# Cellular automata with vanishing particles

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April 28, 2003

**Abstract**

We consider particle weight functions which assign weights to certain words. Given a cellular automaton, we search for particle weight functions, for which the total weights of configurations do not increase with time. In this case the weight of a shift-invariant Borel probability measure does not increase either, so we get a Ljapunov function on the space of measures. We give some conditions which ensure that the weight of a measure converges to zero. In particular we prove that this happens in the elementary cellular automaton rule number 18 and in the Gacs-Kurdyumov-Levin cellular automaton.

## 1 Introduction

The dynamics of many cellular automata can be conveniently described as dynamics of particles moving in a neutral medium. When two or more particles meet, they may change to particles of different types or some of them may disappear. In some cellular automata the density of particles steadily decreases to zero. Such a behaviour has been observed e.g., in the elementary cellular automata rules 18, 184, 62, or 54 (Boccara et al. [1] or Grasberger [9]). This looks like dynamics with an attractor. However, the subspace of particle-free configurations is usually not an attractor if the cellular automaton is regarded as a topological dynamical system.

To describe the phenomenon of vanishing particles, Kůrka and Maass introduced in [12] the concept of a limit set associated to a probability measure. In the present paper we develop a theory which enables to identify the limit sets for a class of cellular automata and initial probability measures. To do it, we generalize number-conserving cellular automata which have been recently investigated in Boccara and Fukś [2, 3], Durand and Formenti [5] and Pivato [17]. In number-conserving cellular automata, each letter has a weight, and the total weight of a configuration is conserved. Moreira et al. [16] consider monotone cellular automata in which the total weight does not increase.

In the present paper we consider particle weight functions $p : A^* \to \mathbb{N}$ which assign non-negative weights to words. Words with positive weight are called particles. The total weight of a configuration is the sum of the weights of all its particles. We say that a configuration is $p$-finite if its total weight is finite. The weight of a probability measure $\mu$ is the $\mu$-mean weight of particles. Given a cellular automaton, we search for particle weight functions for which the total weight of $p$-finite configurations does not increase with time. In this case the weight of a measure does not increase either, so we get a Ljapunov function on the space of measures. A special case is a conserving particle weight function which is conserved along trajectories.

We give some conditions which ensure that the weight of a measure converges to zero. This may happen for all initial Borel probability measures and in this case the subspace of measures with zero weight is an attractor in the space of all shift-invariant measures. This is, however, not always the case. Often there exist shift-periodic configurations which are also periodic with respect to the cellular automaton and occur arbitrarily close to the subspace of measures with zero weight. Nevertheless this subspace can still attract many nonperiodic measures, and we give some sufficient conditions when this happens. As an auxiliary concept we introduce particle identification functions. Each particle in an image configuration is identified as a follower of specified particles in the preimage configuration. The existence of a particle identification can be checked algoritmically and implies that the weight of configurations does not increase.

We show in particular that in the elementary cellular automaton rule number 18, the subspace of measures with zero weight attracts all shift-invariant connected measures. (This includes all Bernoulli measures with positive probability vector and all Markov measures with positive stochastic matrix.) We investigate also the well-known Gacs-Kurdyumov-Levin cellular automaton and show that the subspace of measures with zero weight is an attractor.

## 2 Dynamical systems, subshifts and measures

A dynamical system is a continuous mapping $F : X \to X$, where $X$ is a compact metric space. Denote by $F^n$ the $n$-th iteration of $F$. A set $Y \subseteq X$ is invariant, if $F(Y) \subseteq Y$. If $Y$ is also closed, then it is compact and $F : Y \to Y$ is a dynamical system - a subsystem of $F : X \to X$. The omega-limit of an invariant closed set $Y$ is

$$\Lambda_F(Y) = \bigcap_{n\geq 0} F^n(Y).$$

The omega-limit set is again closed an invariant. A set $V \subseteq X$ is inward, if $F$ maps its closure into its interior, i.e., $F(\overline{V}) \subseteq V^\circ$. A set $Y \subseteq X$ is an attractor, if there exists a closed inward set $V$, such that $\Lambda_F(V) = Y$. The largest attractor is the global attractor $\Lambda_F(X)$.

For a finite alphabet $A$ denote by $A^* = \cup_{n\geq 0} A^n$ the set of words over $A$ and by $A^+ = \cup_{n>0} A^n$ the set of words of nonzero length. The length of a word $u = u_0 \dots u_{n-1} \in A^n$ is denoted by $|u| = n$. The word of zero length is $\lambda$. We say that $u \in A^*$ is a subword of $v \in A^*$ ($u \sqsubseteq v$) if there exists $k$ such that $v_{k+i} = u_i$ for all $i < |u|$. We denote by $u_{[i,j)} = u_i \dots u_{j-1}$ and $u_{[i,j]} = u_i \dots u_j$. The space $A^{\mathbb{Z}}$ of $A$-configurations, or doubly infinite sequences of letters of $A$ is equipped with the metric

$$d(x, y) = 2^{-n}, \text{ where } n = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}.$$

The cylinder

$$[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l,l+|u|)} = u\}$$

of a word $u \in A^*$ located at $l \in \mathbb{Z}$ is a clopen set (closed and open). The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$. For any $u \in A^+$ we have a $\sigma$-periodic configuration $u^\infty \in A^{\mathbb{Z}}$ defined by $(u^\infty)_{k|u|+i} = u_i$ for $k \in \mathbb{Z}$ and $0 \leq i < |u|$. A subshift is any subset $\Sigma \subseteq A^{\mathbb{Z}}$, which is $\sigma$-invariant (i.e., $\sigma(\Sigma) = \Sigma$), and closed. The language

$$\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$$

of a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is the set of words occurring in the configurations of $\Sigma$. The language $\mathcal{L}(\Sigma)$ is extendable. This means that for any $u \in \mathcal{L}(\Sigma)$, any subword $v \sqsubseteq u$ belongs to $\mathcal{L}(\Sigma)$, and there exists a letter $a \in A$ such that $ua \in \mathcal{L}(\Sigma)$. Conversely, any extendable language $L \subseteq A^*$ defines a unique subshift.

A Borel probability measure on $A^{\mathbb{Z}}$ is a nonnegative map defined on the family of Borel sets of $A^{\mathbb{Z}}$, which is countably additive and assignes 1 to the full space. A Borel probability measure $\mu$ is $\sigma$-invariant, if $\mu(U) = \mu(\sigma^{-1}(U))$ for every Borel set $U$. In this case the measure of a cylinder does not depend on its location, so $\mu([u]_i) = \mu([u]_0) = \mu(u)$ for every $u \in A^*$ and $i \in \mathbb{Z}$. Thus a $\sigma$-invariant measure is determined by a map $\mu : A^* \to [0,1]$ subject to conditions

$$\sum_{a\in A} \mu(ua) = \sum_{a\in A} \mu(au) = \mu(u), \quad \mu(\lambda) = 1.$$

For any $\sigma$-periodic configuration $u^\infty$ there exists a unique $\sigma$-invariant measure $\mu_{\{u\}}$ defined by $\mu_{\{u\}}(v) = 1/|u|$ if $v \sqsubseteq u^\infty$ and $\mu_{\{u\}}(v) = 0$ otherwise. The distance of two $\sigma$-invariant measures $\mu, \nu$ is

$$\varrho(\mu,\nu) = \sum_{u\in A^*} |\mu(u) - \nu(u)| \cdot \#A^{-2|u|} \leq \frac{1}{\#A - 1}.$$

This metric yields the topology of weak convergence on the compact space $\mathfrak{M}(A^{\mathbb{Z}})$ of $\sigma$-invariant measures. We have $\lim_{n\to\infty} \mu_n = \mu$ iff $\lim_{n\to\infty} \mu_n(u) = \mu(u)$ for every $u \in A^*$. The support

$$\mathrm{supp}(\mu) = \bigcap\{Y : \ \mu(Y) = 1, Y \subseteq A^{\mathbb{Z}} \text{ is closed}\}$$

of a $\sigma$-invariant measure $\mu$ is a subshift. The support of $\mu_{\{u\}}$ is the orbit of $u^\infty$. We say that $\mu$ has full support if $\mathrm{supp}(\mu) = A^{\mathbb{Z}}$. This means that $\mu(u) > 0$ for every $u \in A^*$. If $\Sigma \subseteq A^{\mathbb{Z}}$ is a subshift, then

$$\mathfrak{M}(\Sigma) = \{\mu \in \mathfrak{M}(A^{\mathbb{Z}}) : \ \mathrm{supp}(\mu) \subseteq \Sigma\}$$

is a closed subspace of $\mathfrak{M}(A^{\mathbb{Z}})$.

