Regular unimodal systems and factors of finite automata

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Abstract

Dynamical systems at the edge of chaos, which have been considered as models of self-organization phenomena, are marked by their ability to perform nontrivial computations. To distinguish them from systems with limited computing power, we formulate two simplicity criteria for general dynamical systems, and apply them to unimodal systems on real interval. We say that a dynamical system is regular, if it yields a regular language when observed through arbitrary almost disjoint cover. Finite automata are regarded as dynamical systems on zero-dimensional spaces and their factors yield another class of simple dynamical systems. These two criteria coincide on subshifts, since a subshift is regular iff it is a factor of a finite automaton (sofic systems). A unimodal system on real interval is regular if it has only a finite number of periodic points, and nonrecursive otherwise. On the other hand each S-unimodal system with finite, periodic or preperiodic kneading sequence is a factor of a finite automaton. Thus preperiodic S-unimodal systems are factors of finite automata, which are not regular.

key words: edge of chaos, computing power, complex dynamics.

1 Introduction

The complexity of a dynamical system might be characterized by the regularity of its visits in different regions of the state space. While the randomness of these visits is measured by statistical properties like topological entropy, the study of their structural properties requires the methods of computer science and formal language theory. This approach is well established in the study of Z-subshifts, which are in one-to-one correspondence with central languages (see Culik II et al. [4]). Regular Z-subshifts are called sofic systems and are factors of subshifts of finite type (see Weiss [14] or Fried [6]). In one-dimensional dynamics formal language methods have been pioneered by Crutchfield and Young in [3], who investigate the regularity of languages generated by logistic system on standard partition of the state space.

In the present paper we generalize these results. Any cover of the state space yields a language, and we are interested in its regularity and recursivity. To get a satisfactory theory we restrict ourselves to covers, which are sufficiently simple. Otherwise the complexity of the obtained language would reflect the complexity of the cover, rather than the complexity of the dynamics. In zero-dimensional spaces the appropriate covers are finite clopen partitions. In the case of a compact real interval we use interval covers consisting of closed intervals, which meet at their endpoints. A general concept is almost disjoint finite cover. It consists of closed sets, which have everywhere the same dimension equal to the dimension of the space, and meet in sets whose dimension is smaller. We say that a dynamical system is regular (recursive), if it generates a regular (recursive) language on each almost disjoint cover.

In formal language theory there is a duality between languages and automata. In particular regular languages are recognized by finite automata. Finite automata can be regarded as dynamical systems on discontinuum, which is a compact, perfect, zero-dimensional space. Dynamical systems on discontinuum are of particular interest, since every dynamical system on a compact metric space is a factor of some dynamical system on discontinuum (generalized Alexandrov Theorem, see Balcar and Simon [1]). Every type of automata studied in computer science corresponds in a natural way to a class of dynamical systems on discontinuum. Since factorization might only lead to simplification, the hierarchy of these automata induces a hierarchy of dynamical systems. Factors of finite automata are at the bottom of this hierarchy.

We show that the simplest dynamical systems, which have only finite attractors, are both regular and factors of finite automata. Equivalence between these two classes occurs also in subshifts. Since we work with noninvertible dynamical systems, the languages we get are in general not central, but only right central, i.e. extendable to the right. Right central languages correspond to N-subshifts in a similar manner as central languages correspond to Z-subshifts (see Culik II et al. [4]). An N-subshift is regular iff it is a factor of a finite automaton. More generally, any zero-dimensional factor of a finite automaton is regular, but not vice versa.

For unimodal systems on real interval we get a reverse situation: If we restrict ourselves to the standard cover consisting of the two intervals meeting at the turning point, we get a regular language iff the kneading sequence is either finite, or periodic or preperiodic (this is the generalization of results of Crutchfield and Young [3]). However, the interval covers with cut points whose itineraries are nonrecursive, yield nonrecursive languages. Such covers occur whenever there exists an infinite number of periodic points. Thus a unimodal system is regular if it has a finite number of periodic points, and nonrecursive otherwise.

These results have been obtained without any assumptions on smoothness.

To investigate unimodal factors of finite automata, we need a stronger theory of S-unimodal systems (negative Schwarzian derivative). We have proved in [10] that any S-unimodal system whose kneading sequence is either preperiodic or periodic odd is a factor of a finite automaton. On the other hand the S-unimodal systems with aperiodic kneading sequences, which occur at the common limit of period doubling and band merging bifurcations, are not factors of finite automata. In the present paper we strengthen the first result and show that any S-unimodal system with finite, periodic or preperiodic kneading sequence is a factor of a finite automaton. Thus preperiodic S-unimodal systems are factors of finite automata, which are not regular.

2 Dynamical systems and languages

We conceive dynamical system as a continuous self map of a topological space, which is compact, metrizable and homogenous, i.e. has at each point the same dimension. Recall $dim(X) = -1$ if $X = 0$, $dim(X) = \sup\{dim_x(X)|x \in X\}$ if $X \neq 0$, $dim_x(X) \leq n+1$ if there exist arbitrary small open neighbourhoods of x whose boundaries have dimension at most n, $dim_x(X) = n+1$ if $dim_x(X) \leq n+1$ and $dim_x(X) \not\leq n$, $dim_x(X) = \infty$, if $(\forall n)(dim_x(X) \not\leq n)$. A cover of space X is a system $\alpha = \{V_a|a \in A\}$ of its subsets whose union is X. A cover α is finer than $\beta = \{V_b|b \in B\}$ $(\alpha \geq \beta)$, if $(\forall a \in A)(\exists b \in B)(V_a \subseteq V_b)$.

Definition 1 *A space X is homogenous, if $(\forall x \in X)(dim_x(X) = dim(X))$. A cover α of X is almost disjoint if it is finite, each V_a is a closed homogenous set with $dim(V_a) = dim(X)$, and $dim(V_a \cap V_{a'}) < dim(X)$ for $a \neq a'$.*

Every zero-dimensional space is homogenous, and a real interval is a homogenous one-dimensional space. A cover of a zero-dimensional compact space is almost disjoint iff it is a finite clopen partition, i.e. if it consists of disjoint clopen (closed and open) sets. A cover $\alpha = \{V_a|a \in A\}$ of a compact real interval is almost disjoint, iff each V_a is a finite union of closed intervals, and $V_a \cap V_b$ consists of endpoints of these intervals. We say that α is an interval cover, if each V_a is a closed interval. The endpoints of these intervals are referred to as cutpoints of α. A space is perfect, if each its point is a limit point. A discontinuum is a compact metrizable space, which is 0-dimensional and perfect. Any two discontinua are homeomorphic, and any countable product of discrete finite spaces is a discontinuum (see e.g. [7]).

Definition 2 *A dynamical system (X, f) is a continuous map $f : X \to X$ of a compact metrizable and homogenous space X to itself. A homomorphism $\varphi : (X, f) \to (Y, g)$ of dynamical systems is a continuous mapping $\varphi : X \to Y$ such that $g\varphi = \varphi f$. We say that (Y, g) is a factor of (X, f), if φ is a factorization (i.e. a surjective map).*

The n-th iteration of a dynamical system (X, f) is defined by $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. A point $x \in X$ is periodic with period $n > 0$, if $f^n(x) = x$ and $f^i(x) \neq x$ for $0 < i < n$. A point x is eventually periodic, if there exists $i \geq 0$, such that $f^i(x)$ is periodic. It is preperiodic, if it is eventually periodic but not periodic. It is aperiodic, if it is not eventually periodic. A subset $A \subseteq X$ is invariant, if $f(A) \subseteq A$, and strongly invariant if $f(A) = A$. The ω-limit set of a set $A \subseteq X$ is $\omega(A) = \bigcap_n \overline{\bigcup_{m>n} f^m(A)}$. If A is nonempty, then $\omega(A)$ is a nonempty, closed, strongly invariant set. If $\varphi : (X, f) \to (Y, g)$ is a factorization, then $\varphi(\omega_f(A)) = \omega_g(\varphi(A))$.

