Vopěnka on Visual Thinking in the History of Geometry from Plato to Lobachevski

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In addition to his mathematical work, Petr Vopěnka made remarkable contributions also to history and philosophy of mathematics. Among the texts in this area the book ***Úhelný kámen evropské vzdělanosti a moci*** (*The Cornerstone of European Education and Power*, 2000, 918 pp.) deserves attention. It is a joint edition of the four-part book series ***Rozpravy s geometrií*** (*Conversations with Geometry*) (1989, 1991, 1995a, 1995b). The first volume, called simply ***Conversations with Geometry*** (1989), was published at the end of the previous regime.

Vopěnka had limited opportunities to lecture and publish, and this may have contributed to his decision of attending the summer schools on mathematics education at Chata pod Suchým, organized by Milan Hejný. That is how I met Peter Vopěnka in the early 1980s.

Vopěnka gave evening lectures on various topics. In 1981, these were *The History of Logic*, in 1983 *The Logic of Euclid's Elements*, *The Phenomenon of Size*, or *The Extension of the Geometric World to Infinity*. Actually, they weren't lectures, but reading of manuscripts he was working on at the time.

When I read my notes from the early 1980s today and compared them with the text of the *Conversations with Geometry*, a shift towards phenomenology is obvious at first glance. In the lectures from the 1980s, we have before us a mathematician who strives for precision also in his philosophical reflection. His goal was, for example, to understand how the Greeks used quantifiers.

In the *Conversations*, he strives more for a psychological understanding of the ways of thinking and their connection to the experiential world.

Perhaps the best way how to approach Vopěnka's *Conversations with Geometry* is to see them as an attempt to answer the fundamental question posed by Edmund Husserl in his article *Die Frage nach dem Ursprung der Geometrie als intentional – historisches Problem*. The article was published in the *Revue internationale de Philosophie* in an issue dedicated to the memory of Husserl, who died on April 27, 1938. In his article, as in other works on the philosophy of mathematics, Husserl seeks to anchor ***authentic*** mathematical knowledge in ***perception***, in immediate intuitive contact with the ideal objects that constitute the subject matter of the given mathematical discipline.

In developing phenomenology as a strict method of examining eidetic perception, Husserl made a fundamental contribution to clarifying the mode of givenness of objects in perception—he introduced the concept of ***horizon*** and many other concepts that are of fundamental importance for clarifying geometric intuition.

Unfortunately, in 1936, when, according to Eugene Fink, Husserl was working on the aforementioned text, he no longer had enough mental well-being (as a Jew, he was forbidden from entering the university grounds and thus its libraries). Therefore, the article is an 18-page fragment in which Husserl formulated the question of the origin of geometry, but only outlined the direction he would take to answer it.

Vopěnka's book can be seen as an elaboration of Husserl's project in the form of a comprehensive interpretation of the development of geometry from the perspective of Husserl's phenomenology.

It is no coincidence that Vopěnka returned to Husserl. Husserl was the teacher of Jan Patočka, who brought his work to Bohemia. As one of the members of the Czech dissident movement after the occupation of Czechoslovakia by the Russian army in 1968, Patočka had deep influence on the intellectual orientation of a generation of Czech intellectuals, including Vopěnka.

I will go through the individual books of *Conversations with Geometry* and highlight the ideas that are characteristic of Vopěnka's approach.

1 *Conversations with Geometry* (1989) – 520 pages

The *Conversations with Geometry* begin with a passage that is one of the most remarkable passages in the entire book:

„*A geometer has before him a sheet of paper covered with lines of various shapes, straight and curved, intertwined and intersecting at various points. His eyes rested on the image, but his gaze leads through the image out of the real world, into the geometric world. For example, behind a straight line, he sees a geometric segment, he sees it in its complete purity and, together with it, he sees perfect straightness. From the moment of this insight, for him, a line segment is forever a geometric segment, and not a line drawn with a ruler.*

There were times when we did not know the geometric world. Children who have not yet learned geometry do not know it. The teacher opens this world to them. His task is seemingly impossible, because he cannot show this world, nor can he find enough words to describe it. He can only evoke it in various ways, for example by drawing lines with a ruler and compass and saying that they resemble line segments and circles, but he can only show what they do not resemble.

