On topological dynamics of Turing machines

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Abstract

We associate to a Turing machine two dynamical systems which we call Turing machine with moving tape (TMT) and Turing machine with moving head (TMH). TMT are equivalent to generalized shifts of Moore [9] and they include two-sided full shifts. TMH are shift-commuting maps of two-sided sofic systems. In both classes we characterize systems with the shadowing property, show that a bijective expansive TMT is conjugate to a subshift of finite type and that topological entropy of every TMH is zero. We conjecture that every TMT has a periodic point.

1 Introduction

Turing machine is a mathematical model of computation with single processor and potentially unlimited memory. It consists of a head with finite set of inner states Q, attached to a doubly infinite tape with letters from a finite alphabet A. Its step by step computation is a dynamical process which may be analyzed with the concepts of topological dynamics. When we look at Turing machines in this way, we are concerned with the computation process itself rather than with its results. This leads to a certain simplification of the concepts involved. Our starting structure is just a transition function $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$, We do not need any initial or final states. The computation may start in any state and proceed indefinitely. The halting may be represented by a fixed point of the dynamics. We need not restrict the computation only to finite tape contents either. The computation works on arbitrary content of the tape, so we need not distinguish any blank letter.

In topological dynamics, a dynamical system is defined by a state space X, which is assumed to be a compact metric space, and a continuous self-map $f : X \to X$ assigning to the present state $x \in X$ of the system the state $f(x)$ in the next time instant. The state of a Turing machine is its configuration, given by the inner state of the head, whole content of the tape and a pointer to a field of the tape. However, there are at least two different ways to make this precise. In the first case, which we call Turing machines with moving tape (TMT), the head is stationary and the tape moves. The state space of a TMT is given by an inner state $q \in Q$ and the content of the tape $u : \mathbb{Z} \to A$, where $\mathbb{Z}$ is the set of integers. It is assumed that the head points to the field with address zero. The move (of the tape) is accomplished by the shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $\sigma(x)_i = x_{i+1}$. Thus the state space of a TMT is $X = Q \times A^{\mathbb{Z}}$, When this space is equipped with the product topology, we get a discontinuum, which is homeomorphic to the Cantor middle third set. This framework for the study of Turing machines as dynamical systems is due to Moore [9], who called them the generalized shifts. Turing machines with more tapes have been considered in Kůrka [4, 5].

To obtain a Turing machine with a moving head (TMH) we could define the state by an inner state $q \in Q$, tape content $u : \mathbb{Z} \to A$, and a pointer to the tape $i \in \mathbb{Z}$. However, the resulting state space $Q \times A^{\mathbb{Z}} \times \mathbb{Z}$ would not be compact, which is a serious drawback in topological dynamics. A compactification of this space may be obtained by putting the inner state on the tape too and admitting, as a configuration, also a tape with no head at all. The configuration is then a map $u : \mathbb{Z} \to Q \cup A$, such that the set $\{i \in \mathbb{Z}; u_i \in Q\}$ has at most one element (Q and A are disjoint). We assume that the head points to the field immediately to its right. The state space $X_1 \subset (Q \cup A)^{\mathbb{Z}}$ is a subshift (indeed a sofic system, see Weiss [12]), i.e. a closed σ-invariant subset of $(Q \cup A)^{\mathbb{Z}}$. The action of the Turing machine commutes with the shift on X_1. We can extend the action in various ways to the full space $(Q \cup A)^{\mathbb{Z}}$ obtaining a cellular automaton. This construction expresses the observation that every Turing machine can be simulated by a cellular automaton (see Smith [10]). In the context of dynamical systems, the extensions of Turing machines to cellular automata have been studied in a slightly different setting by Maass [8].

As dynamical systems, both TMT and TMH have strange but different properties. Some of these properties may be expressed in terms of the position function, which describes the movement of the tape or head. We show that every TMT is locally bijective, has never attracting periodic point, is never positively expansive, and when it is bijective and expansive, then it is conjugate to a subshift of finite type. Moreover we characterize the TMT with the shadowing property. We conjecture that every TMT has a periodic point. TMH do not have attracting periodic points either, and they are neither positively expansive nor expansive. They are equicontinuous if and only if they have the shadowing property, and they have always zero topological entropy.

2 Dynamical systems

Let A be a finite alphabet with at least two elements, $\mathbb{Z}$ the set of integers, and $\mathbb{N}$ the set of non-negative integers. For $n, m \in \mathbb{Z}$ denote $\langle n, m \rangle = \{i \in \mathbb{Z}; n \leq i \leq m\}$. For $n \in \mathbb{N}$ Denote A^n the set of words of letters from A of length n, and $A^* = \cup_{n \in \mathbb{N}} A^n$ the set of finite words. For $u \in A^*$ denote $|u|$ its length, u_i

its i-th letter, and $u_{|i} = u_0 ... u_{i-1}$ its initial segment of length i.

Denote $A^{\mathbb{N}}$ the set of infinite sequences of letters from A, and $A^{\mathbb{Z}}$ the set of doubly infinite sequences of letters from A. Define a metric on $A^{\mathbb{Z}}$ by

$$d(x,y) = 2^{-n} \text{ where } n = \inf\{i \geq 0; x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

Topologically, $A^{\mathbb{Z}}$ is a discontinuum, i.e. a compact, zero-dimensional, perfect metric space. Every two discontinua are homeomorphic (see Hocking and Young [3]). If $K \subset \mathbb{Z}$ is a finite set, and $u : K \to A$ a map, the set

$$[u] = \{x \in A^{\mathbb{Z}}; (\forall i \in K)(x_i = u_i)\}$$

is called a cylinder. It is a clopen (closed and open) set, and every clopen set is a finite union of cylinders.