Let A be a finite alphabet, and $N = \{0, 1, 2, ...\}$ the set of non-negative integers. We frequently use alphabet $2 = \{0, 1\}$. Denote $A^N = \{u = u_0 u_1 ... | u_i \in A\}$ the space of infinite sequences of letters of A, with product topology. For $n \in N$ denote A^n the set of sequences of length n, $A^* = \cup_{n \in N} A^n$ the set of finite sequences, and $\overline{A^*} = A^* \cup A^N$. Denote $|u|$ the length of a sequence $u \in \overline{A^*}$, $(0 \leq |u| \leq \infty)$, and λ the word of zero length so, $u_i = \lambda$ for $i > |u|$. Write $u \trianglelefteq v$ if u is a substring of v, and $u \sqsubseteq v$, if u is an initial substring of v. Denote $u_{|i} = u_0 ... u_{i-1}$ the initial substring of u of length i. If $u \in A^*$, denote $\overline{u} \in A^N$ the infinite repetition of u, defined by $\overline{u}_{k|u|+i} = u_i$. For $u \in \overline{A^*}$ define $\sigma(u) \in \overline{A^*}$ by $\sigma(u)_i = u_{i+1}$. Thus $\sigma(\lambda) = \lambda$. For $u \in A^n$ denote $[u] = \{v \in A^N | v_{|n} = u\} \subseteq A^N$ the cylinder determined by u. It is a clopen set, and $\alpha_n = \{[u] | u \in A^n\}$ is a clopen partition of A^N. A subshift is any nonempty subset $\Sigma \subseteq A^N$, which is closed and σ-invariant. If Σ is a subshift, then (Σ, σ) is a dynamical system.

Definition 3 *Let (X, f) be a dynamical system, and $\alpha = \{V_a | a \in A\}$ a finite cover of X. The language and subshift generated by (X, f) on α are*

$$\mathcal{L}(X, f, \alpha) = \{u \in A^* | V_u \neq 0\}, \quad \mathcal{S}(X, f, \alpha) = \{u \in A^N | V_u \neq 0\},$$

where $V_u = \{x \in X | (\forall i < |u|)(f^i(x) \in V_{u_i})\}$ is the set of points, which visit sets of α according to the sequence u. We say that (X, f) is regular (recursive), if $\mathcal{L}(X, f, \alpha)$ is regular (recursive) for every almost disjoint cover α of X.

Definition 4 *A nonempty language $L \subseteq A^*$ is right central, if*
$(\forall u \in L)(\forall v \trianglelefteq u)(v \in L)$, and
$(\forall u \in L)(\exists a \in A)(ua \in L)$.
The adherence of $L \subseteq A^*$ and the language of $\Sigma \subseteq A^N$ are defined by
$\mathcal{A}(L) = \{u \in A^N | (\forall v \in A^*)(v \trianglelefteq u \Rightarrow v \in L)\}$.
$\mathcal{L}(\Sigma) = \{u \in A^* | (\exists v \in \Sigma)(u \trianglelefteq v)\} = \mathcal{L}(\Sigma, \sigma, \{[a] | a \in A\})$.

If (X, f) is a dynamical system and α a closed cover of X, then $\mathcal{L}(X, f, \alpha)$ is a right central language, $\mathcal{S}(X, f, \alpha)$ is a subshift, $\mathcal{L}(\mathcal{S}(X, f, \alpha)) = \mathcal{L}(X, f, \alpha)$, and $\mathcal{A}(\mathcal{L}(X, f, \alpha)) = \mathcal{S}(X, f, \alpha)$. If $L \subseteq A^*$ is a right central language and $\Sigma \subseteq A^N$ a subshift, then $\mathcal{A}(L)$ is a subshift, $\mathcal{L}(\Sigma)$ is a right central language, $\mathcal{AL}(\Sigma) = \Sigma$, and $\mathcal{LA}(L) = L$. This can be proved by a simple compactness argument (cf. [4]).

Proposition 1 *Let (X, f) be a dynamical system and $\alpha \geq \beta$ almost disjoint covers of X. If $\mathcal{L}(X, f, \alpha)$ is regular (recursive), then so is $\mathcal{L}(X, f, \beta)$.*

Proof: Suppose that there exist $a \in A$, and $b \neq b' \in B$ with $V_a \subseteq V_b \cap V_{b'}$. Then $dim(V_a) \leq dim(V_b \cap V_{b'}) < dim(X)$, which is a contradiction. Thus for each $a \in A$ there exists a unique $h(a) \in B$ with $V_a \subseteq V_{h(a)}$. We prove $V_b = \cup\{V_a | h(a) = b\}$. If not, denote $Y = V_b - \cup\{V_a | h(a) = b\}$. For each $y \in Y$ there exist a', b', with $y \in V_{a'} \subseteq V_{b'}$, thus $Y \subseteq \cup\{V_b \cap V_{b'} | b' \neq b\}$, and $dim(Y) < dim(V_b)$. Since Y is open in V_b, V_b is not homogenous, which is a contradiction. The map h can be extended to a monoid homomorphism $h : A^* \to B^*$, which is λ-free, i.e. $h^{-1}(\lambda) = \lambda$. We get $\mathcal{L}(X, f, \beta) = h(\mathcal{L}(X, f, \alpha))$, and since both regular and recursive languages are closed to λ-free homomorphisms (see Hopcroft and Ullmann [8]), $\mathcal{L}(X, f, \beta)$ is regular (recursive). □

Proposition 2 *A dynamical system (X, f) on a zero-dimensional space X is regular (recursive) iff there exists a sequence α_n of clopen partitions of X such that $\alpha_{n+1} \geq \alpha_n$, $diam(\alpha_n) \to 0$, and $\mathcal{L}(X, f, \alpha_n)$ is regular (recursive).*

Proof: Let β be a finite clopen partition. For each $x \in X$ there exists n_x and $A_x \in \alpha_{n_x}$ with $x \in A_x$, such that $\{A_x | x \in X\} \geq \beta$. Let $\{A_x | x \in K\}$ be a finite subcover of $\{A_x | x \in X\}$, and $n = \max\{n_x | x \in K\}$. Then $\alpha_n \geq \beta$, and we get the result by Proposition 1. □

Proposition 3 *A dynamical system (I, g) on a compact real interval is regular (recursive) iff $\mathcal{L}(I, g, \alpha)$ is regular (recursive) for each interval cover α.*

Proof: If α is an almost disjoint cover of I, then each V_a is a finite union of closed nontrivial intervals, so there exists an interval cover, which refines it, and we apply Proposition 1. □

Definition 5 *A set $A \subseteq X$ is an attractor, of a dynamical system (X, f), if it has a closed neighbourhood U such that $f(U) \subseteq int(U)$, and $A = \omega(U)$ (see [5]). We say that (X, f) is a system with finite attractors, if all its attractors are finite, i.e. if $\omega(X)$ is finite .*

Lemma 1 *Any attractor is stable, i.e. any its neighbourhood contains an invariant neighbourhood.*