We can only lead someone part of the way into the geometric world, we can only bring him to its gate, but the decisive step must be taken by each individual... The geometer draws pictures not to study them, but to look into the geometric world through them.“

This quote is fascinating in its truthfulness. It is an articulation of the experience of encountering the world of mathematics. Vopěnka describes what probably each of us experienced in childhood, that fascinating moment when we realized that mathematics touches on something absolute.

The text is an evocation of Plato’s interpretation of mathematics. Vopěnka does not explain Plato’s theory, but evokes it.

He does not approach it from the outside, from the point of view of philosophy, but from the inside, from the mathematical experience hiding behind Plato’s words. The book is an introduction to geometry written from the perspective of a member of the Platonic Academy.

„*No document sufficiently elaborating on Plato's concept of geometry has survived. ... But even though geometry in its purely Platonic sense was probably never practiced, Plato's influence continues to be felt in mathematics to this day. The temptation to revive this concept is therefore extremely seductive, not only because we will be uncovering something long forgotten, but above all because we will be able to better understand what came later. In this chapter, we will make such an attempt. We will draw on both Platonic philosophy and the traces that the Platonic concept of geometry left in Euclid*"

In the following text, Vopěnka explains some geometric concepts and findings from the perspective of Plato's philosophy.

„*When we observe a geometric object, we can see its various accompanying phenomena. For example, one of the observed objects is a line segment, another is curved, yet another is a square, and so on. Line segments, circles, straightness, curvature, and the like are what can be seen on geometric objects, what can be perceived on them; they are certain appearances, or ideas. ... In any given line segment, the idea of a line segment is present; if it were not there, we would not be able to see it there, and therefore we would not be able to record that this object is a line segment. Conversely, we say that any geometric line segment participates in the idea of a line segment.*“

Vopěnka continues:

„Not every accompanying phenomenon can be considered an idea. An idea must be primary. A primary phenomenon is one that we are able to register without any prior knowledge or evidence of it, as soon as we see it, that is, as soon as it is right in front of our eyes. ... In order to be able to perceive non-redness at all, we must first know redness from somewhere else. Non-redness is therefore not immediately perceptible and is therefore not an idea. On the other hand, redness is an idea, because even if we did not know it before, as soon as we see something red for the first time, we will certainly notice it. ... Straightness and crookedness are ideas. On the other hand, non-straightness and non-crookedness are not ideas, because they are not primary. In order to be able to see indirectness, we must first know what straightness is.“

The distinction between phenomena and ideas is an important part of Vopěnka's interpretation of Plato's conception of mathematics.

„*Ideas are constant and unchanging; they are timeless. We do not perceive individual ideas in objects of the geometric or the real world; rather, we see the world of ideas through these objects. We perceive that a given circle is curved. However, this does not mean that only the particular circle we are observing is curved, but that every circle is curved. As soon as we begin to perceive the idea of a circle and the idea of curvature, we also perceive that the idea of curvature is present in the idea of a circle, that is, that a circle would not be a circle without curvature. ... The participation of one idea in another is unchanging, eternal, timeless, and always the same.*"

From interpreting ideas and explaining the participation of one idea in another, Vopěnka moves on to the concept of truth:

„*The seat of truth is not the real world. There we find only its imprints, but not pure truth in its irrevocability. ... The seat of truth is the world of ideas. Participation, this sole relationship between ideas in their world, is precisely what truth is. ... By placing truth in the world of ideas, Plato's philosophy established the subject matter of science. It is the world of ideas that is a common subject matter for all sciences*"

Then comes the influence of Aristotle who refused to place geometric phenomena in the world of ideas. He considered geometric objects to be abstractions of real objects existing only in our imagination.

„According to Aristotle, a geometric curve should be an abstraction of a curve drawn on paper. If we abstract only the shape and size from such a drawn curve, we have no right to neglect, for example, its thickness, even if it is negligible. We cannot obtain an ideal geometric curve as an abstraction of shape and size from any real object. We do not obtain geometric objects from real objects by abstracting their shape and size, but we must abstract further from these abstractions. Of course, we must know how to abstract further; otherwise we could end up with a whole series of abstractions that have nothing to do with geometric objects. For example, we could have round points, while other points could be square, and so on.

We would not have to arrived at line segments at all, because no ruler is perfectly straight, and so it is not at all clear why we should aim for straightness in additional abstractions, why we should not rather have wavy lines instead of line segments, and so on. No drawn circle touches the ruler at a single point, as Protagoras correctly states. However, Aristotle's influence was so immense that, despite the obvious fallacy of his interpretation of the geometric world, this interpretation was accepted for centuries."