The two-sided shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$, and it is continuous. A two-sided subshift is any nonempty closed subset $\Sigma \subseteq A^{\mathbb{Z}}$ which is σ-invariant i.e. $\sigma(\Sigma) \subseteq \Sigma$. A two-sided subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is of finite type if there exists $k > 0$ such that for every $x \in A^{\mathbb{Z}}$, $x \in \Sigma$ iff for every $n \in \mathbb{Z}$, $[x_n ... x_{n+k-1}] \cap \Sigma \neq \emptyset$. We consider also one-sided subshifts, which are closed σ-invariant subsets of $A^{\mathbb{N}}$. While two-sided subshifts are bijective, one-sided subshifts are not.

Definition 1 *A dynamical system (X, f) is a continuous map $f : X \to X$ of a compact metric space X to itself. A homomorphism $H : (X, f) \to (Y, g)$ of dynamical systems is a continuous map $H : X \to Y$ such that $Hf = gH$. A conjugacy is a homomorphism which is bijective.*

Denote $f^t : X \to X$ the t-th iteration of f defined by $f^0(x) = x$, $f^{t+1}(x) = f(f^t(x))$. If f is bijective, we have also negative iterations $f^{-t} = (f^{-1})^t$, where f^{-1} is the inverse map to f. A point $x \in X$ is periodic if $f^t(x) = x$ for some $t > 0$. It is eventually periodic, if $f^p(x)$ is periodic for some $p \in \mathbb{N}$. Thus a point is eventually periodic iff its orbit $\mathcal{O}(x) = \{f^t(x); t \geq 0\}$ is finite. A set $Y \subseteq X$ is invariant if $f(Y) \subseteq Y$. A set $Y \subseteq X$ is an attractor if there exists an open set $U \subseteq X$ such that $f(\overline{U}) \subseteq U$, and $Y = \omega(U) = \cap_{i \in \mathbb{N}} f^i(\overline{U})$. If X is a discontinuum, then this means that there exists a clopen invariant set U such that $Y = \omega(U)$. A periodic point is attracting, if its orbit is an attractor. An ε-chain from $x^0 \in X$ to $x^n \in X$ is a sequence $x^0, ..., x^n$ such that $n > 0$ and $d(f(x^i), x^{i+1}) < \varepsilon$ for $0 \leq i < n$. A sequence x^i is ε-shadowed by a point $x \in X$, if $d(f^i(x), x^i) < \varepsilon$ for $0 \leq i \leq n$.

A point $x \in X$ is equicontinuous, if the family $(f^t)_{t \in \mathbb{N}}$ is equicontinuous at x, i.e. if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B(x, \delta))(\forall n \geq 0)(d(f^n(y), f^n(x)) < \varepsilon)$$

Here $B(x, \varepsilon) = \{y \in X; d(y, x) < \varepsilon\}$ is the ball with centre x and radius ε. Denote $\mathrm{Eq}(X, f)$ the set of equicontinuity points of (X, f). Every attracting periodic point is equicontinuous.

A clopen partition of a discontinuum X is a surjective continuous function $\alpha : X \to A$, where A is a finite set with the discrete topology. Thus $\{\alpha^{-1}(a); a \in A\}$ is a finite collection of nonempty disjoint clopen sets, whose union is the full space. Denote

$$\mathcal{L}_\alpha(X,f) = \{u \in A^*; (\exists x \in X)(\forall i < |u|)(u_i = \alpha f^i(x))\}$$

the language generated by (X,f) on α, and $\mathcal{L}_\alpha^n(X,f) = \mathcal{L}_\alpha(X,f) \cap A^n$. The topological entropy $h_\alpha(X,f)$ of (X,f) on α is defined by

$$h_\alpha(X,f) = \lim_{n\to\infty} \frac{1}{n} \ln \operatorname{card} \mathcal{L}_\alpha^n(X,f)$$

The topological entropy $h(X,f)$ of (X,f) is defined by

$$h(X,f) = \sup\{h_\alpha(X,f);\ \alpha \text{ is a clopen partition of } X\}$$

Definition 2 *Let (X,f) be a dynamical system.*
1. (X,f) is positively expansive, if there exists $\varepsilon > 0$, such that for every two distinct points $x,y \in X$ there exists $t > 0$ with $d(f^t(x), f^t(y)) > \varepsilon$.
2. A bijective dynamical system (X,f) is expansive, if there exists $\varepsilon > 0$, such that for every two distinct points $x,y \in X$ there exists $t \in \mathbb{Z}$ with $d(f^t(x), f^t(y)) > \varepsilon$.
3. A dynamical system (X,f) is locally bijective, if for every $x \in X$ there exists a neighbourhood U of x, on which f is one-to-one.
4. A dynamical system (X,f) is bounded-to one if there exists $k > 0$ such that for all x, $\operatorname{card}(f^{-1}(x)) \le k$.
5. A dynamical system (X,f) has the shadowing property, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that every δ-chain is ε-shadowed by some point.

It is well-known that every positively expansive dynamical system on discontinuum is conjugate to a one-sided subshift, and every bijective expansive dynamical system on discontinuum is conjugate to a two-sided subshift. A locally bijective map is open, i.e. for every open set $U \subseteq X$, $f(U)$ is open. Moreover for every $x \in X$, the set of its preimages $f^{-1}(x)$ is finite.

