Zero-dimensional dynamical systems and their languages

Petr Kůrka

Faculty of Mathematics and Physics, Charles University in Prague
Malostranské náměstí 25, 118 00 Praha 1, Czechia

Abstract

We measure the complexity of dynamical systems on zero-dimensional compact metric spaces by the complexity of languages which these systems generate on clopen partitions of the state space, i.e. by the complexity of their factor subshifts. We study especially dynamical systems with regular languages, and two its subclasses: dynamical systems with periodic and bounded periodic languages. We relate these language properties to other topological properties like equicontinuity, finite attractors and shadowing property. We show that in every class considered there exists a universal dynamical system, such that every dynamical system in the class is its factor. Universal systems are not unique, but in every class there exists the smallest universal system.

1 Introduction

The behaviour of many a dynamical system is elucidated, when a symbolic model for it is found. In this case, the dynamical system in question is shown to be a factor of a symbolic system, usually a one-sided or a two-sided subshift. This method works in general. According to a generalized Alexandrov Theorem (see e.g. Balcar and Simon [2]), every dynamical system on a compact metric space is a factor of some dynamical system on discontinuum, which is a zero-dimensional perfect compact metric space. For this reason, dynamical systems on discontinuum are quite diversified. Besides subshifts they include for example cellular automata and adding machines (see Buescu and Stewart [5]). In general, they can be represented as automata networks, i.e. countable systems of interconnected finite automata (see e.g. Garzon [6]).

In the present paper we introduce a classification of dynamical systems on zero-dimensional spaces using the concepts of formal language theory. Suppose that we observe the development of a dynamical system in time, but we are unable to distinguish points which are too close to each other. We partition the state space into a finite number of sets, and we are able to distinguish only in which set of the partition the system finds itself. In other words, we have an observable, i.e. a function defined on the state space with finite range, which is regarded as an alphabet or a symbol set. It is natural to assume that this observable is continuous, i.e. that the sets of the partition are clopen (closed and open). Every trajectory of the system then generates a sequence of symbols, and the set of all these sequences is a language.

Given any family $\mathfrak{L}$ of languages (e.g. regular or recursive), we say that a dynamical system has languages of class $\mathfrak{L}$ (or that it is of class $\mathfrak{L}$), if every clopen partition of its state space generates a language in $\mathfrak{L}$. The languages obtained are right central, i.e. closed to subwords and extendable to the right. Right central languages are in one-to-one correspondence with one-sided subshifts, and the corresponding subshifts are factors of the dynamical system in question. Thus the classification is based on the complexity of one-sided subshifts, which can be obtained as factors of a dynamical system in question.

We relate this language complexity of dynamical systems to their topological properties using the concepts of topological entropy, equicontinuity, attractors, shadowing property, etc. In particular, we consider the class of regular languages and two its subclasses. A language is regular, if it can be recognized by a finite automaton. Two-sided subshifts with regular languages have been introduced in Weiss [14] under the name of sofic systems. We say that a regular language is periodic, if its corresponding subshift is countable, and it is bounded periodic, if its corresponding subshift is finite. We show that a dynamical system has bounded periodic languages iff it is equicontinuous, every system with periodic languages has at least one equicontinuity point, and every system with regular languages has the shadowing property. Topological properties of cellular automata with bounded periodic and regular languages have been considered in Kůrka [12]. A generalization of the language classification to dynamical systems on arbitrary compact metric spaces has been considered in Kůrka [10, 11].

In topological dynamics a dynamical system is called universal, if every dynamical system is its factor. This means that it is the most complex dynamical system imaginable, since every other system might be constructed from it by gluing together some states. There exists a universal minimal dynamical system, i.e. a minimal dynamical system on a compact Hausdorff space, such that every minimal dynamical system on compact Hausdorff space is its factor (see Auslander [1]). We say that a dynamical system is universal for a family of languages $\mathfrak{L}$, if it is of class $\mathfrak{L}$, and every dynamical system of class $\mathfrak{L}$ is its factor. We give conditions on a family of languages $\mathfrak{L}$ which guarantee the existence of a universal dynamical system of class $\mathfrak{L}$. Universal systems are not unique, but under these conditions there exists the unique smallest universal system. In particular, the smallest universal systems exist for the classes of bounded periodic, periodic and regular languages. There exists also the smallest universal system for the family of transitive periodic languages, which is just the adding machine with infinite multiplicities.

2 Zero-dimensional spaces and systems

A compact metric space X is zero-dimensional, if for every two different points $x, y \in X$ there exist disjoint clopen (closed and open) sets $x \in U$ and $y \in V$. A space is perfect if it has no isolated points. Zero-dimensional spaces are usually constructed as product or inverse limit spaces. Denote $\mathbb{N} = \{0, 1, ...\}$ the set of non-negative integers. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of finite alphabets, and suppose that every A_i has at least two elements. Their product

$$X = \prod_{i\in\mathbb{N}} A_i$$

is the set of all sequences $x = (x_i)_{i\in\mathbb{N}}$, where $x_i \in A_i$. If every A_i is equipped with the discrete topology, their product is a metrizable topological space. A metric which yields the product topology is for example

$$d(x, y) = 2^{-n} \text{ where } n = \min\{i \in \mathbb{N}, x_i \neq y_i\}.$$

The product space is a discontinuum, i.e. a zero-dimensional, perfect, compact metric space. Every two discontinua are homeomorphic. When all $A_i = A$ are equal, we get a special case of the power space $X = A^{\mathbb{N}}$.

Definition 1 *Suppose that $(A_i)_{i\in\mathbb{N}}$ is a sequence of finite alphabets, and $(h_i : A_{i+1} \to A_i)_{i\in\mathbb{N}}$ is a sequence of mappings. Their inverse limit space is*

$$X = \lim_{i}(A_i, h_i) = \left\{ x \in \prod_{i\in\mathbb{N}} A_i;\ h_i(x_{i+1}) = x_i \right\}$$

With the metric d inherited from the product space, we have $d(x, y) < 2^{-n}$ iff $x_n = y_n$. The inverse limit space is a closed subspace of the product space $\prod_i A_i$, hence it is compact and zero-dimensional. However, it need not be perfect. Every zero-dimensional compact metric space is homeomorphic to some inverse limit space (see Hocking and Young [8]). For example, a product space $X = \prod_i A_i$ can be represented as an inverse limit space $X = \lim_i(A_{|i}, h_i)$, where $A_{|i} = A_0 \times ... \times A_{i-1}$, and $h_i(u_0...u_i) = u_0...u_{i-1}$.

Definition 2 *A clopen partition of a zero-dimensional space X is a surjective continuous function $\alpha : X \to A$, where A is a finite set with the discrete topology. Thus $\{\alpha^{-1}(a); a \in A\}$ is a finite collection of nonempty disjoint clopen sets, whose union is the full space.*

The diameter of a partition $\alpha : X \to A$ is $\mathrm{diam}(\alpha) = \max\{\mathrm{diam}(\alpha^{-1}(a)); a \in A\}$, where the diameter of a set $Y \subseteq X$ is $\mathrm{diam}(Y) = \sup\{d(x, y); x, y \in Y\}$. A clopen partition $\alpha : X \to A$ is finer than a clopen partition $\beta : X \to B$ (we write $\alpha \geq \beta$), if there exists a map $h : A \to B$ with $h\alpha = \beta$. The join $\alpha \vee \beta : X \to C$ of clopen partitions $\alpha : X \to A$ and $\beta : X \to B$ is defined by $(\alpha \vee \beta)(x) = (\alpha(x), \beta(x))$. Its range C is a subset of $A \times B$.

Definition 3 *We say that* $(\alpha_i : X \to A_i)_{i\in I\!N}$ *is an increasing sequence of clopen partitions, if* α_{i+1} *is finer than* α_i*, and* $\lim_{i\to\infty} \mathrm{diam}(\alpha_i) = 0$.

