^p$ be an r-blocking word with offset j. Then $w = vuv$ is a q-blocking word with offset j, where $q = p + m + r$. Since $(A\ , F)$ is nonwandering, there exists $t > 0$ and $x \in [w]_0 \cap F^{-t}([w]_0)$. Put $y = (vu)^\infty$. Since $x, F^t(x), y \in [w]_0$,

$$F^t(y)_{\langle j, j+q)} = F^t(x)_{\langle j, j+q)} = x_{\langle j, j+q)} = y_{\langle j, j+q)}$$

Since y and $F^t(y)$ are σ-periodic with period $p+m \le q$, $F^t(y) = y$ and $y \in [u]_p$ is F-periodic with period t. □



THEOREM 6.5 (Blanchard and Tisseur [6]). *If* $(A \;\; , F)$ *is a surjective equicontinuous CA, then there exists* $p > 0$ *such that* $F^p = \mathrm{Id}$. *In particular,* F *is bijective.*

Proof. By Theorem 5.2, there exists $m \ge 0$ and $p > 0$ such that $F^m = F^{p+m}$. Since every point is eventually periodic and nonwandering, it is periodic and $m = 0$. □

7. Preimages. We study now the number of preimages of points in surjective CA's.

THEOREM 7.1 (Hedlund [19]). *Let* $(A \;\; , F)$ *be a surjective CA with radius* r. *Then for every* $y \in A \;\;$, $|F^{-1}(y)| \le |A|^{2r}$.

Proof. Suppose that $x_0, x_1 \ldots x_{n-1}$ are distinct elements in $F^{-1}(y)$ and $n > |A|^{2r}$. There exists $m > r$, such that for all $i \ne j$, $(x_i)_{\langle -m,m\rangle} \ne (x_j)_{\langle -m,m\rangle}$, so $y_{\langle -m+r,m-r\rangle}$ has at least n preimages and this is a contradiction. □

Although every finite word has exactly $|A|^{2r}$ preimages, infinite words may have strictly fewer preimages.

EXAMPLE 8 (L: A closing CA, Boyle et al. [7]). $(\mathbf{3} \;\; , L)$, *where*

$$L(x)_i = \begin{cases} x_i + x_{i+1} \bmod 2 & \textit{if} \quad x_i \ne 2 \\ 2 & \textit{if} \quad x_i = 2 \end{cases}$$

is a surjective CA with nonconstant number of preimages.

We will show later that $(\mathbf{3} \;\; , L)$ is a closing CA. In Figure 7, two's are represented by a black square, ones by a dot and zeros by empty space. While the only preimage of 2^∞ is 2^∞ itself, 1^∞ has two preimages $(01)^\infty$ and $(10)^\infty$. We show that L is surjective. The radius is $r = 1$ so we show

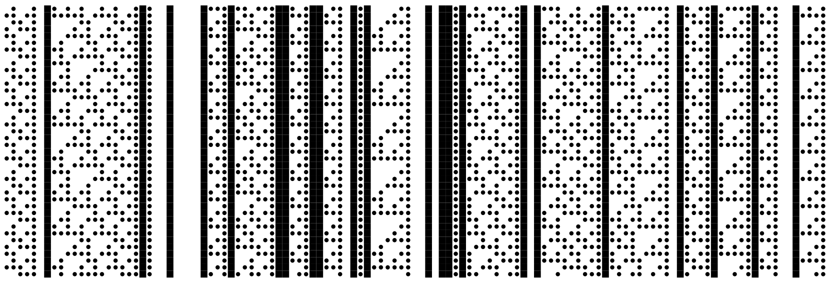


FIG. 7. *L: A closing CA*

that every $v \in \mathbf{2}^n$ has exactly $|A|^{2r} = 9$ preimages. For every $u_{-1}, u_n \in A$ put successively for $i = n-1, \dots, 0$

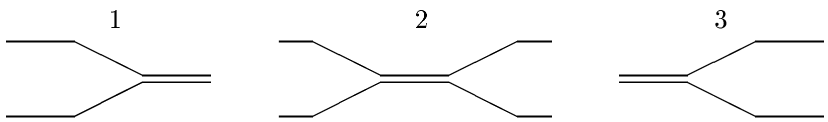
$$u_i = \begin{cases} v_i + u_{i+1} \bmod 2 & \text{if} \quad v_i \neq 2 \\ 2 & \text{if} \quad v_i = 2 \end{cases}$$

Then $f(u_{\langle -1,n \rangle}) = v_{\langle 0,n)}$, so every couple $(u_{-1}, u_n) \in \mathbf{3}^2$ determines a distinct preimage and $|f^{-1}(v)| = |A|^2 = 9$.

We are going to characterize CA with constant number of preimages.

DEFINITION 7.1. *Let $x, y \in A$ and $s \geq 0$.*
1. x, y are totally s-separated if $\forall n \in$, $x_{\langle n,n+s)} \neq y_{\langle n,n+s)}$.
2. x, y are left (right) s-separated if there exists $m \in$, such that $x_{\langle n,n+s)} \neq y_{\langle n,n+s)}$ for all $n \leq m$ (for all $n \geq m$).
3. x, y are left (right) asymptotic, if there exists $m \in$ such that $x_n = y_n$ for all $n \leq m$ (for all $n \geq m$).
4. A point $z \in A$ is σ-transitive, iff for every open $U \subseteq A$ there exist $n < 0 < m$ such that $\sigma^n(z), \sigma^m(z) \in U$.

LEMMA 7.1. *Let $(A$ $, F)$ be a surjective CA with radius r, let $x, y \in A$, $x \neq y$ be such that $F(x) = F(y)$ is a σ-transitive point. Then exactly one of the following three statements is true.*
1. x and y are left $2r$-separated and right asymptotic.
2. x and y are both left and right $2r$-separated.
3. x and y are left asymptotic and right $2r$-separated.



Proof. Assume that x and y are not right $2r$-separated, and that the statement (1) does not hold. Since $x \neq y$ there exists p with $x_p \neq y_p$.

Since x and y are not right $2r$-separated, there exists $m > p$ such that $x_{\langle m,m+2r)} = y_{\langle m,m+2r)}$. Since (1) does not hold, there are two possibilities:
a. x and y are not left $2r$-separated, so there exists $n < p$ with $x_{\langle n,n+2r)} = y_{\langle n,n+2r)}$. In this case $u = x_{\langle n,m+2r)}$, $v = y_{\langle n,m+2r)}$ form a diamond and every point in $\{u,v\}$ would have the same image.
b. x and y are not right asymptotic. In this case there exists $q > m$ with $x_q \neq y_q$ and since x and y are not right $2r$- separated there exists $n > q$ with $x_{\langle n,n+2r)} = y_{\langle n,n+2r)}$. Again, $u = x_{\langle m,n+2r)}$ and $v = y_{\langle m,n+2r)}$ would form a diamond and this is not possible. We have proved that if x and y are not right $2r$-separated, then (1) holds. Similarly, if x and y are not left $2r$-separated then (3) holds. $\square$

LEMMA 7.2. *Let $(A \ ,F)$ be a surjective CA with radius r, $z \in A$ a σ-transitive point, and $x_1, \ldots x_k \in F^{-1}(z)$ pairwise right (left) $2r$-separated points. Then there exist $y_1, \ldots, y_k \in F^{-1}(z)$ which are pairwise totally $2r$-separated.*

Proof. By the assumption there exists $p \in$ such that for all $n \geq p$ and for every distinct $i, j \leq k$, $(x_i)_{\langle n,n+2r)} \neq (x_j)_{\langle n,n+2r)}$. For every $q > 0$ there exists $n_q > p + q$ such that $d(z, \sigma^{n_q}(z)) < 2^{-q}$. Choosing a subsequence if necessary, we can assume that for every $i \leq k$, there exist limits

$$\lim_{q \to \infty} \sigma^{n_q}(x_i) = y_i$$

Clearly $F(y_i) = z$. We show that $(y_i)_{i \leq k}$ are totally $2r$-separated. For $m \in$ put $s = \max\{-m, m + 2r\}$. There exists $q \geq s$ such that for every $i \leq k$, $d(\sigma^{n_q}(x_i), y_i) < 2^{-s}$. Since $-s \leq m \leq m + 2r \leq s$,

$$(y_i)_{\langle m,m+2r)} = \sigma^{n_q}(x_i)_{\langle m,m+2r)} \neq \sigma^{n_q}(x_j)_{\langle m,m+2r)} = (y_j)_{\langle m,m+2r)}$$

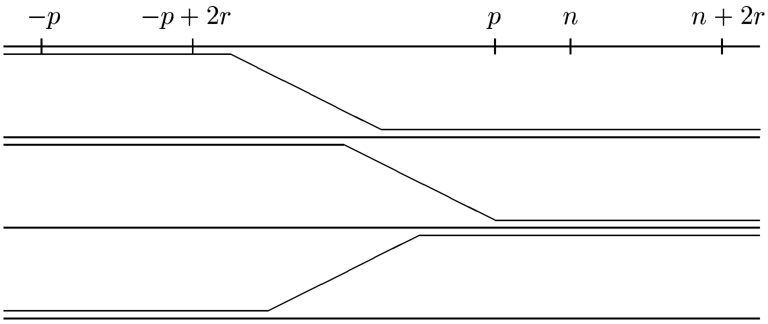
so $(y_i)_{i \leq k}$ are totally $2r$-separated. $\square$