**Definition 1** *A $\sigma$-invariant measure $\mu$ is connected, if there exists a positive constant $c > 0$ such that for every $u, v, w \in A^*$ we have*

$$\mu(uvw) \geq \mu(v) \cdot c^{|u|+|w|}.$$

The set of $\sigma$-invariant connected measures is dense in $\mathfrak{M}(A^{\mathbb{Z}})$. Every Bernoulli measure and every Markov measure with positive stochastic matrix is connected. Let $R = (R(a,b))_{a,b\in A}$ be a positive stochastic matrix, i.e., $R(a,b) > 0$ and for every $a \in A$, $\sum_{b\in A} R(a,b) = 1$. Then there exists a unique positive probability vector $P = (P(a))_{a\in A}$, such that $P \cdot R = P$. The Markov measure associated to $R$ is defined by

$$\mu(u_0 \dots u_{n-1}) = P(u_0) \cdot R(u_0, u_1) \cdots R(u_{n-2}, u_{n-1}).$$

Define a matrix $S$ by $S(b,a) = P(a) \cdot R(a,b)/P(b)$ and let $c$ be the minimum of all entries in the matrices $R$, $S$, and vector $P$. Then $\mu$ is a connected measure with connectivity constant $c$. If the rows of $R$ are equal, then $R(a,b) = P(b)$ and we get a special case of a Bernoulli measure. A $\sigma$-invariant measure $\mu$ is $\sigma$-ergodic, if for every Borel set $U \subseteq A^{\mathbb{Z}}$ such that $\sigma(U) \subseteq U$ either $\mu(U) = 0$ or $\mu(U) = 1$. A $\sigma$-invariant measure $\mu$ is $\sigma$-ergodic iff it is an extremal point in $\mathfrak{M}(A^{\mathbb{Z}})$, i.e., iff it cannot be written as a convex combination $\mu = a_0\mu_0 + a_1\mu_1$ where $0 < a_0 < a_0 + a_1 = 1$, and $\mu_0, \mu_1$ are $\sigma$-invariant measures. For a configuration $x \in A^{\mathbb{Z}}$ and words $u, v \in A^+$ set

$$\begin{aligned} \eta_v(u) &= \#\{i \in [0, |u| - |v|) : u_{[i,i+|v|)} = v\} \\ \eta_v(x) &= \#\{i \in \mathbb{Z} : x_{[i,i+|v|)} = v\} \\ \eta_v^n(x) &= \#\{i \in [-n, n] : x_{[i,i+|v|)} = v\} \\ X_A &= \{x \in A^{\mathbb{Z}} : \forall v \in A^+, \delta_v(x) = \lim_{n\to\infty} \frac{\eta_v^n(x)}{2n+1} \text{ exists}\}. \end{aligned}$$

It follows from the Ergodic theorem (see e.g., Denker et al. [4]), that if $\mu$ is a $\sigma$-invariant measure, then $\mu(X_A) = 1$ and for every $v \in A^*$, the measure of $v$ is the integral of its density $\delta_v$,

$$\mu(v) = \int \delta_v(x)\, d\mu.$$

# 3 Cellular automata

A cellular automaton is a continuous mapping $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, which commutes with the shift, i.e., $F \circ \sigma = \sigma \circ F$. For any cellular automaton $F$ there exists a local rule $f : A^{d+1} \to A$ such that $F(x)_i = f(x_{[i+a,i+a+d]})$ for some anticipation $a \in \mathbb{Z}$ and diameter $d \geq 0$. We extend the local rule to $f : A^{n+d} \to A^n$ by $f(u)_i = f(u_{[i,i+d]})$. If we set $f(u) = \lambda$ for $|u| \leq d$, then $f : A^* \to A^*$. For any cellular automaton $F$, the global attractor $\Lambda_F(A^{\mathbb{Z}})$ is a subshift. Its language is

$$\mathcal{L}_f(A^*) = \mathcal{L}(\Lambda_F(A^{\mathbb{Z}})) = \bigcap_{n\geq 0} f^n(A^*).$$

For any cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ we have $F(X_A) \subseteq X_A$. If $\mu$ is a $\sigma$-invariant measure, then $F\mu$ defined by $(F\mu)(U) = \mu(F^{-1}(U))$ is also a $\sigma$-invariant measure. If $F\mu = \mu$, we say that $\mu$ is $F$-invariant. The map $\mu \mapsto F\mu$ is continuous and affine, i.e., it preserves convex combinations: $F(a_0\mu_0 + a_1\mu_1) = a_0(F\mu_0) + a_1(F\mu_1)$. Thus we obtain a dynamical system $F : \mathfrak{M}(A^{\mathbb{Z}}) \to \mathfrak{M}(A^{\mathbb{Z}})$. The properties of this dynamical system do not depend on the anticipation constant $a$. Cellular automata $F$ and $F \circ \sigma^a$ give rise to the same dynamical system on the space of $\sigma$-invariant measures. If $\mu$ is $\sigma$-ergodic, then $F\mu$ is $\sigma$-ergodic too. However, the subspace of $\sigma$-ergodic measures is not compact, so there is no subsystem of $\sigma$-ergodic measures.

**Definition 2** *(Kůrka and Maass [12]) Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and $\mu$ a $\sigma$-invariant probability measure on $A^{\mathbb{Z}}$. The $\mu$-limit of $F$ is*

$$\Lambda_F(\mu) = \bigcap \{Y \subseteq A^{\mathbb{Z}} : \lim_{n\to\infty} F^n\mu(Y) = 1, Y \textit{ is closed}\}.$$

It is easy to see that $\Lambda_F(\mu) \subseteq \Lambda_F(A^{\mathbb{Z}})$ is a subsfift and its language $\mathcal{L}_F(\mu) = \mathcal{L}(\Lambda_F(\mu))$ can be characterized as $u \notin \mathcal{L}_F(\mu) \Leftrightarrow \lim_{n\to\infty}(F^n\mu)(u) = 0$. Indeed

$$u \notin \mathcal{L}_F(\mu) \quad\Leftrightarrow\quad \Lambda_F(\mu) \subseteq A^{\mathbb{Z}} \setminus [u]_0 \Leftrightarrow \lim_{n\to\infty}(F^n\mu)(A^{\mathbb{Z}} \setminus [u]_0) = 1 \Leftrightarrow \lim_{n\to\infty}(F^n\mu)(u) = 0.$$

**Proposition 1** *For a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ and a measure $\mu \in \mathfrak{M}(A^{\mathbb{Z}})$ we have*

$$\lim_{n\to\infty} \varrho(F^n\mu, \mathfrak{M}(\Sigma)) = 0 \Leftrightarrow \Lambda_F(\mu) \subseteq \Sigma.$$

**Proof:** Assume that $\lim_{n\to\infty} \varrho(F^n\mu, \mathfrak{M}(\Sigma)) = 0$ and let $u \notin \mathcal{L}(\Sigma)$. For every $\varepsilon > 0$ there exists $n$ such that for all $m \geq n$ there exists $\nu \in \mathfrak{M}(\Sigma)$ with $\varrho(F^n\mu, \nu) < \varepsilon$. Since $\nu(u) = 0$, $(F^n\mu)(u) \leq \varepsilon 2^{2|u|}$. Thus $(F^n\mu)(u) \to 0$ and $u \notin \mathcal{L}_F(\mu)$. Conversely assume that $\Lambda_F(\mu) \subseteq \Sigma$. Let $\nu$ be a limit measure of a converging subsequence $F^{n_i}\mu \to \nu$. If $u \notin \mathcal{L}(\Sigma)$, then $(F^n\mu)(u) \to 0$, so $\nu(u) = 0$ and $\nu \in \mathfrak{M}(\Sigma)$. It follows that $\lim_{n\to\infty} \varrho(F^n\mu, \mathfrak{M}(\Sigma)) = 0$. □

If a subshift $\Sigma \subseteq A^{\mathbb{Z}}$ is an attractor of $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, then $\mathfrak{M}(\Sigma)$ need not be an attractor of $F : \mathfrak{M}(A^{\mathbb{Z}}) \to \mathfrak{M}(A^{\mathbb{Z}})$. For example, for the product cellular automaton $F(x) = x_{i-1}x_ix_{i+1}$ in the binary alphabet $A = \{0, 1\}$, the zero configuration $\{0^\infty\}$ is an attractor. However, every convex combination $\mu = \varepsilon\mu_{\{0\}} + (1-\varepsilon)\mu_{\{1\}}$ is $F$-invariant, so it is not attracted to $\mu_{\{0\}}$.

Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with a local rule $f : A^{d+1} \to A$ and an anticipation $a \in \mathbb{Z}$. We say that $u \in A^+$ is a blocking word, if there exist $0 \leq l < r \leq |u|$, and for every $m$ there exists $f_m(u) \in A^{r-l}$ such that for all $x \in [u]_0$, $F^m(x)_{[l,r)} = f_m(u)$. A cellular automaton $F$ has a blocking word iff it has an equicontinuous configuration $x \in A^{\mathbb{Z}}$, i.e., if

$$\forall\varepsilon > 0, \exists\delta > 0, \forall y, (d(y, x) < \delta \Rightarrow \forall n, d(F^n(y), F^n(x)) < \varepsilon).$$

If there exists an equicontinous configuration, we say that $F$ is almost equicontinuous. For a blocking word $u$ we have $f_n(u) \sqsubseteq F^n(u^\infty)$. It follows that any blocking word $u$ has a preperiod $q_u$ and a period $p_u$ such that

$$0 \leq q_u < q_u + p_u \leq \#A^{|u|},$$

and for all $t \geq q_u$, $f_{t+p_u}(u) = f_t(u)$. Denote by $\mathcal{B}(F)$ the set of blocking words of $F$. If $u, v \in \mathcal{B}(F)$, then for any $w \in A^*$, $uwv \in \mathcal{B}(F)$ and $l(uwv) = l(u)$, $r(uwv) = |uw| + r(v)$.

**Theorem 2** *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an almost equicontinuous cellular automaton. If $\mu$ is a $\sigma$-invariant measure with full support, then*

$$\mathcal{L}_F(\mu) = \{u \in A^* : \exists v \in \mathcal{B}(F), \exists m \geq q_v, u \sqsubseteq f_m(v)\}.$$

*Moreover, if $F$ is surjective, then $\mathcal{L}_F(\mu) = A^*$, so $\Lambda_F(\mu) = A^{\mathbb{Z}}$.*

**Proof:** If $u \sqsubseteq f_m(v)$, where $v$ is a blocking word and $m \geq q_v$, then $u$ is a proper subword of $f_m(vvv)$. This shows that $\mathcal{L}_F(\mu)$ is extendable and defines a subshift. Assume that $v$ is a blocking word. For every $n \geq 0$ we have $(F^n\mu)(f_n(v)) \geq \mu\{x \in A^{\mathbb{Z}} : x_{[0,|v|)} = v\} = \mu(v)$. If $m \geq q_v$, then for any $i \geq 0$

$$(F^{m+ip_v}\mu)(f_m(v)) = (F^{m+ip_v}\mu)(f_{m+ip_v}(v)) \geq \mu(v) > 0,$$

so $f_m(v) \in \mathcal{L}_F(\mu)$. Conversely assume that $u$ does not satisfy the condition. For $m > 0$ set

$$X_m = \{x \in A^{\mathbb{Z}} : \exists i, j, k, l, \ -m \leq i < j \leq 0, \ |u| \leq k < l \leq m, \ x_{[i,j)}, x_{[k,l)} \in \mathcal{B}(F)\}.$$

If $x \in X_m$ and $t \geq (\#A)^{2m+1}$, then $F^t(x)_{[0,|u|)} \neq u$, so $(F^t\mu)(u) \leq \mu(X \setminus X_m)$. Since $\mu(X_m) \leq \mu(X_{m+1})$ and $\lim_{m\to\infty} \mu(X_m) = 1$, we get $\lim_{t\to\infty}(F^t\mu)(u) = 0$. If $F$ is surjective, then it is nonwandering (see Kůrka [10, 11]), i.e., for every $x \in A^{\mathbb{Z}}$ and every open $U \ni x$ there exists $n > 0$ such that $F^n(U) \cap U \neq \emptyset$. It follows that for every blocking word $v$, $q_v = 0$, so $\mathcal{L}_F(\mu) = A^*$. □

A cellular automaton with local rule $f : A^{d+1} \to A$ is right-permuting (left-permuting), if for any $u \in A^d$, the map $a \mapsto f(ua)$ ($a \mapsto f(au)$) from $A$ to $A$ is bijective. We say that $F$ is permuting, if it is either right-permuting or left-permuting.

**Theorem 3** *If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a permuting cellular automaton, then for every $\sigma$-invariant connected measure $\mu$, $\Lambda_F(\mu) = A^{\mathbb{Z}}$.*

**Proof:** Let $\mu$ be a connected measure with connectivity constant $c$. Assume that $F$ is right-permuting. If $f(u') = u$, then for any $v \in A^*$ there exists a unique $v' \in A^{|v|}$ such that $f(u'v') = uv$. Since $\mu(u'v') \geq \mu(u')c^{|v|}$, we have

$$(F\mu)(uv) = \sum_{u' \in f^{-1}(u)} \mu(u')c^{|v|} = (F\mu)(u)c^{|v|}.$$

Note that this holds even for $u = \lambda$ with any $u' \in A^d$. It follows that for any $u \in A^*$ and any $n$, $(F^n\mu)(u) \geq c^{|u|}$, so $u \in \mathcal{L}(\Lambda_\mu)$. □

Theorems 2 and 3 have been proved in [12] for a special case of surjective cellular automata and for a related concept of measures with complete connections.

# 4 Particle weight functions

**Definition 3** *A particle weight function over an alphabet $A$ is a map $p : A^* \to \mathbb{N}$, which satisfies the following conditions:*

*(1) $p(\lambda) = 0$.*

*(2) There exists $b_0 > 0$, such that for every $u \in A^*$, $p(u) \leq b_0$.*

*(3) There exists $b_1 > 0$, such that if $x \in A^{\mathbb{Z}}$, $p(x_{[i,j)}) > 0$, and $p(x_{[k,l)}) > 0$, then the interval $[i,j) \cap [k,l)$ has less than $b_1$ elements.*

The set of particles is the support $\mathcal{S}_p = \{u \in A^+ : \ p(u) \neq 0\}$ of $p$. The (total) weight of a word $u \in A^*$, or of a configuration $x \in A^{\mathbb{Z}}$ are defined by

$$\begin{aligned}
\eta_p(u) &= \sum_{v \in A^+} \eta_v(u) \cdot p(v) = \sum_{i=0}^{|u|-1} \sum_{j=i+1}^{|u|} p(u_{[i,j)}) \\
\eta_p(x) &= \sum_{v \in A^+} \eta_v(x) \cdot p(v) = \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} p(x_{[i,j)}) \\
\eta_p^n(x) &= \sum_{v \in A^+} \eta_v^n(x) \cdot p(v) = \sum_{i=-n}^{n} \sum_{j=i+1}^{\infty} p(x_{[i,j)})
\end{aligned}$$

We denote by $\mathcal{F}_p = \{x \in A^{\mathbb{Z}} : \ \eta_p(x) < \infty\}$ the set of $p$-finite configurations. The particle free configurations form a subshift $\Sigma_p$ with language $\mathcal{L}_p$.

$$\Sigma_p = \{x \in A^{\mathbb{Z}} : \ \eta_p(x) = 0\}, \quad \mathcal{L}_p = \{u \in A^* : \ \eta_p(u) = 0\}.$$

Any function $p : A^* \to \mathbb{N}$ with $p(\lambda) = 0$ and finite support is a particle weight function. Its subshift $\Sigma_p$ is a subshift of finite type.