Proof: Let $U \subseteq X$ be closed neighbourhood of A, $f(U) \subseteq int(U)$, and $A = \omega(U)$. Let $V \subseteq U$ be an open neighbourhood of A. Suppose that V does not contain any invariant neighbourhood of A. For $k > 0$ denote $W_k = \cup_{n \in N} f^n(B_{1/k}(A))$, where $B_\varepsilon(A) = \{x \in X | \varrho(y, A) < \varepsilon\}$. W_k is an invariant neighbourhood of A, so $W_k \not\subseteq V$. Thus there exists some $x_k \in B_{1/k}(A)$, and an integer n_k with $f^{n_k}(x_k) \in X - V$. Since $X - V$ is compact, there exists a subsequence k_i, with $f^{n_{k_i}}(x_{k_i}) \to x \in X - V$. For all sufficiently large k, $x_k \in U$, so $x \in \omega(U) = A$, which is a contradiction. □

Theorem 1 *Any dynamical system with finite attractors is regular.*

Proof: Let (X, f) be a dynamical system with finite $\omega(X)$. Since $\omega(X)$ is strongly invariant, it consists of the periodic points of X. Let $\alpha = \{V_a | a \in A\}$ be an almost disjoint cover of X. For each $p \in \omega(X)$ there exists an open set S_p such that $p \in S_p$, $S_p \cap S_q = 0$ for $p \neq q$, and $S_p \cap V_a = 0$ whenever $p \notin V_a$. Then $S = \cup\{S_p | p \in \omega(X)\}$ is a neighbourhood of $\omega(X)$. By Lemma 1 there exists an invariant neighbourhood $U \subseteq S$ of $\omega(X)$. Denote $U_p = U \cap S_p$. Then U_p is a neighbourhood of p, $f(U_p) \subseteq U_{f(p)}$, and $U_p \cap V_a = 0$ whenever $p \notin V_a$. Denote $L = \{u \in A^* | (\exists p \in \omega(X))(\forall i < |u|)(f^i(p) \in V_{u_i})\}$. Since each $p \in \omega(X)$ is periodic, L is a regular language. For each $x \in X$, $\omega(x) \subseteq \omega(X)$, so there exists $p \in \omega(X)$ such that $\lim_{i \to \infty} f^{it_p}(x) = p$, where t_p is the period of p. It follows there exists n_x with $f^{n_x}(x) \in int(U)$, and an open neighbourhood W_x of x with $f^{n_x}(W_x) \subseteq int(U)$. Since X is compact, there exists a finite subcover $\{W_x | x \in K\}$. Let $n = \max\{n_x | x \in K\}$. Then for any $x \in X$ there exists unique $p \in \omega(X)$ with $f^n(x) \in U_p$, and for each $i \geq 0$, $f^{n+i}(x) \in U_{f^i(p)}$. Moreover, if $f^{n+i}(x) \in V_a$, then $f^i(p) \in V_a$. Thus if $u \in A^*$ and $f^n(x) \in V_u$, then $p \in V_u$. It follows that if $u \in \mathcal{L}(X, f, \alpha)$, then $\sigma^n(u) \in L$, so $\mathcal{L}(X, f, \alpha)$ is a regular language. $\Box$

3 Finite automata

A multitape Turing machine is a finite control device connected to several potentially infinite tapes. We can assume that the control device itself occupies a field on a tape, which does not move. The resulting machine is conceptually simpler. It consists of a finite number of doubly infinite tapes, which interact at the scanned positions. The contents of the scanned positions of all tapes determine the move of the machine: the letters which it writes on the tapes and the directions of their shifts. In contrast to computer science we assume that the tapes are actually infinite and have also infinite content. Thus we obtain a dynamical system on discontinuum (cf. Moore [12]). In a similar manner we conceive a multitape finite automaton. It is a finite control attached to input tapes, on which all future inputs are written. Moreover there are output tapes, to which the results are written. Thus the tapes move in one direction only.

Let A be a finite alphabet and denote $Z = \{\ldots -1, 0, 1, \ldots\}$ the set of integers. A tape is a data structure containing the letters of A indexed by Z. The state of a tape is a map $u : Z \to A$, thus its state space is the full shift A^Z, which is a discontinuum. The tape state may be updated by rewriting the zeroth position and shifting the tape left or right. This is accomplished by (continuous) updating functions $\sigma_a^i : A^Z \to A^Z$ defined by $\sigma_a^i(u)_j = a$ if $i + j = 0$, and $\sigma_a^i(u)_j = u_{i+j}$ otherwise. Note that $\sigma^i(u) = \sigma_{u_0}^i(u)$ coincides with the canonical shift action.

$$\begin{aligned}\sigma_a^{-1}(...u_{-2}u_{-1}u_0u_1u_2...) &= ...u_{-3}u_{-2}u_{-1}au_1...\\ \sigma_a^{0}(...u_{-2}u_{-1}u_0u_1u_2...) &= ...u_{-2}u_{-1}au_1u_2...\\ \sigma_a^{1}(...u_{-2}u_{-1}u_0u_1u_2...) &= ...u_{-1}au_1u_2u_3...\end{aligned}$$

Definition 6 *A Turing automaton is a finite system of finite alphabets $(A_t)_{t\in T}$ together with transition functions $\delta_t : Q \to A_t$, $\eta_t : Q \to \{-1,0,1\}$, where $Q = \prod_{t\in T} A_t$ is the set of inner states. A Turing automaton determines a dynamical system (X,f), where $X = \prod_{t\in T} A_t^Z$, $\pi : X \to Q$ is the projection $\pi(u)_t = u_{t0}$, and*

$$f(u)_t = \sigma_a^i(u_t) \quad \textit{where} \quad a = \delta_t\pi(u),\ i = \eta_t\pi(u), \quad u\in X, t\in T.$$

The advance of a tape $t\in T$ on $u\in X$ in n steps is $d_t(u,n) = \sum_{i=0}^{n-1}\eta_t\pi f^i(x)$. A finite automaton is a Turing automaton, whose tapes never move to the right, i.e. $(\forall t\in T)(\forall q\in Q)(\eta_t(q)\geq 0)$.

Theorem 2 *A dynamical system with finite attractors is a factor of a finite automaton.*

Proof: Let (Y,g) be a system with finite attractors. Then $\omega(Y)$ consists of periodic points, and we have an inverse map $g^{-1} : \omega(X) \to \omega(X)$. Denote t_p the period of $p\in\omega(Y)$. By Lemma 1 there exists a g^{t_p}-invariant neighbourhood U of $p\in\omega(Y)$, which does not meet $\omega(Y)-\{p\}$. It follows that $Y_p = \{y\in Y|g^{kt_p}(y)\to p\}$ is an open set, and $\{Y_p|p\in\omega(Y)\}$ is a clopen partition of Y. Construct a two-tape finite automaton (X,f) with alphabets $A_1 = A_2 = \omega(Y)$. Define transition functions by $\eta_1(p,q) = 0$, $\eta_2(p,q) = 1$, $\delta_1(p,q) = g(p)$, $\delta_2(p,q) = p$. Thus the first tape is stationary and only one of its fields is used. Its content changes according to g, and the results are written to the second (output) tape. As the time proceeds, the negative part of the second tape contains ever longer periodic segments. For $p\in\omega(Y)$, $k\geq 0$ define
$X_{pk} = \{(u,v)\in X|u_0 = p\ \&\ \min\{j>0|g^j(v_{-j})\neq u_0\} = k+1\}$,
$X_{p\infty} = \{u\in X|(\forall j>0)(g^j(v_{-j}) = u_0 = p)\}$.
Then $f(X_{pk}) = X_{g(p),k+1}$. For $k\geq 0$ define $\pi_k : X\to A_2^N$ by $\pi_k(u,v) = v_{-k-1}v_{-k-2}...$. Then $\pi_{k+1}f(u,v) = \pi_k(u,v)$ for $(u,v)\in X_{pk}$. Denote
$Y_{p0} = Y_p\cap\overline{Y-g(Y)}$, $Y_{pk} = g^k(Y_{g^{-k}(p),0})$, $Y_{p\infty} = \{p\}$.
Note that some Y_{pk} may be empty. For $p\in\omega(Y)$ there exists a continuous mapping $\psi_p : A_2^N \to Y$ such that $\psi_p(A_2^N) = Y_{p0}$ if $Y_{p0}\neq 0$, and $\psi_p(A_2^N) = \{p\}$ if $Y_{p0} = 0$. Define
$\varphi(u,v) = g^k\psi_{g^{-k}(p)}\pi_k(u,v)$ for $(u,v)\in X_{pk}$, and $\varphi(u,v) = p$ for $(u,v)\in X_{p\infty}$.
Then $\varphi : X\to Y$ is clearly continuous. For any $y\in Y-\omega(Y)$ there exists only a finite chain of preimages, so φ is also surjective. If $(u,v)\in X_{pk}$, then $\varphi f(u,v) = g^{k+1}\psi_{g^{-k-1}(g(p))}\pi_{k+1}f(u,v) = g^{k+1}\psi_{g^{-k}(p)}\pi_k(u,v) = g\varphi(u,v)$. If $(u,v)\in X_{p\infty}$, then $\varphi f(u,v) = g(p) = g\varphi(u,v)$. Thus $\varphi : (X,f)\to(Y,g)$ is a factorization. $\Box$