PROPOSITION 7.1 (Hedlund [19]). *Let $(A \ ,F)$ be a surjective CA and $z \in A$ a σ-transitive point. Then its preimages are pairwise totally $2r$-separated.*

Proof. Let $k_0 > 0$ be the maximum number such that there exists points $x_1, \ldots, x_{k_0} \in F^{-1}(z)$, which are pairwise totally $2r$-separated. Assume that $F(y) = z$ and that y is different from all x_i. If y were right $2r$-separated from all x_i, by Lemma 7.2 there would exist $k_0 + 1$ pairwise totally $2r$-separated preimages of z. Thus there exists $j \leq k_0$ such that y and x_j are not right $2r$-separated and by Lemma 7.1 they are right asymptotic. Similarly we prove that there exists $l \leq k_0$ such that y and x_l are left asymptotic. Let $x_{k_0+1}, \ldots, x_{k_1}$ be all preimages of z different from

$x_1, \ldots, x_{k_0}$. There exists $p > 0$ such that

$$\forall i \in (k_0, k_1\rangle, \exists j \leq k_0, (x_i)_{\langle p,\infty)} = (x_j)_{\langle p,\infty)}$$
$$\forall i \in (k_0, k_1\rangle, \exists l \leq k_0, (x_i)_{(-\infty,-p+2r\rangle} = (x_l)_{(-\infty,-p+2r\rangle}$$



For every $q > 0$ there exists $n_q > p + q$ such that $d(\sigma^{-p}(z), \sigma^{n_q}(z)) < 2^{-q}$. By choosing a subsequence if necessary, we can assume that there exist limits

$$\lim_{q\to\infty} \sigma^{n_q}(x_i) = y_i, \ \ 1 \leq i \leq q_1$$

so $F(y_i) = \sigma^{-p}(z)$. Since $F^{-1}(\sigma^{-p}(z)) = \{\sigma^{-p}(x_i) : \ 1 \leq i \leq k_1\}$, for every $i \leq k_1$ there exists $j_i \leq k_1$ such that $y_i = \sigma^{-p}(x_{j_i})$. There exists $n > p$ such that for all $i \leq k_1$, $d(\sigma^{n+r}(x_i), y_i) < 2^{-r}$, so

$$(x_i)_{\langle n,n+2r)} = (y_i)_{\langle 0,2r)} = (x_{j_i})_{\langle -p,-p+2r)}$$

However, both sets $\{(x_i)_{\langle n,n+2r)} : \ \ 1 \leq i \leq k_1\}$ and $\{(x_{j_i})_{\langle -p,-p+2r)} : 1 \leq i \leq k_1\}$ contain exactly k_0 elements and they are equal. Consider a set of words $\{u_i = (x_i)_{\langle -p,n)}; \ 1 \leq i \leq k_1\}$ and define a binary matrix $M = (M_{ij})_{1\leq i,j\leq k_1}$ by

$$M_{ij} = 1 \ \Leftrightarrow \ (x_i)_{\langle n,n+2r)} = (x_j)_{\langle -p,-p+2r)}$$

For every $i \leq k_1$ there exists at least one $j \leq k_1$ with $M_{ij} = 1$ and for some i there exist at least two j with $M_{ij} = 1$. Dually, for every j there exists at least one, and sometimes at least two i with $M_{ij} = 1$. Let $\Sigma \subseteq A$ be a SFT defined by $w \in \Sigma$ iff for every $i \in$ there exists $j_i \leq k_1$ such that

$$w_{\langle i(n+p),(i+1)(n+p))} = u_{j_i} \ \text{ and } \ M_{j_i,j_{i+1}} = 1$$

Then for every $x, y \in \Sigma$, $F(x) = F(y)$, but Σ is an infinite set and this is a contradiction. ◻

Proposition 7.2 (Hedlund [19]). *Let $(A \ \ , F)$ be a surjective CA and $y, z \in A$. Let z be σ-transitive and $|F^{-1}(z)| = m$. Then $|F^{-1}(y)| \geq m$*

and there exist m *pairwise totally* $2r$*-separated preimages of* y*. In particular every two* σ*-transitive points have the same number of preimages.*

Proof. Let $F^{-1}(z) = \{x_1, \dots, x_m\}$. By Proposition 7.1, $(x_i)_{i\leq m}$ are pairwise totally $2r$-separated. For every q there exists n_q for which $d(\sigma^{n_q}(z), y) < 2^{-q}$. By choosing a subsequence if necessary, we can assume that $\sigma^{n_q}(x_i)$ converge to some w_i as $q \to \infty$. Then $(w_i)_{i\leq m}$ are pairwise totally $2r$-separated and $F(w_i) = y$. □

COROLLARY 7.1. *Let* $(A\ \ , F)$ *be a surjective CA and put*

$$m = \min\{|F^{-1}(x)| : x \in A\ \ \}.$$

The set $\{x \in A\ \ : |F^{-1}(x)| = m\}$ *is residual and contains all* σ*-transitive points.*

In the CA $(\mathbf{3}\ \ , L)$ of Example 8, $m = 1$. Every point which has an infinite number of two's in $\langle 0, \infty)$, has only one preimage.

8. Openness. We characterize CA with constant number of preimages. Recall that a dynamical system (X, F) is open if for every open set $U \subseteq X$, $F(U)$ is open. Clearly every bijective DS is open. By a theorem of Parry [27], a one-sided subshift is open iff it is a SFT. Since the sum CA of Example 3 is conjugated to the full shift of four symbols, it is open. Moreover, every its point has exactly four preimages.

THEOREM 8.1 (Hedlund [19]). *Let* $(A\ \ , F)$ *be a CA. The following conditions are equivalent.*

1. There exists $m > 0$ *such that for all* $x \in A\ \ $, $|F^{-1}(x)| = m$.

2. There exist continuous maps $F_1, \dots, F_m : A\ \ \to A\ \ $ *such that for all* $x \in A\ \ $, $F^{-1}(x) = \{F_1(x), \dots, F_m(x)\}$.

3. F *is open.*

Proof. $1 \Rightarrow 2$: By Proposition 7.2, the preimages of every point are pairwise totally $2r$-separated. Assume that $F^{-1}(y) = \{x_1, \dots, x_m\}$. We show

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y' \in B_\delta(y), \exists i \leq m, \exists x' \in B_\varepsilon(x_i), F(x') = y'$$

If this were not true, there would exist $\varepsilon > 0$ such that for every $\delta = 2^{-n}$ there would exist $y_n \in B_\delta(y)$, and its preimage $z_n \in F^{-1}(y_n)$, such that $d(z_n, x_i) \geq \varepsilon$ for every i. If z were a limit of a subsequence of $(z_n)_{n\geq 0}$, then $F(z) = y$, so y would have at least $m + 1$ preimages and this is a contradiction. Moreover, if $\varepsilon \leq 2^{-r}$, then the m preimages of y' belong to distinct $B_\varepsilon(x_i)$, since they are pairwise totally $2r$-separated and therefore their distance is at least 2^{-r}. Thus we get a stronger condition

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y' \in B_\delta(y), \forall i \leq m, \exists! x'_i \in B_\varepsilon(x_i), F(x'_i) = y'$$

Suppose that δ has this property with $\varepsilon = 2^{-r}$. Let $\mathcal{V} = \{V_b : \ b \in B\}$ be a clopen partition of A with $\mathrm{diam}(\mathcal{V}) < \delta$. Choose from every V_b a point $y_b \in V_b$ and denote its preimages $F^{-1}(y_b) = \{x_{b,1}, \dots, x_{b,m}\}$. Define $F_i : A \to A$ by

$$x \in V_b \ \Rightarrow \ d(F_i(x), x_{b,i}) < \varepsilon, \ F(F_i(x)) = x$$

Then F_i are continuous.
$2 \Rightarrow 3$: Assume that $U \subseteq A$ is an open set such that $F(U)$ is not open, so there exists $x \in U$ such that $F(U)$ is not a neighbourhood of $F(x)$. For some $i \leq m$, $x = F_i(F(x))$. There exists a sequence $(y_n \notin F(U))_{n \geq 0}$ converging to $F(x)$ and $F_i(y_n) \notin U$. $(F_i(y_n))_{n \geq 0}$ has a subsequence converging to some $z \notin U$ and $F(z) = F(x)$. Since the preimages of every y_n are pairwise totally $2r$-separated, z differs from all $F_j(F(x))$ and this is a contradiction.
$3 \Rightarrow 1$: We show first that F is surjective. Suppose that $V = F(A) \neq A$, Since V is clopen and (A, σ) is transitive, there exists $n > 0$ such that $\sigma^n(V) \cap (A \setminus V) \neq \emptyset$ and this is impossible since V is σ- invariant. Put $m = \min\{|F^{-1}(x)| : \ x \in A\}$ and assume that for some $y \in A$, $F^{-1}(y) = \{x_1, \dots, x_p\}$, where $p > m$. Let $U_i \ni x_i$ be pairwise disjoint open sets. Then $V = F(U_1) \cap \dots \cap F(U_p)$ is an open set containing x, and every element of V has at least p preimages. However, σ-transitive points are dense, so V contains a σ-transitive point which has only m preimages. $\square$

The inverse maps F_i are in general not CA. In the sum CA of Example 3, the four inverse maps may be given by conditions

$$F_0(x)_{\langle 0,1 \rangle} = 00, \ \ F_1(x)_{\langle 0,1 \rangle} = 01, \ \ F_2(x)_{\langle 0,1 \rangle} = 10, \ \ F_3(x)_{\langle 0,1 \rangle} = 11$$

Only when $m = 1$, i.e., when F is bijective, the inverse map F^{-1} is a CA. This is the case of identity and of every almost equicontinuous open map.

