Language complexity of unimodal systems

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Abstract

We study the complexity of formal languages which are generated by S-unimodal systems on interval covers. We show that there exist S-unimodal systems with nonrecursive languages, those with recursive but not context-sensitive languages, and those with context-sensitive but not regular languages. We show that S-unimodal systems with regular languages include all systems with finite, periodic and preperiodic kneading sequences, but also all infinitely renormalizable systems. Finally we show that systems with zero topological entropy might be characterized by periodic languages (a subclass of regular languages), and systems with unique fixed points can be characterized by bounded periodic languages.

1 Introduction

The rich variation of dynamics encountered in the simple quadratic family $f_r(x) = rx(1-x)$ has been the source of fascination as well as deep mathematical theorems (see e.g., von Neumann & Ulam [15], May [13], Collet & Eckmann [6], de Melo & van Strien [14]). Better understanding of the quadratic family and more general unimodal families calls for classificating schemes which could evaluate their complexity. Their statistical complexity is well captured by topological entropy, which is nondecreasing with the parameter r (see de Melo & van Strien [14]).

Another approach to the complexity of unimodal systems has been pioneered by Crutchfield & Young [7] (see also Xie [18] and Friedman [8]) and it is based on the theory of formal languages and computational complexity. The decomposition of the real interval into two intervals separated by the critical point yields a language of possible itineraries. The obtained language is regular if the kneading sequence is finite or periodic (when the system has a stable periodic orbit), or when its kneading sequence is preperiodic (when the system is at a

*The work has been done in the Institut de Mathématiques de Luminy - CNRS during the fellowship of the French Ministry of Education and Research.

band merging bifurcation). Thus the language complexity is not monotonous with r: the really complex behaviours can be found only in the interior of the parameter range.

This approach could be generalized. Every closed cover of the state space of a dynamical system yields a language of its finite itineraries and the corresponding subshift of its infinite itineraries. For classification purposes, suitable test covers have to be chosen. For zero-dimensional systems, appropriate test covers seem to be clopen partitions (see Kůrka [11]). In one-dimensional case we use interval covers which consist of intervals which intersect at most in their end points. For a given class $\mathfrak{L}$ of languages we say that a dynamical system on real interval is of class $\mathfrak{L}$, if there exists a separating sequence of interval covers, each of which yields a language belonging to $\mathfrak{L}$ (here separating means that the diameter of the covers tends to 0, and every cover of the sequence refines the preceding one). Using this concept, the language complexity of circle rotations has been investigated in Blanchard & Kůrka [3].

In the present paper we investigate in similar manner unimodal systems. We show that there exist S-unimodal systems with nonrecursive languages, those with recursive but not context-sensitive languages, and also those with context-sensitive but nonregular languages (no results have been obtained for context-free languages). This has been demonstrated on S-unimodal systems with nonrecurrent critical points, where the kneading sequence can be reconstructed from the language of a sufficiently fine interval cover. The systems with regular languages include again systems with finite, periodic or preperiodic kneading sequences. However, in contrast to the results of Crutchfield & Young [7], infinitely renormalizable systems yield regular languages too. This is possible, since the cannonical cover plays no distinguished role in our approach.

We consider also two subclasses of regular languages. We say that a regular language is periodic, if it can be generated by a finite graph whose every communicating class is a cycle. This happens exactly when the corresponding subshift is countable. We say that a periodic language is bounded periodic, if moreover every communicating set is final, i.e., no edge goes out of it. This happens exactly when the correponding subshift if finite. We show that a unimodal system has periodic languages iff it has zero topological entropy, i.e., if it has finitely many periodic points (whose periods must then be powers of 2) or if it is the Feigenbaum system. Finally the system has bounded periodic languages iff it has unique fixed point.

2 Subshifts and languages

A dynamical system (X, f) is a continuous mapping $f : X \to X$, where X is a compact metric space. A homomorphism $H : (X, f) \to (Y, g)$ is a continuous map $H : X \to Y$ such that $Hf = gH$. A surjective homomorphism is called a factor map, a bijective homomorphism is called conjugacy.

A point $x \in X$ is periodic if $f^n(x) = x$ for some $n > 0$. It is eventually periodic if $f^m(x)$ is periodic for some $m \geq 0$ and it is preperiodic if it is eventually periodic but not periodic. It is aperiodic if it is not eventually periodic. A point $x \in X$ is recurrent, if for every neighbourhood U of x there exists $n > 0$ such that $f^n(x) \in U$.

A dynamical system (X, f) is equicontinuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$, if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \geq 0$. A subset $Y \subseteq X$ is an attractor if there exists a closed subset $V \subseteq X$ such that $f(V) \subset \mathrm{int}(V)$, and $Y = \cap_{n \geq 0} f^n(V)$.

A closed cover of a space X is a finite collection $\mathcal{V} = \{V_a, a \in A\}$ of its closed subsets whose union is X. If $V_a \cap V_b = 0$ for $a \neq b$, then all V_a are clopen (closed and open). In this case we say that $\mathcal{V}$ is a clopen partition. A closed cover of a real compact interval is called interval cover, if it consists of closed intervals which overlap at most in their end points. The diameter of $\mathcal{V}$ is $\mathrm{diam}(\mathcal{V}) = \max\{\mathrm{diam}(V_a); a \in A\}$. We say that $\mathcal{V}$ is finer than $\mathcal{W} = \{W_b; b \in B\}$, if there exists a function $h : A \to B$ such that $V_a \subseteq W_{h(a)}$. We say that $(\mathcal{V}_i)_{i \in \mathbb{N}}$ is a separating sequence of closed covers, if $\mathcal{V}_{i+1}$ is finer than $\mathcal{V}_i$, and $\lim_{i \to \infty} \mathrm{diam}(\mathcal{V}_i) = 0$.

If A is a finite alphabet, $n \in \mathbb{N}$, denote by A^n the set of words over A of length n, $A^* = \cup_{n \in \mathbb{N}} A^n$ the set of finite words over A, $A^{\mathbb{N}}$ the set of one-way infinite words, and $\overline{A^*} = A^* \cup A^{\mathbb{N}}$. For $u \in \overline{A^*}$, denote by $|u|$ its length ($0 \leq |u| \leq \infty$). A language over A is any subset $L \subseteq A^*$. A (one-sided) full shift is a dynamical system $(A^{\mathbb{N}}, \sigma)$ on the power space $A^{\mathbb{N}}$ equipped with the product topology, given by $\sigma(u)_i = u_{i+1}$. A subshift (over A) is any subsystem of the full shift, i.e., a dynamical system (Σ, σ), where $\Sigma \subseteq A^{\mathbb{N}}$ is a nonempty closed subset Σ of $A^{\mathbb{N}}$ which is σ-invariant, i.e. $\sigma(\Sigma) \subseteq \Sigma$ (the inclusion might be strict).

Let (X, f) be a dynamical system and let $\mathcal{V} = \{V_a; a \in A\}$ be a closed cover of X. For $u \in \overline{A^*}$ put

$$V_u = \{x \in X; (\forall i < |u|)(f^i(x) \in V_{u_i})\} = \bigcap_{i < |u|} f^{-i}(V_{u_i})$$

$$\mathcal{L}_{\mathcal{V}}(X, f) = \{u \in A^*; V_u \neq \emptyset\}, \quad \Sigma_{\mathcal{V}}(X, f) = \{u \in A^{\mathbb{N}}; V_u \neq \emptyset\}.$$

Then $\Sigma_{\mathcal{V}}(X, f)$ is a subshift and $L = \mathcal{L}_{\mathcal{V}}(X, f)$ is a right central language, i.e., it is closed under subwords (if $u \in L$ and v is a subword of u then $u \in L$) and extendable to the right (if $u \in L$ then there exists $a \in A$ with $ua \in L$). In fact the right central languages are in one-to-one correspondence with subshifts (cf. Blanchard and Hansel [3]), and for a subshift $\Sigma \subseteq A^{\mathbb{N}}$, and partition $\mathcal{V} = \{[a]; a \in A\}$, $\mathcal{L}_{\mathcal{V}}(\Sigma, \sigma)$ is exactly the language of words occurring in points of Σ.