If $X = \lim_i(A_i, h_i)$ is an inverse limit space, denote $\alpha_i : X \to A_i$ the projection $\alpha_i(x) = x_i$. Since $h_i\alpha_{i+1} = \alpha_i$, $(\alpha_i)_{i\in I\!N}$ is an increasing sequence of clopen partitions. The sets $[a] = \alpha_i^{-1}(a) = \{x \in X;\ x_i = a\}$, where $a \in A_i$, are called cylinders. Every clopen set is a finite union of cylinders. Suppose that X is a zero-dimensional, compact metric space, and $(\alpha_i : X \to A_i)_{i\in I\!N}$ is an increasing sequence of clopen partitions with $h_i : A_{i+1} \to A_i$ satisfying $h_i\alpha_{i+1} = \alpha_i$. Define $\alpha : X \to \lim_i(A_i, h_i)$ by $\alpha(x)_i = \alpha_i(x)$. Then α is continuous and bijective, hence it is a homeomorphism.

Definition 4 *A (zero-dimensional) dynamical system* (X, f) *is a continuous map* $f : X \to X$ *of a nonempty zero-dimensional compact metric space* X *to itself. A homomorphism* $\varphi : (X, f) \to (Y, g)$ *of dynamical systems is a continuous map* $\varphi : X \to Y$ *such that* $g\varphi = \varphi f$*. A factorization is a homomorphism which is onto. A conjugacy is a homomorphism which is bijective.*

The i-th iteration $f^i : X \to X$ of f is defined by $f^0(x) = x$, $f^{i+1}(x) = f(f^i(x))$. A point $x \in X$ is periodic, if $f^i(x) = x$ for some $i > 0$. A set $Y \subseteq X$ is invariant, if $f(Y) \subseteq Y$. We refer to a dynamical system (X, f) as an inverse limit system if X is an inverse limit space.

Proposition 1 *For every dynamical system* (X, f)*, where* $X = \lim_i(A_i, h_i)$ *there exists a sequence* $(r_i)_{i\in I\!N}$ *of non-negative integers called radii, and a sequence of maps* $(f_i : A_{r_i} \to A_i)_{i\in I\!N}$ *such that for every* $x \in X$*,* $f(x)_i = f_i(x_{r_i})$*, i.e.* $\alpha_i f = f_i\alpha_{r_i}$.

Proof: Since f is uniformly continuous, for every $\varepsilon = 2^{-i}$ there exists $\delta = 2^{-n}$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Denote r_i the minimum n with this property (radius of the i-th position). Thus if $x_{r_i} = y_{r_i}$, then $f(x)_i = f(y)_i$. □

Dynamical systems on product spaces can be viewed as automata networks, i.e. countable networks of finite automata. The automaton at site $i \in I\!N$ is given by a finite set A_i of its inner states, a finite set of inputs $K_i \subseteq I\!N$, and a transition function $f_i : \prod_{j\in K_i} A_j \to A_i$. The state of an automata network is given by the inner states of all automata, so the state space of the network is $X = \prod A_i$. The global transition function $f : X \to X$, is given by $f(x)_i = f_i(x|K_i)$. Thus X is a product space, and f is a continuous function. Conversely, if f is a continuous self-map on a product space $X = \prod A_i$ and $i, j \in I\!N$, we say that i depends on j $(j \to i)$ iff there exist $u, v \in X$ such that $(\forall k \neq j)(u_k = v_k)$, and $f(u)_j \neq f(v)_j$. For every i, the set $K_i = \{j \in I\!N; j \to i\}$ is finite, and there exists a function

$$f_i : \prod_{j\in K_i} A_j \to A_i \quad \text{such that} \quad f(x)_i = f_i(x|K_i)$$

3 Subshifts, languages, and topological entropy

If A is a finite alphabet, $n \in \mathbb{N}$, denote A^n the set of words over A of length n, $A^* = \cup_{n \in \mathbb{N}} A^n$ the set of finite words over A, and $\overline{A^*} = A^* \cup A^{\mathbb{N}}$. For $u \in \overline{A^*}$, denote $|u|$ its length $(0 \leq |u| \leq \infty)$, and λ the word of zero length. Write $u \trianglelefteq v$ if u is a subword of v. Denote $u_{|i} = u_0 ... u_{i-1}$ the initial subword of u of length i. For $u \in A^*$ denote $u^n \in A^*$ the repetition of u n times, and $\overline{u} \in A^{\mathbb{N}}$ its infinite repetition.

A language over A is any subset $L \subseteq A^*$. A (one-sided) full shift is a dynamical system $(A^{\mathbb{N}}, \sigma)$ on the power space $A^{\mathbb{N}}$ given by $\sigma(u)_i = u_{i+1}$. A subshift (over A) is any subsystem of the full shift, i.e a dynamical system (Σ, σ), where $\Sigma \subseteq A^{\mathbb{N}}$ is a nonempty closed σ-invariant subset of $A^{\mathbb{N}}$. Thus subshifts are (zero-dimensional) dynamical systems.

Definition 5 *A nonempty language $L \subseteq A^*$ is right central, if*
$(\forall u \in L)(\forall v \trianglelefteq u)(v \in L)$*, (every subword of a word in L is in L) and*
$(\forall u \in L)(\exists a \in A)(ua \in L)$ *(every word in L can be extended to a word in L).*
The adherence of a right central language $L \subseteq A^*$ is defined by

$$\mathcal{A}(L) = \{u \in A^N ; (\forall v \in A^*)(v \trianglelefteq u \Rightarrow v \in L)\}$$

The language of a subshift $\Sigma \subseteq A^N$ is defined by

$$\mathcal{L}(\Sigma) = \{u \in A^* ; (\exists v \in \Sigma)(u \trianglelefteq v)\}$$

If $L \subseteq A^*$ is a right central language and $\Sigma \subseteq A^{\mathbb{N}}$ a subshift, then $\mathcal{A}(L)$ is a subshift, $\mathcal{L}(\Sigma)$ is a right central language, $\mathcal{AL}(\Sigma) = \Sigma$ and $\mathcal{LA}(L) = L$. Thus right central languages and subshifts are in one-to-one correspondence (cf. Blanchard and Hansel [3]).

If (X, f) is a dynamical system, and $\alpha : X \to A$ a clopen partition, denote $\alpha^{\mathbb{N}} : (X, f) \to (A^{\mathbb{N}}, \sigma)$ the homomorphism defined by

$$\alpha^{\mathbb{N}}(x)_i = \alpha f^i(x).$$

Denote $\Sigma_\alpha(X, f) = \alpha^{\mathbb{N}}(X) \subseteq A^{\mathbb{N}}$ the subshift generated by α on X, so that $\alpha^{\mathbb{N}} : (X, f) \to (\Sigma_\alpha(X, f), \sigma)$ is a factorization. Denote $\mathcal{L}_\alpha(X, f) = \mathcal{L}(\Sigma_\alpha(X, f))$ the language generated by f on α. For $n > 0$ denote $\mathcal{L}^n_\alpha(X, f) = \mathcal{L}_\alpha(X, f) \cap A^n$. In particular if $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift and $\alpha_0 : A^{\mathbb{N}} \to A$ is the projection $\alpha_0(x) = x_0$, then $\mathcal{L}_{\alpha_0}(\Sigma, \sigma) = \mathcal{L}(\Sigma)$. If $\alpha : (X, f) \to (\Sigma, \sigma)$ is a factorization, where $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift, then $\alpha = \alpha_0^{\mathbb{N}}$, where $\alpha_0 : X \to A$ is defined by $\alpha_0(x) = \alpha(x)_0$. If $X = \lim_i (A_i, h_i)$ is an inverse limit space, denote $\Sigma_n(X, f) \subseteq (A_n)^{\mathbb{N}}$ the subshift determined by the partition $\alpha_n : X \to A_n$.

Definition 6 *Let $\mathfrak{L}$ be a family of right central languages. We say that a dynamical system (X, f) is of class $\mathfrak{L}$ (or that it has languages of class $\mathfrak{L}$), if for every clopen partition $\alpha : X \to A$, the language $\mathcal{L}_\alpha(X, f)$ is in the family $\mathfrak{L}$.*

Let A_1, A_2 be finite alphabets. A map $h : A_1 \to A_2$ extends to a monoid homomorphism $h^* : A_1^* \to A_2^*$ by $h^*(u)_i = h(u_i)$. We say that a language $L_2 \subseteq A_2^*$ is a factor of a language $L_1 \subseteq A_1^*$, if there exists a map $h : A_1 \to A_2$ such that $L_2 = h^*(L_1)$. A family $\mathfrak{L}$ of (right central) languages is closed to factors, if it contains all factors of all its members. (In the formal language theory a more general concept of λ-free homomorphism is used, see e.g. Hopcroft and Ullmann [9].)