**Lemma 1** *If $p : A^* \to \mathbb{N}$ is a particle weight function, then for any $u, v \in A^+$ and $x \in A^{\mathbb{Z}}$ we have*

$$\eta_p(u) \leq b_0 b_1 |u|, \quad \eta_p(uv) \leq \eta_p(u) + \eta_p(v) + b_0 b_1^2,$$
$$\eta_p(x_{[-n,n]}) \leq \eta_p^n(x) \leq \eta_p(x_{[-n,n]}) + b_0 b_1^2 \leq (2n+1+b_1) b_0 b_1.$$

**Proof:** For each $i < |u|$ there are at most $b_1$ integers $j$ such that $u_{[i,j)} \in \mathcal{S}_p$, so $\eta_p(u) \leq b_0 b_1 |u|$. Let $i_0 < |u| < j_0$ be any integers such that $(uv)_{[i_0,j_0)} \in \mathcal{S}_p$. If $0 \leq i < |u| < j$ are such that $(uv)_{[i,j)} \in \mathcal{S}_p$, then either $j \leq j_0$ and then $j - i < b_1$, or $j_0 > j$ and then $j_0 - i < b_1$. In both cases there are at most $b_1^2/2$ possibilities for $[i,j)$ and the statement follows. □

**Definition 4** *Let $p : A^* \to \mathbb{N}$ be a particle weight function. The mean weight of a configuration $x \in X_A$ and the weight of a measure $\mu \in \mathfrak{M}(A^{\mathbb{Z}})$ are defined by*

$$\begin{aligned} \delta_p(x) &= \sum_{v \in A^+} \delta_v(x) \cdot p(v) = \lim_{n \to \infty} \eta_p^n(x)/(2n+1), \\ \delta_p(\mu) &= \sum_{v \in A^+} \mu(v) \cdot p(v). \end{aligned}$$

**Proposition 4** *If $p : A^* \to \mathbb{N}$ is a particle weight function, then the weight map $\delta_p : \mathfrak{M}(A^{\mathbb{Z}}) \to [0, \infty)$ is continuous, bounded and affine. If $x \in X_A$ and $\mu \in \mathfrak{M}(A^{\mathbb{Z}})$, then*

$$\delta_p(x) \leq b_0 b_1, \quad \delta_p(\mu) = \int \delta_p(x)\, d\mu \leq b_0 b_1.$$

**Proof:** For $i \leq b_0$ set $W_i = \bigcup\{[w]_0 : |w| \geq b_1, p(w) = i\}$. Since the cylinders of $W_i$ are pairwise disjoint, we get $\delta_p(\mu)$ as a finite sum of measures of open sets

$$\delta_p(\mu) = \sum_{|v| < b_1} p(v)\mu(v) + \sum_{i=1}^{b_0} i \mu(W_i).$$

It follows that $\delta_p$ is continuous and affine on $\mathfrak{M}(A^{\mathbb{Z}})$. The integral formula follows from the Ergodic Theorem. The bound is a consequence of Lemma 1. □

**Definition 5** *Let $p : A^* \to \mathbb{N}$ be a particle weight function and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a cellular automaton.*

*(1) $F$ is $p$-nonincreasing, if for any $x \in \mathcal{F}_p$, $\eta_p(F(x)) \leq \eta_p(x)$.*

*(2) $F$ is $p$-conserving, if for any $x \in \mathcal{F}_p$, $\eta_p(F(x)) = \eta_p(x)$.*

If $F$ is $p$-nonincreasing, then the subshift $\Sigma_p$ is $F$-invariant, i.e., $F(\Sigma_p) \subseteq \Sigma_p$.

**Proposition 5** *Let $p : A^* \to \mathbb{N}$ be a particle weight function and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a cellular automaton. If $\eta_p(f(u)) \leq \eta_p(u)$ for all $u \in A^+$, then $F$ is $p$-nonincreasing.*

**Proof:** Let $f : A^{d+1} \to A$ be the local rule and assume that the anticipation is $a = 0$. If $x \in \mathcal{F}_p$ then there exists $n$ such that all particles of $x$ occur in the interval $[-n, n+d]$ and all particles of $F(x)$ occur in the interval $[-n, n]$. Then $\eta_p(F(x)) = \eta_p(F(x)_{[-n,n]}) \leq \eta_p(x_{[-n,n+d]}) = \eta_p(x)$. $\square$

**Proposition 6** *Let $p : A^* \to \mathbb{N}$ be a particle weight function, $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a cellular automaton, $x \in X_A$, and $\mu \in \mathcal{M}(A^{\mathbb{Z}})$.*

*(1) If $F$ is $p$-nonincreasing then $\delta_p(F(x)) \leq \delta_p(x)$ and $\delta_p(F\mu) \leq \delta_p(\mu)$.*

*(2) If $F$ is $p$-conserving then $\delta_p(F(x)) = \delta_p(x)$ and $\delta_p(F\mu) = \delta_p(\mu)$.*

**Proof:** (1) Let $x \in X_A$ and for $n > 0$ let $y \in \mathcal{F}_p$ be a configuration such that $y_{[-n,n]} = x_{[-n,n]}$ and $y$ contains no particles in $(-\infty, -n)$ and $(n, \infty)$. Then

$$\eta_p^{n-d}(F(x)) \leq \eta_p(F(x)_{[-n+d,n-d]}) + b_0 b_1^2 \leq \eta_p(F(y)) + b_0 b_1^2 \leq \eta_p(y) + b_0 b_1^2 \leq \eta_p^n(x) + 2b_0 b_1^2$$

and $\delta_p(F(x)) \leq \delta_p(x)$. By the Ergodic theorem,

$$\delta_p(F\mu) = \int \delta_p(F(x))d\mu \leq \int \delta_p(x)d\mu = \delta_p(\mu).$$

(2) If $F$ is $p$-conserving, then

$$\eta_p^{n-d}(F(x)) \geq \eta_p(F(x)_{[-n+d,n-d]}) \geq \eta_p(F(y)) - 2b_0 b_1^2 = \eta_p(y) - 2b_0 b_1^2 \geq \eta_p^n(x) - 2b_0 b_1^2$$

and $\delta_p(F(x)) \geq \delta_p(x)$. By the Ergodic theorem, $\delta_p(F\mu) \geq \delta_p(\mu)$. $\square$

**Definition 6** *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and let $p : A^* \to \mathbb{N}$ be a particle weight function.*

*(1) A word $v \in A^+$ is $(F, p)$-reducing, if there exists $m$ such that for every $x \in \mathcal{F}_p$,*

$$\eta_p(F^m(x)) \leq \eta_p(x) - \eta_v(x)$$

*(2) $F$ is $p$-attracting if there exists $s$ such that if $v \in A^+$ and $\eta_p(v) \geq s$, then $v$ is $(F, p)$-reducing.*

**Theorem 7** *Let $p : A^* \to \mathbb{N}$ be a particle weight function and let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a $p$-nonincreasing and $p$-attracting cellular automaton. Then for every $\sigma$-invariant measure $\mu$ we have $\Lambda_F(\mu) \subseteq \Sigma_p$. This means that $\mathcal{M}(\Sigma_p)$ is a global attractor in $\mathcal{M}(A^{\mathbb{Z}})$.*

**Proof:** Since $\delta_p(F^{n+1}\mu) \leq \delta_p(F^n\mu)$, there exists a limit $d = \lim_{n\to\infty} \delta_p(F^n\mu)$. Assume that $d > 0$ and set $\mu_n = (\mu + F\mu + \cdots + F^{n-1}\mu)/n$. Since $\mathcal{M}(A^{\mathbb{Z}})$ is compact, there exists a converging subsequence $\lim_{i\to\infty} \mu_{n_i} = \nu$. Since $|F\mu_n - \mu_n| \leq |F^n\mu - \mu|/n$, we get $F\nu = \nu$ and $\delta_p(\nu) \geq d$. There exists $v \in A^+$ such that $\nu(v) > 0$ and $\eta_p(v) \geq s$. Let $x \in X_A$ and for $n > 0$, let $y \in \mathcal{F}_p$ be a configuration such that $y_{[-n,n]} = x_{[-n,n]}$ and $y$ contains no particles in $(-\infty, -n)$ and $(n, \infty)$. Then

$$\begin{aligned}
\eta_p^{n-md}(F^m(x)) &\leq \eta_p(F^m y) + b_0 b_1^2 \leq \eta_p(y) + b_0 b_1^2 - \eta_v(y) \leq \eta_p^n(x) + 2b_0 b_1^2 - \eta_v^n(x) + |v| \\
\delta_p(F^m x) &\leq \delta_p(x) - \delta_v(x) \\
\delta_p(F^m \nu) &\leq \delta_p(\nu) - \nu(v)
\end{aligned}$$

the last inequality being obtained by integration. Since $\nu$ is $F$-invariant, this is a contradiction, so $d = 0$, and for every particle $u \in \mathcal{S}_p$, $\lim_{n\to\infty}(F^n\mu)(u) = 0$. $\square$

**Definition 7** *Let $p : A^* \to \mathbb{N}$ be a particle weight function and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a cellular automaton.*