The topological entropy of a dynamical system (X, f) on a cover $\mathcal{V}$ is

$$h_{\mathcal{V}}(X, f) = \lim_{n \to \infty} \frac{1}{n} \ln \mathrm{card}\{u \in A^n; V_u \neq \emptyset\}$$

and the topological entropy of (X,f) is $h(X,f) = \sup_{\mathcal{V}} h_{\mathcal{V}}(X,f)$ where the supremum is take over all open covers of X. If X is a zero-dimensional space and if $(\mathcal{V}_n)$ is a separating sequence of clopen partitions, then

$$h(X,f) = \lim_{n\to\infty} h_{\mathcal{V}_n}(X,f) = \lim_{n\to\infty} h(\Sigma_{\mathcal{V}_n}(X,f),\sigma)$$

3 Complexity classes of languages

We consider the language classes of Chomsky hierarchy. A language is recursive (**REC**), if it can be recognized by a (deterministic or nondeterministic) Turing machine. It is context-sensitive (**CS**), if it can be recognized by a nondeterministic Turing machine in linear space. A language is regular (**REG**) if it can be recognized by a (deterministic or nondeterministic) finite automaton.

A (finite) graph is a fourtuple $G = (V,E,s,t)$ where V is a finite set of vertices, E is a finite set of edges, and $s,t : E \to V$ are the source and target maps such that s is surjective. A path in G is a sequence $u \in \overline{E^*}$ such that $t(u_{i-1}) = s(u_i)$ for every $0 < i < |u|$. A path is simple if it does not contain any edge twice. We say that vertices $a,b \in V$ communicate, if there exists a path from a to b and a path from b to a. The communication relation is an equivalence on the set of those vertices, which communicate with themselves. The equivalence classes are called communicating sets. A communicating set is a cycle, if for every $a,b \in V$ there is exactly one simple path from a to b. A communicating set is final, if no edge leaves it. Every graph has at least one final communicating set.

A labeled graph over an alphabet A is a graph $G = (V,E,s,t)$ together with a labeling function $l : E \to A$. A labeling function extends to paths $l : \overline{E^*} \to \overline{A^*}$. The language $L_{G,l}$ and the subshift $\Sigma_{G,l}$ of a labeled graph are the sets of the labels of its finite or infinite paths respectively. A right-central language L is regular iff $L = L_{G,l}$ for some labeled graph (G,l). A general language L (not necessarily right central) is regular iff there exists a labeled graph (G,l) and an initial vertex $v_0 \in V$, such that L is exactly the set of labels of paths which start in v_0. In both cases we can assume that the graph is right-resolving. This means that for every vertex $v \in V$ and every letter $a \in A$ there exists at most one edge with source v and label a.

For dynamical purposes, two subfamilies of the family of regular languages seem to be useful. We say that a regular language is periodic (**PER**), if it can be presented by a graph whose every communicating set is a cycle. We say that a language is bounded periodic (**BPER**), if moreover every communicating set of the graph is a final cycle.

Proposition 1 *A subhift has bounded periodic languages iff it is finite.*

Proof: Let (G,l) be a labeled graph whose every communicating set is a final cycle. Let p be the least common multiple of the length of all cycles, and let

m be the maximal length of all paths outside of communicating sets. Then for every infinite path u of G, $u_i = u_{i+p}$ for all $i > m$. Thus u is determined by its initial substring of length $m+p$, so the number of infinite paths is finite, and $\Sigma_{G,l}$ is finite too. Conversely if (X,f) is a finite dynamical system, construct a graph $G = (X, X, s, f)$, where s is the identity function, with a labeling function $l : X \to X$ which is identity too. Then G has the required properties and (X,f) is conjugate to $(\Sigma_{G,l}, \sigma)$. $\Box$

Proposition 2 *A subshift with regular languages has periodic languages iff it is countable. In this case it has a nonzero finite number of periodic orbits.*

Proof: Let (G,l) be a labeled graph whose every communicating set is a cycle. Let W be the set of simple paths whose sources and targets coincide. Let U be the set of simple paths whose sources and targets lie in different communicating sets. Then W and U are finite. If x is an infinite path in G, then $x = u_0 w_0^{k_0} \dots u_{m-1} w_{m-1}^{k_{m-1}} \overline{w_m}$ for some $u_i \in U$ and $w_i \in W$. This is a countable set, so $\Sigma_{G,l}$ is countable too. It is clear that the set of periodic points is finite. Conversely suppose that $\Sigma \subset A^{\mathbb{N}}$ is a countable subshift with regular languages. Let (G,l) be a minimal right-resolving presentation for Σ (see Lind & Marcus [12]). We show that every communicating set W of G is a cycle. Indeed if not, then there exist different paths u, v, w such that $s(u) = s(v) = t(v)$ and $t(u) = t(v) = s(w)$. Since (G,l) is right resolving, $l(u) \neq l(v)$. Then any sequence $x = x_0 l(w) x_1 l(w) x_2 \dots$ where $x_i \in \{l(u), l(v)\}$, belongs to $\Sigma = \Sigma_{G,l}$. Thus Σ is not countable. $\Box$

Let A_1, A_2 be finite alphabets. A map $h : A_1 \to A_2$ extends to a monoid homomorphism $h^* : A_1^* \to A_2^*$ by $h^*(u)_i = h(u_i)$. We say that a language $L_2 \subseteq A_2^*$ is a factor of a language $L_1 \subseteq A_1^*$, if there exists a map $h : A_1 \to A_2$ such that $L_2 = h^*(L_1)$. (In the formal language theory a more general concept of λ-free homomorphism is used, see e.g., Hopcroft and Ullmann [9].) For the corresponding subshifts Σ_1, Σ_2 we get a ("block one") factor map $h^{\mathbb{N}} : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$, defined by $h^{\mathbb{N}}(u)_i = h(u_i)$.

Definition 1 *A family $\mathfrak{L}$ of languages is closed under factors, if it contains all factors of all its members. A family of languages $\mathfrak{L}$ is closed under concatenations, if for every language $L \subseteq A^*$ of class $\mathfrak{L}$, and every $n > 0$, the language $L^n = \{(u_0 \dots u_{n-1})(u_1 \dots u_n) \dots (u_{k-1} \dots u_{k+n-2}) \in (A^n)^*; u_0 \dots u_{k+n-2} \in L)\}$ belongs to $\mathfrak{L}$.*

All families of languages considered above are closed both under factors and concatenations. In the sequel, any abstract family of languages considered is supposed to have these properties.