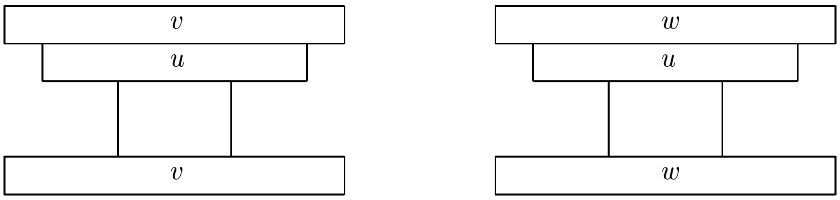
Theorem 8.2. *Every CA which is open and almost equicontinuous is bijective.*

Proof. Let $u \in A^m$ be a $2r$-blocking word with offset j. Suppose that some $z \in [u]_0$ has two distinct preimages x, y. Then $v = x_{\langle -r, m+r)}$, $w = y_{\langle -r, m+r)}$ are distinct (since x and y are totally $2r$-separated), and both $F([v]_{-r})$ and $F([w]_{-r})$ are contained in $[u]_0$. Moreover, v and w are $2r$-blocking words with offsets $j + r$. By Proposition 6.3 , (A, F) is nonwandering, so there exist $p, q > 0$ and $x' \in [v]_{-r} \cap F^{-p}([v]_{-r})$, $y' \in [w]_r \cap F^{-q}([w]_{-r})$. It follows

$$x'_{\langle j, j+2r)} = F^{pq}(x')_{\langle j, j+2r)} = F^{pq-1}(z)_{\langle j, j+2r)} = F^{pq}(y')_{\langle j, j+2r)} = y'_{\langle j, j+2r)}$$

$$x_{\langle j, j+2r)} = x'_{\langle j, j+2r)} = y'_{\langle j, j+2r)} = y_{\langle j, j+2r)}$$

so x and y are not $2r$-separated. Thus z has only one preimage, and since F is open, every point has exactly one preimage. ☐



EXAMPLE 9 (B: A bijective almost equicontinuous CA).
$(A\ \ \ , B)$ where $A = \{000, 001, 010, 011, 100\} = \{0, 1, 2, 3, 4\}$, and

$$B(x, y, z)_i = (x_i, (1 + x_i)y_{i+1} + x_{i-1}z_i, (1 + x_i)z_{i-1} + x_{i+1}y_i)$$

(the addition is modulo 2).

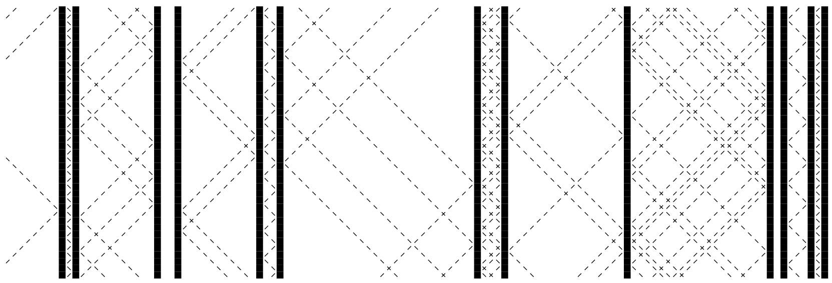


FIG. 8. *B: A bijective CA*

The dynamics is conveniently described as movement of three types of particles, $1 = 001$, $2 = 010$ and $4 = 100$. Four is a wall which neither moves nor changes. A 1 goes to the left and when it hits a wall (4), it changes to 2. A two goes to the right and when it hits the wall, it changes to a one. One's and two's may cross, to form three's. Clearly 4 is a 1-blocking word, so the system is almost equicontinuous. It is bijective, as its inverse is

$$B^{-1}(x, y, z)_i = (x_i, (1 + x_i)y_{i-1} + x_{i+1}z_i, (1 + x_i)z_{i+1} + x_{i-1}y_i)$$

9. Closingness.

DEFINITION 9.1. *A CA $(A\ \ \ , F)$ is right closing (left closing), if for every distinct left asymptotic (right asymptotic) $x, y \in A\ \ \ $, $F(x) \neq F(y)$. A CA is closing if it is either left or right closing.*

Clearly $(A \quad ,F)$ is right closing iff its dual $(A \quad ,\overline{F})$ is left closing.

The CA $(\mathbf{3} \quad ,L)$ of Example 8 is left closing. If $L(x) = z$, then for every $i \in$

$$x_i = \begin{cases} x_{i+1} + z_i \bmod 2 & \text{if} \quad z_i \neq 2 \\ 2 & \text{if} \quad z_i = 2 \end{cases}$$

Thus if $L(x) = L(y)$ and if x and y are right asymptotic, then $x = y$. The CA is, however, not right closing. The points $x = 0^\infty.20^\infty$ and $y = 0^\infty.21^\infty$ are left asymptotic and their images coincide.

Example 10 (R: a surjective CA). *The CA* $(\mathbf{3} \quad \times \mathbf{3} \quad ,R)$, *where* $R = L \times \overline{L}$, *is not closing but it is surjective.* The points $x = (0^\infty.20^\infty, 0^\infty)$ and $y = (0^\infty.21^\infty, 0^\infty)$ are left asymptotic but their images coincide. The points $x = (0^\infty, 0^\infty 2.0^\infty)$ and $y = (0^\infty, 1^\infty 2.0^\infty)$ are right asymptotic but their images coincide.

Proposition 9.1. *Every closing CA is surjective.*

Proof. Let $(A \quad ,F)$ be a CA with radius r which is not surjective. By Theorem 6.2 there exists a diamond, i.e., words $w \in A^{2r}$ and $u \neq v \in A^+$ with $|u| = |v|$ and $f(wuw) = f(wvw)$. Put $x = w^\infty.uw^\infty$, $y = w^\infty.vw^\infty$. Then x, y are both left and right asymptotic and $F(x) = F(y)$, so F is neither right nor left closing. □

Proposition 9.2. *A CA* $(A \quad ,F)$ *is right closing iff there exists* $m > 0$ *such that*

$$x_{\langle -m,0)} = y_{\langle -m,0)} \;\; \& \;\; F(x)_{\langle -m,m\rangle} = F(y)_{\langle -m,m\rangle} \;\Rightarrow\; x_0 = y_0$$

Proof. Suppose that F is right closing. For $m > 0$ let X_m consist of all pairs $(x, y) \in A \quad \times A \quad$ such that

$$x_{\langle -m,0)} = y_{\langle -m,0)}, \;\; x_0 \neq y_0, \;\; F(x)_{\langle -m,m\rangle} = F(y)_{\langle -m,m\rangle}$$

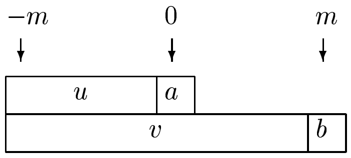
Every X_m is closed and $X_{m+1} \subseteq X_m$. If all X_m were nonempty, their intersection would contain a pair (x, y) of distinct left asymptotic words with $F(x) = F(y)$, so F would not be right closing. Thus for some m, $X_m = \emptyset$.
Conversely, suppose that F is not right closing, so there exist distinct left asymptotic words x, y with $F(x) = F(y)$. Taking the shifts of x and y if necessary, we can assume $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$. Thus for every $m > 0$, the condition is not satisfied. □

Proposition 9.3. *Let* $(A \quad ,F)$ *be a right closing CA. There exists* $m > 0$ *such that if* $u \in A^m$, $v \in A^{2m}$ *and* $F([u]_{-m}) \cap [v]_{-m} \neq \emptyset$, *then*

$$\forall b \in A, \exists! a \in A, F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset.$$

Proof. By Proposition 9.2, for every b there exists at most one a with $F([ua]_{-m}) \cap [vb]_{-m} \neq \emptyset$. Put

$$B = \{(u,v) \in A^m \times A^{2m} : F([u]_{-m}) \cap [v]_{-m} \neq \emptyset\}$$



Consider an oriented graph with vertices B. For every couple of words $u \in A^{m+1}$, $v \in A^{2m+1}$ such that $F([u]_{-m}) \cap [v]_{-m} \neq \emptyset$, put an oriented edge

