Formalism 25

Mikhail Katz, Bar-Ilan University

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Joint work with Karl Kuhlemann, Sam Sanders and David Sherry.

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We aim to demonstrate the incoherence of the position of both Gaifman and Erhardt, and to clarify Robinson's philosophical stance.

Passions and meaning

Abraham Robinson's intellectual profile continues to inspire passions fifty years after his passing.

A number of publications over the past few decades have analyzed his philosophical position, including [Benis-Sinaceur 1988, p. 603], [Gaifman 2012], [Sherry and Katz 2012], [Sanders 2020], [Weir 2024], and most recently [Erhardt 2025].

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- Erhardt following Gaifman reads Robinson as a finitist (see below for a discussion of distinct meanings of the term) who rejects infinite totalities.

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- A majority of commentators take it for granted that Robinson's mature position is best described as mathematical Formalism.
- Erhardt following Gaifman reads Robinson as a finitist (see below for a discussion of distinct meanings of the term) who rejects infinite totalities.

Significantly, there is a key observation in Robinson's 1975 article that is missing from Erhardt's text (though he does cite the article). The observation helps understand Robinson's take on **meaning**.

Robinson's take on meaning and reference

In 1975, Robinson clarified his position as follows:

> *mathematical theories that, allegedly, deal with infinite totalities have no detailed meaning, i.e.* **reference**. *[Robinson 1975, p. 42]; reprinted in Selected Papers [Robinson 1979, p. 557]*

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Namely, the term *infinite totality* has no reference (or referent) in either the physical world or any Platonic realm of mathematical abstracta.

- Robinson's main goal here was to distance himself from mathematical Platonism.
- He did *not* believe that expressions such as *infinite totality* lacked meaning in the sense of being 'pointless' or 'devoid of significance', as we will see below.
- He only intended that such expressions lacked a reference, as we explained.

Reference and direct interpretation

In his earlier philosophical text *Formalism 64* he used the term *direct interpretation* in place of *reference*:

> *I . . . regard a theory which refers to an infinite totality as meaningless in the sense that its terms and sentences cannot possess the*
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Robinson's position that infinite totalities possess no referent amounts to a rejection of mathematical Platonism with regard to such totalities.

Erhardt on Robinson

Question

How does Erhardt present Robinson's position?

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Erhardt misrepresents Robinson's position by quoting Robinson out of context. Thus, Erhardt claims the following:

> *In Formalism 64, Robinson's most fully developed philosophical paper, he writes (Robinson 1964/1979b, pp. 230–231, italics in the original), "[. . .] the notion of a particular class of five elements, e.g. of five particular chairs, presents itself to my mind as clearly as the notion of a single individual [. . .]. By contrast, I feel quite* **unable to grasp** *the idea of an actual infinite totality. To me there appears to exist an unbridgeable gulf between sets or structures of one, or two, or five elements, on one hand, and infinite structures on the other hand [. . .]"*[1]

[1] Robinson as quoted in [Erhardt 2025, p. 431].

Profusion of ellipses

We deliberately reproduced the profusion of ellipses '[. . .]' in Erhardt's quotation from Robinson.

Superficially, the abridged passage may suggest a finitist position.

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Do Erhardt's deletions alter the meaning of the passage as intended by Robinson?

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We will analyze Robinson's passage in its context later in the talk.

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Question

Where did Erhardt get the idea of describing Robinson as a finitist?

Passing the torch

Erhardt thanks **Gaifman** for encouraging him to read Robinson's text *Formalism 64* [Erhardt 2025, p. 446]. Gaifman may have been the original source of a characterisation of Robinson as a finitist.

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> *Abraham Robinson, who was a* **finitist, or something very near to it**, *realized the seriousness of the* **limitations** *that his position implied with regard to syntactic concepts that required quantification over infinite domains. [Gaifman 2012, p. 488]*
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Question

Was Robinson a finitist as claimed by Gaifman, and does Robinson's position entail serious 'limitations' as Gaifman claims?

Broad and narrow finitisms

To address the issue, it is crucial to distinguish between finitism in a narrow sense and finitism in a broad sense.

- Broad finitism: opposition to the use of infinitary concepts at the metamathematical level (as in Hilbert's program).
- In its narrow sense, finitism is characterized by an opposition to the use of infinite totalities at **both** the metamathematical and the mathematical level.

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The fact that Robinson's position is more accurately described as Formalism than finitism is evident from his recommendation:

> *We should continue the business of Mathematics "as usual," i.e., we should act as if infinite totalities really existed. [Robinson 1965, p. 230]*

Robinson vs finitists

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Indeed, Robinson explicitly argued *against* some finitists’ rejection of the use of infinitary terms in mathematics:

> *Those who adopt this attitude [including the Intuitionists] think that a concept, or a sentence, or an entire theory, is acceptable only if it can be understood properly and that a concept, or sentence, or a theory, is understood properly only if* **all terms** *which occur in it can be interpreted directly, as explained.*

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> *By contrast, the formalist holds that* **direct interpretability is not a necessary condition for the acceptability** *of a mathematical theory. [Robinson 1965, p. 234]*

Dummett insists on meaning for all terms

One such intuitionist is Dummett, who claimed the following:

> *Constructivist philosophies of mathematics insist that the meanings of* **all terms**, *including logical constants, appearing in mathematical statements must be given in relation to constructions which we are capable of effecting, and of our capacity to recognise such constructions as providing proofs of those statements.[Dummett 1975, 301]*

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Robinson clearly disagreed with Dummett's claim that *all terms* must be assigned such a direct meaning. Robinson concluded:

> *To sum up, the* **direct interpretability of the terms** *of a mathematical theory is not a necessary condition for its acceptability; a theory which includes infinitary terms is not thereby less acceptable or less rational than a theory which avoids them. [Robinson 1965, p. 235]*

Here Robinson endorsed the Formalist position and moreover contrasted it with finitism in the narrow sense.