3 Turing machines with moving tape

Definition 3 *A transition function for a Turing machine is a triple of maps $\delta = \langle \delta_Q, \delta_A, \delta_Z \rangle : Q \times A \to Q \times A \times \{-1,0,1\}$, where Q is a nonempty finite set, A is a finite set with at least two elements, and $Q \cap A = \emptyset$. We refer to Q as the set of inner states and to A as the alphabet of the Turing machine.*

Given a transition function $\delta : Q \times A \to Q \times A \times \{-1,0,1\}$, consider a space $X = Q \times A^{\mathbb{Z}}$. An element $x \in X$ is given by $x_q \in Q$ and $x_i \in A$ for $i \in \mathbb{Z}$.

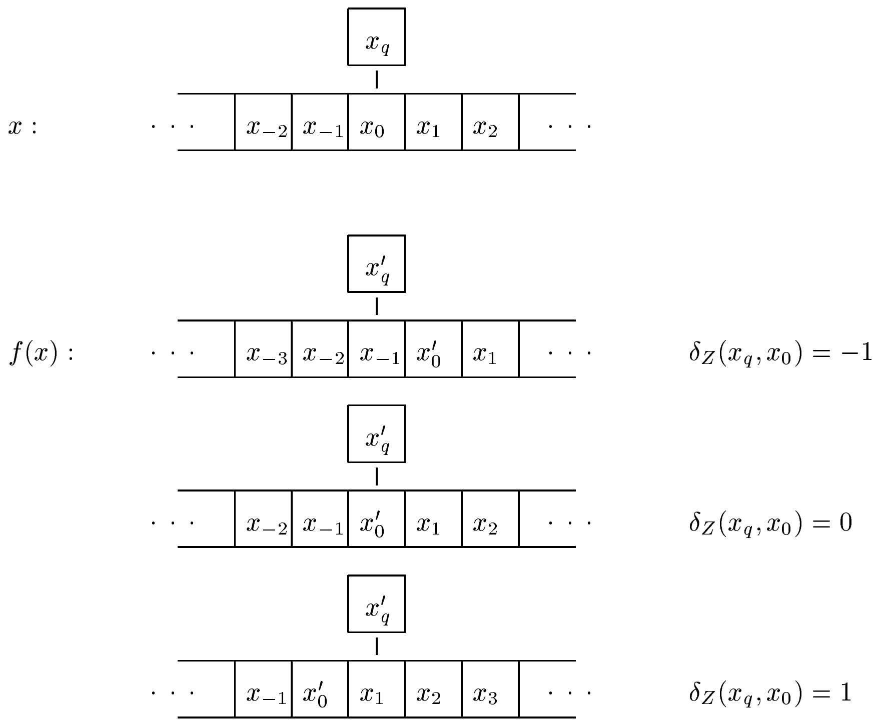
Denote $\pi : X \to Q \times A$ the projection defined by $\pi(x) = (x_q, x_0)$. Define a metric d on X by

$$
\begin{array}{lll}
d(x,y) = 1 & \text{if} & x_q \neq y_q, \\
d(x,y) = 2^{-n} & \text{if} & x_q = y_q \ \text{ and } n = \min\{i \in \mathbb{N}; x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}
\end{array}
$$

Again, X is a discontinuum. We have $d(x,y) < 2^{-n}$ iff $x_q = y_q$ and $x_i = y_i$ for $|i| \leq n$.

Definition 4 *Let $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$ be a transition function. A Turing machine with moving tape (TMT) is a dynamical system (X, f), where $X = Q \times A^{\mathbb{Z}}$, and f is defined by*

$$
\begin{array}{rll}
f(x)_q = & \delta_Q \pi(x) & \\
f(x)_i = & \delta_A \pi(x) & \textit{if} \quad i = -\delta_Z \pi(x) \\
f(x)_i = & x_{i+\delta_Z \pi(x)} & \textit{otherwise}
\end{array}
$$



The new state $f(x)$ is obtained from x by changing first x to x' where $x'_q = \delta_Q(x_q, x_0)$, $x'_0 = \delta_A(x_q, x_0)$, $x'_i = x_i$ otherwise, and then $f(x) = \sigma^j(x')$, where $j = \delta_Z(x_q, x_0)$. This clearly includes full shift on $A^{\mathbb{Z}}$ when we put $Q = \{1\}$ a one point set and $\delta(1,a) = (1,a,1)$. An equivalent concept has been introduced by Moore [9] under the name generalized shift. See also Kůrka [4, 5, 6].

Proposition 1 *Every TMT is locally bijective and bounded-to-one.*

Proof: If $\pi(x) = \pi(y)$, and $f(x) = f(y)$, then $x = y$. Thus for every x, f is one-to-one on $\pi^{-1}\pi(x)$, which is a clopen neighbourhood of x. Moreover, for every $x \in X$, $\mathrm{card}(f^{-1}(x)) \leq \mathrm{card}(Q \times A)$. $\Box$

Definition 5 *The (relative) position function $\wp : X \times I\!\!N \to Z\!\!\!Z$ of a TMT (X, f) is defined by*

$$\wp(x,t) = \sum_{i=0}^{t-1} \delta_Z \pi f^i(x)$$

In case that f is bijective we define for negative t

$$\wp(x,t) = -\sum_{i=t}^{-1} \delta_Z \pi f^i(x)$$

$\wp(x,t)$ is the total shift of the tape in time t. Clearly

$$\wp(x,t+s) = \wp(x,t) + \wp(f^t(x),s); \quad |\wp(x,t)| \leq |t|.$$

Define functions $\underline{\wp}, \overline{\wp} : X \times I\!\!N \to Z\!\!\!Z$ by