Definition 2 *Let $\mathfrak{L}$ be a class of right central languages closed under factors. We say that a zero-dimensional system (X,f) is of class $\mathfrak{L}$, if there exists a* *separating sequence of clopen partitions $\mathcal{V}_i$ of X, such that for every i, $\mathcal{L}_{\mathcal{V}_i}(X,f)$ is in $\mathfrak{L}$. We say that a dynamical system (I,f) on a real interval is of class $\mathfrak{L}$, if there exists a separating sequence of interval covers $\mathcal{V}_i$ of X, such that for every i, $\mathcal{L}_{\mathcal{V}_i}(X,f)$ is in $\mathfrak{L}$.*

Proposition 3 *Every dynamical system (I,f) of class $\mathfrak{L}$ is a factor of a zero-dimensional system of class $\mathfrak{L}$.*

The proof works for a general dynamical system (X,f) possessing a separating sequence of closed covers: Let $\mathcal{V}_i = \{V_a; a \in A_i\}$ be an increasing sequence of closed covers, and let $h_i : A_{i+1} \to A_i$ satisfy $V_a \subseteq V_{h_i(a)}$ for every $a \in A_{i+1}$. We have factor maps $h_i^{\mathbb{N}} : (Y_{i+1},\sigma) \to (Y_i,\sigma)$, where $Y_i = \Sigma_{\mathcal{V}_i}(X,f)$, and $h_i^{\mathbb{N}}(u)_i = h(u_i)$. Let $Y = \{y \in \prod_i Y_i;\ h_i^{\mathbb{N}}(y_{i+1}) = y_i\}$ be their inverse limit, define $g : Y \to Y$ by $g(y)_i = \sigma(y_i)$, and $H : Y \to X$ by $H(y) = \cap_i V_{y_{i0}}$. Then (Y,g) is a zero-dimensional system of class $\mathfrak{L}$, and $H : (Y,g) \to (X,f)$ is a factor map.

Proposition 4 *Every dynamical system with bounded periodic languages is equicontinuous.*

Proof: Since every factor of an equicontinuous system is equicontinuous (see Auslander [1]), it suffices to prove the theorem for zero-dimensional spaces. Given $\varepsilon > 0$, let $\mathcal{V} = \{V_a; a \in A\}$ be a clopen partition of X with $\mathrm{diam}(\mathcal{V}) < \varepsilon$ such that $\mathcal{L}_{\mathcal{V}}(X,f)$ is bounded periodic. There exists δ such that whenever $d(x,y) < \delta$, then x,y belong to the same set of the partition, and therefore for all i, $d(f^i(x), f^i(y)) < \varepsilon$. □

Proposition 5 *Every dynamical system with periodic languages has zero topological entropy.*

Proof: It suffices again to prove the theorem for zero-dimensional systems. Let $(\mathcal{V}_i)_{i\in\mathbb{N}}$ be a separating sequence of clopen partitions of a zero-dimensional space X. By Bowen theorem (see Block & Coppel [4]), the topological entropy of a dynamical system is concentrated on its nonwandering set, so the topological entropy of every $(\Sigma_{\mathcal{V}_n}(X,f),\sigma)$ is zero. It follows that the topological entropy of (X,f) is zero too. □

4 Symbolic dynamics of unimodal systems

With slight modifications we follow the exposition of unimodal systems in Collet & Eckmann, Bruin [5] and de Melo & van Strien [14].

Definition 3 *An S-unimodal system is a dynamical system (I,f) on a closed interval $I = [a,b]$ with negative Schwarzian derivative (and therefore continuus third derivative) and negative second derivative such that there is exactly one critical point $c \in (a,b)$ with $f'(c) = 0$, and $f(a) = f(b) = a$.*

Let A be a finite alphabet and $u, v \in \overline{A^*}$. We write $u \sqsubseteq v$ if u is an initial segment of v, and $u \sqsubset v$ if u is a proper initial segment of v. The initial substring of a word u of length n is denoted by $u_{|n} = u_0 ... u_{n-1}$. We denote by λ the word of zero length, and put by convention $u_i = \lambda$ if $i \geq |u|$, where $|u|$ is the length of u. For binary alphabet $\mathbf{2} = \{0, 1\}$ and $u \in \mathbf{2}$ put $\widehat{u} = 1 - u$. For $u \in \mathbf{2}^*$ with $|u| = n > 0$ put $\widehat{u} = u_0 ... u_{n-2} \widehat{u_{n-1}}$.

If (I, f) is an S-unimodal system an $x \in I$, define its itinerary $\mathcal{I}(x) \in \overline{\mathbf{2}^*}$ by

$$\begin{array}{lll} \mathcal{I}(x)_i = 0 & \text{if} & f^i(x) < c, \\ \mathcal{I}(x)_i = 1 & \text{if} & f^i(x) > c \end{array}$$

for all $i < |\mathcal{I}(x)| = \min\{i \geq 0; f^i(x) = c\} \leq \infty$. The kneading sequence of (I, f) is $\mathcal{K}(f) = \mathcal{I}(f(c))$. We have always $\mathcal{I}(a) = \overline{0}$, $\mathcal{I}(c) = \lambda$, $\mathcal{I}(b) = 1\overline{0}$. For $u, v \in \overline{\mathbf{2}^*}$ define $u \prec v$ iff for $k = \max\{i; u_{|i} = v_{|i}\}$ either

$$\begin{array}{llll} u_0 + ... + u_{k-1} \bmod 2 = 0 & \text{and} & u_k \prec v_k, & \text{or} \\ u_0 + ... + u_{k-1} \bmod 2 = 1 & \text{and} & u_k \succ v_k. & \end{array}$$

Here, if u_k or v_k is λ, which happens when one of the words is a proper initial substring of the other, we put $0 \prec \lambda \prec 1$. We write also $u \overset{\prec}{\sqsubset} v$ if $u \prec v$ or $u \sqsubset v$, and $u \overset{\succ}{\sqsubset} v$ if $u \succ v$ or $u \sqsubset v$. Of course $u \preceq v$ means $u \prec v$ or $u = v$. The order $\prec$ corresponds to the order on the real line so that $\mathcal{I}(x) \prec \mathcal{I}(y)$ implies $x < y$, and $x < y$ implies $\mathcal{I}(x) \preceq \mathcal{I}(y)$.

There is the canonical interval cover $\mathcal{V}_c = \{V_i; i \in \mathbf{2}\}$ with $V_0 = [a, c]$, $V_1 = [c, b]$. Since the intervals V_u overlap, we have a weaker form of the order preserving property:

Proposition 6 *Suppose that $u, v \in \overline{\mathbf{2}^*}$, $|u| = |v|$, $u \prec v$, $x \in V_u$ and $y \in V_v$. Then $x \leq y$.*

Proof: By induction on the common length of u and v. $\Box$

An S-unimodal system has at most one (one-sided) stable periodic orbit, and this happens exactly when its kneading sequence $\mathcal{K}(f)$ is finite or periodic. If $\mathcal{K}(f)$ is infinite aperiodic (but possibly preperiodic), then $\mathcal{V}_c$ is a generator so (I, f) is a factor of $(\Sigma_{\mathcal{V}_c}(I, f), \sigma)$. This yields an upper estimate on the complexity of (I, f).

Theorem 1 *Let $\mathfrak{L}$ be a class of languages closed under factors and concatenations, and let (I, f) be an S-unimodal system whose kneading sequence is neither finite nor periodic. If $\mathcal{L}_{\mathcal{V}_c}(I, f)$, belongs to $\mathfrak{L}$, then (I, f) is of class $\mathfrak{L}$.*

Proof: For $n > 0$ put $A_n = \mathcal{L}_{\mathcal{V}_c}(I, f) \cap \mathbf{2}^n$. By Theorems II.5.4 and II.6.2 in Collet & Eckmann [6], different points have different itineraries, so $\mathcal{V}_n = \{V_u; u \in A_n\}$ is a separating sequence of interval covers. Since $\mathfrak{L}$ is closed under concatenations, $\mathcal{L}_{\mathcal{V}_n}(I, f) = \mathcal{L}^n_{\mathcal{V}}(I, f)$ belongs to $\mathfrak{L}$ for every n. $\Box$

We characterize subshifts generated by the canonical covers. For S-unimodal system (I,f) define the upper kneading sequence $\overline{\mathcal{K}}(f) \in \mathbf{2}^{\mathbb{N}}$ as the maximal $u \in \mathbf{2}^{\mathbb{N}}$ for which $f(c) \in V_u$. If the kneading sequence is infinite, then $\overline{\mathcal{K}}(f) = \mathcal{K}(f)$. If $\mathcal{K}(f) = w_0...w_{n-2}$ is finite, and $w_{n-1} = w_0 + ... + w_{n-2} + 1 \bmod 2$, then $\overline{\mathcal{K}}(f) = \overline{w_0...w_{n-2}w_{n-1}w_0...w_{n-2}\widehat{w_{n-1}}}$. For $w \in \mathbf{2}^{\mathbb{N}}$ put

$$L_w = \{u \in \mathbf{2}^*; (\forall i > 0)(\sigma^i(u) \preceq_{\sqsubset} w)\}, \quad \Sigma_w = \{u \in \mathbf{2}^{\mathbb{N}}; (\forall i > 0)(\sigma^i(u) \preceq w)\}$$

A sequence $w \in \mathbf{2}^{\mathbb{N}}$ is called maximal if $w \in \Sigma_w$. If w is maximal, then it can be reconstructed from Σ_w. Indeed, if $v \in \mathbf{2}^{\mathbb{N}}$ is the maximal sequence with the property that $0v \in \Sigma_w$, then $v = w$.