Culprit: Standard Model

The source of Platonists' discomfort with Formalism (and their proclivity to paint Formalists as finitists) is identified in section 10 of Robinson's text *Formalism 64* in the following terms:

[2] [Robinson 1965, p. 242]. This use of the term *meaningless* was discussed in the Introduction.

Culprit: Standard Model

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> *In particular, I will mention here the assumption that there exists a* **standard or intended model** *of Arithmetic or (alternatively, but relatedly) of Set Theory. Clearly, to the formalist, the entire notion of standardness must be meaningless, in accordance with our first basic principle.*[2]

As we already mentioned, Robinson denied the existence of a referent for such a standard model (a.k.a. intended interpretation) of $\mathbb{N}$ or $\mathbb{R}$, in either the physical or any Platonic realm.

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To Platonists this may appear as a narrow finitist stance, but they may well ponder why Robinson didn't name his text "Finitism 64".

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In Errett Bishop's footsteps

Erhardt goes on to comment as follows:

> *There is, however, another alternative: more strongly committing to finitism. This is the path taken by various constructivists, who rather than play the* **uninterpretable game of symbols** *choose, in a philosophically principled manner, to adopt a weaker form of mathematics. [Erhardt 2025, p. 444]*

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Erhardt may be right about *constructivists* (rather than Robinson) describing classical math as an 'uninterpretable game of symbols':

Example (Exhibit A)

Errett Bishop claims:

> *"The successful formalization of math helped keep math on a wrong course. . . . Math becomes the* **game of sets***, which is a fine game as far as it goes, with rules that are admirably precise" [Bishop 1967, p. 4].*

More Bishop games

Example (Exhibit B)

"If every mathematician occasionally, perhaps only for an instant, feels an urge to move closer to reality, it is not because he believes mathematics is lacking in meaning. He does not believe that mathematics consists in drawing brilliant conclusions from arbitrary axioms, of juggling concepts devoid of pragmatic content, of playing a **meaningless game**" [Bishop 1967, p. viii].

[3] As is evident from Erhardt's comment on Robinson's would-be reaction to Wiles' proof of Fermat's Last Theorem; see below.

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However, Erhardt does not admit in his article that Robinson himself did *not* describe mathematics as a game.
Erhardt's starts by making an unfounded assumption that Robinson was a finitist; Erhardt compounds his error - of attributing to Robinson the position of narrow finitism[3] - by taking it upon himself to offer advice as to how to practice the latter.

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Robinson's position in context

We saw that Erhardt claimed that Robinson was a finitist based on a truncated passage from Robinson's text *Formalism 64*. Examining Robinson's passage in context reveals a rather different picture. Robinson begins by mentioning the traditional positions of mathematical philosophy (Formalism, Intuitionism, Logicism) on page 228 of his text *Formalism 64*.

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On page 230, he mentions a philosophical school of **nominalism**.[4]

Robinson does not treat nominalism in much detail, on the grounds that to him it represents

> *"little depth from the mathematical point of view" [Robinson 1965, p. 231].*

Robinson claims that to a nominalist, there is not much difference, at the philosophical level, between a set of five objects and an infinite set, as both are 'illusory' to a nominalist:

[4]Today the term *nominalism* may refer to any anti-Platonist philosophy of mathematics. Robinson uses the term in a narrower sense.

Robinson vs nominalists

To a **nominalist**, *the existence of a set of five elements is no less illusory than the existence of the totality of all natural numbers. At the other end of the scale are the so-called* **platonic realists** *or platonists who believe in the* **ideal** *existence of mathematical entities in general, including the existence of transfinite sets of arbitrarily large cardinal numbers to the extent to which they can be introduced at all by means of suitable axioms. [Robinson 1965, p. 230]*

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By the time Robinson gets to the "five particular chairs" (as quoted by Erhardt) at bottom of p. 230, it is clear that his goal is to stress the difference between his position and nominalists'.

- After noting the difference between "five particular chairs" and an infinite set, Robinson emphasizes on p. 231 that describing infinite totalities as 'meaningless' **does not mean** that "such a theory is therefore pointless or devoid of significance."

Really or ideally

Recall that Robinson clarified his adjective 'meaningless' in 1975 by stating that what he has in mind is an absence of a *reference*; namely, the term *infinite totality* does not *refer* to anything in either the real or a Platonic realm. Already in *Formalism 64* he emphasized that he rejected the idea that infinite totalities exist

> *"either really or ideally" [Robinson 1965, p. 230].*

- Here 'really' refers to the physical realm, and 'ideally' to a Platonic realm, as he states explicitly in the passage above.

It emerges that Erhardt's presentation of Robinson's position, as when Erhardt claims that it

> *"places Robinson at odds with typical mathematical practice" [Erhardt 2025, p. 432],*

conflates the *generic* meaning of terms such as *to grasp* and *meaningless*, with the precise *technical* sense attributed to such terms by Robinson.

Fermat's Last Theorem: is the proof legitimate?

The conflation of the generic and the technical sense of the term *meaningless* leads Erhardt to a preposterous misrepresentation of Robinson's position, as when Erhardt claims:

> *Though Fermat's Last Theorem is a general result proven by illegitimate methods and that purports to say something about all natural numbers—constituting an illicit reference to actual infinity—one can derive from it a potentially infinite number of legitimate, material claims. Yet the fact that the proof is* **illegitimate on Robinson's view** *is crucial to our appraisal of his view. [Erhardt 2025, p. 441]*

Erhardt's *illegitimacy* claim stems from his mistaken identification of Robinson as a narrow finitist.

Question

Is Wiles' proof of Fermat's Last Theorem 'illegitimate on Robinson's view' as Erhardt claims?