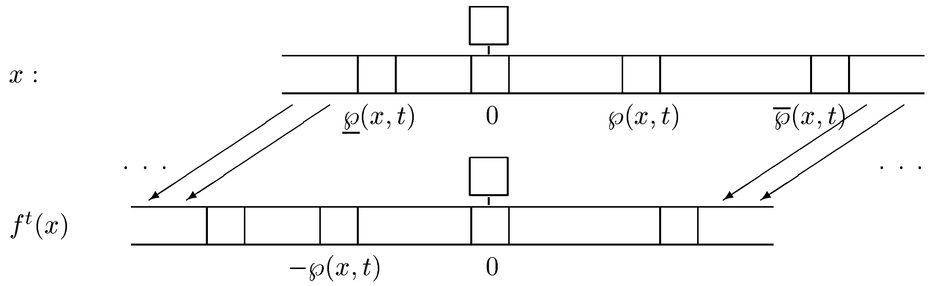
$$\underline{\wp}(x,t) = \min\{\wp(x,s); 0 \leq s \leq t\}; \quad \overline{\wp}(x,t) = \max\{\wp(x,s); 0 \leq s \leq t\}$$

If f is bijective, define for negative t

$$\underline{\wp}(x,t) = \min\{\wp(x,s); t \leq s \leq 0\}; \quad \overline{\wp}(x,t) = \max\{\wp(x,s); t \leq s \leq 0\}$$

For a given $t > 0$, $\langle \underline{\wp}(x,t), \overline{\wp}(x,t) \rangle$ is the interval of relative addresses of the fields on the tape which the head visits up to time t.

Lemma 2 *Let $x, y \in X$, $t \in I\!\!N$, $y_q = x_q$, and $y_i = x_i$ for $\underline{\wp}(x,t) \leq i \leq \overline{\wp}(x,t)$. Then*
1. $\wp(x,t) = \wp(y,t)$
2. $f^t(x)_q = f^t(y)_q$,
3. $f^t(x)_{i-\wp(x,t)} = f^t(y)_{i-\wp(y,t)}$ for $\underline{\wp}(x,t) \leq i \leq \overline{\wp}(x,t)$
4. $f^t(x)_{i-\wp(x,t)} = x_i$ for $i < \underline{\wp}(x,t)$ or $\overline{\wp}(x,t) < i$.



Lemma 3 *Suppose that for every $n \in \mathbb{N}$ there exist $y \in X$, and $t \in \mathbb{N}$ such that $\wp(y,t) = n$. Then there exists $x \in X$ with $\liminf_{t\to\infty} \wp(x,t) = 0$, and $\limsup_{t\to\infty} \wp(x,t) = \infty$.*

Proof: For $n \in \mathbb{N}$ denote

$$V_n = \{x \in X; (\exists t)(\wp(x,t) \geq n \;\&\; (\forall s \leq t)(\wp(x,s) \geq 0))\}$$

If $\wp(y,t) = n$, let $s < t$ be the last integer for which $\wp(y,s) = 0$. Then $\wp(f^s(y), t-s) = \wp(y,t) - \wp(y,s) = n$, so $f^s(y) \in X_n$. Thus V_n is a nonempty closed set, and $V_{n+1} \subseteq V_n$, so there exists $x \in \cap V_n$ by compactness. Clearly $\limsup_{t\to\infty} \wp(x,t) = \infty$. To show that $\liminf_{t\to\infty} \wp(x,t) = 0$, observe that for every n, $x \in V_n$, so there exists t with $\wp(x,t) = n$, and necessarily $t \geq n$, so $\wp(x,n) \geq 0$. $\Box$

Proposition 4 *Let (X,f) be a TMT, and $x \in X$. Then*
1. $x \in \mathrm{Eq}(X,f)$ iff $\wp(x,t)$ is a bounded function of t.
2. If $x \in \mathrm{Eq}(X,f)$, then x is eventually periodic.

Proof: Suppose that $|\wp(x,t)| \leq k$ for all $t \in \mathbb{N}$. Let $n > k$ and $d(x,y) < 2^{-n-k}$. By Lemma 2, $f^t(x)_q = f^t(y)_q$, and $f^t(x)_{i-\wp(x,t)} = f^t(y)_{i-\wp(y,t)}$ for all $|i| \leq n+q$. Thus $f^t(x)_i = f^t(y)_i$ for all $|i| \leq n$, so $d(f^t(x), f^t(y)) < 2^{-n}$. Conversely suppose that $\wp(x,t)$ is not bounded and $x \in \mathrm{Eq}(X,f)$. Then for $\varepsilon = 1$ there exists $\delta = 2^{-k}$ such that if $d(x,y) < 2^{-k}$, then $\pi f^t(x) = \pi f^t(y)$ for all t. Let t is the first number with $|\wp(x,t)| = k+1$. If $\wp(x,t) = k+1$, then $f^t(x)_0 = x_{k+1}$. Pick a point y with $y_i = x_i$ iff $i \neq k+1$. Then $f^t(x)_0 \neq f^t(y)_0$, which is a contradiction. From the boundedness of $\wp(x,t)$ easily follows that x is eventually periodic. $\Box$