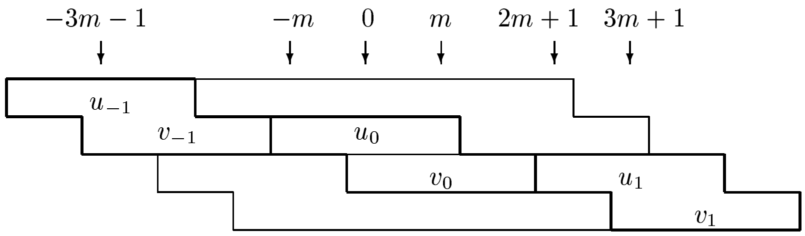
$$(u_{\langle 0,m-1\rangle}, v_{\langle 0,2m-1\rangle}) \to (u_{\langle 1,m\rangle}, v_{\langle 1,2m\rangle})$$

This graph defines a SFT $\Sigma \subseteq B\quad$ of order two, and there is a conjugacy $\varphi : (A\quad, \sigma) \to (\Sigma, \sigma)$ defined by $\varphi(x)_i = (x_{\langle i,i+m-1\rangle}, F(x)_{\langle i,i+2m-1\rangle})$. The topological entropy is $h(\Sigma,\sigma) = h(A\quad,\sigma) = \ln|A|$. In the graph of Σ, from every vertex $(u_{\langle 0,m-1\rangle}, v_{\langle 0,2m-1\rangle})$ there leads at most $|A|$ edges (for every v_{2m} at most one). If there leads fewer than $|A|$ edges from some vertex, then the topological entropy $h(\Sigma,\sigma)$ would be smaller than $\ln|A|$ (Lind and Marcus, [22], Corollary 4.4.9.). □

Theorem 9.1. *If $(A\quad, F)$ is a right closing CA, then for some $p > 0$, $(A\quad, \sigma^p \circ F)$ is a factor of a two-sided full shift.*

Proof. Let $m > 0$ be the number from Proposition 9.3. We can assume $m \geq r$, where r is the radius. Put $p = 3m+1$ and

$$B = \{(u,v) \in A^{2m+1} \times A^{2m+1} : F^{-1}([u]_{-m}) \cap [v]_0 \neq \emptyset\}$$



If (u_0,v_0) and (u_1,v_1) belong to B, then there exists $y \in [u_1]_{2m+1} \cap F^{-1}([v_1]_{3m+1})$. We can change y in interval $\langle 0, 2m\rangle$ to obtain a point $x \in [v_0u_1]_0 \cap F^{-1}([v_1]_{3m+1})$. By Proposition 9.3, x has a preimage in $[u_0]_{-m}$, so

$$F^1([u_0]_{-m}) \cap [v_0u_1]_0 \cap F^{-1}([v_1]_{3m+1}) \neq \emptyset$$

If $(u,v) \in B$, then $X_1 = F^1([v_{-1}u_0]_{-3m-1}) \cap [v_0 u_1]_0 \cap F^{-1}([v_1 u_2]_{3m+1}) \neq \emptyset$, and similarly, for every $q > 0$,

$$X_q = \bigcap_{n=-q}^{q} F^{-n}([v_n u_{n+1}]_{(3m+1)n}) \neq \emptyset$$

By compactness, $\bigcap_{q>0} X_q$ is nonempty. If $x, y \in X_q$, then

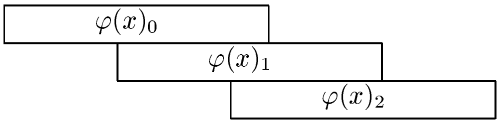
$$x_{\langle -q(3m+1)+mq, q(3m+1)-mq \rangle} = y_{\langle -q(3m+1)+mq, q(3m+1)-mq \rangle}$$

Thus there exists a unique $\varphi(u,v) \in \bigcap_{q>0} X_q$ and $\varphi : (B \;, \sigma) \to (A \;, \sigma^p \circ F)$ is a factor map. □

THEOREM 9.2 (Boyle and Kitchens [8]). *Every closing CA* $(A \;, F)$ *has a dense set of points which are both* F*-periodic and* σ*-periodic.*

Proof. We can assume that F is right closing. Since $(A \;, \sigma^p \circ F)$ is a factor of a two-sided full shift, it has a dense set of periodic points. We can assume $p > m$ where m is the integer from Proposition 9.3. We show that if $x \in A$ is a $(\sigma^p \circ F)$-periodic point, then it is both σ-periodic and F-periodic. Put $B = A^{2p+1}$ and construct a map $\varphi : A \to B$ by

$$\varphi(x)_i = (\sigma^p \circ F)^i(x)_{\langle -p,p \rangle} = F^i(x)_{\langle pi-p, pi+p \rangle}$$



Because F is right closing and $p > m$, $\varphi(\sigma(x))$ is uniquely determined by $\varphi(x)$, in fact, $\varphi(\sigma(x))_i$ depends only on $\varphi(x)_i$ and $\varphi(x)_{i+1}$. If $x \in A$ is $(\sigma^p \circ F)$-periodic, i.e., if $(\sigma^p \circ F)^q(x) = x$, then $\varphi(x)$ is σ- periodic since

$$\varphi(x)_{i+q} = (\sigma^p \circ F)^{i+q}(x)_{\langle -p,p \rangle} = (\sigma^p \circ F)^i(x)_{\langle -p,p \rangle} = \varphi(x)_i$$

It follows that $\varphi(\sigma(x))$ is σ-periodic with period q too. In B , there are at most $|B|^q$ σ-periodic points with period q, so there exist $0 \leq k < k+s < |B|^q$ such that $\varphi(\sigma^{k+s}(x)) = \varphi(\sigma^k(x))$ and for all $l \geq k$, $\varphi(\sigma^{l+s}(x)) = \varphi(\sigma^l(x))$. Applying this argument to $\sigma^{-j}(x)$ where $j = |B|^q$, we get $\varphi(\sigma^{l+s}(x)) = \varphi(\sigma^l(x))$ for all $l \geq 0$. In particular, $\varphi(\sigma^s(x)) = \varphi(x)$, so $\sigma^s(x) = x$ and

$$x = (\sigma^p \circ F)^{qs}(x) = F^{qs} \circ \sigma^{pqs}(x) = F^{qs}(x)$$

Thus x is both σ-periodic and F-periodic. □

THEOREM 9.3. *A CA is open iff it is both left closing and right closing.*

Proof. If $(A\ \ , F)$ is open and $z \in A\ \ $, then every two preimages of z are totally $2r$-separated, so they are neither left nor right asymptotic. This means that F is both right and left closing.
Conversely, Suppose that $(A\ \ , F)$ is a CA with radius r which is both left and right closing. Let $m > 0$ be an integer with property of Proposition 9.3 and $u \in A^{2k+1}$. Then $F([u]_{-k})$ is open, since it is the union of all cylinders $[v]_{-k-m}$ such that $v \in A^{2k+2m+1}$ and $F([u]_{-k}) \cap [v]_{-k-m} \neq \emptyset$. ◻

10. Expansivity.

PROPOSITION 10.1. *Every positively expansive CA $(A\ \ , F)$ with radius r is conjugated to $(\Sigma_{\langle -r,r\rangle}(F), \sigma)$.*

Proof. Suppose that there exist $x \neq y$ such that for all $n \geq 0$, $F^n(x)_{\langle -r,r\rangle} = F^n(y)_{\langle -r,r\rangle}$. Let $x_k \neq y_k$, so $|k| > r$. Assume $k > 0$ and define $z \in A\ \ $ by $z_{(-\infty,0\rangle} = x_{(-\infty,0\rangle}$, $z_{\langle 0,\infty)} = y_{\langle 0,\infty)}$. Then $x \neq z$ and for all $n \geq 0$, $F^n(x)_{(-\infty,r\rangle} = F^n(z)_{(-\infty,r\rangle}$. It follows that for every $k \geq 0$

$$F^n(\sigma^{-k}(x))_{(-\infty,r+k\rangle} = F^n(\sigma^{-k}(z))_{(-\infty,r+k\rangle}.$$

and this is a contradiction. Thus $\varphi : (A\ \ , F) \to (\Sigma_{\langle -r,r\rangle}(F), \sigma)$ is injective, hence a conjugacy. ◻

THEOREM 10.1 (Kůrka [21], Nasu [25]). *Every positively expansive CA is open and conjugated to a onesided SFT.*

Proof. We show that if $(A\ \ , F)$ is positively expansive, then it is both left and right closing. Let $\delta = 2^{-m}$ be the constant of expansivity, so

$$x \neq y \;\Rightarrow\; \exists n \geq 0, F^n(x)_{\langle -m,m\rangle} \neq F^n(y)_{\langle -m,m\rangle}$$

Suppose that x, y are distinct left asymptotic points with $F(x) = F(y)$. By taking a shift of x and y if necessary, we can assume $x_{(-\infty,m\rangle} = y_{(-\infty,m\rangle}$, so $d(F^n(x), F^n(y)) < \delta$ for all $n \geq 0$ and this is a contradiction. By Theorem 9.3, $(A\ \ , F)$ is open, so $(\Sigma_{\langle -r,r\rangle}(F), \sigma)$ is open too. By a Theorem of Parry, [27], a one sided subshift is a SFT iff it is open, so $\Sigma_{\langle -r,r\rangle}(F)$ is a SFT. ◻