Theorem 2 *Let (I,f) be an S-unimodal system. Then $\mathcal{L}_{\mathcal{V}_c}(I,f) = L_{\overline{\mathcal{K}}(f)}$, $\Sigma_{\mathcal{V}_c}(I,f) = \Sigma_{\overline{\mathcal{K}}(f)}$.*

Proof: Suppose that $u \in \mathcal{L}_{\mathcal{V}_c}(I,f)$, $|u| = n$. There exists $x \in V_u$, so $f^i(x) \in V_{\sigma^i(u)}$. If $\sigma^i(u) \preceq_{\sqsubset} \overline{\mathcal{K}}(f)$ were not satisfied, then there would exist $j < n$ such that $w_0...w_{j-i} \prec u_i...u_j$. By Proposition 6 $f(c) \leq f^i(x)$, so $f^i(x) = f(c)$, and $f(c) \in V_{u_i...u_j}$. This is however contradiction, since $w_0...w_{j-i}$ is the maximal string of length $j-i+1$ for which $f(c) \in V_{w_0...w_{j-i}}$. Suppose that $u \in L_{\overline{\mathcal{K}}(f)}$. We proceed by induction on the length $n = |u|$. If $n = 1$, then clearly $u \in \mathcal{L}_{\mathcal{V}_c}(I,f)$. Suppose that the Proposition holds for $n-1$. Then the condition is satisfied for $\sigma(u)$, so it belongs to the language. For $i = 1$ we get $\sigma(u) \preceq_{\sqsubset} \overline{\mathcal{K}}(f)$. If $\sigma(u) \sqsubset \overline{\mathcal{K}}(f)$, then $c \in V_u$. If $\sigma(u) \prec \overline{\mathcal{K}}(f)$, then every $y \in V_{\sigma(u)}$ satisfies $y \leq f(c)$, so there exists $x \in V_u$ with $f(x) = y$. In both cases we have $u \in \Sigma_{\mathcal{V}_c}(I,f)$. The equality of subshifts follows by compactness. □

If $\mathcal{K}(f)$ is finite, then $\overline{\mathcal{K}}(f)$ is preperiodic, and there exists a unimodal system (I,g) with $\mathcal{K}(g) = \overline{\mathcal{K}}(f)$. The two systems are, however, quite different: the former has a superstable periodic orbit, while the latter is topologically transitive on a finite union of closed intervals.

We explore now the relation between the complexity of an infinite maximal sequence $w \in \mathbf{2}^{\mathbb{N}}$ and its corresponding subshift Σ_w. Every maximal w can be split in a unique manner into a sequence of blocks $w = \Delta_0\Delta_1...$ where $|\Delta_0| = 1$, and for every $k > 0$, Δ_k is the initial segment of w up to and including the first difference, so $\Delta_k = w_0...w_{j-1}\widehat{w_j}$ for some $j \geq 0$. If for some k there is no such j, then the last block is infinite $\Delta_k = w$, and the number of blocks $||w|| = k+1$ is finite. Otherwise the number of blocks is infinite, $||w|| = \infty$. If $w_0 = 0$ then $w = \overline{0}$, so the splitting is $w = 0.\overline{0}$, and $||w|| = 2$. In all other cases $\Delta_0 = 1$. For every $k > 0$ we have $\Delta_k \prec w$, so Δ_k is even and since $\Delta_0 = 1$, $\Delta_0...\Delta_k$ is odd. We show now that every block Δ_k ends between two consecutive blocks:

Proposition 7 *Let $w \in \mathbf{2}^{\mathbb{N}}$ be a maxinal sequence. Then for every $0 < k < ||w||$ there exists $Q(k) < k$ such that $\Delta_k = \Delta_0...\Delta_{Q(k)-1}\widehat{\Delta_{Q(k)}}$. Put $Q(0) = -1$. Q is called the kneading function and it completely determines the kneadnig sequence.*

Proof: Suppose that $\Delta_k = \Delta_0 ... \Delta_{p-1}\Delta$ where $0 < |\Delta| < |\Delta_p|$ for some p. Then $\Delta = w_0 ... w_{i-1}\widehat{w_i}$ for some i and since $\Delta \prec w$, Δ is even. On the other hand since $\Delta_k \prec w$, we get $\Delta_0 ... \Delta_{p-1}\Delta \prec \Delta_0 ... \Delta_{p-1} w_0 ... w_i$. Since $\Delta_0 ... \Delta_{p-1}$ is odd, we get $\Delta \succ w_0 ... w_i$, which is a contradiction. $\Box$

Example 1 (Feigenbaum) *The least kneading sequence of a unimodal system is* $0.\overline{0}$. *The corresponding unimodal system has either unique fixed point* a, *or two fixed points both less than the critical point. When the second fixed point passes the critical point, the kneading sequence becomes* $1.\overline{1}$, *and in a series of period doubling bifurcations we get systems with kneading sequences* $1.0.\overline{10}$, $1.0.11.\overline{1011}$,... *All these systtems have finite splittings and kneading function* $Q(k) = k-1$. *At the limit of these systems there is the Feigenbaum system with infinite splitting and kneading function* $Q(k) = k-1$, *so*

$$\mathcal{K}(f) = 1.0.11.1010.10111011.1011101010111010...$$

Example 2 (Fibonacci) *The Fibonacci unimodal system has kneading function* $Q(0) = -1$, $Q(1) = 0$, $Q(k) = k-2$ *for* $k > 1$, *so*

$$\mathcal{K}(f) = 1.0.0.11.101.10010.10011100.1001110110011...$$

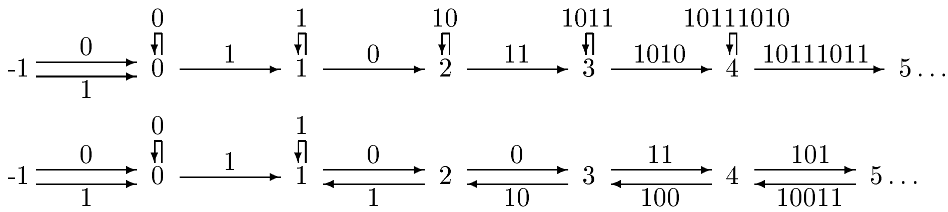
Proposition 8 *Let* $w = \Delta_0\Delta_1 ...$ *be a maximal sequence with infinite splitting* $||w|| = \infty$. *Let* G *be an infinite graph with vertex set* $V_G = \{-1, 0, 1, ...\}$, *and labeled edges* $-1 \stackrel{0}{\to} 0$, $-1 \stackrel{1}{\to} 0$, $k \stackrel{\Delta_k}{\to} k+1$, *and* $k \stackrel{\widehat{\Delta_k}}{\to} Q(k)+1$ *for* $k \geq 0$. *Then* Σ_w *consists exactly of the labels of infinite paths in* G *which start at vertex* -1.