Corollary 1 *The equicontinuity set of a TMT is open.*

Proposition 5 *No TMT has an attracting periodic point.*

Proof: Assume that $f^m(x) = x$, and $\mathcal{O}(x) = \{x, f(x), ..., f^{m-1}(x)\}$ is an attractor. Suppose first that $\wp(x, m) = 0$. Then there exists $k > 0$ such that $|\wp(x,t)| \leq k$ for every t. It follows that if $d(x,y) < 2^{-k}$, then y is a periodic point too. Thus in every clopen invariant set U containing $\mathcal{O}(x)$ there exists a periodic orbit different from $\mathcal{O}(x)$, so $\omega(U) \neq \mathcal{O}(x)$. If $\wp(x,m) \neq 0$, then $\wp(x,t)$ is not a bounded function of t, so $x \notin \mathrm{Eq}(f)$. Again, $\mathcal{O}(x)$ is not an attractor. □

Theorem 6 *A TMT has the shadowing property, iff its returns are of bounded length, i.e. iff*

$$(\exists k > 0)(\forall x \in X)(\forall t > 0)(\wp(x,t) = 0 \Rightarrow (\forall s < t)(|\wp(x,s)| \leq k))$$

Proof: Suppose that (X, f) has the shadowing property, and that every 2^{-n-k}-chain is 2^{-n}-shadowed by some point. Suppose that $x \in X$, $0 < s < t$, $\wp(x,t) = 0$, and $|\wp(x,s)| = k+1$. We can assume without loss of generality that $\wp(x,s) = k+1$, and $\wp(x,u) > 0$ for $0 < u < t$. We construct a 2^{-n-k}-chain x^i with following properties:
$x^i = f^i(x)$ for $i < s$,
$x^s_q = f^s(x)_q$, $x^s_j = f^s(x)_j$ iff $j \neq -n-k-1$
$x^i = f^{i-s}(x^s)$ for $i > s$.
Suppose that x^i is 2^{-n}-shadowed by some $y \in X$. Then $f^t(y)_i = y_i$ for $i \leq 0$, but $x^0_{-n} = x_{-n} = f^s(x)_{-n-\wp(x,s)} \neq x^s_{-n-\wp(x,s)} = x^t_{-n}$, which is impossible. Thus $x^0, ..., x^t$ is not 2^{-n}- shadowed by any point which is a contradiction with our assumptions. Conversely suppose that the condition is satisfied, and let x^i be a 2^{-n-k}-chain. Define its position function by $p(t) = \sum_{s=0}^{t-1} \delta_Z \pi(x^s)$. Define a point $x \in X$ by $x_q = x^0_q$, $x_i = x^t_{i-p(t)}$, if t is the first number for which $|i - p(t)| \leq n$, and arbitrarily if no such t exists. Then x 2^{-n}-shadows x^i. □

Proposition 7 *No TMT is positively expansive.*

Proof: Assume that (X, f) is a positively expansive TMT, so we have $\varepsilon = 2^{-n}$ such that for every $x \neq y$ there exists t with $d(f^t(x), f^t(y)) \geq \varepsilon$. Suppose that for some $x \in X$, $\liminf_{t\to\infty} \wp(x,t) = a > -\infty$. Construct a point $y \in X$ such that $x_i = y_i$ iff $i \geq a - n$. Then $x \neq y$, and $d(f^t(x), f^t(y)) < \varepsilon$ for every $t \geq 0$, which is a contradiction. Thus for every $x \in X$, $\liminf_{t\to\infty} \wp(x,\infty) = -\infty$, and similarly we show that $\limsup_{t\to\infty} \wp(x,\infty) = \infty$ for every $x \in X$. This is, however, in contradiction with Lemma 3. □

Lemma 8 *A TMT is bijective iff the function $\delta_{QA} = \langle \delta_Q, \delta_A \rangle : Q \times A \to Q \times A$ is bijective. If a bijective TMT is expansive, then for every $x \in X$*

$$\liminf_{t\to-\infty} \wp(x,t) = -\infty \quad \textit{or} \quad \liminf_{t\to\infty} \wp(x,t) = -\infty, \quad \textit{and}$$

$$\limsup_{t\to-\infty} \wp(x,t) = \infty \quad \textit{or} \quad \limsup_{t\to\infty} \wp(x,t) = \infty$$

The proof is similar as in Proposition 7.

Theorem 9 *If a bijective TMT is expansive, then it is conjugate to a two-sided subshift of finite type.*

Proof: Every bijective expansive TMT (or more generally dynamical system on discontinuum) is conjugate to a two-sided subshift. By a Theorem of Walters [11], a two-sided subshift is of finite type iff it has the shadowing property (called pseudo-orbit tracing property in Walters). Thus it suffices to prove that a bijective expansive TMT has the shadowing property. Suppose that (X, f) is a bijective TMT which has not the shadowing property. For $n > 0$ denote

$$V_n = \{x \in X; (\exists s < 0 < t)(\wp(x,s) = \wp(x,t) = n \ \& \ (\forall u \in \langle s,t \rangle)(\wp(x,u) \geq 0))\}$$

For $n < 0$ denote

$$V_n = \{x \in X; (\exists s < 0 < t)(\wp(x,s) = \wp(x,t) = n \ \& \ (\forall u \in \langle s,t \rangle)(\wp(x,u) \leq 0))\}$$