Proposition 2 *Let $\mathfrak{L}$ be a family of languages closed to factors. Then a dynamical system (X, f) is of class $\mathfrak{L}$ iff there exists an increasing sequence $(\alpha_i : X \to A_i)_{i\in\mathbb{N}}$ of clopen partitions of X, such that for every i, $\mathcal{L}_{\alpha_i}(X, f)$ is of class $\mathfrak{L}$.*

Proof: Let $\beta : X \to B$ be a finite clopen partition. For every $x \in X$ there exists an integer n_x such that $W_x = \alpha_{n_x}^{-1}\alpha_{n_x}(x) \subseteq \beta^{-1}\beta(x)$. Let $\{W_x; x \in A\}$ be a finite subcover of $\{W_x; x \in X\}$, and $n = \max\{n_x; x \in A\}$. Then $\alpha_n \geq \beta$, so we have a map $h : A_n \to B$ with $h\alpha_n = \beta$, and $\mathcal{L}_\beta(X, f) = h^*(\mathcal{L}_{\alpha_n}(X, f))$. Thus $\mathcal{L}_\beta(X, f)$ is in $\mathfrak{L}$. □

Definition 7 *Let (X, f) be a dynamical system and $\alpha : X \to A$ a clopen partition. The topological entropy $h_\alpha(X, f)$ of (X, f) on α is defined by*

$$h_\alpha(X, f) = \lim_{n\to\infty} \frac{1}{n} \ln \mathrm{card}(\mathcal{L}_\alpha^n(X, f))$$

The topological entropy $h(X, f)$ of (X, f) is defined by

$$h(X, f) = \sup\{h_\alpha(X, f);\ \alpha \text{ is a clopen partition of } X\}$$

Proposition 3 *Let (X, f) be a dynamical system and $(\alpha_i : X \to A_i)_{i\in\mathbb{N}}$ an increasing sequence of clopen partitions. Then*

$$h(X, f) = \lim_{n\to\infty} h_{\alpha_n}(X, f) = \lim_{n\to\infty} h(\Sigma_{\alpha_n}(X, f), \sigma)$$

Proof: Since $\mathcal{L}_\alpha(X, f) = \mathcal{L}_{\alpha_0}(\Sigma_\alpha(X, f), \sigma)$, we have

$$h(X, f) = \sup\{h(\Sigma_\alpha(X, f), \sigma);\ \alpha \text{ is a clopen partition of } X\}$$

If $\beta : X \to B$ is a clopen partition, there exists n such that α_n is finer than β, and this implies $h_{\alpha_n}(X, f) \geq h_\beta(X, f)$. □

4 Classes of regular languages

The language families of Chomsky hierarchy, and especially the families of regular and recursive languages are perhaps the most natural classes to start with (note that these classes are closed to factors). However, in the study of unimodal systems on the real interval, nonrecursive languages occur naturally (see Kůrka [10, 11]). Gilman [7] shows that every cellular automaton has context-sensitive languages. An extensively used language property is topological transitivity.

Definition 8 *A language $L \subseteq A^*$ is transitive, if for every $u, v \in L$ there exists $w \in A^*$ such that $uwv \in L$.*

Dynamical systems with transitive languages are just systems which are topologically transitive. The family of transitive languages is clearly closed to factors. We introduce now regular languages and two its subclasses.

Definition 9 *A graph is a fourtuple $G = (V, A, s, t)$, where V is a finite set of vertices, A is a finite set of arcs and $s, t : A \to V$ are the source and target maps, such that s is surjective.*

The surjectivity of s means that for every vertex there is an arc leading out of it. A path in a graph $G = (V, A, s, t)$ is a finite or infinite sequence of arcs $u \in \overline{A^*}$ such that $t(u_i) = s(u_{i+1})$ for every $i < |u| - 1$. If $u = u_0 ... u_{n-1}$ is a finite path, its source and target are $s(u) = s(u_0)$, and $t(u) = t(u_{n-1})$. A path is simple, if it does not contain any arc twice. We say that vertices $a, b \in V$ communicate, if there exists a path from a to b and from b to a. The communication relation is an equivalence on the set of those vertices, which communicate with themselves. The equivalence sets are called communication sets. A communication set W is cyclic, if for every $a, b \in W$ there is exactly one simple path from a to b. A communication set W is final if no path leaves it, i.e. if for every $u \in A$, if $s(u) \in W$, then $t(u) \in W$. Every graph has at least one final communication set.

Definition 10 *The language $L_G \subseteq A^*$ and subshift $\Sigma_G \subseteq A^{\mathbb{N}}$ of a graph $G = (V, A, s, t)$ are sets of its finite and infinite paths respectively.*

Clearly $\mathcal{A}(L_G) = \Sigma_G$, and $\mathcal{L}(\Sigma_G) = L_G$. A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is called a subshift of finite type if there exists a graph $G = (V, A, s, t)$ such that $\Sigma = \Sigma_G$.

Definition 11 *Let $L \subseteq A^*$ be a right central language. We say that*
1. L is regular, if it is a factor language of some graph,
2. L is periodic, if it is a factor language of a graph, whose every communication set is a cycle.
3. L is bounded periodic, if it is a factor language of a graph, whose every communication set is a final cycle.

Equivalently, every regular language over A is generated by a labelled graph, which is a quintuple $G = (V, E, s, t, h)$, where (V, E, s, t) is a graph, and $h : E \to A$ is a labelling function. Clearly the classes of regular, periodic and bounded periodic languages are closed to factors, so Proposition 2 applies to them. In the sense of Definition 6 we speak about dynamical systems with regular, periodic and bounded periodic languages respectively. Two-sided subshifts with regular languages are called sofic systems (see Weiss [14]).

Proposition 4 *A language $L \subseteq A^*$ is bounded periodic iff there exist integers $m \geq 0$ and $p > 0$ such that for every $u \in L$ and $i \geq m$ we have $u_i = u_{i+p}$. The minimum p with this property is called the period of L. The minimum m with this property is called the preperiod of L.*

Proof: If L is generated by a graph G, denote p the least common multiple of the lengths of all final cycles in G, and m maximum length of all paths outside of communication sets. Then $u_i = u_{i+p}$ for every $u \in L$, and $i \geq m$. Conversely, let m and p be preperiod and period of L. For every sequence $u = u_0 \dots u_{m+p-1} \in L$ construct a graph with vertices $\{0, 1, \dots m+p\}$, arcs from i to $i+1$ labelled by u_i, and an arc from $m+p$ to m labelled by u_m. Take the union of all these graphs. $\Box$

Proposition 5 *A subshift has bounded periodic languages iff it is finite. Every finite dynamical system is conjugate to a subshift with bounded periodic languages.*

Proof: Let $\Sigma \subseteq A^N$ be a bounded periodic subshift, let m, p be its preperiod and period respectively. Then every $x \in \Sigma$ is determined by $x_{|(m+p)} \in A^{m+p}$, and this is a finite set. If $\Sigma \subseteq A^N$ is a finite subshift, then for every $x \in \Sigma$ there exist m, p with $\sigma^{m+p}(x) = \sigma^m(x)$. If (X, f) is a finite dynamical system, construct a graph $G = (X, X, s, f)$, where $s(x) = x$, and $h : X \to \Sigma_G$ defined by $h(x)_i = f^i(x)$. Then $h : (X, f) \to (\Sigma_G, \sigma)$ is a conjugacy. $\Box$

Proposition 6 *A subshift with regular languages has periodic languages iff it is countable. A subshift with periodic languages has a nonzero finite number of periodic points.*

Proof: Let $G = (V, A, s, t)$ be a graph whose every communication set is cyclic. Denote W the set of simple paths whose sources and targets coincide. Denote U the set of simple paths whose source and target lie in different communication sets. Then W and U are finite sets. If $x \in \Sigma_G$, then $x = u_0 w_0^{k_0} \dots u_{m-1} w_{m-1}^{k_{m-1}} u_m \overline{w_m}$ for some $u_i \in U$ and $w_i \in W$. Thus Σ is countable. Conversely, suppose that $G = (V, A, s, t)$ is a graph with a communication set $W \subseteq V$, which is not cyclic. Then there exist different paths u, v, w such that $s(u) = s(v) = t(w) \in W$, and $t(u) = t(v) = s(w) \in W$. Any sequence $x = x_0 w x_1 w x_2 \dots$, where $x_i \in \{u, v\}$ belongs to Σ_G, so Σ_G is uncountable. $\Box$

The following examples show that in every class considered there are many dynamical systems besides subshifts.