Proof: Note that for every $u \in \mathbf{2}^{\mathbb{N}}$ there is at most one path with label u starting at -1. Let $u \in \Sigma_w$. Then u can be split in a unique way into blocks $u = \Gamma_0\Gamma_1 ...$ such that $\Gamma_0 = 10^k$ or $\Gamma_0 = 0^{k+1}$ for some $k \geq 0$, $1 \sqsubseteq \Gamma_1$, and for every $k > 0$, Γ_k is the initial sequence of w up to the last block where there is no difference encountered. In other words there exists $R(k) > 0$ such that $\Gamma_k = \Delta_0 ... \Delta_{R(k)-1}$, and $\Delta_{R(k)} \not\sqsubseteq \Gamma_{k+1}$. If for some k no difference is encountered, then the number of blocks is $k+1$ and Γ_k is infinite. We show that $\widehat{\Delta_{R(k)}} \sqsubseteq \Gamma_{k+1}$. Indeed suppose that $\Delta_0 ... \Delta_{R(k)-1}\Delta \sqsubset \Gamma_k\Gamma_{k+1} ...$ with $\Delta = w_0 ... w_{i-1}\widehat{w_i}$ for some i with $0 < |\Delta| < |\Delta_{R(k)}|$. Then $\Delta \prec w$, but $\Delta_0 ... \Delta_{R(k)-1}\Delta \prec w$, and $\Delta_0 ... \Delta_{R(k)-1}$ is odd, so $\Delta \succ w$ and we get a contradiction. It follows that $\widehat{\Delta_{R(k)}} = \Delta_0 ... \Delta_{Q(R(k))} \sqsubseteq \Gamma_{k+1}$, so there is a path

$$R(k) \stackrel{\widehat{\Delta_{R(k)}}}{\longrightarrow} Q(R(k))+1 \stackrel{\Delta_{Q(R(k))+1}}{\longrightarrow} ... \stackrel{\Delta_{R(k+1)-1}}{\longrightarrow} R(k+1)$$

and $R(k) \stackrel{\Gamma_{k+1}}{\longrightarrow} R(k+1)$. Put $R(0) = 0$. Since $-1 \stackrel{\Gamma(0)}{\to} 0 \stackrel{\Gamma(1)}{\to} R(1)$, $u = \Gamma_0\Gamma_1 .,,,$ corresponds to an infinite path in G. Conversely every infinite path u in G can be split into blocks $u = \Gamma_0\Gamma_1 ...$ such that Γ_0 is the longest path with source -1

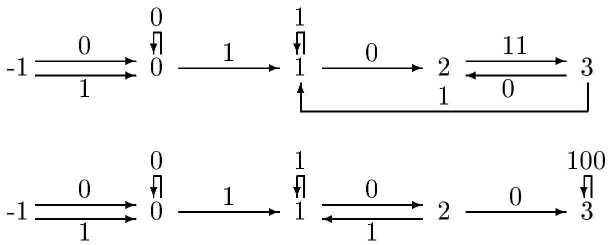
and target $0 = R(0)$. Γ_1 is the longest path with source $R(0)$, whose vertices increase. Denote by $R(1)$ its target. For every $k > 1$, Γ_k is the longest path with source $R(k-1)$, which goes first to $Q(R(k-1))$ and then continues to ever increasing vertices $Q(R(k-1))+1$,... until some $R(k)$. Suppose by contradiction that for some $i > 0$ we have $\sigma^i(u) \succ w$. Then there exists k,l and j such that $\sigma^i(u) \sqsupseteq \Gamma\Gamma_{k+1}...\Gamma_l = w_0...w_{j-1}\widehat{w_j}$, where Γ is a proper final segment of Γ_k or it is empty. But then $\Gamma_k...\Gamma_l$ is an initial segment of w, and $\Gamma_{k+1}...\Gamma_l \succ w$ and this is in contradiction with the maximality of w. $\Box$



The graphs for the Feigenbaum and Fibonacci systems

Proposition 9 *Let $w \in \mathbf{2}^{\mathbb{N}}$ be an infinite maximal sequence which is eventually periodic. Then there exists a finite labeled graph such that Σ_w consists exactly of the labels of infinite paths in G.*

Proof: The splitting of w is either infinite eventually periodic or it is finite. In the former case $w = \Delta_0...\Delta_{q-1}\overline{\Delta_q...\Delta_{q+p-1}}$ and the graph from Proposition 8 works for it. (Note that the period $\Delta_q...\Delta_{q+p-1}$ is even in this case.) For every $r > q$ we can identify vertices r,$r+p$, $r+2p$,... without changing the language. Thus we obtain a finite graph with vertex set $V_G = \{-1, 0, ...q+p\}$, and the same edges except that Δ_q leads to $q+1$ now, $q+p \stackrel{\Delta_q}{\to} q+1$. If the spliting of w is finite $w = \Delta_0...\Delta_q$, then the graph from Proposition 8 works too, but there is a unique path from q with infinite label $\Delta_q = w$. (Note that the period $\Delta_0...\Delta_{q-1}$ is odd in this case.) The same language can be obtained by a finite graph with vertex set $V_G = \{-1, 0, ..., q\}$ and the same edges, except that there is a single edge $q \stackrel{\Delta_0...\Delta_{q-1}}{\longrightarrow} q$ starting at q. $\Box$



The graphs for $\overline{101} = 1.\overline{0.11}$ and $\overline{100} = 1.0.0.\overline{100}$

Definition 4 *Let $\mathfrak{L}$ be a class of right central languages. We say that an infinite sequence $u \in A^{I\!N}$ is of class $\mathfrak{L}$, if the langauge $\{u_{|n}; n \in I\!N\}$ belongs to $\mathfrak{L}$.*

Observe that u is regular iff it is eventually periodic.

Theorem 3 *Let $w \in \mathbf{2}^{I\!N}$ be a maximal sequence. For the classes $\mathfrak{L}$ of regular, context-sensitive and recursive languages, the language L_w is of class $\mathfrak{L}$ iff w is of class $\mathfrak{L}$.*