In the final chapter Vopěnka explains the difference between the Greek concept of ***apeiron***, meaning something indefinite, unbounded, undefined, and therefore also endless, and the modern concept of ***infinity***, which does not have to be indefinite and undefined.

2 Second Conversations with Geometry – *Illuminating the path to infinity* (1991)

In his second *Conversations with geometry*, Vopěnka focuses on the Middle Ages and puts forward a remarkable thesis, according to which medieval theology transformed European culture's relationship to infinity. While for the ancient Greeks, *apeiron* was a negative concept denoting something indefinite and undefined, and therefore not part of mathematics, after its translation into Latin as *infinity*, under the influence of attributing infinity to God, it became a positive concept that gradually found its way into mathematics.

„God is infinite and eternal and cannot be limited. Infinity belongs to God and God alone. ... The study of infinity thus acquired a lofty meaning; it became part of theology.

The infinity attributed to God himself must be absolute. ... Once infinity was attributed to God, it became clear how it should be interpreted.

***First**, absolute infinity is absolutely perfect. Given how perfection was understood, this means that it is sharp, i.e., definite, unambiguously determined, and unchanging.*

***Second**, absolute infinity is insurmountable; in other words, there is nothing greater.*

***Third**, absolute infinity is unattainable by any created being.*“

According to Vopěnka, this theological interpretation remains valid to this day:

“*We interpret absolute infinity in the same way*. *We still interpret infinity present in the infinity of creating natural numbers or in extending a line segment by a given length. Therefore, we will refer to this interpretation as* ***the classical interpretation***, *and when we talk about* ***absolute infinity***, *we will always understand it in this way*.“

The acceptance of the classical interpretation of infinity led to the expansion of the geometric world into absolute infinity. The interpretation that the geometric world continues beyond the horizon into absolute infinity prevailed.

„The identification of real space with geometric space is one of the characteristic features of modern natural science. According to Aristotle, geometric objects are abstractions of shapes and sizes from real objects. But this means that real space determines the feasibility of geometric objects.

Aristotle is not concerned with interpreting the shapes and sizes of real objects through geometric objects, but rather with interpreting geometric objects through real objects. According to him, geometric objects were separated from real ones, and we can best understand them when we return them to where we separated them from, that is, to the natural real world. ...

The emerging modern natural science quite clearly emphasized the opposite direction in this identification, as it was concerned with the interpretation of the real world. ... It interpreted real space as geometric space, thus identifying it with it. This is nothing unusual; after all, these spaces have been identified since Aristotle's time. It is enough to reverse the direction of this identification."

But with this identification the story did not end.

„*However, modern natural science did not merely equate real space with geometric space, but took a much more serious and far-reaching step; a step that was, in the true sense of the word, fateful. It equated real space with classical geometric space. ...*

By recognizing absolute infinity as the only infinity and accepting its classical interpretation, the natural geometric world was so devalued that no geometer dared to talk about it, even though it was right in front of his eyes. The classical geometric world was considered the only correct geometric world. Classical geometric space is probably the very first phenomenon for which an understanding of absolute infinity and, at the same time, completeness was negotiated and generally accepted.“

The classical geometric world, i.e., the geometric world extending into absolute infinity, had a fundamental influence on the understanding of the universe.

„According to Aristotle, the vault of heaven is the boundary of the real world. It is a sharp, solid, and insurmountable boundary; beyond it, there is nothing. This understanding is therefore largely similar to, for example, the understanding we can form of a boundary such as three-dimensionality; thinking that there is nothing beyond this boundary, i.e., that real space is not embedded in any multidimensional space, does not cause us any difficulty. The real world does not disintegrate into apeira in the distance beyond the horizon, but is enclosed within sharp boundaries. Placing the real world in classical geometric space has opened up an unimaginably vast, absolutely infinite space beyond this boundary.“

3 *Geometrization of the Real World* (Third Conversations with Geometry) (1995)

„*By identifying real space with classical geometric space and immediately inserting the real world into classical geometric space, modern European natural science took a fateful step. It embarked on a path of unexpected successes that boosted the confidence of the European people in their own abilities in an unprecedented way*.“

This identification has several consequences. One of them is the relativity of movement. Vopěnka writes about this:

„*There is no doubt that the interpretation according to which motion and rest are relative is one of the first truly significant consequences of placing the real world in geometric space. In this space, where there is no up or down, right or left, where there are no fixed and permanent places, real space floats without it being possible to say where it is located, whether it is moving or remaining in the same place.*“

But it’s not just about the relativity of motion.