Example 1 (adding machines) *Let $k = (k_i)_{i \in \mathbb{N}}$ be a sequence of integers such that $k_i > 1$ for every i. Denote $X = \prod_i X_i$, where $X_i = \{0, 1, \dots, k_i - 1\}$, and define $f : X \to X$ by*
$f(x)_i = x_i$, if there exists $j < i$ with $x_j \neq k_j - 1$, and

$f(x)_i = x_i + 1 \bmod k_i$ *otherwise.*
In the special case that all $k_i = 2$, *we get* $f(x)_i = x_i + x_0 x_1 ... x_{i-1} \mod 2$. *Then* (X, f) *is an infinite, bounded periodic system with no periodic point (cf. Buescu and Stewart [5]).*

Example 2 *Let* $X = \{0,1\}^{\mathbb{N}}$, $f(\overline{0}) = \overline{0}$, *and* $f(0^i 1x) = 0^i x$ *for* $i \geq 0$, $x \in X$. *Then* (X, f) *is a periodic system with a countable number of fixed points.*

Example 3 *Let* $X = \{0,1\}^{\mathbb{N}}$, *and* $f(x)_i = x_{2i}$. *Then* (X, f) *is a regular system with infinite topological entropy.*

5 Topological properties, languages and radii

Let (X, f) be a dynamical system. A point $x \in X$ is equicontinuous ($x \in Eq(X, f)$), if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B_\delta(x))(\forall n > 0)(d(f^n(x), f^n(y)) < \varepsilon)$$

Here $B_\delta(x) = \{y \in X; d(x,y) < \delta\}$. A point $x \in X$ is a non-wandering point ($x \in \Omega(X, f)$), iff for every neighbourhood U of x there exists $n > 0$ such that $f^n(U) \cap U \neq 0$. A δ-chain is any sequence $(x_i \in X)_{i \in \mathbb{N}}$ such that $d(f(x_i), x_{i+1}) < \delta$. A dynamical system (X, f) has the shadowing property, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for every δ-chain $(x_i)_{i \in \mathbb{N}}$ there exists $x \in X$, which ε-shadows x_i, i.e. $d(x_i, f^i(x)) < \varepsilon$ for every i. A dynamical system (X, f) has finite attractors, if $\omega(X, f) = \cap_{i \in \mathbb{N}} f^i(X)$ is a finite set.

Proposition 7 *Every dynamical system is conjugate to some inverse limit system with radii* $r_i = i + 1$.

Proof: Let (X, f) be a dynamical system. Construct by induction a sequence of clopen partitions $\alpha_i : X \to A_i$. Start with arbitrary α_0. Suppose that α_i has been already defined, and let α_{i+1} be any partition which is finer than $\alpha_i \vee \alpha_i f$ and satisfies $\text{diam}(\alpha_{i+1}) \leq \text{diam}(\alpha_i)/2$. Thus we have $h_i, f_i : A_{i+1} \to A_i$ with $h_i \alpha_{i+1} = \alpha_i$, $f_i \alpha_{i+1} = \alpha_i f$, and a homeomorphism $\alpha : X \to \lim_i(A_i, h_i)$ defined by $\alpha(x)_i = \alpha_i(x)$. If we define $f' : \lim_i(A_i, h_i) \to \lim_i(A_i, h_i)$ by $f'(x)_i = f_i(x_{i+1})$, then $\alpha f(x)_i = \alpha_i f(x) = f_i \alpha_{i+1}(x) = f_i(\alpha(x)_{i+1}) = f'(\alpha(x))_i$, so $\alpha : (X, f) \to (\lim_i(A_i, h_i), f')$ is a conjugacy, and f' has radii $r_i = i + 1$. $\square$

$$\begin{tikzcd} X \arrow[d, "\alpha_{i+1}"'] \arrow[dr, "\alpha_i"] & \\ A_{i+1} \arrow[r, "h_i"] & A_i \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_{i+1}"'] & X \arrow[d, "\alpha_i"] \\ A_{i+1} \arrow[r, "f_i"] & A_i \end{tikzcd}$$

Proposition 8 *Let $X = \lim_i(A_i, h_i)$ be an inverse limit system. If (X, f) is a dynamical system with radii $r_i = i + 1$, then $h_i f_{i+1} = f_i h_{i+1}$ for $i > 0$. Conversely, if $(f_i : A_{i+1} \to A_i)_{i \in \mathbb{N}}$ is a sequence satisfying $h_i f_{i+1} = f_i h_{i+1}$ for $i > 0$, and if we define f by $f(x)_i = f_i(x_{i+1})$, then (X, f) is a dynamical system.*

$$\begin{tikzcd}
X \arrow[r, "\alpha_{i+2}"] \arrow[dr, "\alpha_{i+1}"'] & A_{i+2} \arrow[r, "f_{i+1}"] \arrow[d, "h_{i+1}"] & A_{i+1} \arrow[d, "h_i"] \\
& A_{i+1} \arrow[r, "f_i"] & A_i
\end{tikzcd}$$

Proof: Suppose that (X, f) has radii $r_i = i + 1$, so $\alpha_i f = f_i \alpha_{i+1}$. Then $f_i h_{i+1} \alpha_{i+2} = f_i \alpha_{i+1} = \alpha_i f = h_i \alpha_{i+1} f = h_i f_{i+1} \alpha_{i+2}$. Since α_{i+2} is onto, we get $f_i h_{i+1} = h_i f_{i+1}$. Conversely, if $x \in X$, and $f(x)_i = f_i(x_{i+1})$, then $h_i(f(x)_{i+1}) = h_i f_{i+1}(x_{i+2}) = f_i h_{i+1}(x_{i+2}) = f_i(x_{i+1}) = f(x)_i$, so $f(x) \in X$. It is clear that f is continuous. $\Box$

Theorem 1 *Let (X, f) be a dynamical system. The following properties are equivalent:*
1. (X, f) has bounded periodic languages.
2. (X, f) is equicontinuous (i.e. $Eq(X, f) = X$)
3. (X, f) is conjugate to an inverse limit system with radii $r_i = i$.

Proof: $1 \Rightarrow 2$. Let $X = \lim_i(A_i, h_i)$. Given $n > 0$, there exist $m \geq 0$ and $p > 0$ such that for all $i \geq m$ $f^{i+p}(x)_n = f^i(x)_n$. Thus the sequence $(f^i(x)_n)_{i \in \mathbb{N}}$ is determined by its first $p + m$ members. Define a sequence t_j by $t_0 = n$, $t_{j+1} = r_{t_j}$ (here r_k is the radius at position k). Then $f^{m+p-1}(x)_n$ is determined by $f^{m+p-2}(x)_{t_1}$, and by induction this is determined by $x_{t_{m+p-1}}$. Thus for $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-t_{m+p-1}}$ such that if $d(x, y) < \delta$, then $d(f^i(x), f^i(y)) < \varepsilon$ for all i.
$2 \Rightarrow 3$. Let $X = \lim_i(A_i, h_i)$. For $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that if $d(x, y) < \delta$ then $d(f^i(x), f^i(y)) < \varepsilon$ for all i, i.e. if $\alpha_k(x) = \alpha_k(y)$, then $\alpha_n f^i(x) = \alpha_n f^i(y)$. This means that for every i, $\alpha_n f^i : X \to A_n$ is coarser than $\alpha_k : X \to A_k$. Let m be the first integer such that for every $p > m$, $\beta_n = \alpha_n \vee \alpha_n f \vee ... \vee \alpha_n f^{m-1} : X \to B_n$ is equal to $\alpha_n \vee \alpha_n f \vee ... \vee \alpha_n f^{p-1}$. Clearly diam$(\beta_n)$ converges to zero, and β_{n+1} is finer than β_n, so there exist $k_n : B_{n+1} \to B_n$ with $k_n \beta_{n+1} = k_{n+1}$. Thus $\beta : X \to Y = \lim_i(B_i, k_i)$ is a homeomorphism. Since $\alpha_n \vee \beta_n f = \beta_n$, β_n is finer than $\beta_n f$, and there exists $g_n : B_n \to B_n$ with $\beta_n f = g_n \beta_n$. Define $g : Y \to Y$ by $g(x)_i = g(x_i)$. Then $\beta f(x)_i = \beta_i f(x) = g_i \beta_i(x) = g_i(\beta(x)_i) = g\beta(x)_i$, so $\beta : (X, f) \to (Y, g)$ is a conjugation.