Foisting an untenable position on Robinson

While Wiles' proof may seem illegitimate to those finitists who view proofs involving infinite totalities as *meaningless* in the generic sense of the term, it is legitimate on Robinson's view, collapsing what Erhardt describes as "our appraisal of his view" due to Erhardt's conflation of distinct meanings of the term *meaningless*. While discussing Robinson's commitment to potential infinity, Erhardt claims that Robinson is a 'devout realist':

> *In opposition to nominalists (and a fortiori, fictionalists), Robinson is a* **devout realist** *with regard to each individual natural number. This is evidenced by his commitment to potential infinity. [Erhardt 2025, p. 438, note 22]*

Robinson himself would have likely rejected such a description of his position. Being able to grasp a concept (such as a potential infinity of metalanguage integers) does not commit one to a reality of a Platonic object. It emerges that Erhardt is trying to foist an untenable position on Robinson.

An alternative: Leibniz's fictionalism

Leibniz's fictionalism offers an alternative to nominalism.
Leibniz treated

- infinitesimals,
- negatives,
- imaginaries

as well-founded fictions; see [Sherry and Katz 2012] in *Studia Leibnitiana*. As is well known, Robinson was influenced by Leibniz.

- Although he sometimes suggested that infinitesimals are eliminable in the manner of Archimedean exhaustion arguments, he made no such suggestion for negatives and imaginaries.
- In all three cases, it is the contribution to systematicity that establishes the mantle 'well-founded fiction', rather than any sort of nominalist reduction.

From Hilbert to Robinson

Already Erhardt's keyword list is symptomatic of a problem with his text: the keyword *finitism* appears, but *formalism* does not. Erhardt misinterprets Robinson's comment about being 'unable to grasp' actual infinite collections, by viewing it as a finitist stance, whereas Robinson merely sought to distance himself from nominalism and Platonism. Erhardt claims to detect a 'tension' between Robinson's philosophy and his mathematical practice:

> *The foundational position he inherited from David Hilbert undermines not only the use of nonstandard analysis, but also Robinson's considerable corpus of* **pre-logic contributions**. . .

We interrupt the quotation here to note that Erhardt's wording here constitutes a historical error. He makes it appear as though Robinson's work in aeronautics, such as his 1956 book *Wing Theory* [Robinson and Laurmann 1956], *preceded* his work in logic. This is incorrect.

Metamathematics prior to Wing Theory

Example

Robinson's dissertation *The metamathematics of algebraic systems* [Robinson 1949], advised by Paul Dienes, dates from 1949.

Erhardt continues:

> *[undermines Robinson's contributions] to the field in such diverse areas as differential equations and aeronautics. This tension emerges from Robinson's disbelief in the existence of infinite totalities . . . [Erhardt 2025, Astract]*

Erhardt is referring to the book *Non-standard Analysis* [Robinson 1966]. However, Erhardt's argument for his claim is too strong because it would apply to all Formalists. Erhardt's Abstract accuses Robinson of "giving up on a commitment to reconciling" his philosophy and his mathematical practice. Possibly a Platonist would tend to view all Formalists in such a fashion. This may betray Erhardt's own philosophical stance (more on this later).

Twin prime conjecture and infinite totalities

In his introduction, Erhardt claims that

> *Even basic conjectures of number theory, such as the twin prime conjecture, presuppose the existence of* **infinite totalities**. *[Erhardt 2025, p. 431]*

However, such a claim involves a conflation of distinct meanings associated with the term *infinite totality*; namely, of different levels of language or theory.

The twin prime conjecture (TPC) can be formulated in Peano Arithmetic (PA).

Example (Twin Prime Conjecture in PA)

TPC can be expressed in PA by the formula on page 442, line 4 in [Erhardt 2025], which we will express as follows:

$$\forall x \, \exists y \, \phi(x, y), \tag{1}$$

where the formula ϕ says that $y > x$ and y and $y + 2$ are prime.

Peano Arithmetic and biinterpretability

Example

PA is a theory that does not know any infinite sets and that is even biinterpretable with the theory obtained from ZF by replacing the axiom of infinity by its negation, and adding $\in$-induction; see [Ackermann 1937]; [Kaye and Wong 2007].

The claim that the TPC presupposes the existence of infinite totalities is incorrect with respect to the object theory PA in which the TPC is formulated (because PA does not know any infinite totalities).

On the other hand, such a claim is true with respect to a metatheory in which PA is interpreted in an appropriate structure and thus given a semantics. On this reading, the term *infinite totality* refers to quantification over an infinite domain. Since formulation (1) involves such quantification, it can be said to involve *infinite totalities* in such a different sense.

Letter to Gödel

The special finitist status of Σ_1^0 and Π_1^0-sentences will be discussed later.

Erhardt's discussion of **Robinson's letter to Gödel** contains further misrepresentations. Erhardt claims the following:

> *In addressing a claim that someone with Gödel's outlook might make – that he can fathom the infinite even if Robinson himself cannot – Robinson states that this objection "does not permit any further direct debate of the issue."*

Contrary to what Erhardt's quotation (taken out of context) suggests, Robinson is perfectly willing to, and does engage in a debate. In his footnote 4, Erhardt similarly misrepresents Robinson's comment "*Hier stehe ich, ich kann nicht anders*" in his letter to Gödel, by quoting it out of context. Robinson's point is not to concede purported brain limitations but rather to reject Gödel's realist assumptions.

From Robinson to Nelson

Robinson gives specific reasons for his reluctance to accept Gödel's realist views concerning the hyperreals, in the following terms:

> *The present evidence for the uniqueness of a non standard ω-ultrapower [such as hyperreal fields] of the reals is not strong. (Robinson to Gödel)*

Robinson is not the only mathematician whose position is misrepresented by Erhardt.

Example (Erhardt's misrepresentation of Nelson)

Erhardt claims that Edward Nelson "believed there is a largest number" [Erhardt 2025, p. 433]. To support his claim, Erhardt cites four texts by Nelson. We examined all four, including Nelson's publication [Nelson 2011], but found no evidence of such a belief as claimed by Erhardt.