„*Galileo took another step, which was, however, a direct consequence of that fateful step taken by modern natural science. He began to intertwine the real world with the geometric world, so that the phenomena of the geometric world now became real phenomena*. ...

Newton adopted Galileo's insertion of the real world into the geometric world. Just as a geometer saw a perfect geometric line in a straight line drawn in the sand, so now Galileo saw a perfect inclined plane in a street leading downhill. ... It is impossible not to notice certain external similarities between Plato's interpretation of the real world and Galileo's interpretation of the world of modern natural science."

Vopěnka ends his excursion into the history of physics with the words:

„*For us, the most important thing at this moment is that the general theory of relativity has corrected that fateful step in modern natural science, which consisted in placing the real world into classical geometric space*".

4 *The opening of non-Euclidean geometric worlds* (1995)

In the final part of the *Conversations with Geometry* Vopěnka describes the emergence of non-Euclidean geometry.

„*It is possible to imagine only one space, Kant claims. This assertion is the cornerstone of Kant's resolution of the dispute between empiricism and rationalism. If it is correct, the Renaissance did not equate real space with geometric space, but only realized that in both cases it is the same space. The necessities of classical geometric space are therefore the necessities of real space, and knowledge of them is knowledge of the laws of the real world.*“

Vopěnka confronts the non-Euclidean geometry with Kant.

„*It is possible to imagine different spaces. Several bold thinkers came up with this claim in the first half of the nineteenth century. ... In any case, the discovery of non-Euclidean geometric worlds and the successful development of their corresponding non-Euclidean geometries was the first serious challenge to Kant's solution to the above-mentioned dispute*“.

Vopěnka's starts with Girolamo Saccheri (1667–1733), whose book *Euclides ab omni naevo vindicatus* (Euclid stripped of all blemishes) was published in 1733. It was almost 50 years before Immanuel Kant (1724–1804) published the *Critique of Pure Reason* (1781). Saccheri believed that he had succeeded in proving Euclid's fifth postulate and demonstrating the impossibility of any geometry other than Euclid's.

On the way to his supposed proof, however, he discovered a large number of theorems of non-Euclidean geometry. Vopěnka presents fifteen propositions from Saccheri's work with proofs, to introduce the reader to the world of non-Euclidean geometry. He then provides quotations from the correspondence of Schweikart, Gauss, Bolyai, and Lobachevsky, each of whom came independently to the conclusion that non-Euclidean geometry is possible.

„*The hyperbolic geometric world has thus been discovered, and we naturally wonder what Gauss, Schweikart, Bolyai, and Lobachevsky, to whom the discovery of this world is attributed, actually discovered, and whether they discovered anything at all. After all, this world was known long before them; Saccheri, Lambert, and many others knew about it.*

None of the above-mentioned discoverers presented any convincing evidence that this world would not collapse. ... Judged by the standards of objective science, we cannot credit the above-mentioned thinkers with anything more than the discovery of a new attitude towards this world.

„*The discovery of a new attitude toward certain phenomena is also a discovery. In a different attitude, we see the same phenomena differently, and thus, strictly speaking, we see different phenomena. ... The various insights that Gauss, Bolyai, and Lobachevsky gained about the hyperbolic geometric world are therefore only a secondary and less significant part of their discovery. Any experienced geometer who shared their attitude towards this world could have replaced them in acquiring this knowledge*“.

5 Closing remarks

Several fundamental insights can be found in Vopěnka's interpretation of the history of geometry.

1. An interpretation of Plato's theory of ideas as a theory that accurately and faithfully describes authentic geometric experience.

2. An interpretation of Christian theology as a historical factor that changed mathematicians' attitudes toward the concept of infinity.

3. An introduction of classical infinity as something distinct from potential infinity and actual infinity, and the demonstration of its role in the interpretation of geometric space.

4. An interpretation of the Renaissance as a period during which geometric space was interpreted as absolutely infinite.

5. A characterization of the insertion of the real world into absolute geometric space as a key moment in the emergence of modern science.

6. An interpretation of the emergence of non-Euclidean geometry as the emergence of a new attitude of mathematicians towards the world of non-Euclidean geometry.

Thank you