Theorem 3 *If (X,f) is a factor of a finite automaton, then there is an integer $q>0$ such that the set $\{x \in X | card(\omega(x)) \leq q\}$ is dense in X.*

Proof: The property is preserved by factorization, so it suffices to prove it for finite automata. Let (X,f) be a finite automaton and denote q the size of Q. Let $U \subseteq X$ be an open set and $x \in U$. There exists n with $[x_{|n}] \subseteq U$, where $[x_{|n}] = \{u \in X | (\forall t \in T)(\forall |i| \leq n)(u_{ti} = x_{ti})\}$. Denote $T_x = \{t \in T | \lim_{k\to\infty} d_t(x,k) = \infty\}$ (T_x might be empty). There exist m, and $k \leq q$, such that $(\forall t \in T - T_x)(\forall i)(d_t(f^m(x), i) = 0)$, $(\forall t \in T_x)(d_t(x,m) > n)$, $\pi f^m(x) = \pi f^{m+k}(x)$. Denote $d_{t0} = d_t(x,m)$, $d_{t1} = d_t(x, m+k)$ and define a point $y \in X$ by
$y_{ti} = x_{ti}$ if $t \in T - T_x$
$y_{ti} = x_{ti}$ if $t \in T_x$, $i \leq d_{t1}$,
$y_{ti} = x_{tj}$ if $t \in T_x$, $i > d_{t1}$, $d_{t0} < j \leq d_{t1}$, and $d_{t1} - d_{t0}$ divides $i - j$.
Then $y \in [x_{|n}]$, and $f^i(y)$ converges to a periodic point with period $k \leq q$. $\square$

Proposition 4 *Any zero-dimensional factor of a finite automaton is regular.*

Proof: Let (X,f) be a finite automaton and denote $\alpha_n = \{V_v | v \in A\}$ the clopen partition of n-cylinders $V_v = \{u \in X | (\forall t \in T)(\forall |i| \leq n)(u_{ti} = v_{ti})\}$ (here $A = \prod_{t\in T} A_t^{2n+1}$). Let us prove that if $V_{u_0 \ldots u_{m-1}} \neq 0$, and $V_{u_{m-1}u_m} \neq 0$, then $V_{u_0\ldots u_m} \neq 0$. Choose $x \in V_{u_0\ldots u_{m-1}}$, $y \in V_{u_{m-1}u_m}$, and define
$z_{ti} = x_{ti}$ if $i \leq d_t(x,m) + n$,
$z_{ti} = y_{tj}$ if $i = d_t(x,m) + n + 1$, $j = d_t(y,1) + n$.
Then $z \in V_{u_0\ldots u_n}$. Thus $\mathcal{L}(X, f, \alpha_n)$ is regular (it is a finite complement language) and (X,f) is regular by Proposition 2. If $\varphi : (X,f) \to (Y,g)$ is a factorization, and $\alpha = \{V_a | a \in A\}$ a clopen partition of Y, then $\varphi^{-1}(\alpha) = \{\varphi^{-1}(V_a) | a \in A\}$ is a clopen partition of X and $\mathcal{L}(Y,g,\alpha) = \mathcal{L}(X,f,\varphi^{-1}(\alpha))$. Indeed for $u \in A^*$ $\varphi^{-1}(V_u) = \{x \in X | (\forall i < |u|)(g^i\varphi(x) \in V_{u_i})\} = \{x \in X | (\forall i < |u|)(f^i(x) \in \varphi^{-1}(V_{u_i}))\}$. $\square$

Theorem 4 *A subshift is regular iff it is a factor of a finite automaton.*

Proof: By Proposition 4, any subshift which is a factor of a finite automaton is regular. On the other hand if $\Sigma \subseteq A^N$ is a regular subshift, then its language is recognized by a finite accepting automaton, so there exists a set of states Q, initial set $q_0 \in Q$, accepting set $Q_1 \subseteq Q$, and a transition function $\delta : Q \times A \to Q$ such that $u \in \mathcal{L}(\Sigma)$ iff $\delta^*(q_0, u) \in Q_1$. Since $\mathcal{L}(\Sigma)$ is right central, $q_0 \in Q_1$ and for any $q \in Q_1$ there exists some $h(q) \in A$ with $\delta(q, h(q)) \in Q_1$. Define a two-tape automaton (X,f) with alphabets $A_1 = Q_1$, $A_2 = A$, $\eta_1(q) = 0$, $\eta_2(q) = 1$ for any $q \in Q_1$. The transitions are defined by

$$\begin{array}{lll} \delta(q,a) \in Q_1 & \Rightarrow & \delta_1(q,a) = \delta(q,a), \quad \delta_2(q,a) = a, \\ \delta(q,a) \notin Q_1 & \Rightarrow & \delta_1(q,a) = \delta(q,h(q)), \quad \delta_2(q,a) = h(q) \end{array}$$

Define $\varphi : X \to A^N$ by $\varphi(x)_i = \delta_2 \pi f^i(x)$, where $\pi : X \to Q_1 \times A$ is the projection. Then $\varphi : (X, f) \to (\Sigma, \sigma)$ is a factorization. $\Box$

Since the set of the inverse words of a regular language is regular, any Z-subshift of a finite type is a factor of a finite automaton. Since a closed hyperbolic set with a local product structure (see Shub [13]) is a factor of a Z-subshift of finite type, it is also a factor of a finite automaton.

Example 1 *There exists a regular system on* 0*-dimensional space, which is not a factor of a finite automaton.*

Proof: Take $X = 2^N$ and for $n > 0$ denote $U_n = [0^{n-1}1]$. For each $n > 0$ let $\{U_{nm} | 0 \leq m < n\}$ be a clopen partition of U_n. Define a continuous map g so that if $x \in U_{nm}$, then $g(x) \in U_{np}$, where $p = m + 1 \bmod n$, and $g^n(x) = x$. Thus each $x \in U_n$ is a periodic point with period n. Then (X, g) is a regular system, which is by Theorem 3 not a factor of a finite automaton. $\Box$

4 Unimodal systems

Definition 7 *A dynamical system* (I, g) *defined on a real interval* $I = [a, b]$ *is unimodal, if* $g(a) = g(b) = a$*, and if there exists a turning point* $c \in (a, b)$*, such that* g *is increasing in* $[a, c]$*, and decreasing in* $[c, b]$*.*

A powerful tool for investigating unimodal systems is the kneading theory, which is based on the concept of itineraries generated by the partition $\{[a, c), \{c\}, (c, b]\}$. We modify this theory using the cover $\{[a, c], [c, b]\}$, or more general interval covers instead. We lose the uniqueness of the itinerary, but the resulting theory is simpler. In particular a simple compactness argument yields in Theorem 6 a necessary and sufficient condition characterizing the language generated by the standard cover. Moreover, we do not need any differentiability assumptions.