*(1) A word $uvw \in A^+$ is a $\Sigma_p$-extension of $v \in A^+$, if for any $i, j$ such that $(uvw)_{[i,j)} \in \mathcal{S}_p$, we have $|u| \leq i < j \leq |uv|$.*

*(2) $F$ is $\Sigma_p$-permuting, if for every $m$, for every $v, v' \in A^+$ such that $f^m(v') = v$ and for every $\Sigma_p$-extension $uvw$ of $v$ there exist $u'w' \in A^+$ such that $f^m(u'v'w') = uvw$.*

*(3) $F$ is $p$-quasiattracting, if there exists $s \geq 0$, such that for every $v \in A^+$ with $\eta_p(v) \geq s$ there exists a $\Sigma_p$-extension $uvw$ of $v$ which is $(F,p)$-reducing.*

**Theorem 8** *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and let $p : A^* \to \mathbb{N}$ be a particle weight. If $F$ is $p$-nonincreasing, $p$-quasiattracting and $\Sigma_p$-permuting, then for every connected $\sigma$-invariant measure $\mu$, $\Lambda_F(\mu) \subseteq \Sigma_p$.*

**Proof:** Assume that $d = \lim_{n\to\infty} \delta_p(F^n\mu) > 0$. As in the proof of Theorem 7 we construct an $F$-invariant measure $\nu$ and a word $v$ with $\nu(v) > 0$ and $\eta_p(v) \geq s$. Since $F$ is $p$-quasiattracting, there exists a $\Sigma_p$-extension $uvw$ of $v$ and $n > 0$ such that $uvw$ is $(F,p)$-reducing. If $f^m(v') = v$, then, since $F$ is $\Sigma_p$-permuting, there exist $u', w' \in A^+$ such that $f^m(u'v'w') = uvw$ and $\mu(u'v'w') \geq \mu(v')c^{|u|+|w|}$. It follows that

$$(F^m\mu)(uvw) \geq \sum_{v' \in F^{-1}(v)} \mu(v')c^{|u|+|w|} = (F^m\mu)(v)c^{|u|+|w|}.$$

Passing to the limit we get $\nu(uvw) \geq \nu(v)c^{|u|+|w|} > 0$. Since $\delta_p(F^m\nu) < \delta_p(\nu) - \nu(uvw)$, we get a contradiction. $\square$

Assume that $p : A^s \to \mathbb{N}$ is a particle weight function with finite support and that $\Sigma_p$ is a finite subshift (consisting of $\sigma$-periodic configurations). We say that $x \in \mathcal{F}_p$ is a simple particle, if for every $n \geq 0$, $\eta_p(F^n(x)) = 1$. If $x$ is a simple particle, then there exist integers $m \geq 0$, $p > 0$, $n \in \mathbb{Z}$ such that $\sigma^n(F^m(x)) = F^{m+p}(x)$. We call $s(x) = n/p$ the speed of $x$.

# 5 Particle identification

**Proposition 9** *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and let $p : A^s \to \mathbb{N}$ be a particle weight function with finite support. Then $F$ is $p$-nonincreasing iff $\delta_p(F(x)) \leq \delta_p(x)$ for all $\sigma$-periodic $x \in A^{\mathbb{Z}}$.*

**Proof:** The only if part has been proved in Proposition 6. Assume that the condition is satisfied and let $x \in \mathcal{F}_p$ be such that all particles of $x$ are within the interval $[-n-s-d, n+s+d]$. For the $\sigma$-periodic configuration $y = (x_{[-n,n]})^\infty$ we have $\eta_p(F(x)) = \delta_p(F(y)) \leq \delta_p(y) = \eta_p(x)$. $\square$

**Proposition 10** *Let $f : A^{d+1} \to A$ be a local rule and let $p : A^s \to \mathbb{N}$ be a particle weight with finite support. If $u$ is a word of minimal length for which $\delta_p(F(u^\infty)) > \delta_p(u^\infty)$, then $|u| \leq (\#A)^{s+d}$.*

**Proof:** Set $x = u^\infty$. If $|u| > (\#A)^{s+d}$, then there exist $0 \leq i < j \leq |u|$ such that $x_{[i,i+s+d)} = x_{[j,j+s+d)}$ and $F(x)_{[i,i+s)} = F(x)_{[j,j+s)}$. Set $v = u_{[i,j)}$ and $w = u_{[j,i+|u|)}$. Then either $\delta_p(F(v^\infty)) > \delta_p(v^\infty)$ or $\delta_p(F(w^\infty)) > \delta_p(w^\infty)$. $\square$

Proposition 10 shows that it is decidable whether a given cellular automaton is $p$-nonincreasing for a given particle weight function with finite support. However the corresponding algorithm is not feasible even for moderately sized cellular automata and particle weight functions. There is a faster algorithm which is based on the identification of particles. Every preimage of a particle must contain a particle. However, if the preimage contains more particles, they need not be always uniquely assigned to the image particle. Given a preimage of a particle, i.e., a word $u$ with $\eta_p(f(u)) > 0$, a particle identification assigns to any particle $u_{[i,j)}$ a share $q_{ij}(u)$ of the weight of $f(u)$.

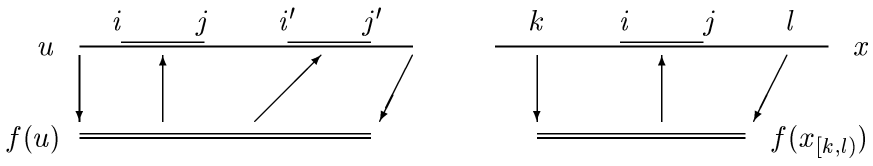



Figure 1: Particle identification

**Definition 8** *A particle identification for $f : A^{d+1} \to A$ and a particle weight function $p : A^* \to \mathbb{N}$ is a system $q = \{q_{ij}(u) : \; u \in A^+, 0 \leq i < j \leq |u|\}$ of integers such that for $u \in A^+$ and $x \in A^{\mathbb{Z}}$,*

$$\begin{aligned} \sum_{i=0}^{|u|-1} \sum_{j=i+1}^{|u|} q_{ij}(u) &= p(f(u)), \\ q(x,i,j) = \sum_{k=-\infty}^{i} \sum_{l=j}^{\infty} q_{i-k,j-k}(x_{[k,l)}) &\leq p(x_{[i,j)}). \end{aligned}$$

*We say that $f$ is $q$-conserving, if $q(x,i,j) = p(x_{[i,j)})$ for all $x \in A^{\mathbb{Z}}$ and $i < j$.*

For $k = 0$ and $u = x_{[k,l)}$ we get in particular $q_{ij}(u) \leq p(u_{[i,j)})$.

**Proposition 11** *Let $p : A^s \to \mathbb{N}$ be a particle weight function with finite support, let $f : A^{d+1} \to A$ be a local rule and $q$ an identification for $p$. Then there exists $b \leq s + d$ such that for every $x \in A^{\mathbb{Z}}$,*

$$q(x,i,j) = \sum_{k=i-b}^{i} \sum_{l=j}^{j+b} q_{i-k,j-k}(x_{[k,l)}).$$

*Thus it is decidable, whether a particle weight function with a finite support has an identification or not.*

**Proof:** If $k < i-b < j-b \leq l-b$ then $l-k > s+d$ and $q_{i-k,j-k}(u_{[k,l)}) \leq p(f(u_{[k,l)}) = 0$. Similar argument works for $k \leq i < j \leq l+b$. The conditions of Definition 8 represent a finite number of verifications, so the problem is decidable. □

**Proposition 12** *Let $F$ be a cellular automaton with local rule $f : A^{d+1} \to A$. If a particle weight function $p : A^* \to \mathbb{N}$ has an identification $q$, then $F$ is $p$-nonincreasing. If $f$ is $q$-conserving, then $F$ is $p$-conserving.*

**Proof:** Assume that $f : A^{d+1} \to A$ and $u \in A^+$. Then

$$\begin{aligned}\eta_p(f(u)) &= \sum_{k=0}^{|u|-d-1} \sum_{l=k+1}^{|u|} p(f(u_{[k,l)})) = \sum_{k=0}^{|u|-d-1} \sum_{l=k+1}^{|u|} \sum_{i=k}^{l-1} \sum_{j=i+1}^{l} q_{i-k,j-k}(u_{[k,l)}) \\ &= \sum_{i=0}^{|u|-1} \sum_{j=i+1}^{|u|} \sum_{k=0}^{i} \sum_{l=j}^{|u|} q_{i-k,j-k}(u_{[k,l)}) \leq \sum_{i=0}^{|u|-1} \sum_{j=i+1}^{|u|} p(u_{[i,j)}) = \eta_p(u).\end{aligned}$$