Is Formalism an inconsistent project?

Erhardt's claim in [Erhardt 2025, p. 433 note 7] that Nelson "believed Peano Arithmetic was inconsistent" is similarly inaccurate. Rather, Nelson *suspected* that PA was inconsistent. When he thought he had a proof he obviously thought it was inconsistent, but when Tao found an error in the proof, Nelson immediately accepted it.

Similar remarks apply to certain verbal excesses, such as Erhardt's attempt to **tie Formalism to inconsistency**:

> *Robinson has already admitted that such totalities are meaningless, and without an argument defending the use of these languages, formalism becomes an* **inconsistent** *project [Erhardt 2025, p. 435].*

Erhardt does not explain how exactly Formalism would become an 'inconsistent' project in the absence of an argument defending the use of infinite totalities. Where is the inconsistency? His claim appears to be untenable.

Kreisel's unrestrained attack on Cohen and Robinson

Georg Kreisel published his "Observations on popular discussions of foundations" in 1971 [Kreisel 1971].
Reviewer James D. Halpern for *Mathematical Reviews* described Kreisel's "Observations" as "an unrestrained attack on P. J. Cohen . . . and on A. Robinson" and noted that

> *Most readers interested in foundations will probably find the previously mentioned papers of Cohen and Robinson profitable reading, the disparagement of these in the paper under review notwithstanding. [Halpern 1971]*

As noted by Robinson,

> *Bernays, in an article published in Dialectica, criticized my attitude in his usual* **gentle** *manner, while Kreisel had stated his disagreement with me previously, also in his usual manner. [Robinson 1973b, p. 515]*

The reference is respectively to [Bernays 1971] and [Kreisel 1971].

Bernays' objectivity

In a critical response to Robinson's text [Robinson 1969], Paul Bernays claimed that
"there is no fundamental obstacle to attributing **objectivity** *sui generis* to mathematical objects" [Bernays 1971, p. 178].
Robinson appears to have responded in 1975 by including such 'objectivity' of infinitary entities alongside 'reference' as the type of **claim that a Formalist would reject**; see [Robinson 1975, p. 49].

- In his note 3, Erhardt seeks to distance himself from Kreisel's verbal excesses, and describes Kreisel's attack as 'bias' and 'prejudice'. It is therefore disappointing to find Erhardt himself engaging in regrettable verbal excesses.
- Erhardt lodges the following claim concerning Robinson's view of mathematics:
  "He explains non-finitary mathematics as a collection of 'uninterpretable games with symbols' (Robinson 1969a, p. 47)." [Erhardt 2025, p. 439]

Attributing the opponent's position to Robinson

Looking up the original, one finds that Erhardt has applied a technique that is already familiar: he attributes Robinson's opponent's position (in this case, 'the intuitionist') to Robinson himself. For the sake of completeness, we reproduce the full passage from Robinson's text *From a formalist's point of view*:

> *The* **intuitionist** *may believe that the classical mathematican, whatever his underlying philosophy, is wasting his time in developing uninterpretable games with symbols, the sin of the formalist being the greater because he does so deliberately and consciously. [Robinson 1969, p. 47]*

It emerges that, according to Robinson, it is the *intuitionist* (not Robinson himself) who is wont to make pejorative remarks about classical mathematicians and formalists allegedly developing 'uninterpretable games with symbols.' Erhardt has again misrepresented Robinson's position.

Platonism and consistency

Erhardt opens his text with the following sweeping claim concerning 'crucial' realist attitudes among mathematicians:

> *The mathematician tacitly adopts a realist outlook, imagining herself to investigate an* **independent realm** *of mathematical objects ... about which she can discern objective truths. This assumption is* **crucial** *to the way mathematicians engage with their subject. [Erhardt 2025, pp. 429–430]*

Erhardt's 'independent realm' becomes an 'independent reality' by page 444. Postulating an 'independent realm of mathematical objects' is the gist of a Platonist position.

Question

Is being a Platonist 'crucial' for the working mathematician, as postulated by Erhardt?

Atiyah on inherent nonsense

How accurate is Erhardt's description? Sir Michael Atiyah, the recipient of both a Fields Medal and an Abel Prize who was evidently not unsuccessful in "engaging with his subject", had the following to say concerning independent realms:

> *The idea that there is a pure world of mathematical objects (and perhaps other ideal objects) totally divorced from our experience, which somehow exists by itself is obviously inherent nonsense. [Atiyah 2006, p. 38].*

Erhardt's opening remarks set the tone (and level of seriousness) for his 20-page text. In the context of a discussion of independence results (such as the Continuum Hypothesis), Erhardt claims that

> *[Robinson]* **does not consider the possibility** *that even if there are objective answers, they may be beyond our ability to know. [Erhardt 2025, p. 432]*

But Robinson does in fact consider such a possibility:

Robinson on the Continuum Hypothesis

> *If X is the continuum hypothesis, . . . , then we know from the complementary results of K. Gödel and P. Cohen that both X and non-X are compatible with all known "natural" assumptions regarding the universe of sets (to use platonic language). While this suggests to the formalist that the entire notion of the universe of sets is meaningless . . . the platonist merely concludes that the basic and* **commonly accepted properties of the universe of sets which are known to us at present are insufficient to decide the continuum hypothesis** *one way or the other. He will maintain, in this and similar cases, that at any rate only one of the alternatives that offer themselves is the correct one, i.e., is in agreement with the truth. [Robinson 1965, p. 232]*

Robinson goes on to analyze the difficulties of the Platonist position with regard to independence results. Erhardt is in error.

More games, consistency

On page 444, Erhardt repeats the unfounded accusation aimed at Robinson, of “deflating mathematics to a game” [Erhardt 2025, p. 444]. There follows a curious paragraph on page 444 in Erhardt concerning Platonism and the problem of consistency:

> *All of this constitutes Robinson giving up on a serious commitment to his finitism. This is not to say that he should have pursued a definitive argument against platonism, but that there are more and less philosophically sophisticated ways of reconciling mathematical practice with the assertion that we cannot grasp infinite collections.*

Question

Does Erhardt believe that Platonism provides a way out for the problem of consistency?