$3 \Rightarrow 1$: There exist $f_i : A_i \to A_i$ with $f(x)_i = f_i(x_i)$. Then every (A_i, f_i) is a finite dynamical system, hence (X, f) has bounded periodic languages. $\square$

Theorem 2 *A dynamical system has finite attractors iff it is conjugate to an inverse limit system with radii $r_0 = 0$, $r_{i+1} = i$.*

Proof: Let (X, f) have finite attractors. Then $\omega(X, f)$ consists of periodic points. Suppose that it consists of a single periodic orbit $\omega(X, f) = \{0, 1, ...p-1\}$ with $f(i) = i + 1 \bmod p$. (The general case is only formally more complicated.) For $i \in \omega(X, f)$ denote $X_i = \{x \in X; \lim_{n\to\infty} f^{np}(x) = i\}$. Then X_i is open, so $\alpha_0 = \{X_i; i \in \omega(X)\}$ is a clopen partition, and $f(X_i) \subseteq X_{f(i)}$. For at least one index i, this inclusion is strict, since otherwise $\omega(X, f)$ would be the full space. Suppose $f(X_{p-1}) \neq X_0$. Choose a point $y \in X_0 - f(X_{p-1})$. For every $x \in f(X_{p-1})$ there exist disjoint clopen sets $x \in V_x$, $y \in W_x$. Since $f(X_{p-1})$ is compact, there exists a finite subcover $\{V_x; x \in K\}$. Denote $X_{00} = \cup\{V_x, ; x \in K\}$, $X_{01} = X_0 - X_{00}$. Then $\alpha_1 = \{X_{00}, X_{01}, X_1, ..., X_{p-1}\}$ is a clopen partition of X, and $f(X_{p-1}) \subseteq X_{00}$. Similarly there exists a clopen partition $\alpha_2 = \{X_{00}, X_{01}, X_{10}, X_{11}, X_{12}, X_2, ..., X_{p-1}\}$ such that $f(X_{00}) \subseteq X_{10} \subseteq X_1$, $f(X_{01}) \subseteq X_{11} \subseteq X_1$, $X_{12} \subseteq X_1$ (X_{12} may be empty). It may be necessary to split X_{01} and X_{12} to several sets, to ensure that the diameter of α_n converges to zero. More formally, the construction proceeds by induction. In the first step we construct clopen partitions $\alpha_0 : X \to A_0$, $\alpha_1 : X \to A_1$, where $A_0 = \omega(X, f) = \{0, 1, ...p-1\}$, $A_1 = \{00, 01, 1, ..., p-1\}$. Define $f_0 : A_0 \to A_0$ by $f_0(i) = f(i)$, $f_1 : A_0 \to A_1$ by $f_1(p-1) = 00$, and $f(i) = i+1$ otherwise. Define $h_0 : A_1 \to A_0$ by $h_0(00) = h_0(01) = 0$, $h_0(i) = i$ for $i > 0$. Then $\alpha_0 f = f_0 \alpha_0$, $\alpha_1 f = f_1 \alpha_0$, and $h_0 \alpha_1 = \alpha_0$. Thus, with respect to α_0 and α_1, we have radii $r_0 = r_1 = 0$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_0"] & X \arrow[d, "\alpha_0"] \\ A_0 \arrow[r, "f_0"] & A_0 \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "\alpha_n"'] \arrow[d, "\alpha_{n-1}"] & X \arrow[d, "\alpha_n"] \\ A_n \arrow[r, "h_{n-1}"] & A_{n-1} \arrow[r, "f_n"] & A_n \end{tikzcd}$$

Suppose that in the n-th induction step we have constructed partitions $\alpha_{n-1} : X \to A_{n-1}$, $\alpha_n : X \to A_n$, and maps $h_{n-1} : A_n \to A_{n-1}$, $f_n : A_{n-1} \to A_n$, satisfying $f_n \alpha_{n-1} = \alpha_n f$, $h_{n-1}\alpha_n = \alpha_{n-1}$. Given $a \in A_n$, denote $B_a = \{b \in A_n; f_n h_{n-1}(b) = a\}$. If $b \in B_a$, then $f(\alpha_n^{-1}(b)) \subseteq \alpha_n^{-1}(a)$. Indeed if $x \in \alpha_n^{-1}(b)$, then $\alpha_n f(x) = f_n \alpha_{n-1}(x) = f_n h_{n-1} \alpha_n(x) = f_n h_{n-1}(b) = a$, so $f(x) \in \alpha_n^{-1}(a)$. There exists a clopen partition $\{Y_{ab}; b \in C_a\}$ of $\alpha_n^{-1}(a)$, such that $B_a \subseteq C_a$, if $b \in B_a$, then $f(\alpha_n^{-1}(b)) \subseteq Y_{ab}$, and if $b \in C_a - B_a$, then $\mathrm{diam}(Y_{ab}) \leq 2^{-n}$. Put $A_{n+1} = \{(a, b); a \in A_n, b \in C_a\}$ and define $\alpha_{n+1} : X \to A_{n+1}$ by $\alpha_{n+1}(Y_{ab}) = (a, b)$, $h_n : A_{n+1} \to A_n$ by $h_n(a, b) = a$, and $f_{n+1} : A_n \to A_{n+1}$ by $f_{n+1}(b) = (f_n h_{n-1}(b), b)$. If $x \in \alpha_n^{-1}(b)$, then $f(x) \in Y_{ab}$, and

$\alpha_{n+1}f(x) = (a,b)$. Thus for every $x \in X$, $\alpha_{n+1}f(x) = (f_n h_{n-1}\alpha_n(x), \alpha_n(x))$ $= f_{n+1}\alpha_n(x)$. Clearly $h_n\alpha_{n+1} = \alpha_n$, so the induction step is accomplished. We have $\lim \mathrm{diam}(\alpha_n) = 0$, so (X,f) is conjugate to $(\lim_n(A_n,h_n), f')$ where $f'(x)_0 = f_0(x_0)$, and $f'(x)_{i+1} = f_{i+1}(x_i)$. Conversely suppose that (X,f) has radii $r_0 = 0$, $r_{i+1} = r_i$. Then (A_0, f_0) is a finite dynamical system whose ω-limit consists of periodic points. Suppose that it consists of a single periodic orbit $\omega(A_0,f_0) = \{0,1,...,p-1\}$. Define $x \in X$ by $x_0 = 0$, $x_1 = f_1(p-1)$, $x_2 = f_2f_1(p-2)$,... Then $\omega(X,f) = \{x, f(x), ..., f^{p-1}(x)\}$. □

Theorem 3 *Every equicontinuous system has the shadowing property.*

Let $X = \lim_i(A_i,h_i)$. For every $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that if $x_k = y_k$, then $f^i(x)_n = f^i(y)_n$ for all i. Let $x^{(0)}, x^{(1)}, ...$ be a δ-chain and $m > 0$. Then $f(x^{(i)})_k = x_k^{(i+1)}$ for $0 \le i < m$. It follows $f^{m-i}(x^{(i)})_n = f^{m-i-1}(x^{(i+1)})_n$, so $f^m(x^{(0)})_n = x_n^{(m)}$. Thus $x^{(0)}$ ε-shadows the δ-chain $x^{(i)}$. □