Proof: Note that w can be reconstructed from Σ_w. If v is the maximal sequence with the property that $0v$ belongs to Σ_w, then $v = w$. Suppose that L_w is a regular language. Let G with vertex set $|G|$ be a right resolving labeled graph and $y_0 \in |G|$ be an initial vertex such that L_w is the set of labels of paths in G which start at y_0. Construct sequences $y_i \in |G|$, $x_i, v_i \in \mathbf{2}$, $i > 0$ as follows. y_1 is the vertex for which there exists an edge $y_0 \stackrel{0}{\to} y_1$, and $x_1 = 0$. Suppose that y_i, x_i have been already constructed. Then there exists the unique couple v_i, y_{i+1}, such that there exists an edge $y_i \stackrel{v_i}{\to} y_{i+1}$, and if there exists the other outcoming edge $y_i \stackrel{\widehat{v_i}}{\to} y'$, then either $x_i = 0$ and $\widehat{v_i} \prec v_i$ or $x_i = 1$ and $\widehat{v_i} \succ v_i$. Put $x_{i+1} = x_i + v_i \bmod 2$, Thus at each step x_i is the parity of $v_0 ... v_{i-1}$. Since there are only finitely many possibilities for the pairs (y_i, x_i), v is either periodic or preperiodic. Since v is the maximal sequence with the property that $0v \in \Sigma_w$, $v = w$, so w is regular. Suppose now that L_w is context-sensitive. Given $u \in \mathbf{2}^*$, construct a word $v \in \mathbf{2}^*$ with length $|u|$ which is maximal with the property that $0v$ belongs to L_w. Since L_w is context-sensitive, this construction might be done in linear space, and $v \sqsubset w$. Then it suffices to verify whether $u = v$. If L_w is recursive, then we get instead a recursive procedure for recognition of w. Conversely if w is regular, then Σ_w is regular by Proposition 9. Suppose that w is context-sensitive. We describe a Turing machine which recognizes in linear space whether given $u \in \mathbf{2}^*$ belongs to L_w. Suppose that u is written on the tape 1 of the machine. Using the machine for the recognition of $\{w_{|n}; n \geq 0\}$ we construct on tape 2 (using as many additional tapes as this machine requires) the initial substring of w of length $|u|$. Consider the alphabet $A = \{0, 1, \bullet\}$ and the word $w^\bullet = \Delta_0 \bullet \Delta_1 \bullet ... \in A^{\mathbb{N}}$ obtained from the splitting of w by inserting $\bullet$ between the blocks. Construct on tape 3 the initial substring of $w^\bullet$ obtained from tape 2. Its length is at most $2|u|$ and it can be constructed in linear space. Finally verify whether u corresponds to a path in the graph associated to w using the tapes 2 and 3. First skip u_0 and compare $\sigma(u)$ bit by bit with the string on tape 3 ignoring the bullets. If a discrepancy is encountered in the middle of the block, the input word is rejected. If a discrepancy is encountered at the end of a block Δ_k, the machine finds on tape 3 the beginning of the block Δ_k. Then it compares Δ_k with the initial string on tape 2 (ignoring the bullets) until the first discrepancy is encountered, which is at the end of block $\Delta_{Q(k)}$. Mark this position on tape 2, reposition the pointers on both tapes 2 and 3 to the beginning, and advance them simultaneously, until the marker on tape 2 is encountered. In this situation, the pointer at tape 3 is at the end of block $\Delta_{Q(k)}$, and the interrupted comparison of tapes 1 and 3 can be resumed. If no difference is found, u is accepted. If w is recursive, then we get a recursive procedure for recognition of Δ_w. $\Box$

5 Periodic languages

In conformity with Definition 2 we consider general interval covers. Every interval cover of $I = [a, b]$ is determined by an increasing sequence $a = d_0 < ... < d_n = b$, and we can write then $\mathcal{V} = \{V_i; i \in A\}$, where $A = \{0, 1, ...n-1\}$, and $V_i = [d_i, d_{i+1}]$.

Theorem 4 *An S-unimodal system (I, f) has bounded periodic languages iff it has unique fixed point.*

Proof: If the only fixed point of (I, f) is a, then $f(x) < x$ for every $x \in I$. If $\mathcal{V}$ is an interval cover, with $a \in V_0$, then there exists m such that for every $u \in \Sigma_{\mathcal{V}}(I, f)$ and every $i > m$ $u_i = 0$, so $\mathcal{L}_{\mathcal{V}}(I, f)$ is bounded periodic. On the other hand if (I, f) has a fixed point different from a then it is not equicontinuous and by Proposition 4 it does not have bounded periodic languages. $\Box$

Theorem 5 *Let (I, f) be an S-unimodal system. Then following conditions are equivalent.*
1. (I, f) has periodic languages.
2. (I, f) has zero topological entropy.
3. The period of every periodic point of (I, f) is a power of 2.
4. $\mathcal{K}(f) \preceq W = 1.0.11.1010...$ (Feigenbaum system)

Proof: $1 \Rightarrow 2$: Proposition 5.
$2 \Leftrightarrow 3$: a theorem of Misiurewicz (Proposition VIII.34 in Block & Coppel [4]).

$3 \Rightarrow 4$: Suppose by contradiction that $\mathcal{K}(f) \succ W$, let $\mathcal{K}(f) = \Delta_0\Delta_1 ...$ be its splitting and let Δ_k be the first block different from the splitting of W. Then $Q(k) = j-1$ for some $j < k$ and there exists a periodic point $\overline{\widehat{W_j}\widehat{W_j}W_j...W_{k-1}}$ of period $2^{k-1}+2^{j-1}$.

$4 \Rightarrow 1$: Suppose first $\mathcal{K}(f) \prec W$. Then (I, f) has a finite number of periodic points and for every $x \in I$, $f^n(x)$ converges to a periodic orbit (see Proposition VI.10 in Block & Coppel [4]). Denote by P the set of periodic points of (I, f). There exists a system of closed disjoint intervals $\{V_p; p \in P\}$ such that if p is stable, then $f(V_p) \subset V_{f(p)}$, and if p is unstable, then $f(V_p) \supset V_{f(p)}$. The system $\{V_p; p \in P\}$ can be extended into an interval cover $\{V_a; a \in A\}$ such that $P \subset A$ and for every $a \in A$ which is not an unstable periodic point there exists a unique $b \in A$ with $f(V_a) \subseteq V_b$. Consider a labeled graph G with vertex set $|G| = A$ and lebeled edges $a \xrightarrow{a} b$ iff $f(V_a) \cap V_b \neq \emptyset$. Then every communicating set of G is a cycle, and G presents $\mathcal{L}_{\mathcal{V}}(I, f)$. Clearly a separating sequence of interval covers can be constructed in this way. Suppose finally that (I, f) is the Feigenbaum system. Then there exists a system of periodic points $\{p_u; u \in \mathbf{2}^*\}$, a system of closed intervals $\{J_u; u \in \mathbf{2}^*\}$ such that $J_u \cap J_v = \emptyset$ for $|u| = |v|$ and $u \neq v$, $J_u \subset J_v$ for $u \sqsubset v$, $p_u \in J_u - (J_{u0} \cup J_{v1})$, there exists a length preserving function $g : \mathbf{2}^* \to \mathbf{2}^*$ such that for every n, g restricted to $\mathbf{2}^n$ is a permutation and $f(p_u) = p_{g(u)}$, $f(J_u) \subseteq J_{g(u)}$. Moreover for every n the union $\cup\{J_u; u \in \mathbf{2}^n\}$ is an attractor and finally for every $u \in \mathbf{2}^{\mathbb{N}}$ the intersection $\cap\{J_v; v \sqsubset u\}$ is a one- point set. Given $n > 0$ we construct an interval cover $\mathcal{V}_n$ as follows. Since $\cup\{J_u; u \in \mathbf{2}^n\}$ is an attractor, there exists a system of disjoint closed intervals $\{V_u; u \in \mathbf{2}^n\}$ such that $J_u \subset \mathrm{int}(V_u)$, $f(V_u) \subset \mathrm{int}V_{g(u)}$. Since for every u with $|u| < n$, p_u is a repelling periodic point, there exists a system of disjoint closed intervals $\{V_u; |u| < n\}$ such that $p_u \in \mathrm{int}(V_u)$, $f(V_u) \supset V_{f(u)}$. We can extend again $\{V_u; |u| \leq n\}$ into an interval cover $\mathcal{V}_n = \{V_a; a \in A_n\}$ so that for every $a \in A_n$ which is not an unstable periodic point there exists unique $b \in A_n$ with $f(V_a) \subset V_b$. Thus $\mathcal{L}_{\mathcal{V}_n}(I, f)$ is a periodic language. It is easy to see that $\mathcal{V}_n$ can be constructed as a separating sequence. □

6 Regular languages

By Theorems 1, 2 and 3, every S-unimodal system with preperiodic kneading sequence has regular languages. In the last section we have seen that systems with zero topological entropy have regular languages too. Besides these systems there are many others with stable periodic orbits, and at their limits there are infinitely renormalizable systems. All these systems have regular languages too.