Second Incompleteness

Erhardt's paragraph continues:

> *As it stands, suggesting that mainstream mathematics continue is tantamount to an admission, despite a reliance on infinitary notions, that Robinson believes it is consistent.*

This is an allusion to Gödel's second incompleteness result, asserting the impossibility of proving Con(PA) (as formalized by Gödel) within Peano Arithmetic itself (and similar results for stronger systems), as Erhardt continues:

> *If not – for instance, if nonstandard analysis led to contradiction – then anything could be proven from it.*

Here's the thing, though:

Question

Wouldn't such a conclusion hold for just about any piece of mathematics, and of any philosophical stance?

Eluding risks

Erhardt continues:

> *Robinson does not venture to explain why he takes infinitary mathematics to be consistent, though a compelling answer might have provided a rationale for accepting the finitary results it produces. In the absence of such an explanation,* **relying upon infinitary mathematics is itself a source of great risk**, *for there is no assurance that the derived concrete implications are valid. [Erhardt 2025, p. 444]*

Erhardt holds that relying on infinitary mathematics is a source of great risk. His tone suggests that Erhardt believes that Platonism is somehow capable of eluding such a 'risk'.

- The reasoning presumably is that if a Platonic realm of mathematics exists, of necessity it could be neither contradictory nor undecidable.

Issues of faith

Erhardt goes on to comment on issues of faith:

> *Robinson is here relying on* **faith** *... if not about the universe of sets, then about basic facts concerning the natural numbers. This hunch is not a sufficient warrant to stand in place of a satisfying mathematical result, like the completion of Hilbert's program, and so his instrumentalism is unjustified.*

We leave the following question to the reader:

Question

Which position involves a greater leap of faith:

1. *relying on infinitary mathematics without a commitment that it possesses a* **reference**, *or*
2. *placing one's faith in a Platonic heaven to ensure the consistency of infinitary mathematics?*

Finally admits

Halfway through his text (p. 10 of 20), Erhardt finally admits that he is familiar with the philosophical concept of *not referring*, when he describes Robinson's approach to non-finitary math as

> *[An] activity that demands no ontological commitments because so many of the terms in its expressions* **do not refer**. *[Erhardt 2025, p. 439].*

Note that Erhardt's clause "do not refer" occurs nowhere in Robinson's texts, and therefore constitutes Erhardt's own paraphrase of Robinson's position. This particular claim of Erhardt's happens to be an accurate description of Robinson's take on the lack of *reference* (in the sense detailed in Section 1),[5] unlike Erhardt's claims regarding Robinson's purported finitism.

[5] To be sure, symbols such as $\mathbb{N}$ and $\mathbb{R}$ can *refer* to the corresponding mathematical entities in the context of a traditional set theory such as ZF. However, Robinson specifically speaks of absence of *reference* to entities in the physical or a putative Platonic realm.

Grand justification wherefore?

Erhardt finds fault with Robinson's strategy as follows:

> *[Robinson] neglects to pursue a* **grand justification** *for infinitary mathematics by – in lieu of attempting to offer 'a satisfactory intellectual motivation' – deflating mathematics to a game. [Erhardt 2025, p. 444]*

The fact that Erhardt's *game* claim involves a misattribution has already been addressed: Robinson didn't say it; rather, he mentioned the intuitionist's use of this pejorative term to criticize both classical mathematicians and formalists.

Let's examine Erhardt's claimed 'neglect' of 'grand justification'.

- The fact is not merely that Robinson had no such intention, but that moreover such a project would have been contrary to his philosophical stance.
- Erhardt fails to appreciate the pragmatic nature of Robinson's philosophical stance, as we now elaborate.

Robinson's enhanced list

In his 1969 text [Robinson 1969], Robinson amplifies his list of two items that he already presented in his *Formalism 64*:

(i) 'infinite totalities' lack reference (in the sense explained in the Introduction), and

(ii) *useful fictions*: a mathematician should continue acting as if they really existed.

In 1969 Robinson adds a third item:

(iii) *against bottomless pragmatism*: there is an identifiable core of logical thought that is common to all of mathematics.

Even more importantly, Robinson uses these three items to eliminate what he diagnoses as **philosophical dead-ends**: items (i), (ii) and (iii) eliminate the positions of, respectively, the platonist, the intuitionist, and the logical positivist.

Incongruity?

It emerges that Robinson is decidedly not in the business of developing 'grand justification' schemes. He is merely ruling out what seem to him to be common philosophical misconceptions, to avoid ending up in a philosophical dead-end. To reproach Robinson for not pursuing a 'grand justification scheme' is to miss entirely the spirit of Robinson's philosophical posture.

Erhardt claims that

> *While Robinson repudiates actual infinity, the mathematics of his career revolves around it. This* **incongruity** *mirrors the move from (i) to (ii); Robinson says the same, noting that* **his position** *suffers a serious drawback because 'the gap due to the absence of consistency proofs for the major mathematical theories appears to be inevitable and we have learned to live with it'. [Erhardt 2025, p. 439]*

But there is no 'incongruity' here, and Robinson certainly did *not* "say the same" as Erhardt claims.

Absence of consistency proofs

This is yet another misrepresentation of Robinson's position, involving a quotation out of context.

The full quotation makes it clear that Robinson viewed 'the absence of consistency proofs' as a difficulty common to *all* schools in the Philosophy of Mathematics, and not merely 'his position' as Erhardt claims.

Thus, Robinson wrote:

> . . . **all known positions** *in the Philosophy of Mathematics, including my own, still involve serious gaps and difficulties. Among these, the gap due to the absence of consistency proofs for the major mathematical theories appears to be inevitable and we have learned to live with it. [Robinson 1965, p. 236]*

Recently the inevitability of such a gap has been challenged in [Artemov 2025].