By Proposition 5, $F$ is $p$-nonincreasing. If $f$ is $q$-conserving, then

$$\begin{aligned}\eta_p(F(x)) &= \sum_{k=-\infty}^{\infty} \sum_{l=k+1}^{\infty} p(f(x_{[k,l)})) = \sum_{k=-\infty}^{\infty} \sum_{l=k+1}^{\infty} \sum_{i=k}^{l-1} \sum_{j=i+1}^{l} q_{i-k,j-k}(x_{[k,l)}) \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} \sum_{k=-\infty}^{i} \sum_{l=j}^{\infty} q_{i-k,j-k}(x_{[k,l)}) = \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} p(x_{[i,j)}) = \eta_p(x).\end{aligned}$$

□

The concept of a particle weight function and its identification can be generalized. If $F$ is not surjective, particle weights and their identifications can be defined only for the words $u \in f^m(A^*)$ for some $m > 0$, or even for words from the omega limit $u \in \mathcal{L}_f(A^*)$. The Theorems 7 and 8 then remain valid. However, the decidability result of Proposition 11 does not hold if $p$ and $q$ are restricted to $\mathcal{L}_f(A^*)$.

# 6 Examples

## 6.1 Just gliders

The alphabet is $A = \{L, 0, R\}$. Define $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$\begin{array}{ll} F(x)_i = R & \text{if } x_{i-1} = R,\ x_i \neq L,\ x_{i+1} \neq L, \\ F(x)_i = L & \text{if } x_{i+1} = L,\ x_i \neq R,\ x_{i-1} \neq R, \\ F(x)_i = 0 & \text{otherwise.} \end{array}$$

Thus $R$ is a glider which moves to the right with velocity 1 and $L$ moves to the left with velocity -1. When two gliders meet, they annihilate. Gilman [8] shows, that if $\mu$ is a Bernoulli measure and $\mu(R) > \mu(L)$, then $\{0, R\}^{\mathbb{Z}}$ is a $\mu$-attractor, i.e.,

$$\mu\{x \in A^{\mathbb{Z}} : \lim_{n\to\infty} F^n(x) \in \{0, R\}^{\mathbb{Z}}\} = 1.$$

```
0 R 0 R 0 0 R 0 0 0 L 0 0 L R 0 0 L R 0 R R 0 L 0 L 0 R L 0 L R 0 R L 0 0 L 0 L    5
0 0 R 0 R 0 0 R 0 L 0 0 L 0 0 R L 0 0 R 0 R 0 0 L 0 0 0 0 L 0 0 R 0 0 0 L 0 L 0    4
0 0 0 R 0 R 0 0 0 0 0 L 0 0 0 0 0 0 0 0 R 0 R L 0 0 0 0 L 0 0 0 0 R 0 L 0 L 0 0    3
0 0 0 0 R 0 R 0 0 0 L 0 0 0 0 0 0 0 0 0 0 R 0 0 0 0 0 L 0 0 0 0 0 0 0 0 L 0 0 0    2
0 0 0 0 0 R 0 R 0 L 0 0 0 0 0 0 0 0 0 0 0 0 R 0 0 0 L 0 0 0 0 0 0 0 0 L 0 0 0 0    2
0 0 0 0 0 0 R 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 R 0 L 0 0 0 0 0 0 0 0 L 0 0 0 0 0    1
0 0 0 0 0 0 0 R 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 L 0 0 0 0 0 0    1
0 0 0 0 0 0 0 0 R 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 L 0 0 0 0 0 0 0    1
```

Figure 2: Just gliders

For any Bernoulli measure with $\mu(R) = \mu(L)$, Kůrka and Maas show in [13], that $\Lambda_F(\mu) = \{0^\infty\}$. Consider a particle weight function $p(L) = p(R) = 1$. Then $F$ is clearly $p$-nonincreasing. However, $F$ is neither $p$-attracting, nor $p$-quasiattracting, so our theorems cannot be applied. Instead we consider another particle weight function defined by

$$p(R0^nL) = 1, \quad n \geq 0$$

and we obtain $\Sigma = \Sigma_p \cap \Lambda_F(A^{\mathbb{Z}}) = \{0, L\}^{\mathbb{Z}} \cup \{0, R\}^{\mathbb{Z}}$. The particles can be seen in Figure 2 with the rightmost column giving the total weight of the configurations. There exists an identification

$$q_{0,n+2}(R0^nL) = 1, \quad q_{m+1,m+k+3}(R0^mR0^kLuL) = 1$$

where $n \geq 2$, $m \geq 0$, $k \leq 1$, and $f(00u) = 0^{|u|}$. Thus $q$ identifies the leftmost particle in any preimage of a particle. It follows that $F$ is $p$-nonincreasing. We show that $F$ is $p$-attractting with $s = 2$. Assume that $x \in A^{\mathbb{Z}}$ and $q(x, i, j) = 1$. Then $x_{[i,j)}$ is a particle and it has a unique follower in $F(x)$. This is the particle $F(x)_{[k-a,l-a-d)}$, where $k \leq i$, $l \geq j$ are unique integers which make $q(x, i, j)$ positive, i.e., $q_{i-k,j-k}(x_{[k,l)}) = 1$. If $j - i \geq 4$ then the follower of $x_{[i,j)} = R0^{j-i-2}L$ is $F(x)_{[i+1,j-1)} = R0^{j-i-4}L$. If $j - i < 4$, then the follower of $x_{[i,j)}$ is $F(x)_{[i',j']}$ where $x_{[i'-1,j'+1)} = L0^{i-i'}L0^{j-i-2}R0^{j'-j}R$ (Figure 2). Assume now that $v$ has two particles and that $x \in [v]_0$ is a $p$-finite configuration. It is impossible that both particles in $v$ have followers in all iterates of $x$ since there is only a finite number of $L$ and $R$ letters between these two particles. Thus there exists $m$ such that at least one of the particles in $v$ does not have a follower in $F^m(x)$ and therefore $\eta_p(F^m(x)) \leq \eta_p(x) - \eta_v(x)$. By Theorem 7, $\mathcal{M}(\Sigma)$ is an attractor. It consists of convex combinations of $\sigma$-invariant measures on $\{0, L\}^{\mathbb{Z}}$ and $\sigma$-invariant measures on $\{0, R\}^{\mathbb{Z}}$. Moreover, every $\mu \in \mathcal{M}(\Sigma)$ is $F$-invariant, i.e., $F\mu = \mu$.

### 6.2 Gacs-Kurdyumov-Levin cellular automaton

The alphabet $A = \{0, 1\}$ is binary, $d = 6$ and $a = -3$. The local rule is given by the following table. The first applicable rule is always used.

| 0011 | 001 | 011 | 0010 | 1011 | 110 | 100 | 101 | 010 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 01 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |

We have modified slightly the original rule of Gacs et al. [6], (see also de Sá and Maes [18]), which reads

$$F(x)_i = \begin{cases} \mathbf{m}(x_{i-3}, x_{i-1}, x_i) & \text{if} \quad x_i = 0 \\ \mathbf{m}(x_i, x_{i+1}, x_{i+3}) & \text{if} \quad x_i = 1 \end{cases}$$

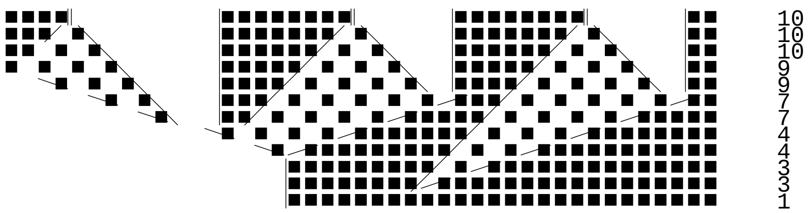


Figure 3: Gacs-Kurdyumov-Levin cellular automaton

where $\mathbf{m}$ is the majority function. Our modification homogenizes also short regions. Note that the iterates of any finite perturbation of $0^\infty$ eventually attain $0^\infty$ and likewise the iterates of any finite perturbation of $1^\infty$ eventually attain $1^\infty$.