Then for every $n > 0$, $V_n \cup V_{-n}$ is a nonempty closed set, $V_{n+1} \subseteq V_n$, and $V_{-n-1} \subseteq V_{-n}$. Thus at least one of the sets $\cap_{n>0} V_n$, $\cap_{n<0} V_n$ is nonempty. Let $x \in \cap_{n>0} V_n$. We have then $\liminf_{t \to -\infty} \wp(x,t) = 0$, $\limsup_{t \to \infty} \wp(x,t) = \infty$. Thus (X, f) is not expansive by Lemma 8 and this is a contradiction. $\Box$

Conjecture 1 *Every TMT has a periodic point.*

4 Turing machines with moving head

Let $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$ be a transition function for a Turing machine. For $n \geq 0$ denote

$$X_n = \{x \in (Q \cup A)^{\mathbb{Z}}; \mathrm{card}\{i \in \mathbb{Z}; x_i \in Q\} \leq n\}$$

Then $X_n \subseteq (Q \cup A)^{\mathbb{Z}}$ is a subshift, indeed a sofic system (see Weiss [12]). We are concerned mostly with the space X_1. We use the standard metric on X_1, so $d(x,y) < 2^{-k}$ iff $x_i = y_i$ for all $|i| \leq k$.

Definition 6 *Let $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$ be a transition function. A Turing machine with moving head (TMH) is a dynamical system (X_1, g), where*

$$\begin{array}{lll}
g(x)_i = \delta_A(x_{i-1}, x_i) & \textit{if} & x_{i-1} \in Q, \quad \delta_Z(x_{i-1}, x_i) \leq 0 \\
g(x)_i = \delta_Q(x_{i-1}, x_i) & \textit{if} & x_{i-1} \in Q, \quad \delta_Z(x_{i-1}, x_i) = 1 \\
g(x)_i = x_{i-1} & \textit{if} & x_i \in Q, \quad \delta_Z(x_i, x_{i+1}) = -1 \\
g(x)_i = \delta_Q(x_i, x_{i+1}) & \textit{if} & x_i \in Q, \quad \delta_Z(x_i, x_{i+1}) = 0 \\
g(x)_i = \delta_A(x_i, x_{i+1}) & \textit{if} & x_i \in Q, \quad \delta_Z(x_i, x_{i+1}) = 1 \\
g(x)_i = \delta_Q(x_{i+1}, x_{i+2}) & \textit{if} & x_{i+1} \in Q, \quad \delta_Z(x_{i+1}, x_{i+2}) = -1 \\
g(x)_i = x_i & \textit{otherwise} &
\end{array}$$

x :	$\cdots$	a	b	q	c	d	$\cdots$	
$g(x)$:	$\cdots$	a	q'	b	c'	d	$\cdots$	$\delta_Z(q,c)=-1$
	$\cdots$	a	b	q'	c'	d	$\cdots$	$\delta_Z(q,c)=0$
	$\cdots$	a	b	c'	q'	d	$\cdots$	$\delta_Z(q,c)=1$

Thus the head is on the tape too, and it points to the field immediately to its right. Clearly g is a shift-commuting map, since every $g(x)_i$ depends only on x_{i-1}, x_i, x_{i+1} and x_{i+2}. If $x \in X_0$, i.e. if there is no head on the tape, then $g(x) = x$ is a fixed point. We say in this case that x is a quiescent state.

Definition 7 *The (absolute) position function* $\mathfrak{p} : X_1 \to \overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ *of a TMH* (X_1, g) *is defined by*

$$\mathfrak{p}(x) = n \textit{ if } x_n \in Q, \quad \mathfrak{p}(x) = \infty \textit{ otherwise}$$

Lemma 10 *Let* δ *be a transition function for a Turing machine, let* (X, f) *be the corresponding TMT and* (X_1, g) *the corresponding TMH. There is a continuous map* $H : X \to X_1$ *given by*

$$\begin{array}{lll} H(x)_i = x_i & \textit{for} & i < 0 \\ H(x)_0 = x_q & & \\ H(x)_i = x_{i-1} & \textit{for} & i > 0 \end{array}$$

Let $\wp : X \times \mathbb{N} \to \mathbb{Z}$, $\mathfrak{p} : Y \to \overline{\mathbb{Z}}$ *be the corresponding (relative and absolute) position functions. Then for every* $x \in X$, $\wp(x,t) = \mathfrak{p} g^t H(x)$, *and* $Hf^t(x) = \sigma^{\wp(x,t)} g^t H(x)$ *(thus* H *is not a homomorphism).*

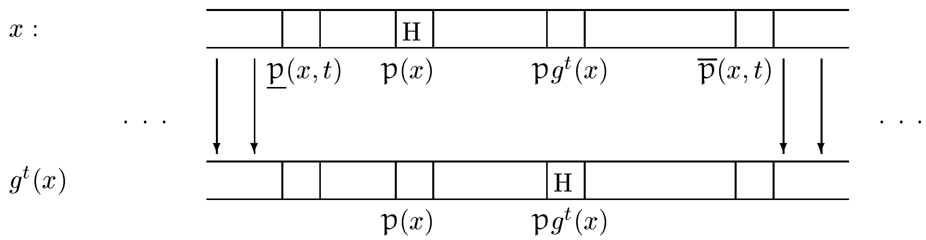
Define $\underline{\mathfrak{p}}, \overline{\mathfrak{p}} : X \times \mathbb{N} \to \mathbb{Z}$ by

$$\underline{\mathfrak{p}}(x,t) = \min\{\mathfrak{p} g^s(x); 0 \le s \le t\}, \quad \overline{\mathfrak{p}}(x,t) = \max\{\mathfrak{p} g^s(x); 0 \le s \le t\} + 1$$

Then $\langle \underline{\mathfrak{p}}(x,t), \overline{\mathfrak{p}}(x,t) \rangle$ is the interval of the absolute addresses on the tape which the head visits or may change up to time t.