Throughout this section we assume that α is an interval cover, which contains among its cut points the turning point c. Thus let $a_0 < ... < a_n$ be an increasing sequence such that a_0, a_n are the endpoints of I, and $a_m = c$ is the turning point, where $0 < m < n$. Denote $V_j = [a_j, a_{j+1}]$, and $\alpha = \{V_j | j \in A\}$, where $A = \{0, ..., n-1\}$. For $j \in A$ denote $\tau(j) = 0$ if $j < m$, $\tau(j) = 1$ if $j \geq m$. For $u \in A^*$, $k \leq |u|$, denote $\tau_k(u) = \sum_{i=0}^{k-1} \tau(u_i) \bmod 2$. We say that $u \in A^*$ is even (odd), if $\tau_{|u|}(u)$ is zero (one).

Definition 8 *For* $u, v \in \overline{A^*}$*, define* $u \prec v$ *iff there exists* k *such that* $u_{|k} = v_{|k}$*, and either* $\tau(u_{|k}) = 0$ *and* $u_k < v_k$*, or* $\tau(u_{|k}) = 1$ *and* $u_k > v_k$*. (This includes the case* $|u| = k$ *or* $|v| = k$*, when* $u_k = \lambda$ *or* $v_k = \lambda$*. We put in this case* $0 < ... < m-1 < \lambda < m < ... < n-1$*.) Define* $u \preceq v$ *iff either* $u \prec v$*, or* $u \sqsubseteq v$ *or* $v \sqsubseteq u$*.*

For example if $n = 2$, we get $00 \prec 0 \prec 01 \prec \lambda \prec 11 \prec 1 \prec 10$. If $v \not\preceq u$ then $u \prec v$, and $u \preceq v$. If $v \not\prec u$, then $u \preceq v$. Thus $\prec$ is a strict ordering on $\overline{A^*}$, and $\preceq$ is an ordering on A^N.

Proposition 5 *If $u, v \in \overline{A^*}$, and $u \prec v$, then $(\forall x \in V_u)(\forall y \in V_v)(x \leq y)$.*

Proof: Let $u \prec v$, $k < \min(|u|, |v|)$, $u_{|k} = v_{|k}$, $u_k \neq v_k$, $x \in V_u$, $y \in V_v$. We prove $x \leq y$ by induction on k. If $k = 0$, then $u_0 < v_0$, and $x \leq y$. If $k > 0$, we apply the Proposition to $\sigma(u)$ and $\sigma(v)$. If $\tau(u_0) = \tau(v_0) = 0$, then $\sigma(u) \prec \sigma(v)$, so $g(x) \leq g(y)$, and $x \leq y$ since $x, y \leq c$. If $\tau(u_0) = \tau(v_0) = 1$, then $\sigma(u) \succ \sigma(v)$, so $g(x) \geq g(y)$, and $x \leq y$ since $x, y \geq c$. $\square$

Definition 9 *The upper and lower itineraries of $x \in I$ are defined by*

$$\mathcal{I}^\alpha(x) = \max\{u \in A^N | x \in V_u\}, \quad \mathcal{I}_\alpha(x) = \min\{u \in A^N | x \in V_u\}$$

Here max and min are meant with respect to $\preceq$. The upper and lower itineraries might be obtained by inductive definition:

$$\begin{array}{rcl}
a_j < g^i(x) < a_{j+1} & \Rightarrow & \mathcal{I}^\alpha(x)_i = \mathcal{I}_\alpha(x)_i = j \\
g^i(x) = a_j, \;\; \tau(\mathcal{I}^\alpha(x)_{|i}) = 0 & \Rightarrow & \mathcal{I}^\alpha(x)_i = \min\{j, n-1\} \\
g^i(x) = a_j, \;\; \tau(\mathcal{I}^\alpha(x)_{|i}) = 1 & \Rightarrow & \mathcal{I}^\alpha(x)_i = \max\{j-1, 0\} \\
g^i(x) = a_j, \;\; \tau(\mathcal{I}_\alpha(x)_{|i}) = 0 & \Rightarrow & \mathcal{I}_\alpha(x)_i = \max\{j-1, 0\} \\
g^i(x) = a_j, \;\; \tau(\mathcal{I}_\alpha(x)_{|i}) = 1 & \Rightarrow & \mathcal{I}_\alpha(x)_i = \min\{j, n-1\}
\end{array}$$

Proposition 6 *$x \in V_{\mathcal{I}^\alpha(x)} \cap V_{\mathcal{I}_\alpha(x)}$. If $x < y$, then $\mathcal{I}^\alpha(x) \preceq \mathcal{I}_\alpha(y)$.*

Proof: If $\mathcal{I}^\alpha(x) \succ \mathcal{I}_\alpha(y)$, then $x \geq y$ by Proposition 5.

Definition 10 *For $w \in A^N$, $j \in A$ denote*

$$\begin{array}{l}
L^j(w) = \{u \in \overline{A^*} | (\forall i)(u_i = j \Rightarrow \sigma^{i+1}(u) \preceq w)\} \\
L_j(w) = \{u \in \overline{A^*} | (\forall i)(u_i = j \Rightarrow w \preceq \sigma^{i+1}(u))\}
\end{array}$$

Proposition 7 *If $u \in \overline{A^*}$, then $u \in \mathcal{L}(I, g, \alpha) \cup \mathcal{S}(I, g, \alpha)$ iff*

$$\begin{array}{l}
u \in L_j(\mathcal{I}_\alpha(g(a_j))) \cap L^j(\mathcal{I}^\alpha(g(a_{j+1}))) \;\; \textit{for each } \; j < m, \;\; \textit{and} \\
u \in L^j(\mathcal{I}^\alpha(g(a_j))) \cap L_j(\mathcal{I}_\alpha(g(a_{j+1}))) \;\; \textit{for each } j \geq m.
\end{array}$$

Proof: Let $u \in \mathcal{L}(I, g, \alpha) \cup \mathcal{S}(I, g, \alpha)$, and $u_i = j$. Pick some $x \in V_u$. Then $g^i(x) \in V_{\sigma^i(u)} \subseteq V_j$. If $j < m$, then $g(a_j) \leq g^{i+1}(x) \leq g(a_{j+1})$. Suppose that $\mathcal{I}_\alpha(g(a_j)) \succ \sigma^{i+1}(u)$. Then $g(a_j) \geq g^{i+1}(x)$ by Proposition 5, so $g(a_j) = g^{i+1}(x) \in V_{\sigma^{i+1}(u)}$. This is however impossible, as $\mathcal{I}_\alpha(g(a_j))$ is the least sequence v, for which $g(a_j) \in V_v$. Thus $\mathcal{I}_\alpha(g(a_j)) \preceq \sigma^{i+1}(u)$, and $u \in L_j(\mathcal{I}_\alpha(g(a_j)))$. Similarly we prove the other inequalities, so the condition is necessary. To prove the sufficiency, we proceed by induction on the length of u. If $|u| \leq 1$, then $V_u \neq 0$. If $|u|$ is finite and the Proposition holds for $|u| - 1$, we apply it to $\sigma(u)$. Thus there exists some $y \in V_{\sigma(u)}$. If $u_0 = j < m$, then $\mathcal{I}_\alpha(g(a_j)) \preceq \sigma(u) \preceq \mathcal{I}^\alpha(g(a_{j+1}))$. If $\mathcal{I}_\alpha(g(a_j)) \not\prec \sigma(u)$, then $\mathcal{I}_\alpha(g(a_j)) \sqsupseteq \sigma(u)$, and $a_j \in V_u \neq 0$. Similarly if $\sigma(u) \not\prec \mathcal{I}^\alpha(g(a_{j+1}))$, then $a_{j+1} \in V_u$. If $\mathcal{I}_\alpha(g(a_j)) \prec \sigma(u) \prec \mathcal{I}^\alpha(g(a_{j+1}))$, then $g(a_j) \leq y \leq g(a_{j+1})$ by Proposition 5, and there exists a unique $x \in V_j$ with $g(x) = y$. Thus $x \in V_u \neq 0$. Similarly we proceed if $j \geq m$. If $|u|$ is infinite, $v \sqsubseteq u$ and $v \in 2^*$, then v satisfies the condition of the Proposition, so $V_v \neq 0$. Thus $V_u = \cap\{V_v | v \sqsubseteq u, v \in A^*\}$ is nonempty by compactness. □