Types of standardness

- Some logicians are skeptical about the philosophical implications of Artemov's result.
- Regardless of the outcome of that particular debate, there seems to be little reason to assume that incompleteness is less of a problem for Platonism than for Formalism, and Robinson certainly did not claim such a thing.

We now examine some issues related to **types of standardness**, as well as some realists' take on the relation of knowledge and truth.

- We take a closer look at some conflations of terms that lead Erhardt to dubious conclusions about Robinson's work.
- We also examine whether the realism proposed by Erhardt and Gaifman has more convincing answers than Formalism when it comes to questions of mathematical knowledge and applicability.

Standardness and nonstandardness

As already mentioned at the end of Section 4, Erhardt appears to have difficulty keeping apart the generic and the technical meaning of terms like *meaningless* and *to grasp*. Following a discussion of Skolem's nonstandard models and Robinson's nonstandard analysis, involving the *technical* distinction between standard and nonstandard numbers, Erhardt claims the following:

> *"Adding to the irony – and irresolvable tension in Robinson's work and philosophy – is that, due to its predication on the transfinite, Robinson himself says that 'the* **entire notion of standardness must be meaningless** *[...]' (Robinson 1964/1979b, p. 242)." [Erhardt 2025, p. 442 note 33]*

- The irony, according to Erhardt, seems to be that Robinson himself talks 'standard and nonstandard', and then comes out swinging against 'the entire notion of standardness'.

Importance of context

However, the context - not clarified by Erhardt - of Robinson's comment on 'the entire notion of standardness' is a discussion of the Platonist idea of a standard or intended model of arithmetic or set theory, as analyzed in the Introduction.

- To reject such a Platonist idea, Robinson used the term 'standard' in its *generic* sense.
- Meanwhile, Erhardt misleadingly presents it as if what is involved is the *technical* sense used in nonstandard analysis.

In more detail, theories of nonstandard analysis exploit a mathematical concept called the *standardness predicate*, which distinguishes between standard and nonstandard numbers (and more general sets).

Since this distinction can be thought of as a formalisation of Leibniz's distinction between assignable and inassignable numbers, the predicate could be referred to also as the

assignability predicate.

Assignability predicate and Intended Interpretation

Meanwhile, Platonists believe in (and Robinson rejects) the existence of standard models a.k.a. **intended interpretations** of $\mathbb{N}$, $\mathbb{R}$, and other mathematical entities.

When the issue is translated (avoiding the term *standard*) into the terminology of

> *the assignability predicate and the intended interpretation,*

it is obvious that there is no connection between them, contrary to the impression Erhardt seeks to create.

We now turn to the root of the problem:

Question

Is the Gaifman–Erhardt misreading of Robinson due to their own philosophical partis pris?

Varieties of realism

Not all mathematical realists hold identical views. We will examine Erhardt's position in a framework proposed by Hamkins.

In an influential 2012 article, Hamkins provides the following useful perspective:

> *The* **multiverse** *view in set theory ... is the view that there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe. The universe view, in contrast, asserts that there is an absolute background set concept, with a corresponding absolute set-theoretic universe in which every set-theoretic question has a definite answer. The multiverse position, I argue, explains our experience with the enormous range of set-theoretic possibilities, a* **phenomenon that challenges the universe view***. [Hamkins 2012, Abstract]; see also [Hamkins 2021, Section 8.14]*

Hamkins on the Continuum Hypothesis as a *switch*

Hamkins goes on to state his position with regard to the Continuum Hypothesis (CH):

> *In particular, I argue that the continuum hypothesis is settled on the multiverse view by our extensive knowledge about how it behaves in the multiverse, and as a result it can no longer be settled in the manner formerly hoped for. (Ibid.)*

- Hamkins views CH as a "**switch**" that can be turned on and off as it were, by passing to a suitable *forcing extension* of the current instance of a set-theoretic universe.
- Hamkins considers himself a realist of the multiverse.
- Hamkins discards the assumption of the **uniqueness** of a set-theoretic universe as sometimes held by other realists.

Applying the multiverse framework to evaluate Erhardt

We will now examine Erhardt's stance, in the context of Hamkins' dichotomy. Erhardt claims the following:

> *The mathematician* **tacitly adopts a realist outlook**, *imagining herself to investigate an independent realm of mathematical objects – one that includes infinite totalities like the natural numbers – about which she can discern objective truths. [Erhardt 2025, p. 430].*

This passage suggests the following:

Corollary

Erhardt's sympathies are with the realist.

But it does not indicate whether or not this constitutes realism about a *unique* set-theoretic universe.
Reading further in Erhardt, we find the following comment concerning Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) and the CH:

Erhardt on CH

> *Consider Cantor's continuum problem: Is there an intermediate cardinality between the naturals and the reals? If you are a realist about infinite totalities, then the continuum problem is an obvious question to ask. However, ZFC+CH and ZFC+CH [sic]* [6] *are both consistent, which Robinson takes as implying that there must be no universe of sets. A platonist will respond that* **we have simply not yet found** *the axioms stating the basic properties of the universe of sets sufficient to decide CH one way or the other. This is a* **valid objection**, *but it does not sway Robinson. [Erhardt 2025, p. 432 note 5]*

Here Erhardt is sympathetic to the idea that CH ought to possess a definite truth value yet to be determined, and considers such reasoning as 'valid'.

[6] The second occurrence of ZFC+CH is a typo and should be "ZFC+$\neg$CH".

Diagnosis of Erhardt's position

Erhardt's position concerning CH is clearly at odds with Hamkins' multiverse-realism, and specifically with Hamkins' position with regard to CH (namely, that it need *not* possess a definite truth value yet to be determined).

Corollary

Erhardt is a universe-realist.