THEOREM 10.2 (Blanchard and Maass [3]). *Every positively expansive CA is mixing.*

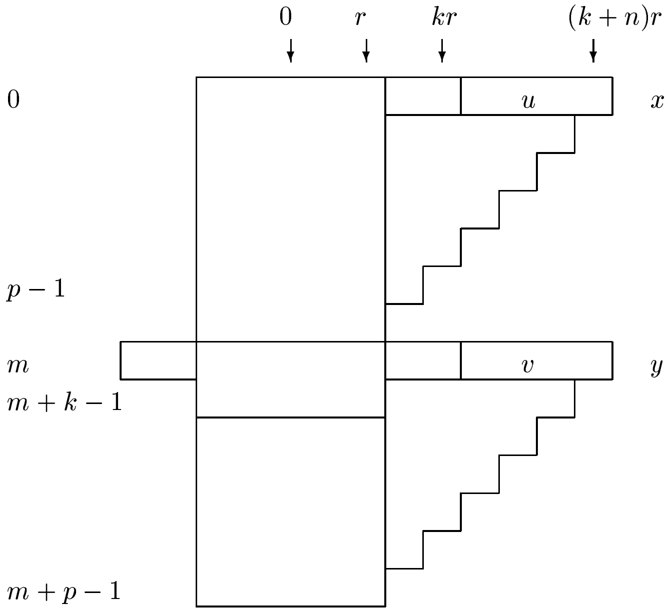
Proof. Let $(A\ \ , F)$ be a positively expansive CA with radius r, put $B = A^{2r+1}$ and let k be the order of SFT $\Sigma_{\langle -r,r\rangle}(F)$, so $w \in \Sigma_{\langle -r,r\rangle}(F)$ iff for all $n \geq 0$, $w_{\langle n,n+k)} \in \mathcal{L}(\Sigma_{\langle -r,r\rangle}(F))$. Let $u, v \in A^{nr}$. There exists $p > 0$ such that for every x, y,

$$\forall i < p, F^i(x)_{\langle -r,r\rangle} = F^i(y)_{\langle -r,r\rangle} \quad\Rightarrow\quad x_{\langle -(k+n)r,(k+n)r\rangle} = y_{\langle -(k+n)r,(k+n)r\rangle}$$

We will show that for all $m \geq p$, $F^m([u]_0) \cap [v]_0 \neq \emptyset$. Choose any $x \in A\ \ $, such that $x_{\langle kr+1,(k+n)r\rangle} = u$. Since F is surjective, there exists $y \in A$

such that

$$F^m(y)_{\langle -kr,kr\rangle} = F^m(x)_{\langle -kr,kr\rangle},\ F^m(y)_{\langle kr+1,(k+n)r\rangle} = v$$



It follows that for $m \le i < m+k$, $F^i(x)_{\langle -r,r\rangle} = F^i(y)_{\langle -r,r\rangle}$. Construct $w \in B^{m+p}$ by

$$w_i = \begin{cases} F^i(x)_{\langle -r,r\rangle} & \text{if} \quad 0 \le i \le m+k-1 \\ F^i(y)_{\langle -r,r\rangle} & \text{if} \quad m \le i \le m+p-1 \end{cases}$$

Since both $w_{\langle 0,m+k)}$ and $w_{\langle m,m+p)}$ belong to the language of the subshift, w belongs to this language too and there exists $z \in A$ such that for $i < m+p$, $F^i(z)_{\langle -r,r\rangle} = w_i$. Thus $z_{\langle kr+1,(k+n)r\rangle} = u$ and $F^m(z)_{\langle kr+1,(k+n)r\rangle} = v$, so $\sigma^{kr+1}(z) = [u]_0 \cap F^{-m}([v]_0)$. $\Box$

There is a large class of permutive local rules which yield expansive, or at least open CA.

Definition 10.1. *Let $s \ge 1$ and $f : A^s \to A$ be a local rule.*
1. f is left permutive iff $\forall u \in A^{s-1}, \forall b \in A, \exists! a \in A, f(au) = b$.
2. f is right permutive iff $\forall u \in A^{s-1}, \forall b \in A, \exists! a \in A, f(ua) = b$.

Proposition 10.2. *Let $(A\ \ , F)$ be a CA and let $f : A^{a-m+1} \to A$ be its local rule. Here $m \le a$ is the memory and anticipation respectively.*
1. If f is left permutive, then F is left closing.
2. If f is right permutive, then F is right closing.
3. If f is both left and right permutive and $m < 0 < a$, then F is expansive.

The sum CA of Example 3 is both left and right permutive with $m = -1$, $a = 1$. The closing CA of Example 8 is left permutive with $m = 0$ and $a = 1$. It is not right permutive. However, the bijective and hence open CA of Example 9 is neither left nor right permutive.

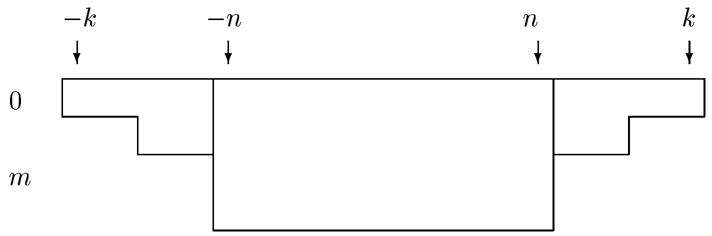
11. Shadowing property. If $(x_i)_{i\geq 0}$ is a 2^{-m}-chain in a CA $(A\ \ , F)$, then for all i, $F(x_i)_{\langle -m,m\rangle} = (x_{i+1})_{\langle -m,m\rangle}$, so $u_i = (x_i)_{\langle -m,m\rangle}$ satisfy $F([u_i]_{-m}) \cap [u_{i+1}]_{-m} \neq \emptyset$. Conversely, if a sequence $(u_i \in A^{2m+1})_{i\geq 0}$ satisfies this property and $x_i \in [u_i]_{-m}$, then $(x_i)_{i\geq 0}$ is a 2^{-m}-chain.

THEOREM 11.1 (Kůrka [21]). *Let $(A\ \ , F)$ be a CA. If for every sufficiently large n, $\Sigma_{\langle -n,n\rangle}(F)$ is a SFT, then F has the shadowing property.*

Proof. For a given $\varepsilon = 2^{-n} \leq 2^{-r}$, let $m + 1 > 1$ be the order of $\Sigma_{\langle -n,n\rangle}(F)$, so $w \in \Sigma_{\langle -n,n\rangle}(F)$ iff for every $i \geq 0$, $w_{\langle i,i+m\rangle} \in \mathcal{L}(\Sigma_{\langle -n,n\rangle}(F))$. Put $k = n + rm$ and $\delta = 2^{-k}$. We show that every δ-chain is ε-shadowed by some point. Let $(x_i)_{i\geq 0}$ be a δ-chain, so $F(x_i)_{\langle -k,k\rangle} = (x_{i+1})_{\langle -k,k\rangle}$. We get

$$F^2(x_i)_{\langle -k+r,k-r\rangle} = F(x_{i+1})_{\langle -k+r,k-r\rangle} = (x_{i+2})_{\langle -k+r,k-r\rangle}$$

and $\forall j \leq m, F^j(x_i)_{\langle -n,n\rangle} = (x_{i+j})_{\langle -n,n\rangle}$. Since $\Sigma_{\langle -n,n\rangle}(F)$ is a SFT of order $m+1$, the sequence $((x_i)_{\langle -n,n\rangle})_{i\geq 0}$ belongs to $\mathcal{L}(\Sigma_{\langle -n,n\rangle}(F))$, so there exists a point $x \in A\ \ $, such that for all $i \geq 0$, $F^i(x)_{\langle -n,n\rangle} = (x_i)_{\langle -n,n\rangle}$, so x ε-shadows the sequence $(x_i)_{i\geq 0}$. $\square$



Every expansive CA has the shadowing property since it is conjugate to a SFT. It is easy to see that the factor subshifts of the product CA and the majority CA are SFT, so these CA have the shadowing property.