The model has been considered as an example of a computation which is stable with respect to random errors. Suppose that after each (deterministic) step, each site of the configuration is flipped with a given probability $\varepsilon$. This deterministic and random updating define together a dynamical system in the space $\mathcal{M}(A^{\mathbb{Z}})$. It has been conjectured that for sufficiently small $\varepsilon$, the resulting dynamical system has two invariant measures, one close to $\mu_{\{0\}}$, the other close to $\mu_{\{1\}}$. The conjecture is still open but a much more complex cellular automaton with these properties has been constructed in Gacs [7]. Mitchell et al. [15] consider more general class of cellular automata which solve the density classification task, i.e., decide whether a given configuration has more zeros or ones. We show in Proposition 13 that the Gacs-Kurdyumov-Levin CA does not have this property. A configuration with large majority of 1's may ultimately attain the zero configuration $0^\infty$. There exists a particle weight function given by $p(1100) = 2$,

$$\begin{array}{lllll} p(1011) = 1, & p(1101) = 1, & p(0011) = 1, & p(0100) = 1, & p(0010) = 1, \\ s(1011) = -3, & s(1101) = -1, & s(0011) = 0, & s(0100) = 1, & s(0010) = 3, \end{array}$$

where $s$ gives the speeds of these particles. The particle 1100 has no speed as it immediately changes to a pair of particles 1101 and 0100. The algorithm of Proposition 11 shows that there exists no identification for this particle weight function. However, there does exist a particle identifcation restricted to the first image subshift with language

$$f(A^*) = \{u \in A^* : \ \forall v \in S, v \not\sqsubseteq u\},$$

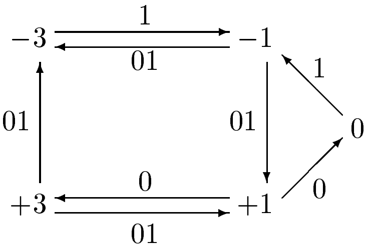
where $S = \{1100, 100100, 110110\}$ is the set of forbidden words. One such identification is given in the following table (the strokes in $u$ give positions $i, j$ in $q_{ij}(u)$).

<table>
<tr><th>image</th><th>preimages</th></tr>
<tr><td>1011</td><td>$\mathrm{xxx110}|1011|, 101|0011|\mathrm{1xx}, \mathrm{x11010}|1011|, 101|0011|011, 101010|1011|$</td></tr>
<tr><td>1101</td><td>$\mathrm{xxx1}|1101|\mathrm{0x}, \mathrm{xx10}|1101|\mathrm{0x}$</td></tr>
<tr><td>0011</td><td>$\mathrm{xx}|0010|\mathrm{11xx}, \mathrm{x}|0010|\mathrm{1011x}, |0010|101011, u|0011|v$</td></tr>
<tr><td>0100</td><td>$\mathrm{x1}|0100|\mathrm{0xxx}, \mathrm{x1}|0100|\mathrm{10xx}$</td></tr>
<tr><td>0010</td><td>$|0010|\mathrm{100xxx}, \mathrm{xx0}|0011|010, |0010|\mathrm{10100x}, 001|0011|010, |0010|101010$</td></tr>
</table>

where $u \in \{\mathrm{xx0}, 001\}$ and $v \in \{\mathrm{1xx}, 011\}$. When two particles meet they annihilate or change as follows:

$$-1, -3 \to \lambda, \quad 0, -1 \to 3, \quad 3, -3 \to 0, \quad 1, 0 \to -3, \quad 3, 1 \to \lambda.$$

It is also possible that three particles $1, 0, -1 \to \lambda$ meet simultaneously and disappear. The particles cannot occur in a configuration in an arbitrary order. Any possible sequence of their occurrence is represented by a path in the following graph. The particles are represented by their speeds and the edges between them represent one of the periodic words $0^\infty$, $1^\infty$, $(01)^\infty$, which separate them.




The path $-3 \to -1 \to 1 \to 3$ is the longest path with increasing speeds and represents a configuration $(01)^\infty 1^i (01)^j 0^k (01)^\infty$ in which no particle ever disappears. Any configuration which contains at least $s = 5$ particles contains a pair of neighbouring particles which approach one another and finally either disappear or are transformed into another particle. This means that the cellular automaton is $p$-attracting, so $\mathcal{M}(\Sigma_p)$ is an attractor. The subshift

$$\Sigma_p = \{0^\infty, 1^\infty, (01)^\infty, (10)^\infty\}$$

is finite and $\mathcal{M}(\Sigma_p)$ is a triangle consisting of convex combinations of $\mu_{\{0\}}$, $\mu_{\{1\}}$ and $\mu_{\{01\}}$. Every measure in this triangle is $F$-invariant. However, there are trajectories which go from the vicinity of any measure of $\mathcal{M}(\Sigma_p)$ to any other measure in $\mathcal{M}(\Sigma_p)$.

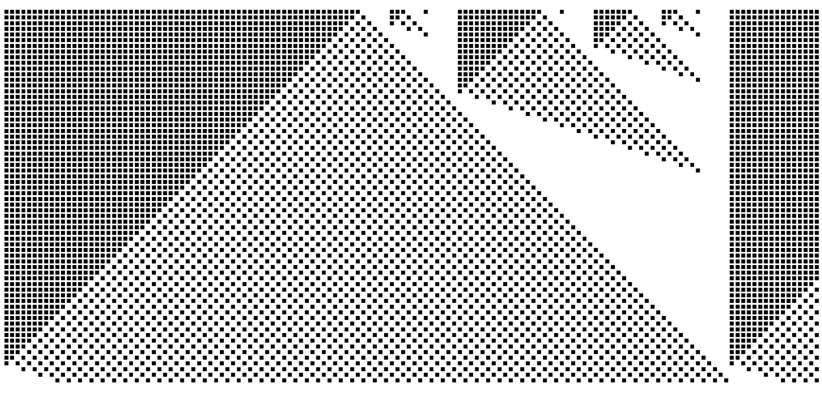


Figure 4: Extinction of majority

**Proposition 13** *For every $\varepsilon > 0$ and every $\nu_0, \nu_1 \in \mathcal{M}(\Sigma_p)$ there exists a measure $\mu$ such that $\varrho(\mu, \nu_0) < \varepsilon$ and $F^t\mu = \nu_1$ for some $t > 0$.*

**Proof:** We start with ergodic measures in $\mathcal{M}(\Sigma_p)$. For a given $n > 0$ construct a $\sigma$-periodic point $x = x^{(n)}$ with period $2^{n+1}$ as follows. First set $x_{[k2^{n+1},(k+1)2^{n+1})} = 1^{2^n-1}0^{2^n+1}$. Now replace each zero gap of length $2^n + 1$ by the word $0^{2^{n-2}+1}1^{2^{n-2}-1}0^{2^{n-1}+1}$ and repeat the procedure for remaining zero gaps of lengths $2^{n-1}+1$ and $2^{n-2}+1$. The procedure ends when the gaps of length 5 and 3 are reached (see Figure 4 with $n=6$). Let $\mu_n$ be the measure on the $\sigma$-orbit of $x^{(n)}$. Then $\lim_{n\to\infty}\mu_n = \mu_{\{1\}}$ but for every $n$, $F^t\mu_n = \mu_{\{0\}}$, where $t = 2^{n+1}-3$. We now modify the construction by setting $x_{[k(2^{n+1}+2),(k+1)(2^{n+1}+2))} = 1^{2^n+1}0^{2^n+1}$ and continue as in the preceding case, i.e., we replace successively each zero gap of length $2^m+1$ by the word $0^{2^{m-2}+1}1^{2^{m-2}-1}0^{2^{m-1}+1}$. Then $x = x^{(n)}$ has period $2^{n+1}+2$. For the corresponding measure $\mu_n$ we have again $\lim_{n\to\infty}\mu_n = \mu_{\{1\}}$, but $F^t\mu_n = \mu_{\{01\}}$, where $t = 2^n - 1$. From symmetry we get also trajectories from the vicinity of $\mu_{\{0\}}$ to $\mu_{\{1\}}$ or to $\mu_{\{01\}}$. It is easy to construct trajectories from the vicinity of $\mu_{\{01\}}$ to either $\mu_{\{0\}}$ or $\mu_{\{1\}}$. Using convex combinations we construct trajectories from the vicinity of any measure of $\mathcal{M}(\Sigma_p)$ to any other measure in $\mathcal{M}(\Sigma_p)$. $\square$

### 6.3 Elementary cellular automaton rule 18

The alphabet is $A = \{0,1\}$ and $F(x)_i = 1$ iff $x_{[i-1,i+1]} \in \{001, 100\}$. Consider the particle weight function given by

$$p(10^{2m}1) = 1, \quad m \geq 0.$$

We show that $F$ is $p$-nonincreasing, $\Sigma_p$-permuting and $p$-quasiattracting, so for any shift-invariant connected measure $\mu$, $\lim_{n\to\infty} F^n\mu \in \mathcal{M}(\Sigma_p)$.