Proposition 8 *Let $w \in A^N$ be eventually periodic, and $j \in A$. If $w \in L^j(w)$, then $L^j(w) \cap A^*$ is regular. If $w \in L_j(w)$, then $L_j(w) \cap A^*$ is regular.*

Proof: Suppose $w \in L^j(w)$ and for $u, v \in A^*$ define $u \approx v$ iff $(\forall t \in A^*)(ut \in L^j(w) \Leftrightarrow vt \in L^j(w))$. We prove that there exist integers $0 < r < s$ such that $jw_{|s} \approx jw_{|r}$. If $w = \overline{w_{|n}}$ is periodic even, we put $r = n$, $s = 2n$. Let $t \in A^*$ and denote $u = jw_{|2n}t$, $v = jw_{|n}t$. Suppose $u \in L^j(w)$ and $v_i = j$. If $i = 0$, then $u_i = j$, so $\sigma(u) = w_{|2n}t \preceq w$. Since $w_{|n}$ is even, we get $\sigma(v) = \sigma^n(w_{|2n}t) \preceq \sigma^n(w) = w$. If $i > 0$, then $u_{i+n} = j$, and $\sigma^{i+1}(v) = \sigma^{i+n+1}(u) \preceq w$, thus $v \in L^j(w)$. Conversely suppose $v \in L^j(v)$ and $u_i = j$. If $i = 0$, then $v_i = j$, so $w_{|n}t \preceq w$, and therefore $w_{|2n}t \preceq w$. If $0 < i < n$, then $w_{i-1} = j$ and $\sigma^i(w) \preceq w$, since $w \in L^j(w)$. Since $\sigma^i(w) \neq w$, we get $\sigma^i(w)_{|n} \prec w$, so $\sigma^{i+1}(u) \prec w$. If $i \geq n$, then $v_{i-n} = j$, so $\sigma^{i+1}(u) = \sigma^{i+1-n}(v) \preceq w$. Thus we have proved $jw_{|n} \approx jw_{|2n}$. If $w = \overline{w_{|n}}$ is periodic odd, similar arguments show that $jw_{|n} \approx jw_{|3n}$. If w is preperiodic, then we can write $w = w_0 ... w_{m-1}\overline{w_m ... w_{m+n-1}}$, where $w_m ... w_{m+n-1}$ is even. Since the set $\{\sigma^i(w) | i \in N\}$ is finite, there exists an integer p such that $\sigma^{i+1}(w)_{|p} \prec w_{|p}$ whenever $w_i = j$. Let k be an integer with $nk > p$ and put $r = m + nk$, $s = m + (n+1)k$. Then again $jw_{|r} \approx jw_{|s}$. We show now that if $u \in L^j(w)$, and $|u| \geq s$, then there exists some $v \approx u$ with $|v| < |u|$. If $jw_{|s} \not\sqsubseteq u$, then $u \approx \sigma(u)$. If $jw_{|s} \sqsubseteq u$, then $jw_{|r}\sigma^s(u) \approx u$. Thus there is only a finite number of equivalence classes contained in $L^j(w)$, so $L^j(w)$ is regular by Nerode Theorem (see Hopcroft and Ullmann [8]). □

Theorem 5 *$\mathcal{L}(I, g, \alpha)$ is a regular (recursive) language iff all $\mathcal{I}_\alpha(g(a_j))$ and $\mathcal{I}^\alpha(g(a_j))$ are eventually periodic (recursive) sequences.*

Proof: If the condition is satisfied, then $L = \mathcal{L}(I, g, \alpha)$ is regular (recursive) since it is an intersection of a finite number of regular (recursive) languages (Propositions 7 and 8). To prove the converse, suppose that $w = \mathcal{I}^\alpha(g(a_j))$ is aperiodic and L is regular. Assume $j \geq m$. Since $a_j \in V_{jw}$, $jw_{|k} \in L$ for any $k > 0$. By Nerode Theorem there exists a right congruence $\approx$ of finite index on A^*, such that L is a union of its equivalence classes. There exists an infinite set $M \subseteq N$ such all $\{w_{|m} | m \in M\}$ have the same parity. Assume they are all even. Since $\approx$ is of finite index, there exist $m, n \in M$, $m \neq n$, such that $jw_{|m} \sim jw_{|n}$. Since $\approx$ is a right congruence, $jw_{|(m+k)} \approx jw_{|n}\sigma^m(w)_{|k} \in L$ for any k, so

$w_{|n}\sigma^m(w)_{|k} \preceq w$, and $\sigma^m(w)_{|k} \preceq \sigma^n(w)_{|k}$, since $w_{|n}$ is even. Interchanging m and n in the above argument we get $\sigma^n(w)_{|k} \preceq \sigma^m(w)_{|k}$ for any k, and therefore $\sigma^n(w) = \sigma^m(w)$. This is, however, in contradiction with the aperiodicity of w. If all $\{w_{|m}|m \in M\}$ are odd, or if $j < m$, the proof is similar. For the case of recursivity suppose that $w = \mathcal{I}^\alpha(g(a_j))$ is nonrecursive and L is recursive. Again $jw_{|k} \in L$ for any k. If $j \geq m$ and $m_{|k}$ is even, or if $j < m$ and $m_{|k}$ is odd, then $w_k = \max\{i|jw_{|k}i \in L\}$. In other cases $w_k = \min\{i|jw_{|k}i \in L\}$. Since L is recursive, we get a recursive procedure for the computation of w, which is a contradiction. $\square$

Definition 11 *Denote $\alpha_g = \{[a,c],[c,b]\}$ the standard cover of a unimodal system (I,g) with endpoints a, b, and turning point c. The itinerary $\mathcal{I}(x) \in \overline{2^*}$ of $x \in I$ is defined by*

$$\mathcal{I}(x)_i = 0 \ \Leftrightarrow\ g^i(x) < c, \quad \mathcal{I}(x)_i = 1 \ \Leftrightarrow\ g^i(x) > c$$

for $0 \leq i < |\mathcal{I}(x)| = \inf\{n \geq 0|g^n(x) = c\}$. The kneading sequence of (I,g) is $\mathcal{K}(g) = \mathcal{I}(g(c))$. Denote also $\mathcal{I}_g(x) = \mathcal{I}_{\alpha_g}(x)$, $\mathcal{I}^g(x) = \mathcal{I}^{\alpha_g}(x)$, $\mathcal{K}_g = \mathcal{I}_g(g(c))$, $\mathcal{K}^g = \mathcal{I}^g(g(c))$.