THEOREM 11.2 (Kůrka [21]). *If $(A\ \ , F)$ is a CA with the shadowing property, then every its factor subshift is sofic.*

Proof. Let $(A\ \ , F)$ be a CA. We show that for every $n \geq 0$, $\Sigma_{\langle -n,n\rangle}(F)$ is a sofic subshift. For $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that every δ-chain is ε-shadowed by some point. We can assume $k \geq n$. Put $B = A^{2k+1}$, $C = A^{2n+1}$ and define a SFT $\Sigma \subseteq B\ \ $ of order 2 so that for $u, v \in B$,

$$uv \in \mathcal{L}(\Sigma) \iff F([u]_{-k}) \cap [v]_{-k} \neq \emptyset$$

Define a map $\varphi : \Sigma \to C$ by $\varphi(u)_i = (u_i)_{\langle k-n,k+n\rangle}$. For $u \in \Sigma$ consider any sequence of points $(x_i \in A\ )_{i\geq 0}$ with $x_i \in [u_i]_{-k}$. Then $(x_i)_{i\geq 0}$ is a δ-chain so there exists a point $x \in A$ which ε-shadows it. We get

$$F^i(x)_{\langle -n,n\rangle} = (x_i)_{\langle -n,n\rangle} = (u_i)_{\langle k-n,k+n\rangle}$$

so $\varphi(u) \in \Sigma_{\langle -n,n\rangle}(F)$. Thus $\varphi(\Sigma) \subseteq \Sigma_{\langle -n,n\rangle}(F)$. Conversely, if $v \in \Sigma_{\langle -n,n\rangle}(F)$, then there exists $x \in A$ with $v_i = F^i(x)_{\langle -n,n\rangle}$ and $v = \varphi(u)$, where $u_i = F^i(x)_{\langle -k,k\rangle}$. Thus $\Sigma_{\langle -n,n\rangle}(F) = \varphi(\Sigma)$ is sofic. ☐

Proposition 11.1. *Every equicontinuous DS on A has the shadowing property.*

Proof. By the assumption for every $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that if $x_{\langle -k,k\rangle} = y_{\langle -k,k\rangle}$, then $F^i(x)_{\langle -n,n\rangle} = F^i(y)_{\langle -n,n\rangle}$ for all $i \geq 0$. Let $(x_i \in A\ )_{i\geq 0}$ be a δ- chain. Then $F(x_i)_{\langle -k,k\rangle} = (x_{i+1})_{\langle -k,k\rangle}$, so for every $m > 0$,

$$F^m(x_0)_{\langle -n,n\rangle} = F^{m-1}(x_1)_{\langle -n,n\rangle} = \cdots F(x_{m-1})_{\langle -n,n\rangle} = (x_m)_{\langle -n,n\rangle}$$

Thus x_0 ε-shadows $(x_i)_{i\geq 0}$. ☐

12. Attractors.

Theorem 12.1 (Hurley [20]). *Let $(A\ ,F)$ be a CA and $x \in A$ an attracting periodic point. Then $\sigma(x) = x$ and $F(x) = x$.*

Proof. Let p be the period of x, so $F^p(x) = x$. By the assumption, there exists a clopen set U with $(x) = \omega(U)$. There exists a nonempty open set $U_0 \subseteq U$ such that if $y \in U_0$, then $\lim_{n\to\infty} F^{np}(y) = x$. Since $(A\ ,\sigma)$ is mixing, for k large enough, both $\sigma^k(U_0)\cap U_0$ and $\sigma^{k+1}(U_0)\cap U_0$ are nonempty. For $y \in \sigma^k(U_0)\cap U_0$ and $z \in \sigma^{k+1}(U_0)\cap U_0$ we get

$$\sigma^k(x) = \lim_{n\to\infty} F^{np}(y) = x = \lim_{n\to\infty} F^{np}(z) = \sigma^{k+1}(x)$$

It follows $\sigma(x) = x$, so $x = a^\infty$ for some $a \in A$. Then $F(x) = b^\infty$ for some $b \in A$ and $a^\infty = \lim_{n\to\infty} F^{np}(a^\infty.b^\infty) = b^\infty$. Thus $a = b$ and therefore $p = 1$. ☐

In the product CA of Example 1, 0^∞ is an attracting fixed point. There is also another larger attractor $\omega(\mathbf{2}\ )$.

Theorem 12.2 (Hurley [20]). *If a CA has two disjoint attractors, then every its attractor contains two disjoint attractors and a continuum of quasi-attractors.*

Proof. Let Y_1 and Y_2 be two disjoint attractors and U_1, U_2 clopen invariant sets with $\omega(U_1) = Y_1$, $\omega(U_2) = Y_2$. If $U_1 \cap U_2$ were nonempty, then $\omega(U_1 \cap U_2)$ would be an attractor contained in both Y_1 and Y_2. Thus $U_1 \cap U_2 = \emptyset$. Let Y be an attractor and U a clopen invariant set with $\omega(U) = Y$. Since $(A\quad, \sigma)$ is mixing, there exists $n > 0$ such that $V_1 = \sigma^n(U_1) \cap U \neq \emptyset$ and $V_2 = \sigma^n(U_2) \cap U \neq \emptyset$ are clopen invariant sets. It follows that $\omega(V_1)$ and $\omega(V_2)$ are disjoint attractors contained in Y. □

In the majority CA of Example 2, $[00]_0$ and $[11]_0$ are disjoint invariant sets, so their omega-limits are disjoint attractors. While $\omega(\mathbf{2}\quad)$ and $Y = \omega([00]_0 \cup [11]_0)$ are subshifts, other attractors are not.

THEOREM 12.3 (Hurley [20]). *If a CA has a minimal attractor then it is a subshift, it is contained in every other attractor and its basin of attraction is a dense open set.*

Proof. Let Y be a minimal attractor and let Z be another attractor. If $Y \cap Z$ were empty, then Y would contain two disjoint attractors. Since Y is a minimal attractor, $Y \subseteq Z$. Let V be a clopen invariant set with $\omega(V) = Y$. Then $\sigma^{-1}(V)$ is a clopen invariant set, so $Y \subseteq \omega(\sigma^{-1}(V)) = \sigma^{-1}(Y)$. Thus $\sigma(Y) \subseteq Y$ and similarly, $\sigma^{-1}(Y) \subseteq Y$, so Y is a subshift. The basin $\mathcal{B}(Y)$ is an open σ-invariant set. Since $(A\quad, \sigma)$ is transitive, for every nonempty open $U \subseteq A$ there exists $n > 0$ with

$$\emptyset \neq \sigma^n(\mathcal{B}(Y)) \cap U = \mathcal{B}(Y) \cap U$$

and $\mathcal{B}(Y)$ is dense. □

Of course an attracting fixed point is an example of a minimal attractor. To see an example of a minimal attractor which is not a fixed point consider

EXAMPLE 11 (A minimal attractor). $(\mathbf{2}\quad \times \mathbf{2}\quad, P \times \sigma)$*, has a minimal attractor* $\{0^\infty\} \times \mathbf{2}$.

COROLLARY 12.1 (Hurley [20]). *For any CA exactly one of the following statements holds.*

1. There exists two disjoint attractors and a continuum of quasi-attractors.

2. There exists a unique minimal quasi-attractor. It is a subshift and it is contained in every attractor.

3. There exists a unique minimal attractor which is contained in every other attractor.

Proof. If there do not exist disjoint attractors, then every two attractors are comparable by inclusion. The number of attractors is at most countable, so the intersection of all attractors is either a quasi-attractor or an attractor and it is σ-invariant. □

EXAMPLE 12 (Q : A quasi-attractor, Hurley [20]). $(\mathbf{2}\quad, Q)$ *where*

$$Q(x)_i = x_i x_{i+1}$$

has a quasi-attractor $\{0^\infty\}$.

FIG. 9. *Q: A quasi-attractor*

The point 0^∞ is equicontinuous but not attracting. Like in Example 1, we have

$$Y = \omega(\mathbf{2}\quad) = \{x \in \mathbf{2}\quad : \forall k > 0, 10^k 1 \not\sqsubseteq x\}$$

For every $m \in\ $, $[0]_m$ is a clopen invariant set and

$$Y_m = \omega([0]_m) = Y \cap \{x \in \mathbf{2}\quad : \forall i \leq m, x_i = 0\}$$

is an attractor. For example Y_0 consists of all points $0^\infty.1^n 0^\infty$ where $0 \leq n \leq \infty$. We have $Y_{m+1} \subset Y_m$ and $\bigcap_{m \geq 0} Y_m = \{0^\infty\}$ is the unique minimal quasi-attractor.

13. Unique attractor. Recall that a DS (X, F) has a unique attractor iff $\omega(X)$ is chain transitive (Akin [1], p.66).