Lemma 11 *Let* (X_1, g) *be a TMH,* $x, y \in X_1 - X_0$ *be nonquiescent states,* $t \in \mathbb{N}$, *and* $y_i = x_i$ *for* $\underline{\mathfrak{p}}(x,t) \le i \le \overline{\mathfrak{p}}(x,t)$. *Then*

1. $\mathfrak{p} g^t(x) = \mathfrak{p} g^t(y)$
2. $g^t(x)_i = g^t(y)_i$ *for* $\underline{\mathfrak{p}}(x,t) \le i \le \overline{\mathfrak{p}}(x,t)$
3. $g^t(x)_i = x_i$ *for* $i < \underline{\mathfrak{p}}(x,t)$ *or* $\overline{\mathfrak{p}}(x,t) < i$.



Lemma 12 *Suppose that for every $n \in \mathbb{N}$ there exists $y \in X$, and $t \in \mathbb{N}$ such that $\mathfrak{p}g^t(y) \geq n$. Then there exists $x \in X$ with $\liminf_{t\to\infty} \mathfrak{p}g^t(x) = 0$, and $\limsup_{t\to\infty} \mathfrak{p}g^t(x) = \infty$.*

Proof: This follows from Lemma 10 and Lemma 3.

Proposition 13 *No TMH is positively expansive. No bijective TMH is expansive.*

Proof: In the case of positive expansivity, the proof is the same as in Proposition 7 using Lemma 12. If a TMH were bijective and expansive, then the head must visit all the fields on the tape, but because of its finite capacity it must do so several times. We obtain again arbitrary long returns, so Lemma 12 applies and we get a contradiction. □

Proposition 14 *If a TMH is not bijective, then it is not locally bijective.*

Proof: Again , a TMH is bijective iff $\delta_{QA} : Q \times A \to Q \times A$ is bijective. Suppose that δ_{QA} is not bijective. Take any quiescent state $x \in X$. If U is an arbitrary neighbourhood of x, then there exists different $y, z \in U$ with $\mathfrak{p}(y) = \mathfrak{p}(z) \neq 0$, and $g(y) = g(z)$. □

Proposition 15 *Let (X_1, g) be a TMH and $x \in X_1 - X_0$. Then*
1. $\mathfrak{p}gt(x)$ is a bounded function of t iff x is eventually periodic.
2. If x is eventually periodic, then $x \in \mathrm{Eq}(X_1, g)$.

Proof: 1 is clear. To prove 2, suppose $|\mathfrak{p}g^t(x)| \leq k$ for every t. Then for every $y \in X$ with $d(x,y) < 2^{-k-1}$ we have $d(g^t(x), g^t(y)) < 2^{-k}$ for every t. □

Example 1 *The converse of Proposition 15(2) is not true. Consider $Q = \{0\}$, $\delta(0,a) = (0,a,1)$. Then $x \in \mathrm{Eq}(X_1, g)$ iff $\mathfrak{p}(x) \neq \infty$.*

Proposition 16 *No TMH has an attracting periodic point.*

Proof: If x is a non-quiescent periodic point, then $\mathfrak{p}g^t(x)$ is a bounded function of t, and all points sufficiently close to x are periodic too. If x is a quiescent point, then there is arbitrarily close different quiescent state which is a fixed point too. □

Theorem 17 *Let (X_1, g) be a TMH. Then the following properties are equivalent.*
1. (X_1, g) is equicontinuous.
2. (X_1, g) has the shadowing property.
3. For every nonquiescent $x \in X_1$, $\mathfrak{p}g^t(x)$ is bounded.

Proof: $1 \Rightarrow 2$: see Kůrka [7] for a more general assertion that every equicontinuous dynamical system on zero-dimensional space has the shadowing property.
$2 \Rightarrow 3$: Suppose that (X_1, g) is a TMH for which there exists $x \in X$ such that $\mathfrak{p}g^t(x)$ is not bounded from above. We can assume without loss of generality that $\mathfrak{p}(x) = 0$, and $\mathfrak{p}g^i(x) \geq 0$ for $i \geq 0$. Let $\varepsilon = 2^{-n}$, $\delta = 2^{-n-k}$. We construct a δ-chain, which is not ε-shadowed by any point. Let t be the first integer for which $\mathfrak{p}g^t(x) = n+k+1$, and let $s < t$ be the last integer for which $\mathfrak{p}g^s(x) = n$. Construct a δ-chain x^i as follows

$$
\begin{array}{ll}
x^0_i = x_i & \text{for } i > -n. \\
x^0_i = g^s(x)_{i+2n+k+1} & \text{for } -n-k \leq i \leq -n \\
x^j = g^j(x^0) \text{ for } 0 < j < t & \\
x^t_{-n-k-1} = g^s(x)_n & \\
x^t_{n+k+1} \in A & \text{arbitrary} \\
x^t_i = g^t(x^0)_i \text{ for } |i| \neq n+k+1 & \\
x^j = g^{j-t}(x^t) & \text{for } j > t.
\end{array}
$$