Definition 5 (cf. de Melo & van Strien [14]) *An S-unimodal system (I, f) is renormalizable with period $n > 1$, if there exists a proper (restrictive) subinterval $J \subset I$, such that the interiors of J, $f(J)$,...,$f^{n-1}(J)$ are disjoint, $f^n(J) \subseteq J$, $f^n(\partial J) \subseteq \partial J$, $f^{n-1}(J)$ contains the critical point, and J is maximal with these* *properties. An S-unimodal system is infinitely renormalizable if it has restrictive subintervals of arbitrary large periods n.*

Let J be a proper restrictive subinterval of period n. Put $J = J_0 = [a_0, b_0]$, $J_i = f^i(J)$, $a_i = f^i(a_0)$, $b_i = f^i(b_0)$ for $i < n$. Put $w_i = \mathcal{K}(f)_i$ for $i < n-1$, and $w_{n-1} = w_0 + ... + w_{n-2} + 1 \bmod 2$, so that $w = w_0...w_{n-1}$ is odd. Then for $i < n-1$, $f : J_i \to J_{i+1}$ is increasing if $w_i = 0$, and decreasing if $w_i = 1$, $\partial J_i = \{a_i, b_i\}$, and there exists unique points $c_i \in J_i$ with $f^{n-i-1}(c_i) = c \in J_{n-1}$. If $w_{n-1} = 1$, then $f^{n-1} : J_0 \to J_{n-1}$ is increasing, so $a_{n-1} < c < b_{n-1}$, and $f^n : J_0 \to J_0$ is increasing on $[a_0, c_0]$ and decreasing on $[c_0, b_0]$. If $w_{n-1} = 0$, then $f^{n-1} : J_0 \to J_{n-1}$ is decreasing, so $b_{n-1} < c < a_{n-1}$, and $f^n : J_0 \to J_0$ is increasing on $[a_0, c_0]$ and decreasing on $[c_0, b_0]$. Thus in both cases (J_0, f^n) is S-unimodal. It follows $\mathcal{I}(a_{n-1}) = \widehat{w_{n-1}}\overline{\widehat{w}}$, $\mathcal{I}(b_{n-1}) = w_{n-1}\overline{\widehat{w}}$, $\mathcal{I}(a_0) = \overline{\widehat{w}}$, $\mathcal{I}(b_0) = w\overline{\widehat{w}}$. For the upper kneading sequences we have

$$\begin{array}{lll} \overline{\mathcal{K}}(f)_{jn+i} = w_i & \text{for} & j > 0,\ 0 \le i < n-1, \\ \overline{\mathcal{K}}(f)_{jn+n-1} = \overline{\mathcal{K}}(f^n)_j + w_{n-1} + 1 \bmod 2 & \text{for} & j \ge 0 \end{array}$$

We consider the renormalization interval cover $\mathcal{W}_r = \{W_i; i \in C\}$, where $C = \{0, c, 1\}$, and

$$\begin{array}{lllll} W_0 = [a, a_{n-1}] & W_c = [a_{n-1}, b_{n-1}] & W_1 = [b_{n-1}, b] & \text{if} & a_{n-1} < b_{n-1} \\ W_0 = [a, b_{n-1}] & W_c = [b_{n-1}, a_{n-1}] & W_1 = [a_{n-1}, b] & \text{if} & b_{n-1} < a_{n-1} \end{array}$$

Lemma 1 *Let (I, f) be renormalizable with period $n > 1$. Let $u \in C^*$ and put $k = \min\{i < |u|; u_i = c\}$ or $k = |u|$ if $u \in \mathbf{2}^*$. Then $u \in \mathcal{L}_{\mathcal{W}_r}(I, f)$ iff the following conditions are satisfied.*
1. $u_{k+nj+i+1} = w_i$ for $j \ge 0$, $0 \le i < n-1$, $k + nj + i + 1 < |u|$
2. $u_{k+nj} \in \{\widehat{w_{n-1}}, c\}$ for $j \ge 0$, $k + nj + i + 1 < |u|$,
3. There exists $v_k \in \mathbf{2}$ such that for $v = u_0...u_{k-1}v_k w_0...w_{n-2}$ (or for $v = u$ if $k = |u|$) we have $\sigma^i(v) \precsim_{\sqsubset} \mathcal{I}(a_0) = \overline{\widehat{w}}$ for $0 < i < k$.

Proof: Suppose for simplicity that $a_{n-1} < b_{n-1}$. Let $u \in \mathcal{L}_{\mathcal{W}_r}(I, f)$ and pick a point $x \in V_u$. We can assume that $f^i(x) \ne c$ for $i < |u| + n$, since otherwise there exists in the vicinity of x a point which belongs to V_u too. Put $v_k = 0$ if $f^k(x) < c$, and $v_k = 1$ if $f^k(x) > c$. Then $v \sqsubset \mathcal{I}(x)$, so the conditions 1 and 2 are satisfied. Suppose that $\sigma_i(v) \succ \mathcal{I}(a_0)$ for some $0 < i < k$. Then $f^i(x) > a_0$, and therefore $a_{n-1} < f_{i-1}(x) < b_{n-1}$, and this is a contradiction. Suppose conversely that the conditions 1,2,3 are satisfied. Assume that there exists a minimal $j < |u|$ such that $\sigma^j(v)$ is an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$. Put $x_j = a_{n-1}$ in the former case and $x_j = b_{n-1}$ in the latter case. Obviously, if $k < |u|$, then $j \le k$. Then succesively for $i = j-1, j-2, ...$ we construct $x_i \in V_{\sigma^i(v)}$ with $f(x_i) = x_{i+1}$. This is always possible since $\sigma^i(v) \prec \mathcal{I}(a_0) \preceq \mathcal{K}(f)$ as otherwise $\sigma^{i-1}(v)$ would be an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$. Thus $x_0 \in V_v$, and since $f^i(x_0) \notin (a_{n-1}, b_{n-1})$

for $0 < i < j$, $x_0 \in W_u$. If there exists no $j < |u|$ such that $\sigma^j(v)$ is an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$, then for all $i > 0$, $\sigma^i(u) \prec \mathcal{I}(a_0) \preceq \mathcal{K}(f)$, so $u \in \mathcal{L}_{\mathcal{V}_c}(I, f)$, and there exists $x \in V_u$. Again $f^i(x) \notin (a_{n-1}, b_{n-1})$ for $0 < i$, so $x \in W_u$. $\Box$

Proposition 10 *If (I, f) is renormalizable, then $\mathcal{L}_{\mathcal{W}_r}(I, f)$ is a regular language.*

Proof: By Proposition 9 there exists a finite labeled graph G for $\Sigma_{\overline{\widehat{w}}}$. Extend G as follows. Let i be a vertex such that there exists a path with source i and label $\overline{w_{n-1}w_0...w_{n-2}}$ or $\overline{\widehat{w_{n-1}}w_0...w_{n-2}}$. Add to G new vertices $v_0, ..., v_{n-1}$ and labeled edges $i \overset{c}{\to} v_0 \overset{w_0}{\to} ... \overset{w_{n-2}}{\to} v_{n-1} \overset{\widehat{w_{n-1}}}{\to} v_0$ and $v_{n-1} \overset{c}{\to} v_0$. Then the extended graph is a presentation for $\mathcal{L}_{\mathcal{W}_r}(I, f)$. $\Box$

Theorem 6 *Every S-unimodal system whose kneading sequence is either finite or periodic has regular languages.*