Theorem 4 *Every system with periodic languages has a point of equicontinuity.*

Proof: Let (X,f) be a system with periodic languages and $X = \lim_i(A_i,h_i)$. For every n there exists $x_n \in X$, such that $\alpha_n(x_n)$ is in a final cycle. Moreover, we can choose x_i so that $\alpha_n(x_m) = \alpha_n(x_n)$ for all $m \ge n$. It follows there exists $x = \lim_{n\to\infty} x_n$. If $y \in X$ is such that $\alpha_n(y) = \alpha_n(x) = \alpha_n(x_n)$, then for all k we have $\alpha_n(f^k(y)) = \alpha_n(f^k(x))$, so $x \in Eq(X,f)$. □

Theorem 5 *Every system with periodic languages has zero topological entropy.*

Proof: Suppose first that (Σ,σ) is a subshift with periodic languages. It follows from the proof of Proposition 6 that the set $\Omega(\Sigma,\sigma)$ of nonwandering points is finite and consists of periodic points. By Bowen's Theorem (see e.g. Block [4]), the topological entropy of a dynamical system is concentrated on its non-wandering set, so $h(\Sigma,\sigma) = 0$. If (X,f) is a periodic system, and α_i an increasing sequence of clopen partitions, then by Proposition 3, $h(X,f) = \lim_{n\to\infty} h(\Sigma_{\alpha_n}(X,f),\sigma) = 0$. □

Proposition 9 *Every dynamical system with the shadowing property has regular languages.*

Proof: Let (X,f) be a dynamical system with the shadowing property and $X = \lim_i(A_i,h_i)$. We show that for every n, $\Sigma_n(X,f)$ is regular. By the assumption, for $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k}$ such that every δ-chain is ε-shadowed by a point. For k there exists $f_k : A_{r_k} \to A_k$ such that $f(x)_k = f_k(x_{r_k})$. Define $\Sigma \subseteq (A_{r_k})^{\mathbb{N}}$ by the condition $u = u^{(0)}u^{(1)}... \in \Sigma$ iff $u_i^{(j+1)} = f_i(u_{r_i}^{(j)})$ for $i \le k$, and $j \ge 0$. Thus Σ is a subshift of finite type. Define $H : \Sigma \to (A_n)^{\mathbb{N}}$ by $H(u) = u_n^{(0)}u_n^{(1)}....$ We claim $H(\Sigma) = \Sigma_n(X,f)$. Indeed if $u \in \Sigma$, then any sequence $x^{(j)}$ with $x^{(j)} \in [u^{(j)}]$ is a δ-chain, so there exists $x \in X$ which ε-shadows $x^{(j)}$, i.e. $f^{(j)}(x)_n = x_n^{(j)}$. Thus $H(u) = H_n(x)$, so $H(\Sigma) \subseteq \Sigma_n(X,f)$. The other inclusion is clear. □

6 Universal systems

Definition 12 *Let $\mathfrak{L}$ be a family of languages. We say that a dynamical system (X, f) is a universal system of class $\mathfrak{L}$, if it is of class $\mathfrak{L}$, and every dynamical system of class $\mathfrak{L}$ is its factor.*

Universal system are constructed using infinite products and inverse limits of metric spaces and dynamical systems. Let $(X_i, d_i)_{i\in\mathbb{N}}$ be a sequence of metric spaces with bounded diameter $\mathrm{diam}(X_i) \leq c$. Their product is the metric space (X, d), where

$$X = \prod_{i\in\mathbb{N}} X_i, \quad d(u, v) = \sum_{i=0}^{\infty} d_i(u_i, v_i)2^{-i}$$

If all X_i are zero-dimensional compact metric spaces, then so is their product. If $(X_i)_{i\in\mathbb{N}}$ is a sequence of metric spaces with bounded diameter, and $h_i : X_{i+1} \to X_i$ are continuous mappings, then their inverse limit is

$$X = \lim_{i}(X_i, h_i) = \left\{ x \in \prod_{i\in\mathbb{N}} X_i ; h_i(x_{i+1}) = x_i \right\}$$

Again, if all X_i are zero-dimensional and compact, then so is their inverse limit. If $(X_i, f_i)_{i\in\mathbb{N}}$ is a sequence of dynamical systems, their product is the system (X, f), where $X = \prod_i X_i$, and $f(x)_i = f_i(x_i)$. If $(h_i : (X_{i+1}, f_{i+1}) \to (X_i, f_i))_{i\in\mathbb{N}}$, is a sequence of homomorphisms, their inverse limit is the dynamical system $(X, f) = \lim_i(X_i, f_i, h_i)$, where $X = \lim_i(X_i, h_i)$, $f(x)_i = f_i(x_i)$.

A homomorphism $h : (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$, where $\Sigma_A \subseteq A^{\mathbb{N}}$ and $\Sigma_B \subseteq B^{\mathbb{N}}$ are subshifts, is called radius zero, if there exists $h' : A \to B$ such that for every $x \in \Sigma_A$, $h(x)_i = h'(x_i)$. (We write $h = (h')^{\mathbb{N}}$; in general homomorphisms of radius r, $h(x)_i = h'(x_{i-r}, ..., x_{i+r})$ for some h'.)

Proposition 10 *Let $\mathfrak{L}$ be a family of languages closed to factors. Then every dynamical system of class $\mathfrak{L}$ is the inverse limit of subshifts of class $\mathfrak{L}$ and radius zero homomorphisms.*

Proof: Let (X, f) be a system of class $\mathfrak{L}$, and let $(\alpha_i : X \to A_i)_{i\in\mathbb{N}}$ be an increasing sequence of clopen partitions with $h_i : A_{i+1} \to A_i$ satisfying $h_i\alpha_{i+1} = \alpha_i$. We have homomorphisms $\alpha_i^{\mathbb{N}} : (X, f) \to (\Sigma_{\alpha_i}(X, f), \sigma)$, $h_i^{\mathbb{N}} : (\Sigma_{\alpha_{i+1}}(X, f), \sigma) \to (\Sigma_{\alpha_i}(X, f), \sigma)$ with $h_i^{\mathbb{N}}\alpha_{i+1}^{\mathbb{N}} = \alpha_i^{\mathbb{N}}$. Define $\alpha : (X, f) \to \lim_i(\Sigma_{\alpha_i}(X, f), \sigma, h_i^{\mathbb{N}})$ by $\alpha(x)_i = \alpha_i^{\mathbb{N}}(x)$. Then α is a conjugacy. □

Let A_1, A_2 be finite alphabets, $\pi_i : A_1 \times A_2 \to A_i$ the projections, and $L_1 \subseteq A_1^*$, $L_2 \subseteq A_2^*$ languages. We denote

$$L_1 \otimes L_2 = \{u \in (A_1 \times A_2)^*; \;\; \pi_1^*(u) \in L_1 \;\;\&\;\; \pi_2^*(u) \in L_2\}$$

their product. If $A_1 \subset A_2$, and $L_1 = L_2 \cap A_1^*$, we say that L_1 is the restriction of L_2 to A_1. A family $\mathfrak{L}$ of (right central) languages is closed to products if $L_1 \otimes L_2$ belongs to $\mathfrak{L}$ whenever L_1, L_2 belong to $\mathfrak{L}$. $\mathfrak{L}$ is closed to restrictions, if for every L_2 in $\mathfrak{L}$ and every right central restriction L_1 of L_2, L_1 belongs to $\mathfrak{L}$.

It is easy to see that the families of recursive, regular, periodic and bounded periodic languages are closed to products and restrictions. On the other hand, the family of transitive languages is closed neither to products nor to restrictions.