THEOREM 13.1. *An equicontinuous CA has either two disjoint attractors or an attracting fixed point, which is the unique attractor.*

Proof. Suppose that $(A\quad, F)$ is an equicontinuous CA which has not disjoint attractors. By Theorem 5.2, there exists a preperiod $m \geq 0$ and a period $p > 0$, such that $F^{m+p} = F^m$. Assume that there exist two distinct attractors $Z \subset Y \subseteq A\quad$. For $y_1 \in Y \setminus Z$ construct a sequence $(y_n \in Y)_{n \geq 1}$ with $F(y_{n+1}) = y_n$. This is possible, since $F(Y) = Y$, and clearly, $y_n \in Y \setminus Z$. There exists a converging subsequence $x = \lim_{i \to \infty} y_{n_i} \in Y$. It follows that x is a nonwandering point. Since it is eventually periodic, it must be periodic and $F^p(x) = x$. There exists $\varepsilon > 0$ such that $B_\varepsilon(F^i(x)) \cap Z = \emptyset$ for all $i < p$. Since x is equicontinuous, there exists $\xi > 0$ such that if $d(z, x) < \xi$, then $d(F^n(z), F^n(x)) < \varepsilon$ for all n. By Proposition 11.1, $(A\quad, F)$ has the shadowing property, so there exists $\delta > 0$ such that every δ-chain which starts in x, is ξ-shadowed by some point, so it remains ε-close to the orbit of x. Let $C_\delta(x)$ be the set of points which can be reached by δ-chains from x, and $V = \overline{C_\delta(x)}$ be its closure. Then $V \cap Z = \emptyset$. We show that V is inward, i.e., $F(V) \subseteq V^\circ$. There exists $\eta > 0$ such that for all $v, w \in A\quad$,

$$d(v, w) < \eta \ \Rightarrow\ d(F(v), F(w)) < \delta/2.$$

Given $z \in V$, there exists $w \in C_\delta(x)$ with $d(z,w) < \eta$ and a δ-chain $x = x_0, \dots, x_{n-1}, w$. If $d(v, F(z)) < \frac{\delta}{2}$, then

$$d(F(w), v) \le d(F(w), F(z)) + d(F(z), v) < \delta$$

so $x = x_0, \dots, x_{n-1}, v, w$ is a δ-chain, and $v \in V$. Thus $B_{\frac{\delta}{2}}(F(z)) \subseteq V$, so $F(V) \subseteq V^\circ$ and $\omega(V)$ is an attractor disjoint from Z and this is a contradiction. Thus $(A\ \ , F)$ has unique attractor $Y = \omega(A\ \ )$. Since Y is a minimal attractor it is chain transitive. By Proposition 11.1, (Y, F) has the shadowing property, so it is transitive. By Theorem 5.3, Y consists of a single periodic orbit, which must be a fixed point by Theorem 12.1. ◻

The zero CA of example 5 is equicontinuous and has a unique attractor. The identity CA of Example 4 is equicontinuous and has disjoint attractors. There is also an example which is not surjective.

EXAMPLE 13 (J: disjoint attractors). $(\mathbf{3}\ \ , J)$ *where* $J(x)_i = \lfloor \frac{x_i+1}{2} \rfloor$, *is equicontinuous, has disjoint attractors and is not surjective.*

THEOREM 13.2. *If a CA has an attracting fixed point which is a unique attractor, then it is equicontinuous.*

Proof. Let a^∞ be the attracting fixed point, so $\omega(A\ \ ) = \{a^\infty\}$. If $b \in A$, $b \ne a$, then $b \notin \mathcal{L}(\{a^\infty\})$, so for some $n > 0$, $b \notin F^n(A\ \ )$. Taking the maximum of these n for all $b \in A \setminus \{a\}$, we get $F^n(A\ \ ) = \{a^\infty\}$ and clearly F is equicontinuous. ◻

PROPOSITION 13.1. *Let* $(A\ \ , F)$ *be a surjective CA. If* $U \subseteq A\ \ $ *is an invariant clopen set, then* $F^{-1}(U) = U$ *and* $F(U) = U$.

Proof. If $F(U) \subset U$, then $U \subseteq F^{-1}F(U) \subseteq F^{-1}(U)$. Suppose that there exists $x \in F^{-1}(U) \setminus U$. There exists an open set $V \ni x$ with $V \cap U = \emptyset$ and $F(V) \subseteq U$. There exists $n > 0$ and words $u_0, \dots, u_{m-1}, u_m \in A^{2n+1}$ such that

$$U = [u_0]_{-n} \cup \cdots \cup [u_{m-1}]_{-n}, \ \ [u_m]_{-n} \subseteq V$$

Define an integer matrix of the type $(m+1) \times m$ by

$$M_{ij} = |\{w \in A^{2n+2r+1} : w_{\langle r, r+2n \rangle} = u_i,\ w \in f^{-1}(u_j)\}|$$

Here r is the radius and $f : A^{2r+1} \to A$ the local rule. Since F is surjective, for every $j < m$, $|f^{-1}(u_j)| = |A|^{2r}$ by Theorem 6.1, so

$$\sum_{i=0}^{m} M_{ij} \le |A|^{2r} \ \text{ and } \ \sum_{i=0}^{m} \sum_{j=0}^{m-1} M_{ij} \le m|A|^{2r}$$

Since $F(U) \subseteq U$ and $F(V) \subseteq U$, for every $i \le m$,

$$\sum_{j=0}^{m-1} M_{ij} = |A|^{2r} \ \text{ and } \ \sum_{i=0}^{m} \sum_{j=0}^{m-1} M_{ij} = (m+1)|A|^{2r}$$

and this is a contradiction. Thus $F^{-1}(U) = U$ and $F(U) = U$. □

THEOREM 13.3. *A surjective CA has either a unique attractor or a pair of disjoint attractors.*

Proof. Suppose that a surjective CA $(A\ \ ,F)$ has at least two attractors. There exists a nonempty clopen invariant set $U \neq A$ and by Proposition 13.1, $V = A\ \ \setminus U$ is a clopen invariant set too. Thus $\omega(U)$ and $\omega(V)$ are disjoint attractors. □

EXAMPLE 14 (U: A unique attractor, Gilman [15]). $(\mathbf{2}\ \ ,U)$ *where*

$$U(x)_i = x_{i+1}x_{i+2}$$

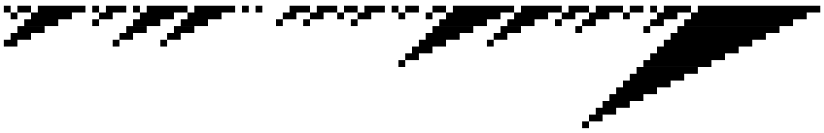


FIG. 10. *U: A unique attractor*

Similarly as in Example 1 we show

$$\omega(\mathbf{2}\ \ ) = \{x \in \mathbf{2}\ \ : \forall n > 0, 10^n 1 \not\sqsubseteq x\}$$

$\omega(\mathbf{2}\ \ )$ is not transitive. If $x \in [10]_0 \cap \omega(\mathbf{2}\ \ )$, then $x_{(1,\infty)} = 0^\infty$, so for every $n > 0$, $F^n(x) \notin [11]_0$. The following chains show that $\omega(\mathbf{2}\ \ )$ is chain transitive

0000	0001	1111	01	1111	0
0001	11	1110	0	1100	0
0011	11	1000	0	0000	
0111	11	0000			
1111					

It follows that the CA does not have the shadowing property and by Theorem 11.2, its factor subshifts $\Sigma_{\langle -n,n\rangle}(U)$ are not sofic.

EXAMPLE 15 (C: Coven's CA [9]). $(\mathbf{2}\ \ ,C)$ *where*

$$C(x)_i = x_i + x_{i+1}(x_{i+2} + 1) \bmod 2$$

is almost equicontinuous, has a unique attractor and does not have the shadowing property (Blanchard and Maass [4]). It is left closing but not open.

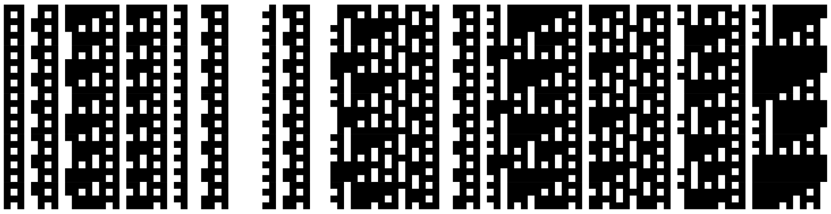


FIG. 11. *Coven's CA*

The CA is left closing since it is left permutive. It is not right closing, since it has not constant number of preimages. 0^∞ has the only preimage itself, while 1^∞ has two preimages $(01)^\infty$ and $(10)^\infty$. To show other properties, observe first that for every $a, b \in \mathbf{2}$, $f(1a1b) = 1c$ where $c = a+b+1$ (here f is the local rule and the addition is modulo 2). Define a CA $(\mathbf{2}\;\;, G)$ by $G(x)_i = x_i + x_{i+1} + 1 \bmod 2$ and a map $\varphi : \mathbf{2}\;\; \to \mathbf{2}\;\;$ by

$$\varphi(x)_{2i} = 1,\ \ \varphi(x)_{2i+1} = x_i$$

Then $\varphi : (\mathbf{2}\;\;, G) \to (\mathbf{2}\;\;, C)$ is an injective homomorphism, so $(\mathbf{2}\;\;, G)$ is a transitive subsystem of $(\mathbf{2}\;\;, C)$.

LEMMA 13.1. *For every $n \geq 0$,*

$$x_{\langle 0,1\rangle} = 10 \ \Rightarrow\ \forall i \leq n, C^i(x)_{\langle 0,1\rangle} \in \{10, 11\}$$

Proof. The statement clearly holds for $n = 0$. Assume that it holds for n and let us prove it for $n+1$. Let $m > 0$ be the first integer for which $C^m(x)_{\langle 0,1\rangle} \neq 10$, so $C^m(x)_{\langle 0,1\rangle} = 11$ and $C^{m-1}_{\langle 0,3\rangle} = 1010$. By the induction hypothesis

$$C^n(x)_{\langle 0,3\rangle} \in \{1010, 1011, 1110, 1111\}$$

so $C^{n+1}(x)_{\langle 0,1\rangle} \in \{10, 11\}$. □

LEMMA 13.2. *The word* 000 *is a* 2*-blocking word with offset* 0*.*

Proof. Let $x_{\langle 0,2\rangle} = 000$ and let $n > 0$ be the first integer with $C^n(x)_{\langle 0,2\rangle} \neq 000$ so $C^n(x)_{\langle 0,2\rangle} = 001$ and $C^{n-1}(x)_{\langle 0,4\rangle} = 00010$. By Lemma 13.1, $C^m(x)_3 = 1$ for all $m \geq n$, so $C^k(x)_{\langle 0,1\rangle} = 00$ for all $k \geq 0$. □

We show now that for every $n > 0$ there are chains $1^{n+1} \to 01^n \to 1^{n+1}$. This follows from the fact that a transitive system $(\mathbf{2}\;\;, G)$ is a subsystem of $(\mathbf{2}\;\;, C)$. The chains differ slightly when n is even or odd.