Proof: We can suppose that $\overline{\mathcal{K}}(f)$ is neither of λ, 1, $\overline{0}$, $\overline{1}$, as in these cases Theorem 5 applies. Let n be the period of $\mathcal{K}(f)$ if $\mathcal{K}(f)$ is periodic, and $n = |\mathcal{K}(f)| + 1$ if it is finite. Then $n > 1$ and there exists a stable or semistable periodic orbit of period n which attracts the critical point. It follows that there exists an interval J containing the critical point such that $f^n(J) \subseteq J$, and J, $f(J)$,...,$f^{n-1}(J)$ are disjoint. By Lemma 5.1. in de Melo & van Strien [14], (I, f) is renormalizable and the renormalized unimodal system (W_c, f^n) has a stable fixed point. Given $m > 0$, there exists an interval cover $\mathcal{V}_m$ of W_r, such that $\mathrm{diam}(\mathcal{V}_m) < 2^{-m}$ and $\mathcal{L}_{\mathcal{V}_m}(W_c, f^n)$ is regular. There exists p such that $\mathcal{U}_m = (\mathcal{V}_m \cup \{W_0, W_1\})^p$ has diameter less than 2^{-m}. This follows from a theorem of Misiurewicz (Thm II.5.2 in Collet & Eckmann [6]), saying that for a unimodal system with stable periodic orbit the only homtervals are those that are eventually mapped into $V_{\mathcal{K}(f)}$. By a choice of $\mathcal{V}_m$, $\mathcal{U}_{m+1}$ will be a refinement o6f $\mathcal{U}_m$ and $\mathcal{L}_{\mathcal{U}_m}(I, f)$ is regular. $\Box$

Theorem 7 *Every infinitely renormalizable S-unimodal system has regular languages.*

Proof: We have a decreasing sequence of intervals $I = J_0 \supset J_1...$, such that (J_k, f^{n_k}) is S-unimodal. Let $\mathcal{V}_k$ be an interval cover consisiting of J_k and connected components of $J_i - J_{i+1}$ for $i < k$. Then $\mathcal{L}_{\mathcal{V}_k}(I, f)$ is regular. There exists an increasing sequence m_k such that $\mathcal{V}_k^{m_k}$ is an separating sequence of intervals cover which all yield regular languages. $\Box$

Corollary 1 *Every S-unimodal system with finite or eventually periodic kneading sequence has regular languages. Every infinitely renormalizable S-unimodal system has regular languages.*

7 Nonrecurrent critical point

For S-unimodal systems whose kneadnig sequence is neither finite nor periodic, the language complexity of the kneading sequence yields an upper estimate on the language complexity of the system in question (Theorem 1). For the S-unimodal systems whose critical point is not recurrent, we have a lower estimate as well. This is a variation on Theorem 3.

Theorem 8 *Let (I, f) be an S-unimodal system with nonrecurrent critical point. Then for the classes $\mathfrak{L}$ of regular, context-sensitive and recursive languages, if (I, f) is of class $\mathfrak{L}$, then $\mathcal{K}(f)$ is of class $\mathfrak{L}$ too.*

Proof: Let $\varepsilon > 0$ be such that for all $n > 0$ we have $|f^n(c) - c| > \varepsilon$. There exists an interval cover $\mathcal{V} = \{V_i; i \in A\}$ with $\mathrm{diam}(\mathcal{V}) < \varepsilon$ such that $\mathcal{L}_{\mathcal{V}}(I, f)$ is in $\mathfrak{L}$. If $\mathcal{V}$ contains c among its end points, then it is a refinement of the canonical cover $\mathcal{V}_c$, so $\mathcal{L}_{\mathcal{V}_c}(I, f)$ is in $\mathfrak{L}$ too. If not, then there exist real numbers $a = d_0 < ... < d_k < c < d_{k+1} < ... < d_m = b$, such that $A = \{0, ..., m-1\}$, and $V_i = [d_i, d_{i+1}]$. Put $B = A - \{k\}$ and define map $\nu : B \to \mathbf{2}$ by $\nu(i) = 0$ if $i < k$ and $\nu(i) = 1$ if $i > k$. Define order $\prec$ on $\overline{B^*}$ by $u \prec v$ iff for $k = \max\{i; u_{|i} = v_{|i}\}$ either
$\nu(u_0) + ... + \nu(u_{k-1}) \bmod 2 = 0$ and $u_k \prec v_k$, or
$\nu(u_0) + ... + \nu(u_{k-1}) \bmod 2 = 1$ and $u_k \succ v_k$.
Here $0 \prec ... \prec k-1 \prec \lambda \prec k+1 \prec ... \prec m-1$. Let $w \in B^{\mathbb{N}}$ be the maximal sequence satisfying $kw \in \Sigma_{\mathcal{V}}(I, f)$. We claim $\nu(w) = \mathcal{K}(f)$. Indeed consider alphabet $C = B \cup \{k_0, k_1\}$ and the interval cover $\mathcal{W} = \{W_i; i \in C\}$ which is the common refinement of $\mathcal{V}$ and $\mathcal{V}_c$. We have again an order $\prec$ on $\overline{C^*}$ which corresponds to the order on the real line. Suppose that $v \in C^{\mathbb{N}}$ be the maximal sequence with the property that both $k_0 v$ and $k_1 v$ belong to $\Sigma_{\mathcal{W}}(I, f)$. Then $f(c) \in W_v$, and since $f^i(c)$ never visits neither V_{k_0} nor V_{k_1}, we have $v = w \in B^{\mathbb{N}}$, and $\nu(v) = \mathcal{K}(f)$. Suppose that $\mathcal{L}_{\mathcal{V}}(I, f)$ is a regular language. Let G with the vertex set $|G|$ be a right resolving labeled graph and $y_0 \in |G|$ be an initial vertex such that $\mathcal{L}_{\mathcal{V}}(I, f)$ is the set of labels of paths which start at y_0. We construct sequences $y_i \in |G|$, $v_i \in B$, $x_i \in \mathbf{2}$ as follows. y_1 is the vertex for which there exists an edge $y_0 \xrightarrow{k} y_0$, and $x_1 = 0$. Suppose that y_i, x_i have been already constructed. Then there exists the unique couple v_i, y_{i+1}, such that there exists an edge $y_i \xrightarrow{v_i} y_{i+1}$, and if $y_i \xrightarrow{v'} y'$ is another outcoming edge from y_i, with $v_i \neq v' \in B$, then either $x_i = 0$ and $v' \prec v_i$ or $x_i = 1$ and $v' \succ v_i$. Put $x_{i+1} = x_i + v_i \bmod 2$. Thus at each step, x_i is the parity of $v_0 ... v_{i-1}$ and $\mathcal{K}(f) = \nu(v_0 v_1 ...)$. Since there is only finitely many possibilities for y_i, x_i, $\mathcal{K}(f)$ is either periodic or preperiodic, and therefore regular. Suppose that $\mathcal{L}_{\mathcal{V}}(I, f)$ is context-sensitive. For given $u \in \mathbf{2}^*$ construct a word $v \in B^*$ with length $|v| = |u|$ which is maximal with the property that kv belongs to $\mathcal{L}_{\mathcal{V}}(I, f)$, and verify whether $\nu(v) = u$. If $\mathcal{L}_{\mathcal{V}}(I, f)$ is recursive, then we get instead a recursive procedure for recognition of $\mathcal{K}(f)$. $\square$

Example 3 *Every sequence* $w = 1.0.0.\Delta_3.\Delta_4...$ *with* $\Delta_i \in \{11, 101\}$ *is maximal, and its correponding unimodal system has nonrecurrent critical point. Thus there exist* S*-unimodal systems which are context-sensitive but not regular, those that are recursive but not context-sensitive and also those that are not recursive.*

For the systems with recurrent critical point which are not infinitely renormalizable, no lower estimate has been obtained. These systems include the Fibonacci systems from Example 2 whose kneadnig sequence is easily shown to be contex-sensitive. The regularity of the Fibonacci system remains an open question.

Acknowledgement I wish to thank François Blanchard for fruitful discussions and comments and to the Institut de Mathématiques de Luminy - CNRS in Marseille for its kind hospitability while this work has been done.

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