As argued above, this may color his evaluation of Robinson's philosophical position.

Applicability of mathematics

The far-reaching applicability of mathematics in diverse areas is certainly a famous problem without easy answers. Erhardt claims that

> *[Robinson's] foundational position* **undermines** *not only the use of nonstandard analysis, but also Robinson's considerable corpus of pre-logic contributions to the field in such diverse areas as differential equations and aeronautics [Erhardt 2025, Abstract]*

How would Formalism "undermine" such contributions? And how exactly does the postulation of a Platonic set-theoretic universe would help here?

Question (Platonism and applicability)

How does such a postulation explain the applicability of mathematics in "such diverse areas as differential equations and aeronautics"?

Robinson on the comforting feeling

In other words, how would the putative existence of a mind-independent realm of mathematical abstracta help explain the ability of an embodied brain to apply the said abstracta to the solution of concrete problems of airplane wing design (cf. [Robinson and Laurmann 1956])? It would seem that the difficulty involved is no less challenging than the applicability of what Leibniz already referred to as *useful fictions*.[7]

Robinson proposed an apt diagnosis of the Platonist's aversion to the Formalist's position:

> *It is perhaps natural that a mathematician should resent suggestions which deprive him of the* **comforting feeling** *that he, like the physicist or biologist, spends his life in the exploration of some* **form of reality**. *[Robinson 1969, p. 46].*

[7] See [Katz and Sherry 2013]; [Katz et al. 2024]; [Ugaglia and Katz 2024]; [Kuhlemann 2025].

Robinson on the applicability

As far as the criterion of the applicability of mathematics in the natural sciences, Robinson explains:

> *This criterion does not imply that the terms of the theory should be interpretable directly and in detail. It is sufficient that we should have rules which tell how to apply certain relevant parts of our theory to the empirical world. [Robinson 1965, p. 234].*

Corollary

Infinitary terms may be referenceless, but their applications can nonetheless be meaningful.

Gaifman's realism

We already mentioned that Erhardt appears to derive his ideas about Robinson from Gaifman. [Gaifman 2012] claims to use the term *realism* in the sense of *realism in truth-value* (considered to be more sophisticated than Platonism).
In relation to the natural numbers, this means assuming that every PA sentence is a factual statement, i.e., has an objective truth value (even if it is undecidable in PA itself).
But as Robinson pointed out almost half a century earlier,

> *[A]s a matter of empirical fact the platonists believe in the objective truth of mathematical theorems* **because** *they believe in the objective existence of mathematical entities. [Robinson 1965, p. 230]*

There is internal evidence in Gaifman's article that such is indeed his position, as illustrated by his snide comment about Hamkins and Robinson:

Gaifman's snide comment

> *Another possibility has emerged from the multiverse conception proposed by Hamkins (2011). On this view, the model of natural numbers depends on the set-theoretic universe containing the model. What it comes to is that we have a clear enough conception of models of PA, or of some extensions of PA, but we have no clear distinction between standard and non-standard models.* **Perhaps Abraham Robinson, the founder of non-standard mathematics, who was not a set theoretician, would agree to that.** *[Gaifman 2012, p. 489]*

That wasn't a nurturing remark.

Gaifman's professed truth-value realism appears to be consistent with the belief that the 'standard or intended model of arithmetic' is a meaningful notion.

Recall that Robinson rejects such a notion.

Finitistic semantics, PRA and undecidability

Note that "ϕ is provable" and "T is consistent" are respectively Σ^0_1 and Π^0_1 sentences (when encoded in PA).

Gaifman apparently holds that one commits oneself to realism in truth value (in relation to the natural numbers) when one uses the terms *provable* and *consistent* in metamathematics.

He seems to find it **philosophically unacceptable** to assume that metamathematical statements about the provability of an arithmetic sentence or the consistency of a theory could be undecidable.

Meanwhile, for a formalist practicing finitistic metamathematics in Primitive Recursive Arithmetic (PRA) such a posture toward undecidability is a natural consequence.

'Finitistic semantics' of PRA is naturally *incomplete* [Tait 1981]:

Finitistic semantics, PRA and undecidability II

- A Π^0_1 sentence is 'true' if it can be proved by induction; it is 'false' if one can provide a counterexample.
- A Σ^0_1 sentence is 'true' if one can provide an example; it is 'false' if one can provide an inductive proof of the corresponding Π^0_1 sentence expressing its negation.

See further in [Simpson 2009].

Finitistic semantics of PRA: Similarity to Intuitionism

- Similarly to intuitionism, *truth* in this semantics is defined by provability, and *falsity* by refutability. Consequently, in metamathematics, on the basis of potential infinity, there is no *tertium non datur*.
- Such a position is unacceptable for Gaifman.

Nevertheless, such a metamathematics can prove meaningful statements:

Example

As explained in [Hrbacek and Katz 2021] the conservativity of the theory SPOT over ZF can be proved in WKL_0 (which is not finitistic). But this conservativity result is a Π^0_2 sentence, and according to the results of reverse mathematics [Simpson 2009], there is therefore a proof in PRA, where the result is formulated as $\phi(m, f(m))$ with a primitive recursive function f. Such a formulation is finitistic.