11111	10	01111	10	111111	0	011111	0
11110	0	01110	0	111101	11	011101	11
11010	10	01010	10	110101	0	010101	0
01111		11111		011111		111111	

Since $(\mathbf{2}\quad, C)$ is almost equicontinuous, it is not transitive. Since it is chain transitive, it does not have the shadowing property. However, its factor subshifts are sofic, so the converse of Theorem 11.2 is not true. For example

$$\Sigma_{\langle 0,1\rangle}(C) = \{10, 11\} \quad \cup \{11, 01\} \quad \cup \{01, 00\}$$

14. Classification. We have seen that many topological properties of CA are closely interrelated while other are independent. The following two tables summarize some of the results obtained.

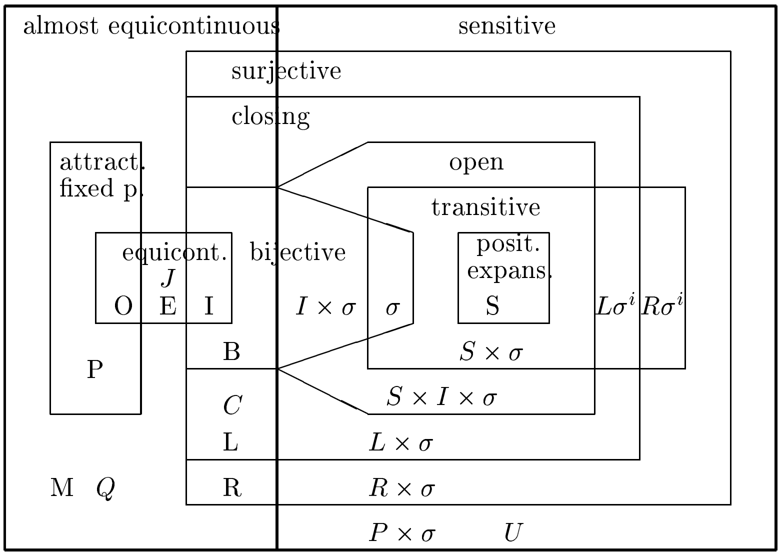


FIG. 12. *Equicontinuity classes*

COROLLARY 14.1.

1. Every positively expansive CA is open and transitive (Theorems 10.1, 10.2).
2. Every transitive CA is surjective and sensitive (Corollary 5.1).
3. Every open CA is closing (Theorem 9.3).
4. Every closing CA is surjective (Proposition 9.1).
5. Every open and almost equicontinuous CA is bijective (Theorem 8.2).
6. Every surjective equicontinuous CA is bijective (Theorem 6.5).
7. Every CA is either sensitive or almost equicontinuous (Corollary 5.1).
8. Every CA with an attracting fixed point is almost equicontinuous.
9. Every transitive system is chain transitive.

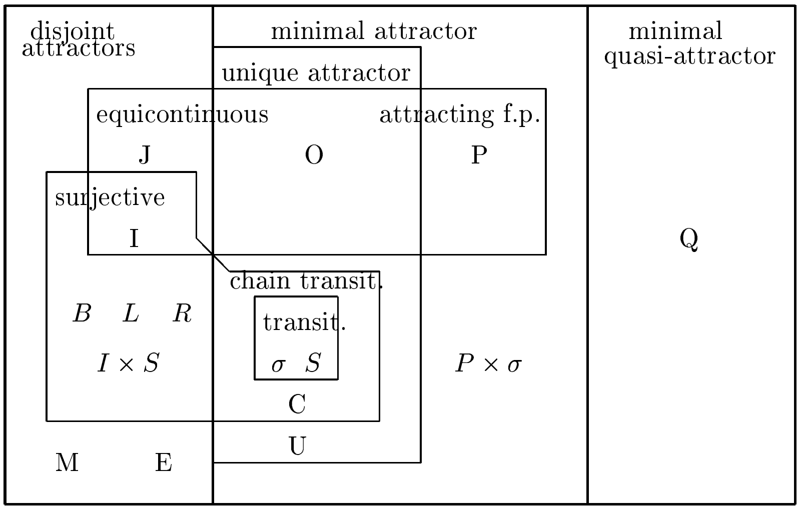


FIG. 13. *Attractor classes*

10. Every equicontinuous CA has either two disjoint attractors or an attracting fixed point, which is the unique attractor (Theorem 13.1).
11. Every CA whose unique attractor is an attracting fixed point is equicontinuous (Theorem 13.2).
12. Every surjective CA has either a unique attractor or a pair of disjoint attractors (Theorem 13.3).
13. Every CA has either two disjoint attractors or a unique minimal attractor or a unique minimal quasi-attractor (Corollary 12.1)

The nonemptiness of various CA classes is shown by following examples:
P: The product CA of Example 1 has an attracting fixed point, it is almost equicontinuous but not equicontinuous.
M: The majority CA of Example 2 is almost equicontinuous. It is not equicontinuous and it does not have an attracting fixed point.
S: The sum CA of Example 3 is positively expansive.
I: The identity CA of Example 4 is equicontinuous, bijective and has disjoint attractors.
O: The zero CA of Example 5 is equicontinuous and has an attracting fixed point.
E: Example 6 is equicontinuous, not bijective and does not have an attracting fixed point.
L: Example 8 is right closing but not left closing, so it is not open.
B: Example 9 is bijective and almost equicontinuous but not equicontinuous.
R: Example 10 is neither left nor right-closing but it is surjective
J: Example 13 has disjoint attractors, it is equicontinuous and not surjec-

tive.
U: Example 14 has a unique attractor, it is not surjective.
C: The Coven's CA of Example 15 is almost equicontinuous, has a unique attractor, it is surjective but not transitive.
σ: The shift CA is bijective, transitive, surjective and sensitive.
$L \circ \sigma^i$: is transitive (for high enough i) and not open.
$R \circ \sigma^i$: is transitive (for high enough i) and not closing.
Q: Example 12 has a quasi-attractor.

15. Open problems. Relations between some CA classes are still unknown. One interesting class form CA which are chain transitive without being transitive. The Coven's CA of Example 15 belongs to this class and it is closing. Does there exist an open or even bijective CA which is chain transitive without being transitive?

A long standing problem is density of periodic points in surjective CA. For two particular classes this property has been recently established: almost equicontinuous surjective CA (Theorem 6.4, Blanchard and Tisseur [6]) and closing CA (Theorem 9.2, Boyle and Kitchens [8]). For general surjective CA, the problem seems to be still open.

Another hard problem has been considered in Nasu [25] as an analogue of Theorem 10.1 which says that every positively expansive CA is conjugated to a onesided SFT. A bijective DS (X, F) is said to be **expansive**, if

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists n \in \quad, d(F^n(x), F^n(y)) \geq \varepsilon$$

Every bijective expansive DS on A is conjugated to a twosided subshift. Is any bijective expansive CA conjugated to a twosided SFT? Or is it at least transitive?

Finally there are many particular CA whose topological properties are unknown. One of the most interesting CA is Coven's CA of Example 15 generalized in Blanchard and Maass [4] to a class of aperiodic Coven CA. The classification of periodic Coven's automata seems to be even more difficult. While they are closing, it is unknown whether they are transitive, sensitive or chain transitive. One particular case is

Example 16 (V: Periodic Coven's CA). $(\mathbf{2}\quad, V)$ *where*

$$V(x)_i = x_i + x_{i+1}x_{i+2} \bmod 2$$

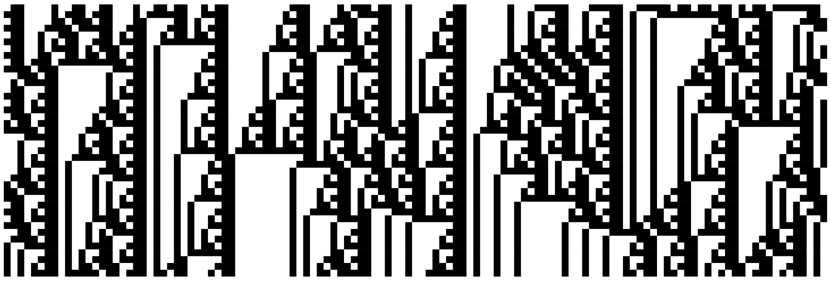


FIG. 14. *V: Periodic Coven's CA*

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