Definition 13 *We say that a universal system (X, f) of class $\mathfrak{L}$ has the extension property, if for every subshifts $\Sigma_A \subseteq A^{\mathbb{N}}$, $\Sigma_B \subseteq B^{\mathbb{N}}$ of class $\mathfrak{L}$, every radius zero factorization $h : (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$, and for every factorization $\beta : (X, f) \to (\Sigma_B, \sigma)$ there exists a factorization $\alpha : (X, f) \to (\Sigma_A, \sigma)$ with $h\alpha = \beta$.*

Theorem 6 *Let $\mathfrak{L}$ be a family of (right central) languages closed to factors, products and restrictions, such that there exists up to conjugacy only countably many subshifts of class $\mathfrak{L}$. Then there exists a universal system of class $\mathfrak{L}$ with the extension property.*

Proof: Consider first a sequence $(\Sigma_i, \sigma)_{i \in \mathbb{N}}$, where $\Sigma_i \subseteq A_i^{\mathbb{N}}$, which contains up to conjugacy all subshifts of class $\mathfrak{L}$. We assume that if $i > j$, then $\mathrm{card}(A_i) \geq \mathrm{card}(A_j)$. Thus (Σ_0, σ) is the trivial one-point system. We construct now a sequence $(X_i, f_i)_{i \in \mathbb{N}}$ of subshifts of class $\mathfrak{L}$, a sequence of integers $(j_i)_{i>0}$ satisfying $0 \leq j_i < i$, and a sequence of radius zero factorizations

$$h_i : (X_i, f_i) \to (X_{j_i}, f_{j_i}); \quad i > 0$$

(In the sense of the theory of categories, this will be a diagram in the category of dynamical systems and their homomorphisms. The desired universal system is the limit of this diagram. See MacLane [13]) The construction proceeds by induction. At every induction step we have integers m, n, designating the number of Σ's used, and the number of X's constructed. In the first induction step put $m = n = 1$, and $(X_0, f_0) = (\Sigma, \sigma)$. In general induction step with m, n given, suppose we have already constructed (X_i, f_i) for $0 \leq i < n$, and j_i, h_i for $0 < i < n$. Consider all radius zero factorizations from (Σ_m, σ) to some (X_i, f_i) already constructed. Suppose there are p such factorizations, namely

$$h_i : (\Sigma_m, \sigma) \to (X_{j_i}, f_{j_i}); \quad \text{where} \quad n \leq i < n + p,\ 0 \leq j_i < n$$

Then for $n \leq i < n + p$ we put $(X_i, f_i) = (\Sigma_m, \sigma)$, $n' = n + p$, $m' = m + 1$, and repeat the induction step. Define now (X, f) by

$$X = \left\{ x \in \prod_{i=0}^{\infty} X_i;\ (\forall i > 0)(h_i(x_i) = x_{j_i}) \right\}; \quad f(x)_i = f_i(x_i).$$

We claim that (X,f) is a universal system of class $\mathfrak{L}$. For $n>0$ denote

$$X_{|n}=\left\{x\in\prod_{i=0}^{n-1}X_i;\ (\forall i\in(0,n))(h_i(x_i)=x_{j_i})\right\}$$

Since $\mathfrak{L}$ is closed to products and restrictions, $(X_{|n},\sigma)$ is a subshift of class $\mathfrak{L}$. If $\alpha:X\to A$ is a clopen partition, then α is uniformly continuous, so there exist n and $\alpha_{|n}:X_{|n}\to A$, such that $\alpha(x)=\alpha_{|n}(x_0,...,x_{n-1})$. Since $\mathfrak{L}$ is closed to factors, $\Sigma_\alpha(X,f)=\alpha_{|n}(X_{|n},\sigma)$ is of class $\mathfrak{L}$. Thus (X,f) has languages of class $\mathfrak{L}$. To show that (X,f) is universal, consider a dynamical system (Y,g) of class $\mathfrak{L}$. By Proposition 10, there exist subshifts (Y_i,g_i) of class $\mathfrak{L}$, and radius zero factorizations $k_i:(Y_{i+1},g_{i+1})\to(Y_i,g_i)$ such that (Y,g) is conjugate to $\lim_i(Y_i,k_i,g_i)$. By the construction of X_i, the sequence $(k_i:(Y_{i+1},g_{i+1})\to(Y_i,g_i))_{i\in\mathbb{N}}$ can be embedded into the diagram of (X_i,f_i), i.e. there exists a sequence $(p_i)_{i\in\mathbb{N}}$ such that $k_i=h_{p_i}$, and therefore $(X_{p_i},f_{p_i})=(Y_{i+1},g_{i+1})$, $(X_{j_{p_i}},f_{j_{p_i}})=(Y_i,g_i)$. We have a projection $\pi:(X,f)\to\lim_i(X_{p_i},f_{p_i},h_{p_i})$, and $\lim_i(X_{p_i},f_{p_i},h_{p_i})$ is conjugate to (Y,g). Thus (Y,g) is a factor of (X,f).

To get a universal system with the extension property, we modify the preceding construction so that the diagram constructed contains not only all sequences of factorizations but also all finite commuting diagrams. We construct a sequence $(X_i,f_i)_{i\in\mathbb{N}}$ of subshifts of class $\mathfrak{L}$, a sequence of positive integers $(s_i)_{i>0}$, a system of integers $(j_{ik})_{i>0,0\le k<s_i}$ with $0\le j_{ik}<i$ and a system of radius zero factorizations

$$h_{ik}:(X_i,f_i)\to(X_{j_{ik}},f_{j_{ik}});\ \ i>0,\ 0\le k<s_i$$

such that the composition of every two sequences of factorizations with the same domain and range coincide. (Thus for example $h_{i_1k_1}h_{i_0k_0}=h_{i_0k_2}$ whenever $j_{i_0k_0}=i_1,\ j_{i_1k_1}=j_{i_0k_2}$.) The construction proceeds again by adjoining successively all subshifts, but this time together with every finite system of factorizations to already existing subshifts. Define (X,f) by

$$X=\left\{x\in\prod_{i\in\mathbb{N}}X_i;\ (\forall i)(\forall k<s_i)(h_{ik}(x_i)=x_{j_{ik}})\right\};\ \ f(x)_i=f_i(x_i)$$

Then (X,f) is again a universal system of class $\mathfrak{L}$. For $n>0$ denote

$$X_{|n}=\left\{x\in\prod_{i<n}X_i;\ (\forall i<n)(\forall k<s_i)(h_{ik}(x_i)=x_{j_{ik}})\right\}$$

and $\pi_{|n}:X\to X_{|n}$ the projection. Then $(X_{|n},\sigma)$ is a subshift of class $\mathfrak{L}$. To prove that (X,f) has the extension property, assume that $\Sigma_A\subseteq A^{\mathbb{N}}$, $\Sigma_B\subseteq B^{\mathbb{N}}$ are subshifts of class $\mathfrak{L}$, $\beta:(X,f)\to(\Sigma_B,\sigma)$ is a factorization, and $h:(\Sigma_A,\sigma)\to(\Sigma_B,\sigma)$ is a radius zero factorization. Since β is uniformly continuous, there exists n and $\beta_n : X_{|n} \to B$ such that $\beta(x) = \beta_n(x_0 ... x_{n-1}) = \beta_n \pi_{|n}(x)$. Denote

$$\Sigma_C = \{(x_{|n}, a) \in X_{|n} \times A;\ \beta_n(x_{|n}) = h(a)\}$$

and $\nu : \Sigma_C \to X_{|n}$, $\pi_A : \Sigma_C \to \Sigma_A$ the projections. (Σ_C is the pullback of β_n and h.) Then (Σ_C, σ) is a subshift of class $\mathfrak{L}$, and $\nu : (\Sigma_C, \sigma) \to (X_{|n}, \sigma)$ is a factorization. For $i < n$ denote $\pi_{ni} : X_{|n} \to X_i$ the projections. By the construction of X, there exists $m > n$ such that $X_m = \Sigma_C$, $s_m = n$, $j_{mi} = i$, and $h_{mi} = \pi_{ni}\nu$. Thus $\nu(x_m) = (x_0 ... x_{n-1})$, so $\nu\pi_m = \pi_{|n}$. Define $\alpha : X \to \Sigma_A$ by $\alpha(x) = \pi_A(x_m) = \pi_A\pi_m(x)$. Then $h\alpha = h\pi_A\pi_m = \beta_n\nu\pi_m = \beta_n\pi_{|n} = \beta$. $\square$

$$\begin{tikzcd}
X \arrow[r, "\pi_m"] \arrow[d, "\pi_i"'] \arrow[dr, "\pi_{|n}"] & X_m \arrow[r, "\pi_A"] \arrow[d, "\nu"] & \Sigma_A \arrow[d, "h"] \\
X_i & X_{|n} \arrow[l, "\pi_{ni}"'] \arrow[r, "\beta_n"] & \Sigma_B
\end{tikzcd}$$

Corollary 1 *In the classes of dynamical systems with bounded periodic, periodic, regular and recursive languages there exist universal systems with the extension property.*

Note that while the universal systems with periodic and bounded periodic languages have zero topological entropy, universal systems with regular and recursive languages have necessarily infinite topological entropy.