The realist obstruction

The fact that realist positions (like Gaifman's) make it harder to appreciate the hidden resources of the theory of the real number line was noted by Massas in the following terms:

> *The conservativity results in [Hrbacek and Katz]* [8] *would likely fall short of convincing anyone who thinks that any existence claim regarding Robinsonian infinitesimals is simply false. [Massas 2024, p. 263]*

Paul Cohen seemed to regard it as a prerequisite (for his progress in foundational research) to abandon the idea that there is a unique (intended) interpretation of $\mathcal{P}(\mathbb{N})$ that conforms to our intuition:

> *I can assure that, in my own work, one of the most difficult parts of proving independence results was to overcome the* **psychological fear of thinking about the existence of various models of set theory as being natural objects** *in mathematics about which one could use natural mathematical intuition. [Cohen 2002, p. 1072]*

Gaifman's knowledge-transcendent truth

With regard to the relation between independence results and mathematical realism, Gaifman mentions 'knowledge-transcendent truth':

> *If an independence result indicates that some mathematical truths outstrip our capacities of knowing them, then it points to* **knowledge-transcendent truth**, *hence to realism. [Gaifman 2012, p. 498]*

Objectively speaking, the implication would seem to go the other way around: it is only *if* one wishes to defend realism that one may be led to postulate knowledge-transcendent truth. Such 'knowledge-transcendent truth' is mentioned twice in his article (the second occurrence is in the concluding section). It emerges that when Gaifman claims that something points toward realism, what is he has in mind is that it points toward *transcendent truth*. However, Gaifman's argument appears to be circular:

The Full Gaifmanian circularity

Namely, the starting assumption that "independence results indicate that **some truths outstrip our capacity**" itself depends on a realist standpoint.
An anti-realist has little reason to assume that independence results would indicate such a thing. So the full Gaifmanian circularity would run as follows:

$$\begin{aligned} \text{realism} &\Rightarrow \text{independence results indicate that truths outpace our capacities} \\ &\Rightarrow \text{there exist transcendent truths} \\ &\Rightarrow \text{realism.} \end{aligned}$$

Gaifman claims further on page 509:

> *I am mostly concerned with* **independence results for first-order arithmetical statements**. *My general view is that these results do not affect our conception of the standard natural numbers. Hence, they* **reinforce any realism** *that is based on this conception.*

The Full Gaifmanian unfalsifiability

Question

How does such 'reinforcement' work exactly?

Gödel's second incompleteness theorem as applied to PA is a classical arithmetic independence result. Gaifman appears to claim that such a result would "reinforce realism." Accordingly, the independence of Con(PA) of PA would reinforce realism.

1. One wonders what Gaifman would make of Artemov's recent proof of $\mathrm{Con}^S(\mathrm{PA})$ within PA. [9]
2. If even independence reinforces realism, lack of independence (in the sense of Artemov's result) would seem to also reinforce realism.
3. This would suggest that Gaifman's variety of realism is **unfalsifiable**.

[9] [Artemov 2025] proposed an alternative formalisation $\mathrm{Con}^S(\mathrm{PA})$ of consistency following Hilbert. Giaquinto quotes Robinson in [Robinson 1965,

Escape route not taken

Currently, all known undecidable first-order arithmetic sentences have a preferred truth value.
For example, Con(PA) is generally considered to be true (and is of course provable in a stronger system such as ZFC), unlike higher-order statements such as CH.
Some mathematicians view this as an argument in favor of realism about Peano Arithmetic.
However, such an argument does not appear in [Gaifman 2012].

Conclusion

1. While Erhardt goes on eventually to touch on Robinson's mathematical Formalism, his Gaifman-inspired presentation of Robinson's position as *finitism* is off the mark, and his attempt to present Robinson's opponent's position as Robinson's own does not inspire sufficient confidence to give credence to his speculations concerning Robinson's analysis of potential infinity.
2. Erhardt's endorsement of a naive realism about a standard model of ZFC, including his faith in a determinate truth value of CH (yet to be 'discovered'), fails to take into account the recent literature in the philosophy of the foundations of mathematics, such as the multiverse perspective of [Hamkins 2012]; see also [Hamkins 2021, Section 8.14].
3. Due to his misreading of Robinson's terms *meaningless*, *to grasp*, and *standard*, Erhardt sees an irony and an unresolvable tension between Robinson's use of nonstandard models of the real numbers and his statement that "the entire notion of standardness must be meaningless."

Robinson's visionary comparison

However, there is neither irony nor tension here, since Robinson is referring to his previous statement that infinite sets, including both standard and non-standard models, lack reference. He did not mean that they are pointless or devoid of significance.

In his 1969 text, Robinson envisions

> *... a solution which involves the equal acceptance of* **several kinds of set theory**. *At this point, it is natural to recall the historical development of Euclidean geometry whose analogy with the recent development of axiomatic set theory is perhaps closer than is generally appreciated. [Robinson 1969, pp. 46-47]*

The comparison between the existence of distinct set theories and the existence of distinct geometries was pursued in more detail by [Cohen and Hersh 1967]. Different set theories of this type were indeed developed shortly after Robinson's passing.

Axiomatic nonstandard analysis

[Hrbacek 1978] and [Nelson 1977] independently developed axiomatic approaches to nonstandard analysis as an alternative to model-theoretic approaches.

In fact, Robinson's visionary insight is arguably more likely to emerge from a Formalist than a Platonist philosophical stance.

Our conclusion here is at variance with Kreisel's. Kreisel claimed the following:

> *"Having accepted the formalist scheme, one is not merely unwilling to appreciate (or study!) further work in foundations, but simply unable to do so without reexamining one's ideas." [Kreisel 1971, p. 191]*

However, Hrbacek and Nelson were indeed *able* to appreciate, study, and make progress in foundations without adopting Platonist ideas.

Enrichment

While traditional set theory is formulated in the $\in$-language, the axiomatic approaches enrich the language of set theory by means of the introduction of a one-place predicate 'standard' or 'st', and are thus formulated in the

st-$\in$-language.

Such theories are conservative over ZFC. Suitable axioms govern the interaction of the new predicate with the traditional ZFC axioms. In the axiomatic approaches, infinitesimals are found within $\mathbb{R}$ itself (rather than in an extension, as in the model-theoretic approaches to nonstandard analysis), and nonstandard integers in $\mathbb{N}$ itself.
This challenges Platonist notions about $\mathbb{N}$ and $\mathbb{R}$ as determinate (or even mind-independent) entities. Abraham Robinson's Formalism remains a viable alternative to mathematical Platonism.

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THANK YOU!

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