Thus $\mathcal{I}(x)$ is finite iff the orbit of x contains c, e.g. $\mathcal{I}(c) = \lambda$ is the empty word. If $g(c) = c$, then $\mathcal{K}(g) = \lambda$. In general

$$\begin{array}{llll}
\mathcal{K}_g = \mathcal{K}(g), & \mathcal{K}^g = \mathcal{K}(g) & \text{if} & \mathcal{K}(g) \text{ is infinite,}\\
\mathcal{K}_g = \overline{\mathcal{K}(g)1}, & \mathcal{K}^g = \mathcal{K}(g)0\overline{\mathcal{K}(g)1} & \text{if} & \mathcal{K}(g) \text{ is finite odd,}\\
\mathcal{K}_g = \overline{\mathcal{K}(g)0}, & \mathcal{K}^g = \mathcal{K}(g)1\overline{\mathcal{K}(g)0} & \text{if} & \mathcal{K}(g) \text{ is finite even}\\
\mathcal{I}_g(x) = \mathcal{I}(x), & \mathcal{I}^g(x) = \mathcal{I}(x) & \text{if} & \mathcal{I}(x) \text{ is infinite,}\\
\mathcal{I}_g(x) = \mathcal{I}(x)1\mathcal{K}_g, & \mathcal{I}^g(x) = \mathcal{I}(x)0\mathcal{K}_g & \text{if} & \mathcal{I}(x) \text{ is finite odd,}\\
\mathcal{I}_g(x) = \mathcal{I}(x)0\mathcal{K}_g, & \mathcal{I}^g(x) = \mathcal{I}(x)1\mathcal{K}_g & \text{if} & \mathcal{I}(x) \text{ is finite even.}
\end{array}$$

Definition 12 *For $w \in \overline{2^*}$ denote $\Sigma_w = \{u \in 2^N|(\forall i > 0)(\sigma^i(u) \preceq w)\} = L^0(w) \cap L^1(w) \cap 2^N$. Then Σ_w is a subshift. We say that w is maximal, if $(\forall i)(\sigma^i(w) \preceq w)$.*

Theorem 6 *$\mathcal{L}(I,g,\alpha_g) = \mathcal{L}(\Sigma_{\mathcal{K}^g})$ is regular iff $\mathcal{K}(g)$ is either finite or eventually periodic.*

Proof: Proposition 7 and Theorem 5.

Proposition 9 $\Sigma_{\mathcal{K}g} = \{\mathcal{I}_g(x), \mathcal{I}^g(x)|x \in I\}$.

Proof: If $x \in I$ and $i > 0$, then either $\sigma^i(\mathcal{I}_g(x)) = \mathcal{I}_g(g^i(x))$, or $\sigma^i(\mathcal{I}_g(x)) = \mathcal{I}^g(g^i(x))$. If $g^i(x) < g(c)$, $\sigma^i(\mathcal{I}_g(x)) \preceq \mathcal{K}_g$ by Proposition 6. If $g^i(x) = g(c)$, then $\sigma^i(\mathcal{I}_g(x)) = \mathcal{K}_g$. Thus $\mathcal{I}_g(x) \in \Sigma_{\mathcal{K}_g}$. For the upper itinerary is the proof similar. If $u \in \Sigma_{\mathcal{K}_g}$, then $V_u \neq 0$ by Proposition 7. If there exists $x \in V_u$, whose orbit does not contain c, then $u = \mathcal{I}^g(x) = \mathcal{I}_g(x)$. If $g^i(x) = c$, then $\sigma^{i+1}(u) = \mathcal{K}_g$, so either $u = \mathcal{I}_g(x)$, or $u = \mathcal{I}^g(x)$. $\square$

The first nonperiodic kneading sequence can be obtained by successive applications of the doubling operator. The double of $s \in 2^* - \{\lambda\}$ is $\mathcal{D}(s) = s\hat{s}$, where $\hat{s}$ is the sequence obtained from s by changing the last bit. The doubling operator can be iterated, and since $s \sqsubseteq \mathcal{D}(s)$, $\mathcal{D}^\infty(s) = \lim_{i\to\infty} \mathcal{D}^i(s)$ is well defined. In particular $\mathcal{D}^\infty(1) = 1011\ 1010...$.

Theorem 7 *A unimodal system (I, g) is regular if $\mathcal{K}_g \prec \mathcal{D}^\infty(1)$, and nonrecursive otherwise.*

Proof: If $\mathcal{K}_g \prec \mathcal{D}^\infty(1)$, then the itinerary of every point $x \in I$ is either finite or eventually periodic. Thus if α is an interval cover with the turning point among its cut points, then all $\mathcal{I}^\alpha(g(a_j))$ and $\mathcal{I}_\alpha(g(a_j))$ are eventually periodic, so $\mathcal{L}(I, g, \alpha)$ is regular by Theorem 5. If α is an arbitrary interval cover, then there exists a finer interval cover with cut point c, and we get the result by Proposition 1. On the other hand suppose that $\mathcal{K}(g) \succeq \mathcal{D}^\infty(1)$. For a sequence n_i of positive integers denote $w = 1^{n_0}(10)^{n_1}(1011)^{n_2}...(\mathcal{D}^i(1))^{n_i}....$ Then $w \in \mathcal{L}(I, g, \alpha_g)$, and there exists a point x with $\mathcal{I}(x) = w$. If the sequence n_j is not recursive, then neither is w. $\square$

Thus regularity and recursivity differentiate unimodal systems similarly as topological entropy, since the topological entropy of (I, g) is zero iff $\mathcal{K}_g \preceq \mathcal{D}^\infty(1)$ (see Milnor and Thurston [11]).

5 S-unimodal systems

Definition 13 *A unimodal system (I, g) is S-unimodal, if it has continuous third derivation, $g''(c) < 0$, $g'(a) > 1$, and $Sg(x) < 0$ for all $x \neq c$, where $Sg(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}(\frac{g''(x)}{g'(x)})^2$ is the Schwarzian derivative.*

Theorem 8 *Let (I, g) be S-unimodal system. If $\mathcal{K}(g)$ is neither finite nor periodic, then V_u has empty interior for any $u \in 2^N$. If $\mathcal{K}(g)$ is either finite or periodic, and $u \in 2^N$, then V_u has nonempty interior iff $u \in \Sigma_{\mathcal{K}_g}$ and there exists $k \geq 0$ with $\mathcal{K}(g) \sqsubseteq \sigma^k(u)$.*

See Guckenheimer [9] or Collet and Eckmann [2] for a proof.