Then $\mathfrak{p}(x^{2t-s}) = -n$, thus the head left the strip $\langle -n, n \rangle$ to the right at the time s, and returned at the time $2t-s$ from the left, which is impossible. Thus x^i is not ε-shadowed by any point.
$3 \Rightarrow 1$. By Proposition 15, every nonquiescent state is equicontinuous. Moreover, there exists k such that for every y, if $\mathfrak{p}(y) = 0$, then $|\mathfrak{p}g^t(y)| \leq k$ for every t. If x is a quiescent state, then for every $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-n-k}$ such that if x is a quiescent state, and $d(x, y) < \varepsilon$, then $d(g^t x, g^t y) < \delta$ for every t. Thus quiescent states are equicontinuous too. □

Theorem 18 *Every TMH has zero topological entropy.*

Proof: For $m > 0$ consider the clopen partition $\alpha_m : X_1 \to (Q \times A)^{\langle -m,m \rangle}$ given by $\alpha_m(x) = (x_{-m} ... x_m)$. Consider a fixed word $u \in \mathcal{L}_{\alpha_m}(X_1, g)$ generated by a point $x \in X_1$, so $u_i = g^i(x)_{-m} ... g^i(x)_m$. The position function may be extended to $u_i \in (Q \times a)^{\langle -m,m \rangle}$ by $\mathfrak{p}(u_i) = j$ if $(u_i)_j \in Q$, and $\mathfrak{p}(u_i) = \infty$ if no such j exists. Denote s_0 the first nonnegative integer such that $\mathfrak{p}(u_{s_0}) \neq \infty$. Denote

$$M = \{i > s_0; (\exists v \in \mathcal{L}_{\alpha_m}(X_1, g))(v_{|i} = u_{|i} \;\&\; v_i \neq u_i)\}$$

Then

$$i \in M \;\;\Rightarrow\;\; \mathfrak{p}(u_{i-1}) = \infty \;\; \& \;\; |\mathfrak{p}(u_i)| = m$$

Let $(s_i)_{i>0}$ be an increasing sequence enumerating M, so $\{s_i; i > 0\} = M$. For $i > 0$ denote $t_i = s_i - s_{i-1}$. Denote $v^i \in \mathcal{L}_{\alpha_m}(X_1, g)$ the word from the definition of M with $v^i_{|s_i} = u_{|s_i}$, $v^i_{s_i} \neq u_{s_i}$. Thus $v^i \neq v^j$ for $i \neq j$. Since in both time instants s_{i-1} and s_i the head is within interval $\langle -m, m \rangle$, in time interval between s_{i-1} and s_i, it may visit only fields with addresses $|j| \leq m + t_i/2$. It follows that v_i is uniquely determined by u_0 and by $\{g^{s_0}(x)_j; m - t_i/2 \leq j \leq m + t_i/2\}$. For a given time t, let n be such that $n \log_a n \geq t > (n-1)\log_a(n-1)$, where $a = \mathrm{card}(A)$. Denote $k = \max\{t_i; i \leq n\}$. For $i \leq n$ put

$$V_i = \{x \in A^{\langle -m-k, m+k \rangle}; (\forall j \in \langle s_0, s_n \rangle)(g^{j-s_0}(x)_{-m} \dots g^{j-s_0}(x)_m = v^i_j)\}$$

Since all v^i are different, all V_i are disjoint. For every $x \in V_i$, $x_{-m} \dots x_m = u_0$, and as shown above, every v^i depends only on at most t_i sites form $\langle -m - k, -m-1 \rangle \cup \langle m+1, m+k \rangle$. Thus $\mathrm{card}(V_i) \geq a^{2k - t_i}$, and $a^{2k-t_1} + \dots + a^{2k-t_n} \leq \mathrm{card}(V_1) + \dots + \mathrm{card}(V_n) \leq a^{2k}$, so $a^{-t_1} + \dots + a^{-t_n} \leq 1$. It follows $t_1 + \dots + t_n \geq n \log_a n \geq t$. Thus there is at most n branchings up to time t. Every branch must correspond to a different inner state $q \in Q$, so we get $\mathrm{card}(\mathcal{L}^t_{\alpha_m}(X_1, g)) \leq (\mathrm{card}(Q))^n$. It follows

$$\frac{1}{t} \ln \mathrm{card}(\mathcal{L}^t_{\alpha_m}(X_1, g)) \leq \frac{n \ln \mathrm{card}(Q)}{(n-1)\log_a(n-1)}$$

and the right-hand side converges to zero as t converges to infinity. Thus $h_{\alpha_m}(X_1, g) = 0$, and therefore $h(X_1, g) = 0$. $\Box$

5 Discussion

Every TMH may be extended to a cellular automaton i.e. to a shift commuting dynamical system with the state space $(Q \cup A)^{\mathbb{Z}}$. It is only necessary to extend the definition of g to the case that two, three or four heads appear among x_{i-1}, x_i, x_{i+1} and x_{i+2}. The heads may annihilate one another, reducing their total number on the tape, or conversely, two heads may produce a third one when they encounter. The results obtained by Maass [8] indicate that the cellular automaton obtained may have properties quite independent from its underlying TMH, so it seems to be difficult to obtain general results in this direction. In the special case that at least one of the heads disappears when several heads encounter, we get that every X_n is an invariant set, and typically the basin of attraction of X_1 might be quite large. This seems to be the mechanism behind the disappearance of dislocations studied extensively on the rule 18 of Wolfram classification (see e.g. Hanson and Crutchfield [2]).

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