**Proof:** We have $\Sigma_p = \{x \in A^{\mathbb{Z}} : \forall i, x_{2i} = 0\} \cup \{x \in A^{\mathbb{Z}} : \forall i, x_{2i+1} = 0\}$. If $x_{2i} = 0$ for all $i \in \mathbb{Z}$, then

$$F(x)_{2i+1} = 0, \quad F(x)_{2i} = x_{2i-1} + x_{2i+1} \bmod 2,$$

and analogously if $x_{2i+1} = 0$. It follows that $F$ is $\Sigma_p$-permuting. The only preimage of 11 is 1001 which is a particle too. If $n > 0$, any preimage of $10^{2n}1$ must contain either $10^{2n+2}1$ or $00(10)^i11(01)^j00$ with $i + j = n - 1$ which again contains a particle. Thus $F$ is $p$-nonincreasing. In fact there exists a particle identification defined by $q_{0,2n+2}(10^{2n}1) = 1$, $q_{2i+2,2i+4}(00(10)^i11(01)^j00) = 1$. However, $F$ is not $p$-attracting so Theorem 7 cannot be used. To apply Theorem 8 we show that $F$ is $p$-quasiattracting.

```
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     0 0 0 0 1 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1     0 0 0 1 0 0 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     0 0 1 0 1 0 0 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1     0 1 0 0 0 1 1 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     0 0 1 0 1 0 0 0 0 0 0 1 0 1 0 0 1 0 1 0 0 0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1     0 1 0 0 0 1 0 0 0 0 1 0 0 0 1 1 0 0 0 1 0 1
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     0 0 1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 1 0 0 0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1     0 1 0 0 0 0 0 1 1 0 0 0 0 0 1 1 0 0 0 1 0 1
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     1 0 1 0 0 0 1 0 0 1 0 0 0 1 0 0 1 0 1 0 0 0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 0 0 0 1 0 1     0 0 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 0 0 1 0 1
1 0 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 0 1 0 0 0     1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0
```

Figure 5: Elementary cellular automaton rule 18

Consider a $\Sigma$-finite configuration $x$ with at least $s = 2$ particles and let $l, r \in \mathbb{Z}$, $n, m \geq 0$ be such that $x_{[l-n,l+n+1]} = 10^{2n}1$ is its leftmost particle and $x_{[r-m-1,r+m]} = 10^{2m}1$ is its rightmost particle. We show that we can change $x$ outside the particle interval $[l-n, r+m]$ so that eventually all but one particles vanish. Call $l = l(x)$ the left boundary of $x$ and $r = r(x)$ the right boundary of $x$. Set $l_t = l(F^t(x))$ and $r_t = r(F^t(x))$ provided there are still particles at time

$t$. If $n > 0$ and $x_{[l-n,l+n+1]} = 10^{2n}1$, then $F(x)_{[l-n+1,l+n]} = 10^{2n-2}1$, so the left boundary does not move, i.e., $l(F(x)) = l(x)$. Similarly, if $m > 0$ then $r(F(x)) = r(x)$. Assume now that $m = 0$ and $x_{[r-1,r]} = 11$. If there exists $i$ such that $x_{[r-2i-2,r]} = 11(01)^i1$, then $x_{[r-1,r]}$ annihilates with its left neighbour particle $x_{[r-2i-2,r-2i-1]} = 11$. If $x_{[r-1,r]}$ does not annihilate with its left neighbour, then the right boundary can move to the left or to the right. If $x_{[r-2i-3,r+2j+2]} = 00(10)^i11(01)^j00$, then $F(x)_{[r-2i-2,r+2j+1]} = 10^{2i+2j+2}1$, so $r(F(x)) = r(x) + j - i$. Since $F$ is $\Sigma_p$-permuting, we can modify the configuration $x$ in the interval $(r+m,\infty)$, so that whenever $F^t(x)_{[r_t-1,r_t]} = 11$, then $F^t(x)_{[r_t-1,r_t+2]} = 1100$ and we get $r_{t+1} \leq r_t$ for all $t$. Similarly we can modify $x$ in the interval $(-\infty, l-n)$, so that $l_{t+1}(x) \geq l_t(x)$ for all $t$. It is possible, however, that ultimately we get a $F$-periodic configuration. Assume that for five consecutive steps, $l_t$ does not move, so $x_{[l,l+1]} = 11$ and $l_t(x)) = l(x)$ for $t \leq 5$ (Figure 5 left). It follows that $x_{[l-6,l+7]} = 10100011000101$. We now modify further the left part to $x_{[l-6,l+7]} = 00001011000101$ and get $l(F(x)) = l(x)-1$, but $l(F^4(x)) = l(x)+1$ (Figure 5 right). Thus we can always advance the left boundary at least one position to the right until it merges with its right neighbouring particle. □

### 6.4 Random walk

We consider now a variant of a cellular automaton analyzed in Kůrka and Maass [12]. Left and right-going particles travel through white and grey regions without interactions. A boundary between two regions changes according to the particles present at this boundary. Thus the boundaries between regions shift randomly to the left or right. The random variables which determine the direction of these shifts are provided by the left shift and right shift, which go in the background. The alphabet is $A = \{0,1\}^3$ and the CA $F = \langle F_u, F_v, F_w\rangle$ is given by $F_v(u,v,w)_i = v_{i-1}$, $F_w(u,v,w)_i = w_{i+1}$, and

$$F_u(u,v,w)_i \neq u_i \quad \text{if} \quad \begin{array}{l} u_{i-1} \neq u_i \text{ and } v_{i-1}+v_i \neq w_{i-1}+w_i, \quad \text{or} \\ u_i \neq u_{i+1} \text{ and } v_i + v_{i+1} = w_i + w_{i+1} \end{array}$$

Here the addition is modulo 2. Consider the particle weight function $p : A^2 \to \{0,1\}$ with support

$$\mathcal{S}_p = \{(u,v,w) \in A^2 :\ u_0 \neq u_1\}$$

Then $F$ is $p$-nonincreasing but not $p$-attracting. While $F$ is $p$-quasiattracting, it is not $\Sigma_p$-permuting, so Theorem 8 cannot be applied. Nevertheless the statement of Theorem 8 holds even in this case. This follows from the fact that the measures in the second and third component are preserved. If $\pi_i : A^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ are the projections, where $i = 0,1,2$, then for $i = 1,2$ we have $\pi_i F^n \mu = \pi_i \mu$ for any $n$. As in the proof of Theorem 8 we construct a measure $\nu$ such that $F\nu = \nu$ and there exists a word $v \in A^+$ which has at least two particles and $\nu(v) > 0$. There exist words $u, w$ such that $\nu(uvw) > 0$ and the two particles in $v$ annihilate some time $m$.

**Acknowledgement.** A part of this paper has been written during my stay at the Universidad de Chile. I thank Alejandro Maass for inspiring discussions and acknowledge the support of Millennium Nucleus in Information and Randomness, Programa Iniciativa Cientifica Milenio P01-005.

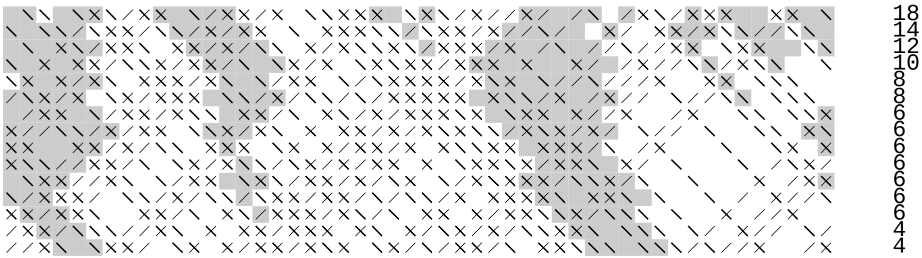



Figure 6: Random walk

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