Proposition 10 *If (I, g) is S-unimodal, then there exists a homomorphism $\psi : (\Sigma_{\mathcal{K}_g}, \sigma) \to (I, g)$, with $(\forall u \in \Sigma_{\mathcal{K}_g})(\psi(u) \in V_u)$, which maps eventually periodic points to eventually periodic points with the same period.*

Proof: If $\mathcal{K}(g)$ is neither finite nor periodic, then by Theorem 8, V_u is a singleton for each $u \in \Sigma_{\mathcal{K}_g}$, so the condition $\psi(u) \in V_u$ defines ψ uniquely, and ψ is clearly continuous. By Proposition 9, it is surjective. Suppose now that $\mathcal{K}(g) = \overline{w}$ is periodic odd, and denote $n = |w|$. Then g^n is decreasing on $V_{\overline{w}}$ and has therefore a unique g^n-fixed point $\psi(\overline{w}) \in V_{\overline{w}}$. If $\sigma^i(u) = \overline{w}$, then there is a unique $\psi(u) \in V_u$ with $g^i\psi(u) = \psi(\overline{w})$. Thus ψ is first defined on the orbit of $\overline{w}$, and then on its preimages. So defined ψ is continuous, since each u for which $\sigma^i(u) = \overline{w}$, is an isolated point. Indeed if $wwu \in \Sigma_{\mathcal{K}_g}$, then $\sigma^{2n}(wwu) = u \preceq \overline{w}$ by definition, and $\sigma^n(wwu) = wu \preceq \overline{w}$, so $u \succeq \sigma^n(\overline{w}) = \overline{w}$, since w is odd. Thus $u = \overline{w}$, and $\overline{w}$ is an isolated point of $\Sigma_{\mathcal{K}_g}$. If $\sigma^i(u) = \overline{w}$, then u is isolated too, since σ is continuous, and every point has only a finite number of preimages under σ. Finally we assume that $\mathcal{K}(g)$ is either finite or periodic even. Then $\mathcal{K}_g = \overline{w}$ is periodic even too. If $w = 0$, then $\Sigma_{\mathcal{K}_g} = \{\overline{0}, 1\overline{0}\}$, and the Proposition is trivial. If $w \neq 0$, then $10 \sqsubseteq w$, and $w^p\overline{1} \in \Sigma_{\mathcal{K}_g}$ for any p. Thus $\overline{w}$ is a limit point in this case and $V_{\overline{w}} \neq V_u$ whenever u is an initial substring of $\overline{w}$. Denote $W = \cap\{\overline{V_u - V_{\overline{w}}} | u \in 2^*, u \sqsubseteq \overline{w}\}$. Then W is a nonempty invariant set which does not contain any inner point of $V_{\overline{w}}$. Suppose that W contains both endpoints of $V_{\overline{w}}$. Then they would be unstable fixed points for g^{2n}, and therefore there would exist a stable fixed point between them. This is, however, in contradiction with $Sg < 0$. Thus W consists of exactly one periodic point with period n and we define $\psi(\overline{w}) \in W$. The construction then proceeds as in the preceding case. It follows from the construction that ψ is continuous at $\overline{w}$, and also at all its preimages. □

Theorem 9 *Let (I, g) be S-unimodal system such that $\mathcal{K}(g)$ is neither finite nor periodic. Then (I, g) is a factor of $(\Sigma_{\mathcal{K}_g}, \sigma)$.*

Proof: Theorem 8 and Proposition 10

Theorem 10 *If (I, g) is S-unimodal and $\mathcal{K}(g)$ is preperiodic, then (I, g) is a factor of a finite automaton.*

Proof: Theorems 6, 4 and 9.

Proposition 11 *Let $g : I \to I$ be a monotone (i.e. either nondecreasing or nonincreasing) continuous map on a compact real interval I. Then (I, g) is a factor of a finite automaton.*

See [10] for a proof.

Theorem 11 *If (I, g) is S-unimodal and $\mathcal{K}(g)$ is either finite or periodic, then (I, g) is a factor of a finite automaton.*

Proof: Denote $\mathcal{K}_g = \overline{w}$, $|w| = n$. Consider an alphabet $A = \{0, 1, 2, 3, 4, 5\}$, a projection $\nu : A \to \{0, 1\}$ defined by $\nu(a) = a \bmod 2$, and a subshift $\Sigma \subseteq A^N$

defined by $x \in \Sigma$ iff either $x \in \Sigma_{\overline{w}}$, or $x = uv$, where u, v satisfy following conditions: $u \in \{0,1\}^*$, $v \in \{2,...,5\}^N$, $\nu(uv) \in \Sigma_{\overline{w}}$, $w_{n-1}w_0...w_{n-2}$ is not a final segment of u, $\nu\sigma(v) = \overline{w}$, and $v_i \in \{4,5\}$ iff n divides i. Since $\Sigma_{\overline{w}}$ is regular, Σ is regular too, and there is a factorization $\varphi' : (X', f') \to (\Sigma, \sigma)$. By Proposition 10 there is a homomorphism $\psi : (\Sigma_{\overline{w}}, \sigma) \to (I, g)$. Thus we have a homomorphism $\psi\nu\varphi' : (X', f') \to (I, g)$, which is, however, not surjective. By Proposition 11 there is a factorization $\varphi'' : (X'', f'') \to (V_{\overline{w}}, g^n)$. We construct a finite automaton (X, f), where $X = X' \times X''$, $f(u', u'') = (f'(u'), f''(u''))$ if $\varphi'(u')_0 \in \{4,5\}$ (this depends only on u'_0 and u''_0), and $f(u', u'') = (f'(u'), u'')$ otherwise. Then $\sigma\nu\varphi'(u') = \overline{w}$ provided $\varphi'(u')_0 \in \{4,5\}$. Define $\varphi : X \to I$ by

$$\begin{array}{lll}
\varphi(u', u'') = \psi\varphi'(u') & \text{if} & \varphi'(u') \in 2^N \\
\varphi(u', u'') = g_{\nu\varphi'(u')_0} g^n \varphi''(u'') & \text{if} & \varphi'(u')_0 \in \{4,5\} \\
\varphi(u', u'') = g_{\nu\varphi'(u')_0} \varphi(f'(u'), u'') & \text{if} & \varphi(f'(u'), u'') \text{ has been defined}
\end{array}$$

(Here $g_i : [a, g(c)] \to V_i$, are the two inverses of g.) Then $\varphi(u', u'') \in V_{\nu\varphi'(u')}$, and φ is a surjection. If $\varphi'(u')_0 \in \{4,5\}$, then $\varphi f(u', u'') = \varphi(f'(u'), f''(u'')) = g_{\nu\varphi' f'(u')_0} \varphi((f')^2(u'), f''(u'')) = \dots\ g_{\nu\varphi'(u')_1} \cdots g_{\nu\varphi'(u')_{n-1}} \varphi((f')^n(u'), f''(u'')) = g_{\nu\varphi'(u')_1} \cdots g_{\nu\varphi'(u')_n} g^n \varphi'' f''(u'') = \varphi'' f''(u'') = g^n \varphi''(u'') = g\varphi(u', u'')$. If $\varphi'(u')$ is eventually periodic, and if $\varphi'(u')_0 \notin \{4,5\}$, then $\varphi f(u', u'') = \varphi(f'(u'), u'') = g\varphi(u', u'')$. Thus φ is a factorization. $\square$

$$\begin{tikzcd}
X' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & \Sigma \arrow[r, "\nu"] \arrow[d, "\sigma"'] & \Sigma_{\overline{w}} \arrow[r, "\psi"] \arrow[d, "\sigma"'] & I \arrow[d, "g"'] \\
X' \arrow[r, "\varphi'"] & \Sigma \arrow[r, "\nu"] & \Sigma_{\overline{w}} \arrow[r, "\psi"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
X'' \arrow[r, "\varphi''"] \arrow[d, "f''"] & V_{\overline{w}} \arrow[d, "g^n"] \\
X'' \arrow[r, "\varphi''"] & V_{\overline{w}}
\end{tikzcd}$$

Corollary 1 *An S-unimodal system with either finite or eventually periodic kneading sequence is a factor of a finite automaton. Thus S-unimodal systems with preperiodic kneading sequences are factors of finite automata, which are not regular.*

Example 2 *If $\mathcal{K}(g) = \mathcal{D}^\infty(1)$, then (I, g) is not a factor of a finite automaton.*

Proof: For any $k > 0$ there exists an open invariant neighbourhood of $\omega(c)$, which does not contain periodic points with periods less than 2^k. By Theorem 3, (I, g) is not a factor of a finite automaton (see [10] for details).

We conjecture that an S-unimodal system with aperiodic kneading sequence is not a factor of a finite automaton.

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