Proposition 11 *Let $\mathfrak{L}$ be a family of languages closed to factors, let (X, f), (Y, g) be universal systems of class $\mathfrak{L}$, and suppose that (X, f) has the extension property. Then every factorization $\psi : (Y, g) \to (X, f)$ has a cross-section, i.e. there exists a homomorphism $\varphi : (X, f) \to (Y, g)$, with $\psi\varphi = Id_X$.*

Proof: Let $(\alpha_i : X \to A_i)_{i \in \mathbb{N}}$ be an increasing sequence of clopen partitions with $h_i : A_{i+1} \to A_i$ satisfying $h_i \alpha_{i+1} = \alpha_i$. Let $\beta_0 : Y \to B_0$ be any partition finer than $\alpha_0\psi$ with $\mathrm{diam}(\beta_0) < 1$, so we have a map $\nu_0 : B_0 \to A_0$ with $\nu_0\beta_0 = \alpha_0\psi$. We continue the construction by induction. Suppose we have constructed a partition $\beta_{i-1} : Y \to B_{i-1}$ and a map $\nu_{i-1} : B_{i-1} \to A_{i-1}$ with $\nu_{i-1}\beta_{i-1} = \alpha_{i-1}\psi$ and $\mathrm{diam}(\beta_{i-1}) < 2^{-i+1}$. Define

$$B'_i = \{(a, b) \in A_i \times B_{i-1};\, h_{i-1}(a) = \nu_{i-1}(b)\}$$

and $\beta'_i : Y \to B'_i$ by $\beta'_i(y) = (\alpha_i\psi(y), \beta_{i-1}(y))$. Let $\beta_i : Y \to B_i$ be any clopen partition finer than β'_i with $\mathrm{diam}(\beta_i) < 2^{-i}$. Then we have maps $\nu_i : B_i \to A_i$ and $k_{i-1} : B_i \to B_{i-1}$ with $\nu_{i-1}k_{i-1} = h_{i-1}\nu_i$ and $\nu_i\beta_i = \alpha_i\psi$. Thus $(\beta_i : Y \to B_i)_{i \in \mathbb{N}}$ is an increasing sequence of clopen partitions, so (Y, g) is conjugate to $\lim_i(\Sigma_{\beta_i}(Y,g),\sigma,k_i^{\mathbb{N}})$. Since (X,f) has the extension property, there exist clopen partition $\gamma_i : X \to B_i$, such that $\gamma_i^{\mathbb{N}} : (X,f) \to (\Sigma_{\beta_i}(Y,g),\sigma)$ is a factorization, and $\nu_i\gamma_i = \alpha_i$. It follows that there exists a homomorphism $\varphi : (X,f) \to (Y,g)$ with $\beta_i\varphi = \gamma_i$. Then for every i we have $\alpha_i = \nu_i\gamma_i = \nu_i\beta_i\varphi$ $= \alpha_i\psi\varphi$, so $\psi\varphi = Id_X$. □

$$\begin{tikzcd}
Y \arrow[r, "\beta_i"] \arrow[d, shift left, "\psi"] & B_i \arrow[r, "k_{i-1}"] \arrow[d, "\nu_i"] & B_{i-1} \arrow[d, "\nu_{i-1}"] \\
X \arrow[u, shift left, "\varphi"] \arrow[r, "\alpha_i"] \arrow[ur, "\gamma_i"] & A_i \arrow[r, "h_{i-1}"] & A_{i-1}
\end{tikzcd}$$

Theorem 7 *Let $\mathfrak{L}$ be a family of languages closed to factors. Then every two universal systems of class $\mathfrak{L}$ with the extension property are conjugate.*

Proof: Suppose that (X,f), (Y,g) are universal systems of class $\mathfrak{L}$ with the extension property. Let $\alpha_0 : X \to A_0$ be any clopen partition with $\mathrm{diam}(\alpha_0) < 1$. Since (Y,g) is universal, there exists a clopen partition $\beta_0 : Y \to A_0$ with $\Sigma_{\beta_0}(Y,g) = \Sigma_{\alpha_0}(X,f)$. We continue the construction by induction. Let $\alpha_n : X \to A_n$, $\beta_n : Y \to A_n$ be clopen partitions with $\Sigma_{\beta_n}(Y,g) = \Sigma_{\alpha_n}(X,f)$. If n is even, we pick a clopen partition $\beta_{n+1} : Y \to A_{n+1}$ finer than β_n with $\mathrm{diam}(\beta_{n+1}) < 2^{-n-1}$, so we have a map $h_n : A_{n+1} \to A_n$ with $h_n\beta_{n+1} = \beta_n$. Since (X,f) has the extension property, there exists a clopen partition $\alpha_{n+1} : X \to A_{n+1}$ with $h_n\alpha_{n+1} = \alpha_n$, and $\Sigma_{\beta_{n+1}}(Y,g) = \Sigma_{\alpha_{n+1}}(X,f)$. If n is odd, we construct first a partition $\alpha_{n+1} : X \to A_{n+1}$ with $\mathrm{diam}(\alpha_{n+1}) < 2^{-n-1}$, and then $\beta_{n+1} : Y \to A_{n+1}$. It follows that both $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are increasing sequences of clopen partitions, and both (X,f) and (Y,g) are conjugate to $\lim_n(\Sigma_{\alpha_n}(X,f),\sigma,h_n)$. □

For transitive systems, the above theorems fail for several reasons. The set of transitive subshifts is not countable, and the family of transitive languages is closed neither to products nor to restrictions. Nevertheless, the adding machine (see Example 1) with infinite multiplicities is a universal system with the extension property for the family of transitive periodic languages. Given a sequence of integers $k = (k_i)_{i\in\mathbb{N}}$ with $k_i > 1$ for all i, define the multiplicity $m(p)$ of a prime p as the sum of the powers to which p occurs as a factor in all the k_i, allowing $m(p)$ to be infinite, if p occurs as a factor in infinitely many k_i. Buescu and Stewart [5] prove, that two adding machines are conjugate iff they have the same multiplicity functions.

Proposition 12 *A transitive language is periodic iff it is bounded periodic. A subshift with transitive periodic language consists of a single periodic orbit.*

Theorem 8 *The adding machine whose all multiplicities are infinite is the universal system with the extension property for the class of transitive periodic languages.*

Proof: Let (X, f) be the adding machine with infinite multiplicities, and (Y, g) a system with transitive periodic languages. Then for every clopen partition $\beta : Y \to B$, $\Sigma_\beta(Y, g)$ consists of a single periodic orbit, so it is a factor of $X_{|n}$ for high enough n. □

Theorem 9 *In the classes of systems with bounded periodic, periodic and regular languages, universal systems are not unique up to conjugacy.*

Proof: In the appropriate class $\mathfrak{L}$ of languages, let (X, f) be the universal system with the extension property, let (Z, h) be the universal transitive periodic system from Theorem 8, and let $(Y, g) = (X \cup Z, f \cup h)$ be their disjoint union. We show that there exists a homomorphism from (Z, h) to (X, f). If $\alpha_i : X \to A_i$ is an increasing sequence of clopen partitions, with $h_i : A_{i+1} \to A_i$ satisfying $h_i \alpha_{i+1} = \alpha_i$, then $\Sigma_{\alpha_i}(X, f)$ has a periodic point, and there exists a homomorphism $\gamma_i : (Z, h) \to \Sigma_{\alpha_i}(X, f)$. Moreover, we can choose γ_i so that $h_i^{\mathbb{N}} \gamma_{i+1} = \gamma_i$. Thus $\gamma : (Z, h) \to (X, f)$ defined by $\alpha_i^{\mathbb{N}} \gamma = \gamma_i^{\mathbb{N}}$ is a homomorphism. Thus $\langle Id_X, \gamma \rangle : (Y, g) \to (X, g)$ is a factorization, so (Y, g) is a universal system of class $\mathfrak{L}$ too. However, if $\psi : (Y, g) \to (\Sigma_B, \sigma)$ is a factorization with transitive range, and if $h : (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$ is a factorization with nontransitive domain, then there is no φ with $h\varphi = \psi$. Thus (Y, g) does not have the extension property, so it is not conjugate to (X, f). □

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